Algebraic L-theory II

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The surgery obstruction?

Recall

The group $L_n(R)$ is the cobordism group of n-dimensional quadratic Poincaré complexes.

Question

How to associate such a complex with a degree one normal map without doing preliminary surgery below middle dimension?

Idea

Use stable homotopy theory.

Remark

The idea goes back to Browder, see III.4 in Surgery on simply-connected manifolds.

The suspension maps I

Question

What is the behaviour of the structures under suspension?

Problem

Given an n-dimensional symmetric complex (C, φ),
is there an $(n+1)$-dimensional symmetric complex $S(C, \varphi)$
which corresponds to the geometric situation?

Geometry

Given an n-dimensional symmetric complex $(C(X), \varphi_X[X])$,
there is an $(n+1)$-dimensional symmetric complex $(C(\Sigma X), \varphi_{\Sigma X}\Sigma[X])$.

Idea

Use products

The suspension maps II

Recall the 1-chain $\omega \in W^{\%}(I)$ alias $\omega \colon I \to W^{\%}(I)$ from Talk 1.

Given $\varphi \in W^{\%}(C)$ we have $\varphi \otimes \omega \in W^{\%}(C \otimes I)$.

Consider the collapse map $c \colon C \otimes I \to \Sigma C$

Recall the homotopy invariance map

$$W^{\%}(C) \otimes I \xrightarrow{-\otimes\omega} W^{\%}(C) \otimes W^{\%}(I) \xrightarrow{-\otimes-} W^{\%}(C \otimes I) \xrightarrow{c^{\%}} W^{\%}(\Sigma C)$$

It factors as

$$W^{\%}(C) \otimes I \to \Sigma W^{\%}(C) \xrightarrow{S} W^{\%}(\Sigma C)$$

The other products give the suspension maps:

$$S \colon \Sigma W^{\%}(C) \to W^{\%}(\Sigma C)$$
$$S \colon \Sigma W_{\%}(C) \to W_{\%}(\Sigma C)$$
$$S \colon \Sigma \widehat{W}^{\%}(C) \to \widehat{W}^{\%}(\Sigma C)$$

Properties of the suspension maps I

Proposition

For every space X there exists a natural chain homotopy Γ_X as follows:

$$\begin{array}{ccc} \Sigma C(X) & \xrightarrow{S \circ \Sigma(\varphi_X)} & W^{\%}(\Sigma C(X)) \\ \Sigma \downarrow & \overset{\Gamma_X}{\rightsquigarrow} & \downarrow \Sigma^{\%} \\ C(\Sigma X) & \xrightarrow[\varphi_{\Sigma X}]{} & W^{\%}(C(\Sigma X)) \end{array}$$

Properties of the suspension maps I

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We have (recall cup products on suspensions):

$$\begin{aligned} (S\varphi)_{s+1} &= \pm\varphi_s \quad (\varphi_{-1} = 0) \\ (S\psi)_s &= \pm\psi_{s+1} \\ (S\theta)_{s+1} &= \pm\theta_s \end{aligned}$$

Properties of the suspension maps II

Proposition

The suspension map

$$S : \Sigma \widehat{W}^{\%}(C) \to \widehat{W}^{\%}(\Sigma C)$$

is a chain equivalence.

Proof.

Let

$$0 \to C \xrightarrow{f} D \xrightarrow{e} D/\mathrm{im} f \to 0$$

be a cofibration sequence. The failure of

$$C \otimes C \xrightarrow{f \otimes f} D \otimes D \xrightarrow{e \otimes e} D/\mathrm{im} f \otimes D/\mathrm{im} f$$

to be short exact is $\ker e \otimes e/\mathrm{im} f \otimes f \neq 0$ which is coinduced. □

Properties of the suspension maps III

Corollary

We have a chain homotopy equivalence

$$\widehat{W}^{\%}(C) \simeq \mathrm{hocolim}_{k \to \infty} \Sigma^{-k} W^{\%}(\Sigma^k C)$$

Proof.

