# THE ALGEBRAIC THEORY OF SURGERY READING SEMINAR

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Abstract. The total surgery obstruction $s(X)$ of a finite Poincaré complex $X$ of formal dimension $n$ is an element of a certain abelian group $\mathbb{S}_n(X)$ with the property that $s(X) = 0$ if and only if $X$ is homotopy equivalent to a closed $n$-dimensional topological manifold.

Both the definition of $s(X)$ and the proof of the above property require a substantial amount of technology known as algebraic theory of surgery due to Ranicki. In this reading seminar we would like to understand the technology in reasonable detail.

The idea is that we all try to read at least the principal references. People will be assigned to talks and will give the talks, but we expect more discussion than usual. Please think about which talk would you like to give.

As a broad outline I suggest to follow the book [Ran92], the other principal references could be [Ran01a], [Ran02] and [Ran08]. The references [Ran01a], [Ran02] are good for the beginning, as a supplement to chapter 1 of [Ran92], which is too condensed. The reference [Ran08] could be good throughout since it presents some shortcuts in comparison to [Ran92] and also most of us have attended the lecture, so we have some acquaintance with it. Nevertheless, I suggest to use slightly different approach than [Ran08], mostly by rearranging the order of things. This is possible since we are already little more experienced. For example in [Ran08] Ranicki first only defined symmetric chain complexes and did all the possible constructions with them - pairs, cobordisms, algebraic surgery. It was only much later that he introduced quadratic chain complexes and started to study the difference between the symmetric and quadratic chain complexes. I suggest a different approach, namely the other way round. I think we can master things faster that way and we should not have problems with the motivation, since we are already motivated enough. When needed we also have plenty of other literature which we can use as a supplement, some of which is listed below.

Although we have the whole semester, we might not be able to cover all the aspects of algebraic theory of surgery, which we would hope for. For example, there is an issue that the quadratic structures on a kernel chain complex of a highly connected degree one map agrees with the quadratic form defined in terms of geometry. This is certainly an interesting topic (see [Ran80b]), but I think it would take us too far off our goal. Similarly, I think we will not need to discuss algebraic Wu classes which are kind of omnipresent in Ranicki's papers. Again, an interesting topic, but not really necessary for us.

There might also be considerable overlap with the talks in the Oberseminar, but I think that is OK, the idea is that we will go much more into detail here.

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**1. Chain complexes.** (Steffen Sagave, 20/04)

The model category structures and constructions on chain complexes.

**2. Structures on chain complexes.** (27/04)

Symmetric, quadratic and hyperquadratic structures on a chain complex over a ring with involution $R$. The $Q$-groups. Poincaré complexes. Symmetric and quadratic construction: how to obtain such structures from various geometric situations.

It might be good to say explicitly that a symmetric structure on a chain complex concentrated in a single dimension is the same as a symmetric bilinear form and similarly a quadratic structure is the same as a quadratic form. Perhaps say something about highly connected chain complexes.

Literature: [Ran01a], sections 1,2,3, and [Ran08], lectures from 04/11/08 and 05/11/08. Important: the exact sequence with all three types of the $Q$-groups. From that it follows that a symmetric chain complexes has a quadratic refinement if and only if its $i$-suspension is null-homologous as a hyperquadratic complex for some $i \geq 0$. This is then used to show that a certain chain complex associated to a degree one normal map can be endowed with a quadratic structure which refines a symmetric structure which it already has.

These things are also covered in the original articles [Ran80a, section 1] and [Ran80b, section 1,2], but especially the second part can be made much easier in the way it is done in the lectures.

**3. Algebraic surgery over $R$, the groups $L_*(R)$.** (04/05)

Pairs and cobordisms of symmetric and quadratic chain complexes, algebraic surgery, $L$-groups, still over a ring with involution $R$.

Besides the definitions we should try to understand how are these constructions analogs of the respective geometric constructions. We should also try to convince ourselves that (at least in even dimensions) the $L$-groups defined via quadratic chain complexes agree with the $L$-groups defined via quadratic forms.

