Automorphisms of manifolds and algebraic theory of surgery

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6th NRW Topology Meeting Düsseldorf
25.11.2006

Goal

To present an algebraic model $\mathbb{S}(X)$ of the automorphism space $\widetilde{\mathcal{S}}^{\mathrm{TOP}}(X)$.
It has many good properties which clarify the surgery theory class. of top. mfds.

Application to mfds SV/Γ (j.w. with M. Weiss)

A homotopy fibration sequence for all f.d. $\mathbb{R}$-vector spaces V

$$\Omega^\infty((S(V)_+ \wedge \mathbb{L}_0)_{hO(1)}) \to \mathbb{L}_0(\mathbb{Z}[\mathbb{Z}_2]) \to \mathbb{S}(SV/\mathbb{Z}_2).$$

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Outline

- *Automorphisms of manifolds*
- *The geometric surgery fibration sequence* $\rightsquigarrow \widetilde{\mathcal{S}}^{\mathrm{TOP}}(X)$
- *L-theory (version of A. Ranicki)*
- *The algebraic surgery fibration sequence* $\rightsquigarrow \mathbb{S}(X)$ *(A. Ranicki)*
- *Application to manifolds* SV/Γ

Automorphisms of manifolds

CAT = DIFF, PL, TOP

Space	k-simplex	
$\mathrm{CAT}(X)$	$h \colon X \times \Delta^k \xrightarrow{\cong} X \times \Delta^k$	$pr_2 = pr_2 \circ h$
$\widetilde{\mathrm{CAT}}(X)$	$h \colon X \times \Delta^k \xrightarrow{\cong} X \times \Delta^k$	$h(X \times \sigma) \subset X \times \sigma$
$\mathrm{G}(X)$	$h \colon X \times \Delta^k \xrightarrow{\simeq} X \times \Delta^k$	$pr_2 = pr_2 \circ h$
$\widetilde{\mathrm{G}}(X)$	$h \colon X \times \Delta^k \xrightarrow{\simeq} X \times \Delta^k$	$h(X \times \sigma) \subset X \times \sigma$

$\mathrm{G}(X) \simeq \widetilde{\mathrm{G}}(X)$

$$\widetilde{\mathrm{CAT}}(X)/\mathrm{CAT}(X) \longrightarrow \mathrm{G}(X)/\mathrm{CAT}(X) \longrightarrow \mathrm{G}(X)/\widetilde{\mathrm{CAT}}(X)$$

$\widetilde{\mathrm{CAT}}(X)/\mathrm{CAT}(X) \downarrow$ (assembly map in) algebraic K-theory

$\mathrm{G}(X)/\widetilde{\mathrm{CAT}}(X) \downarrow$ (assembly map in) algebraic L-theory

The geometric surgery fibration sequence

$\mathrm{G}(X)/\widetilde{\mathrm{CAT}}(X) \simeq \widetilde{\mathcal{S}}^{\mathrm{CAT}}(X)_1$

$\widetilde{\mathcal{S}}^{\mathrm{CAT}}(X)$ = the block structure space of X
"(block)moduli space of mfds in the htpy type of X"

k-simplex: (M, f), $f\colon (M, \partial_i M) \xrightarrow{\simeq} (X \times \Delta^k, X \times \partial_i \Delta^k)$

$\widetilde{\mathcal{S}}^{\mathrm{CAT}}(X)_1$: k-simplex (M, f) where $M \cong X \times \Delta^k$

The geometric surgery fibration sequence for X ($\dim(X) \geq 5$)

$$\widetilde{\mathcal{S}}^{\mathrm{CAT}}(X) \to \mathcal{N}^{\mathrm{CAT}}(X) \xrightarrow{\theta} \mathcal{L}(X)$$

$\mathcal{N}^{\mathrm{CAT}}(X)$ = space of degree 1 normal maps
k-simplex $(f, b)\colon (M, \partial_i M) \to (X \times \Delta^k, X \times \partial_i \Delta^k)$ of deg 1, $b\colon \nu_M \to \nu_{X \times \Delta^k}$

$\mathcal{L}(X)$ = surgery obstruction space

$(f, b)\colon (M, \partial M) \to (X \times \Delta^k, X \times \partial \Delta^k)$ degree 1 normal map s.t. $f|_{\partial M}$ is an $\cong$
representing an element in $\pi_k \mathcal{N}^{\mathrm{CAT}}(X)$
$\theta_*(f, b) = 0 \in \pi_k \mathcal{L}(X) \iff (f, b) \sim (f', b')$ rel ∂ s.t. f' is a $\simeq$, $\dim(X) \geq 5$

Properties of the GSFS

- studied for SV/Γ, $K(\Gamma,1)$,...
- surgery exact sequence = LES of π_* of GSFS - on π_0 ex seq of ptd sets
- $\mathcal{N}^{\mathrm{CAT}}(X)$ is a cohlgy theory - Thom-Pontryagin construction
- $\pi_k\mathcal{L}(X)$ depend only on $\pi_1(X)$, $w\colon\pi_1(X)\to\{\pm 1\}$, $n+k$ mod 4, $n=\dim(X)$

  $L_{k+n}(\mathbb{Z}[\pi_1 X]^w) := \pi_k\mathcal{L}(X)$, $\exists$ a spectrum $\mathbb{L}_n(X)$, $\pi_k\mathbb{L}_n(X)=\pi_k\mathcal{L}(X)$.
- $\theta_*\colon\pi_0\mathcal{N}^{\mathrm{CAT}}(X)\to L_n(\mathbb{Z}[\pi_1 X]^w)$ is NOT a homomorphism in general
- $\theta_*\colon\pi_0\mathcal{N}^{\mathrm{TOP}}(X)\to L_n(\mathbb{Z}[\pi_1 X]^w)$ is a homomorphism!