Study

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W^{\%}(C) & \longrightarrow & \widehat{W}^{\%}(C) & \longrightarrow & \Sigma W_{\%}(C) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle S^k} & & \downarrow{\scriptstyle S^k} & & \downarrow{\scriptstyle S^k} & & \\
0 & \longrightarrow & \Sigma^{-k} W^{\%}(\Sigma^k C) & \longrightarrow & \Sigma^{-k} \widehat{W}^{\%}(\Sigma^k C) & \longrightarrow & \Sigma^{-k+1} W_{\%}(\Sigma^k C) & \longrightarrow & 0
\end{array}$$

and observe

$$\mathrm{hocolim}_{k \to \infty} \Sigma^{-k} W_{\%}(\Sigma^k C) \simeq 0$$

□

The quadratic construction I

Recall

We have the cofibration sequence

$$W_{\%}(C) \longrightarrow W^{\%}(C) \longrightarrow \widehat{W}^{\%}(C)$$

And we know

$$\widehat{W}^{\%}(C) \simeq \operatorname{hocolim}_{k\to\infty} \Sigma^{-k} W^{\%}(\Sigma^k C)$$

Recipe for quadratic cooking

To construct a quadratic structure it is enough to construct a symmetric structure and find a reason for its vanishing after some suspension.

The quadratic construction II

Important: The symmetric construction φ_X is natural with respect to maps of spaces, but not with respect to arbitrary chain maps.

Consider $F\colon \Sigma^p X \to \Sigma^p Y$ for some $p \geq 0$
We have the equation

$$F^{\%} \circ \varphi_{\Sigma^p X} - \varphi_{\Sigma^p Y} \circ F_* = 0$$

but when we consider the composition chain map

$$f\colon C(X) \xrightarrow{\Sigma^p} \Sigma^{-p} C(\Sigma^p X) \xrightarrow{F_*} \Sigma^{-p} C(\Sigma^p Y) \xrightarrow{\Sigma^{-p}} C(Y)$$

then we may have

$$f^{\%} \circ \varphi_X - \varphi_Y \circ f \neq 0.$$

The quadratic construction III

Therefore:

Given a cycle $[X] \in C_n(X)$ we have a cycle

$$(f^{\%} \circ \varphi_X - \varphi_Y \circ f)[X] \in W^{\%}(C(Y))$$

such that

$$S^p(f^{\%} \circ \varphi_X - \varphi_Y \circ f)[X] \sim$$
$$(F^{\%} \circ \varphi_{\Sigma^p X} - \varphi_{\Sigma^p Y} \circ F_*)\Sigma^p[X] \sim 0 \in W^{\%}(C(\Sigma^p Y))$$

Specifying the null-homotopy (recall the recipe for quadratic cooking) produces the quadratic construction map

$$\psi_F \colon H_n(X) \to Q_n(C(Y)).$$

The quadratic construction IV

More details:

$$
\begin{array}{ccccccc}
\Sigma^{-p} C(\Sigma^p X) & & & \longrightarrow & & & \Sigma^{-p} C(\Sigma^p Y) \\
 & \nwarrow & & & & \nearrow & \\
\Big\downarrow & & C(X) & \longrightarrow & C(Y) & & \Big\downarrow \\
 & & \Big\downarrow & & \Big\downarrow & & \\
 & & W^{\%}(C(X)) & & W^{\%}(C(Y)) & & \\
 & \swarrow & & & & \searrow & \\
\Sigma^{-p} W^{\%}(C(\Sigma^p X)) & & & \longrightarrow & & & \Sigma^{-p} W^{\%}(C(\Sigma^p Y))
\end{array}
$$

S-duality I

Question

Degree one normal map $\rightsquigarrow$ stable homotopy theory?

