**Seminar S4D2** **Wintersemester 2012/13**

# Cohomology of groups

Wednesdays, 10:15 — 12:00 Uhr, Seminarraum 1.007

Prof. Dr. C.-F. Bödigheimer, Dr. T. Macko

**Organizational meeting:** Monday, 09.07.2012, 17:15 - 18:00, Hausdorff-Raum
**Signing-in (also after the organizational meeting):** via instructors

This seminar is an introduction to the (co-)homology theory of discrete groups. To any natural number $n = 0, 1, \ldots$ and any group $G$ we associate abelian groups $H_n(G)$ and $H^n(G)$ in a functorial way. Like the homology groups for spaces, these also satisfy certain axioms. The cohomology of a group $G$ arizes whenever we study an operation of $G$ on a space or a module and we are interested in the fixed point sets of the action or the quotients of the action.

Prerequisites for the seminar are courses Topologie I and II.

Literature: We will be mostly using the book of K. Brown [Bro94], but also other sources as described for each talk in detail.

The talks are supposed to be 90 minutes. That means, prepare ca. 70 minutes and be prepared for questions during the talk. The talks should be discussed with the instructors two weeks before the talk date.

## Talks

(1) **Definition of $H_*(G)$ and $H^*(G)$** ............ Anna Hermann
10.10.2012
Group rings, $G$-modules, $G$-invariants and $G$-coinvariants, projective resolutions, the functors Tor and Ext. Definition of the homology $H_n(G; M)$ as the $n$-th derived functor of $G$-coinvariants $M \mapsto M_G$. Definition of the cohomology $H^n(G; M)$ as the $n$-th derived functor of $G$-invariants $M \mapsto M^G$. In particular we obtain a definition for "trivial coefficients", i.e. $M = \mathbb{Z}$ with the trivial operation.
[Bro94, III], [Wei94, 6.1], [HS97, VI.2]

(2) **Classifying spaces of groups** ............ Britta Kleberger
17.10.2012
For any discrete group $G$ there is a well-defined homotopy type $BG$, sometimes also denoted $K(G, 1)$. Homotopy theoretic definition and characterization of $BG$ (the general construction is postponed to Talk 5). Geometric examples: graphs (free groups), tori (finitely generated free abelian groups), infinite lens spaces (finite cyclic groups), some knot complements. Classification theorem for regular $G$-coverings, the bijection $[X, BG] = \mathrm{Hom}(\pi_1(X), G))$.

[Hat02, 1.B], [Bro94, I.Appendix]

(3) **Group homology and classifying spaces** .................................... Tilman Becker
24.10.2012
Idenitification $H_*(G;\mathbb{Z}) = H_*(BG;\mathbb{Z})$. Here the left hand side is the group homology from Talk 1 and the right hand side is the singular homology of the classifying space $BG$. Local coefficients on spaces and covering spaces. Borel construction. Identification of $H_*(G;M) = H_*(BG;\underline{M})$ with $\underline{M}$ the local coefficients system over $BG$ associated to $M$. Homology of a covering space. Homology with local coefficients for manifolds, the existence of the fundamental class.
[Bro94, I.4, III.1], [Hat02, 3.H]

(4) **Homology and cohomology of the cyclic groups** ................................ Malte Leip
31.10.2012
Norm-element and periodic resolutions. Calculation of $H_n(\mathbb{Z}/m;M)$ and $H^n(\mathbb{Z}/m;M)$ for trivial and non-trivial coefficients. Application: if $BG$ is finite-dimensional then $G$ is torsionfree.
[Bro94, I.6], [Eve91, 2.1], [HS97, VI.7], [Wei94, 6.2]

(5) **Bar resolution and** $BG$ .................................................. Frank Zickenheiner
07.11.2012
In this talk a general construction for the space $BG$ should be presented, the bar resolution, homogeneous and inhomogeneous version. Normalization. Comparison with the cellular chain complex of $EG$ and $BG$. Application: rational homology $H_n(G;\mathbb{Q}) = 0$ for finite groups $G$ and $n > 0$. Discuss the functoriality of $BG$.
[Bro94, II.3,6.], [Hat02, 1.B], [Wei94, 6.5], [Eve91, 2.3]

