THE ATIYAH-SINGER INDEX THEOREM
SEMINAR PROGRAM, WS 2009/2010

TIBOR MACKO, ROMAN SAUER

Tuesdays, 14:00-16:00, SR 5

The aim of the seminar is to discuss the theorem of Atiyah and Singer that for an ellipitic operator on a smooth closed manifold the analytical index equals the topological index. We are also interested in the cohomological formulation which enables to express the index in terms of characteristic numbers. We would like to discuss the proof, but perhaps more importantly the examples, which include the Hirzebruch signature theorem, a version of Gauss-Bonnet theorem, the Hirzebruch-Riemann-Roch theorem and results about the $\widehat{A}$-genus for spin manifolds. There are many applications, some of which we want to discuss as well. This depends to some extent on the interests of the participants.

The principal reference we will use is [LM89]. All the references to chapters below are to this book. In case we want to recommend other references these are explicitly stated. These might include the original paper [AS68], which is very readable, but somehow condensed for our purposes. Nevertheless, we recommend to everyone to read at least the introduction. Other reference might be [Roe98], which contains a different proof, but also a friendly discussion of some background material.

Spin geometry

1. Clifford algebras. (Helge Bösche, 20.10.)
Sections I.1.-I.4.

2. Representations of Clifford algebras. (Christoph Winges, 27.10.)
Sections I.5.-I.6.

3. K-theory. (Jan Essert, 3.11.)
Section I.9.

4. Spin structures on manifolds. (Daniel Kasprowski, 10.11.)
Sections II.1.-II.2.

5. Clifford and spinor bundles. (Adam Mole, 17.11.)
Sections II.3.-II.4. It is probably not necessary to give the full account of section II.4. The speaker should consult with the speakers of talks 6 and 13 what is needed for their talks.

6. Dirac operators. (Noé Bárcenas-Torres, 1.12.)
Sections II.-5.-II.6.

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Analysis

The chapter III contains much more material than we plan to cover. For example we plan to skip the generalizations and the heat kernel aspects of the proof.

7. Differential operators and Sobolev spaces. (Paul Striewski, 8.12)
Sections III.1.-III.2. Very important are Corollary 2.9. and Theorem 2.15.

8. Pseudodifferential operators, elliptic operators and parametrices. (Christian Wegner, 15.12.)

Sections III.3.-III.4. The main outcome of this talk should be Theorem 4.6. This requires to find an 'inverse' of an elliptic operator which is in general only a pseudo-differential operator. This is the reason that these operators are used. The main technical theorem about them is Theorem 3.17 What is also important for later is the discussion after Theorem 3.19 on page 188. Clearly it will not be possible to give all the details. We suggest that the speaker concentrates on the definitions and on the statements of the main theorems and black-boxes the proofs which are not possible to give because of time.

9. Index of elliptic operators, definition and topological invariance. (Malte Röer, 5.1.)

Section III.5. up to Corollary 5.7 (inclusive), and section III.7.

Proof

10. Proof of the main theorem. (Henrik Rüping, 12.1.)

Section III.13. up to the end of the proof of the Index Theorem on page 254. Here it might be useful to look at the original paper [AS68] as well. Some of the discussion there is simpler.

11. Multiplicative sequences and the Chern character. (19.1.)
Sections III.11-III.12.

12. Cohomological version of the main theorem, including examples. (26.1.)

Section III.13 from page 254 to the end. The speaker may have to recall all the necessary notions from talk 5 for the statements and the proofs.

Applications

13. Manifolds of positive scalar curvature. (2.2 .)
Section IV.4. Here background from section II.4. is needed.

References

[AS68] M. F. Atiyah and I. M. Singer. The index of elliptic operators. I. *Ann. of Math. (2)*, 87:484–530, 1968.

[LM89] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.

[Roe98] John Roe. *Elliptic operators, topology and asymptotic methods*, volume 395 of *Pitman Research Notes in Mathematics Series*. Longman, Harlow, second edition, 1998.