Idea

S-duality

Definition

Let X, Y be pointed spaces. A map $\alpha \colon S^N \to X \wedge Y$ is an N-dimensional S-duality map if the slant product maps

$$\alpha_*([S^N]) \backslash - \colon \tilde{C}(X)^{N-*} \to \tilde{C}(Y) \quad \text{and} \quad \alpha_*([S^N]) \backslash - \colon \tilde{C}(Y)^{N-*} \to \tilde{C}(X)$$

are chain equivalences. We say the spaces X, Y are S-dual.

S-duality II

Example

Let X be an n-dimensional geometric Poincaré complex with the k-dimensional Spivak normal fibration $\nu_X \colon X \to \mathrm{BSG}(k)$. Then $\mathrm{Th}(\nu_X)$ is an $(n+k)$-dimensional S-dual to X_+.

Proof.

Embed $X \subset \mathbb{R}^N$ with the regular neighborhood $X \subset W \subset \mathbb{R}^N$.
Consider the collapse $\rho \colon S^N \to W/\partial W \simeq \mathrm{Th}(\nu_X)$
Take

$$\alpha \colon S^N \xrightarrow{\rho} \mathrm{Th}(\nu_X) \simeq \frac{W}{\partial W} \xrightarrow{\Delta} \frac{W \times W}{W \times \partial W} \simeq \mathrm{Th}(\nu_X) \wedge X_+$$

Then

$$\alpha_*([S^N]) \backslash - \simeq ([X] \cap -) \circ (u(\nu_X) \cap -) \colon C^{N-*}(\mathrm{Th}(\nu_X)) \to C(X)$$

□

S-duality III

Proposition

1. *For every finite CW-complex X there exists an N-dimensional S-dual, which we denote X^*, for some large $N \geq 1$.*
2. *If X^* is an N-dimensional S-dual of X then its reduced suspension ΣX^* is an $(N+1)$-dimensional S-dual of X.*
3. *For any space Z we have isomorphisms*

$$S\colon [X, Z] \cong [S^N, Z \wedge X^*] \quad \gamma \mapsto S(\gamma) = (\gamma \wedge \mathrm{id}_Y) \circ \alpha,$$
$$S\colon [X^*, Z] \cong [S^N, X \wedge Z] \quad \gamma \mapsto S(\gamma) = (\mathrm{id}_X \wedge \gamma) \circ \alpha.$$

4. *A map $f\colon X \to Y$ induces a map $f^*\colon Y^* \to X^*$ for N large enough via the isomorphism*

$$[X, Y] \cong [S^N, Y \wedge X^*] \cong [Y^*, X^*].$$

The quadratic signature of a degree one normal map I

A degree one normal map $(f, b)\colon M \to X$ of Poincaré complexes $\rightsquigarrow$

$$\mathrm{Th}(b)\colon \mathrm{Th}(\nu_M) \to \mathrm{Th}(\nu_X) \rightsquigarrow$$

$$F := \mathrm{Th}(b)^*\colon \Sigma^k X_+ \to \Sigma^k M_+$$

The map F induces the Umkehr map

$$f^!\colon\, C(\tilde{X}) \to \Sigma^{-k} C(\Sigma^k \tilde{X}_+) \xrightarrow{F} \Sigma^{-k} C(\Sigma^k \tilde{M}_+) \to C(\tilde{M})$$

The quadratic construction gives an n-dimensional quadratic structure

$$\psi_F([X]) \in Q_n(C(\widetilde{M}))$$

The quadratic signature of a degree one normal map II

Consider the mapping cone $\mathcal{C}(f^!)$ with $e\colon C(\tilde{M}) \to \mathcal{C}(f^!)$.

Definition/Proposition

The n-dimensional quadratic complex over $\mathbb{Z}[\pi_1(X)]$

$$(\mathcal{C}(f^!), e_{\%}\psi_F([X]))$$

is Poincaré and defines the quadratic signature of (f, b)

$$\mathbf{sign}^{\mathbf{L}\bullet}(f, b) \in L_n(\mathbb{Z}[\pi_1(X)])$$

Remark

The quadratic signature $\mathbf{sign}^{\mathbf{L}\bullet}(f, b)$ coincides with the surgery obstruction defined in Spiros' lectures, but the proof is beyond the scope of these lectures. See Ranicki, The algebraic theory of surgery II.