The reference [Ran02] is very helpful for the intuitive idea, also [Ran08], lectures 23/10/08-04/11/08 (in the last one the 4-periodicity of the quadratic $L$-groups is interesting in comparison to symmetric $L$-groups which are not 4-periodic). However, the presentation in the talk should better follow more rigorous references: [Ran92], chapter 1, [Ran01a], section 5, until line 6 on page 16, [Ran80a], sections 3-5. Do not forget: the cobordism is an equivalence relation, the definition and meaning of the boundary operation from Proposition 1.15 of [Ran92].

Another reference for those especially interested is [Ran01b], but I think we should not go into such depth at this stage.

**4. Normal complexes.** (11/05)

Definition and explanation how normal complexes come from a geometric situation. They are very important for two reasons. One is that they measure the difference between the symmetric and quadratic $L$-groups. Explain this purely algebriac result (Propositions 2.8, 2.11 in [Ran92]). The second reason - a geometric Poincaré complex is locally not Poincaré, but it still is locally a normal complex - hint on this. As a preparation it might be good to recall the Spivak normal fibration of a Poincaré complex, the $S$-duality and the fact that the Thom space of the Spivak normal fibration of a Poincaré complex is the $S$-dual to that complex.

The lectures on 11/11, 18/11 and 19/11 from [Ran08] are a good source for the preparation as well as some of the definitions. A more rigorous and also very good source is [Ran92, chapter 2], very important: the definition and Proposition 2.8, resp 2.11, see also [Ran01a], section 5 from page 16, sections 6,7.

Another references for the preparations are [Lüc02], section 3.2, and [Bro72], section 1.4.

**5. Algebraic surgery over $\mathbb{A}$, the groups $L_*(\mathbb{A})$, the $L$-spectra.** (18/05)

The definition of an additive category with chain duality $\mathbb{A}$. Explain quickly that the chain duality is enough to define all the concepts that have been defined so far in this more general setting, that means: symmetric, quadratic and hyperquadratic structure on a chain complex over $\mathbb{A}$, $Q$-groups, Poincaré complexes, pairs, cobordisms, $L$-groups $L_*(\mathbb{A})$. All these things are easy at this stage (considering what we have done), so one can go through this quite fast, but it is good to mention these things, because it might not be completely obvious when it is not said explicitly.

The main part should be examples! In particular $\mathbb{A}(R)$, $\mathbb{A}^*(X)$, $\mathbb{A}_*(X)$. This is quite tricky, so it should be explained really well. When time permits, explain the symmetric and quadratic construction: how to obtain such structures from various geometric situations in this setting.

The benefit of explaining these examples will be that we will put ourselves in a position to give a fast and easy definition of the $L$-spaces and $L$-spectra. This should definitely be included. Literature: [Ran92], chapters 1,4,5,11,13

**6. Exact sequences and assembly maps.** (08/06)

Explain the technology which sets up the exact sequences in algebraic surgery. This is explained in [Ran92] chapter 3, especially Proposition 3.9., maybe have a look at chapter 14.

The simply connected and the non-simply connected assembly maps, [Ran92], chapters 6,9. This is quite straightforward.

Explain in detail what are the terms in the exact sequences obtained via this technology, when one of the maps is the assembly map. It is also important how the elements are represented. This is explained in [Ran92], chapters 14.

If time permits perhaps start talking about the visible $L$-groups?

**7. Homology theories and Poincaré duality.** (15/06)

Explain the isomorphism $L_n(\mathbb{A}_*(X)) \cong H_n(X; \mathbf{L}_\bullet)$ - it is explained in [Ran92], chapters 12,13

Visible $L$-theory: [Ran92], chapter 9.

Review all the possible exact sequences that have been constructed, explain the braid of Proposition 14.7. [Ran92].

**Possible topics for the further talks:**

One could continue with the second part of the book [Ran92], where the topics include the detailed definition of the total surgery obstruction $s(X)$ and the proof of its properties. Various versions relate to various connectivity assumptions, which geometrically correspond to various categories of manifolds: topological or homology manifolds. One could also discuss examples of Poincaré complexes which are not homotopy equivalent to manifolds and which can be detected by the total surgery obstruction. Finally one could try to understand the identification of the algebraic and geometric surgery exact sequences.

## References


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