(6) **Interpretation of** $H_1(G)$**,** $H_2(G)$ **and** $H^2(G)$ ............................. Emanuel Reinecke
14.11.2012
Abelianization of $G$. Hopf-formula for $H_2(G)$. Extensions and their classification via $H^2(G)$.
[Bro94, II.5, Excs 1], [HS97, VI.4, VI.9+10]

(7) **Mayer-Vietoris sequence for amalgamated products** .................. Sebastian Kremer
21.11.2012
For some groups $G$ it is possible to decompose $BG$ into classifying spaces of easier groups and use the Mayer-Vietoris sequence to calculate the homology of a group. Free products and free amalgamated products. Examples: free groups, groups acting on trees, nice gemoetric example $\mathrm{SL}_2(\mathbb{Z})$.
[Bro94, II.7 + Appendix], [Wei94, 6.2], [HS97, VI.8+14], [Eve91, 2.2], [Ser03]

(8) **Products, universal coefficient and the Künneth theorems** ... Katarzyna Jankiewicz/ Tomasz Przezdziecki
28.11.2012
The cross-product in the homology and the cup-product in the cohomology. The cohomology ring $H^*(G)$. Example: the cohomology ring of the cyclic groups. Example for the Künneth sequence: $G = \mathbb{Z}/m \times \mathbb{Z}/l$.
[Bro94, V.1-4], [Eve91, 3], [Bro94, III.1: Exc 3: V.5], [HS97, VI.15]

(9) **Pontrjagin product** ......................................................... Fabian Henneke
05.12.2012
Shuffle-product for the homology of abelian groups. Application: Homology of finitely-generated abelian groups.
[Bro94, V.5+6]

(10) **Restriction, induction and transfer** ........................................ Anna Hermann
12.12.2012
Restriction to subgroups, induction of coefficient modules. Transfer. Cartan-Eilenberg double-coset formula. Application: detection of cohomology by the Sylow subgroups. Example: symmetric groups.
[Bro94, III.3.,5.,6.,9.,10], [HS97, Vi.16], [AM04, II.5+6]

(11) **Spectral sequences I: basics** .................................................... Felix Boes
19.12.2012
Basic definitions. Spectral sequence of a double complex. Spectral sequence of a filtered complex. Application: Gysin seqeunce, Wang sequence. Example: Künneth spectral sequence.
[Bro94, VII], [Wei94, 5.1-5.6], [Eve91, 7]

(12) **Spectral sequences II: Lyndon-Hochschild-Serre spectral sequence** ... Sebastian Opper
09.01.2013
Spectral sequence of a groups extension $1 \to H \to G \to Q \to 1$, equivalently of a fibration $BH \to BG \to BQ$. Example: homology of dihedral groups and wreath-products (Theorem of Nakaoka).
[Bro94, VII.6], [Wei94, 6.8], [Eve91, 5.3, 7]

(13) **Cohomology theory of finite groups** .................................. Christian Wimmer
16.01.2013
For finite groups $G$ it turns out that the homology and cohomology groups of $G$ have certain similar properties. These enable to organize them together in what is called the Tate cohomology groups $\widehat{H}^*(G)$. Definition and basic properties. Coordinate with the next speaker.
[Bro94, VI.1-5]

(14) **Free actions on spheres** ................................................ Andreas Mihatsch
23.01.2013
One of the most impressive applications of group homology is related to the question which groups act freely on finite-dimensional spheres. Why can $\mathbb{Z}/2$ act freely, but $\mathbb{Z}/2 \times \mathbb{Z}/2$ or a symmetric group $\Sigma_n$ for $n \geq 4$ cannot? The answer is that such a group must have periodic homology. There exists a characterization of such groups.
[Bro94, I.6,VI.6-9]

## Literatur


[AM04] Alejandro Adem and R. James Milgram. *Cohomology of finite groups*, volume 309 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2004.

[Bro94] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original.

[Eve91] Leonard Evens. *The cohomology of groups*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1991. Oxford Science Publications.

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[HS97] P. J. Hilton and U. Stammbach. *A course in homological algebra*, volume 4 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997.

[Ser03] Jean-Pierre Serre. *Trees*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.