Algebraic surgery I

Recall

The groups $L_n(R)$ are the cobordism group of n-dimensional quadratic Poincaré complexes.

Question

How to show that these are the same as the groups defined via forms and formations?

Ideas

1. Use algebraic surgery to replace (C, ψ) by (C', ψ') with C' highly connected.
2. Show that highly connected complexes correspond to forms/formations and highly connected cobordisms of such complexes yield the same equivalence relations as are those given by hyperbolic forms/boundary formations.

Algebraic surgery II

We concentrate on the symmetric case, the quadratic case is analogous.

The starting data in geometric surgery is $x\colon S^k \times D^{n-k} \hookrightarrow M$.

The effect of the surgery on M along x is

$$M' = (M \smallsetminus \mathrm{int}(x)) \cup D^{k+1} \times S^{n-k-1}$$

and the trace of the surgery is the cobordism

$$W = M \times [0,1] \cup D^{k+1} \times D^{n-k}$$

between M and M'.

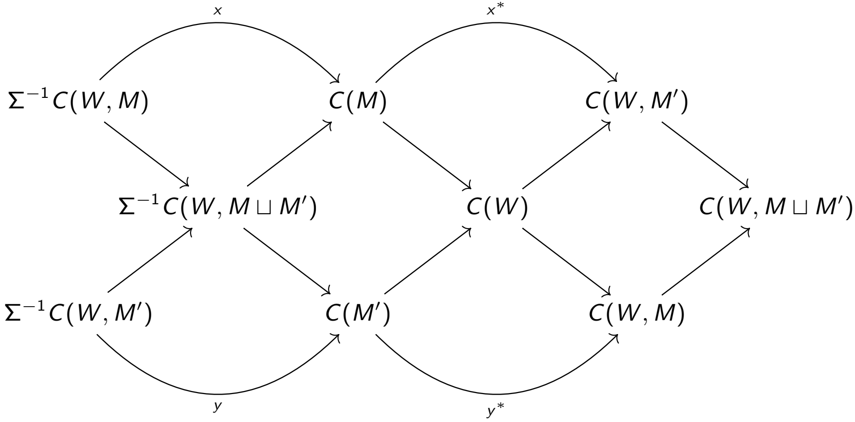
From the homotopy theoretic point of view we have that

$$M \cup_x D^{k+1} \simeq W \simeq M' \cup_y D^{n-k}$$

for some embedding $y\colon D^{k+1} \times S^{n-k-1} \hookrightarrow M'$.

Algebraic surgery III

Remember: $M \cup_x D^{k+1} \simeq W \simeq M' \cup_y D^{n-k}$



Algebraic surgery IV

Ideas

1. Cone:
$$C_*(W, M \sqcup M') \simeq \mathcal{C}(x^* \colon C(M) \to C(W, M'))$$
2. Duality:
$$- \cap [W] \colon C^{n+1-*}(W, M') \xrightarrow{\simeq} C_*(W, M)$$
3. Fiber:
$$C(M') \simeq \Sigma^{-1}\mathcal{C}(\mathrm{incl} \circ (- \cap [W]) \colon C^{n+1-*}(W, M') \to C_*(W, M \sqcup M'))$$

Algebraic surgery V

Let (C, φ) be an n-dimensional symmetric Poincaré complex.

The starting data for an algebraic surgery on (C, φ) is an $(n+1)$-dimensional symmetric pair $(f \colon C \to D, (\delta\varphi, \varphi))$.

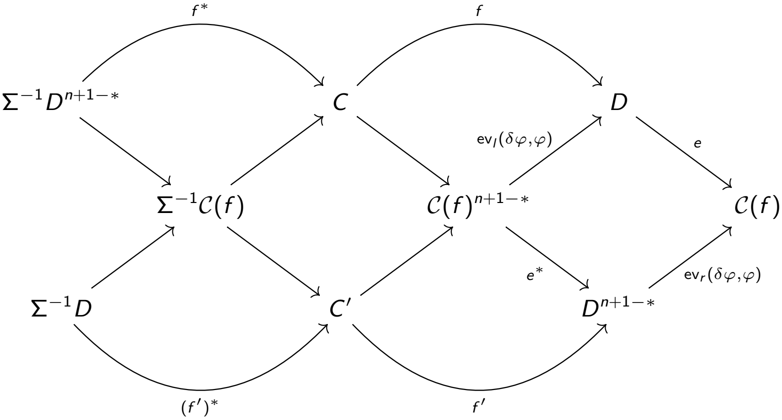
The effect of the algebraic surgery on (C, φ) along $(f \colon C \to D, (\delta\varphi, \varphi))$ is the n-dimensional symmetric complex (C', φ') defined as follows:

The underlying chain complex is defined by

$$C' = \Sigma^{-1}\mathcal{C}(\mathrm{ev}_r(\delta\varphi, \varphi))$$

so that we have the diagram

Algebraic surgery VI



The Algebraic Thom Construction I

To define the structure φ' is harder.

We need the algebraic Thom construction and its reverse.

Let $f : C \to D$ be a chain map.

Question

$\mathcal{C}(f^{\%})$ versus $W^{\%}(\mathcal{C}(f))$?

Remark/Idea/Slogan

This is like cap products in (X, A) versus cap products in X/A.

The Algebraic Thom Construction II

We have a cofibration sequence of chain complexes

$$C \xrightarrow{f} D \xrightarrow{e} \mathcal{C}(f), \qquad j : ef \simeq 0 \; (C \to \mathcal{C}(f))$$

which induces

$$\begin{array}{ccccc} W^{\%}(C) & \xrightarrow{f^{\%}} & W^{\%}(D) & \longrightarrow & \mathcal{C}(f^{\%}) \\ & & & \searrow^{e^{\%}} & \Big\downarrow{\scriptstyle \Phi_{j^{\%}}} \\ & & & & W^{\%}(\mathcal{C}(f)) \end{array}$$

$$\Phi_{j^{\%}} : (\delta\varphi, \varphi) \mapsto e^{\%}(\delta\varphi) + j^{\%}(\varphi) =: \delta\varphi/\varphi. \tag{1}$$

The Algebraic Thom Construction III

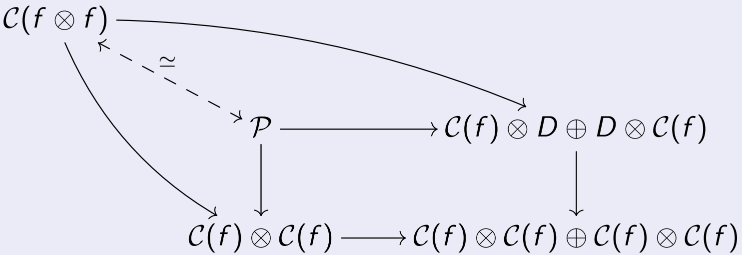
Proposition

We have a homotopy equivalence

$$\mathcal{C}(f^{\%}) \simeq W^{\%}(\mathcal{C}(f)) \times_{\mathcal{C}(f)\otimes\mathcal{C}(f)} D \otimes \mathcal{C}(f)$$

Proof.

Apply $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, -)$ to the diagram



The algebraic Thom construction IV

An $(n+1)$-dimensional symmetric pair $(f\colon C \to D, (\delta\varphi, \varphi))$ gives:

$$\begin{array}{ccccccc}
C^{n-*} & \xrightarrow{\partial^*} & \mathcal{C}(f)^{n+1-*} & \xrightarrow{e^*} & D^{n+1-*} & \xrightarrow{f^*} & C^{n+1-*} \\
\downarrow{\scriptstyle \mathrm{ev}(\varphi)} & & \downarrow{\scriptstyle \mathrm{ev}_l(\delta\varphi,\varphi)} & \searrow{\scriptstyle (\delta\varphi,\varphi)_0} & \downarrow{\scriptstyle \mathrm{ev}_r(\delta\varphi,\varphi)} & & \downarrow{\scriptstyle \mathrm{ev}(\varphi)} \\
C & \xrightarrow[f]{} & D & \xrightarrow[e]{} & \mathcal{C}(f) & \xrightarrow[\partial]{} & \Sigma C
\end{array}$$

and

$$W^{\%}(C) \to W^{\%}(D) \to \mathcal{C}(f^{\%}) \xrightarrow{\text{proj}} \Sigma W^{\%}(C) \to \Sigma W^{\%}(D)$$

with

$$\text{proj}\colon W^{\%}(\mathcal{C}(f)) \times_{\mathcal{C}(f)\otimes\mathcal{C}(f)} D \otimes \mathcal{C}(f) \to \Sigma W^{\%}(C)$$
$$(\delta\varphi/\varphi, (\delta\varphi/\varphi)_0, \mathrm{ev}_r((\delta\varphi,\varphi)) \mapsto \Sigma(\varphi)$$

Algebraic surgery VII

The cofibration sequence

$$C' \xrightarrow{f'} D^{n+1-*} \xrightarrow{\mathrm{ev}_l(\delta\varphi,\varphi)} \mathcal{C}(f)$$

gives

$$W^{\%}(C') \to W^{\%}(D^{n+1-*}) \to \mathcal{C}((f')^{\%}) \xrightarrow{\mathrm{proj}} \Sigma W^{\%}(C') \to \Sigma W^{\%}(D^{n+1-*})$$

with

$$\mathrm{proj} \colon \mathcal{C}((f')^{\%}) \simeq W^{\%}(\mathcal{C}(f)) \times_{\mathcal{C}(f) \otimes_S \mathcal{C}(f)} D^{n+1-*} \otimes \mathcal{C}(f) \to \Sigma W^{\%}(C')$$

Note

$$(\delta\varphi/\varphi, (\delta\varphi/\varphi)_0, e) \in \mathcal{C}((f')^{\%})$$

Define

$$\varphi' = \Sigma^{-1}\mathrm{proj}(\delta\varphi/\varphi, (\delta\varphi/\varphi)_0, e).$$

Finally get

$$(C' \xrightarrow{f'} D^{n+1-*}, (\delta\varphi', \varphi')).$$

Algebraic surgery VIII

Proposition

Let (C, φ) *be an n-dimensional symmetric Poincaré complex and let* $(C \xrightarrow{f} D, (\delta\varphi, \varphi))$ *be data for algebraic surgery with the effect the n-dimensional symmetric Poincaré complex* (C', φ')*. Then there exists a cobordism*

$$(C \oplus C' \xrightarrow{g\ g'} E, (\delta\varphi_E, \varphi \oplus -\varphi'))$$

between (C, φ) *and* (C', φ')*.*

Proposition

The equivalence relation generated by surgery and homotopy equivalence is the same as the equivalence relation given by cobordism.

The boundary

Definition/Proposition

Let (C, φ) be an n-dimensional symmetric complex.

View it as an n-dimensional symmetric pair $(0 \to C, (0, \varphi))$.

Let $(\partial C, \partial \varphi)$ denote the effect of algebraic surgery on th $(n-1)$-dimensional symmetric Poincaré complex $(0, 0)$ along $(0 \to C, (0, \varphi))$.

Then $(\partial C, \partial \varphi)$ is an $(n-1)$-dimensional symmetric Poincaré complex, it is called the boundary of (C, φ).

Note that $\partial C = \Sigma^{-1} \mathcal{C}(\varphi_0 \colon C^{n-*} \to C)$.

There exists $\overline{\varphi} \in W^{\%}(C^{n-*})$ such that

$$(\partial C \to C^{n-*}, (\overline{\varphi}, \partial \varphi))$$

is an n-dimensional symmetric Poincaré pair.