  (for a new groups str. on $\pi_0\mathcal{N}^{\mathrm{TOP}}(X)$)
- $\theta_*\colon\pi_k\mathcal{N}^{\mathrm{TOP}}(*)\xrightarrow{\cong}L_k(\mathbb{Z}) \rightsquigarrow \mathcal{N}^{\mathrm{TOP}}(X)\simeq\Omega^{\infty+n}(X_+\wedge\mathbb{L}_0)$

Questions:

1. Is $\pi_0\widetilde{\mathcal{S}}^{\mathrm{TOP}}(X)$ a group?
2. Is $\widetilde{\mathcal{S}}^{\mathrm{TOP}}(X)\simeq\Omega^\infty\mathbb{L}_0(\text{something})$?

Definitions and Conventions

R - a ring with involution $\bar{\ }: R \to R$, $\epsilon \in \{\pm 1\}$.
E.g. $R = \mathbb{Z}G$, $w: G \to \{\pm 1\}$, $\bar{g} = w(g)g^{-1}$.
P - a f.g. free left R-module
$P^* = \mathrm{Hom}_R(P, R)$ a f.g. free left R-module via the involution

A bilinear form on P

$$\varphi \in \mathrm{Hom}_R(P, P^*) \xrightarrow{\cong} (P \otimes_R P)^*$$
$$\varphi \mapsto (\lambda\colon (x,y) \mapsto \varphi(x)(y)))$$

A bilinear coform on P

$$\varphi \in \mathrm{Hom}_R(P^*, P) \xrightarrow{\cong} P \otimes_R P$$
$$(f \mapsto f(x)y) \leftarrow\!\shortmid x \otimes y$$

A non-degenerate bilinear form on P is an iso $P \to P^*$,
it defines an iso between bilinear forms and coforms on P.

$L(R)$ via f.g. free R-modules

Let $\mathrm{ev}\colon P \to (P^*)^*$ be an iso $\mathrm{ev}(p) = (f \mapsto \overline{f(p)})$.

Define $T\colon \mathrm{Hom}_R(P, P^*) \to \mathrm{Hom}_R(P, P^*)$ by $T(\varphi) = \varphi^* \circ \mathrm{ev}$.
Note $T\colon (P \otimes_R P)^* \to (P \otimes_R P)^*$ is $T(\lambda)\colon (x, y) \mapsto \overline{\lambda(y, x)}$.

Let $\mathbb{Z}_2$ act on $\mathrm{Hom}_R(P, P^*)$ by $\epsilon \cdot T$.

An ϵ-symmetric bilinear form on P is

$$\varphi \in Q^\epsilon(P) = \mathrm{Hom}_R(P, P^*)^{\mathbb{Z}_2} = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\mathbb{Z}, \mathrm{Hom}_R(P, P^*)),$$

An ϵ-quadratic form on P is

$$\psi \in Q_\epsilon(P) = \mathrm{Hom}_R(P, P^*)_{\mathbb{Z}_2} = (\mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}_R(P, P^*)).$$

$\varphi \in Q^\epsilon(P)$ is non-degenerate if it represents an iso $P \to P^*$.

The symmetrization map

$$(1 + \epsilon \cdot T)\colon Q_\epsilon(P) \to Q^\epsilon(P)$$

$\psi \in Q_\epsilon(P)$ is non-degenerate if $(1 + \epsilon T) \cdot \psi$ is non-degenerate.

X - $2k$-dim'l mfd, then $\varphi([X])\colon x \mapsto (y \mapsto x(y \cap [X]))$ is a non-degenerate $(-1)^k$-symmetric bilinear form on $H^k_c(\widetilde{X})$ over $\mathbb{Z}[\pi_1(X)]^w$.

$(f,b)\colon M \to X$ a k-ctd degree one normal map of $2k$-dim'l mfds, then $\exists$ a non-degenerate $(-1)^k$-quad. form $\psi(f)$ on $K_k(f) = Ker(\widetilde{f}_*\colon H_k(\widetilde{M}) \to H_k(\widetilde{X}))$.

Operations $(P,\varphi) \oplus (P',\varphi')$, $(P,\psi) \oplus (P',\psi')$.

The standard hyperbolic ϵ-sym. bil. form $H^\epsilon(P)$ on $P \oplus P^*$ is $\begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}$.

Definition ($L^{2k}(R)$)

$L^{2k}(R) =$ Grothendieck group of $[(P,\varphi)]$ with non-degenerate $\varphi \in Q^{(-1)^k}(P)$ modulo the hyperbolic forms - forms iso to finite sums of $H^{(-1)^k}(P)$.

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Definition ($L_{2k}(R)$)

$L_{2k}(R) =$ Grothendieck group of $[(P,\psi)]$ with non-degenrate $\psi \in Q_{(-1)^k}(P)$ modulo the hyperbolic forms - forms iso to finite sums of $H_{(-1)^k}(P)$.

Properties of $L_n(R)$.

surgery obstruction

For $(f, b)\colon M \to X$ a k-ctd degree one normal map of $2k$-dim'l mfds,
$(f, b) \sim (f', b')$ with $f' \simeq \iff (K(f), \psi(f)) = 0$ in $L_{2k}(\mathbb{Z}[\pi_1(X)]^w)$.

$L_n(R) = L_{n+4}(R)$
$L_0(\mathbb{Z}) = \mathbb{Z}$ given by the signature/8.
$L_1(\mathbb{Z}) = 0$.
$L_2(\mathbb{Z}) = \mathbb{Z}_2$ given by the Arf invariant.
$L_3(\mathbb{Z}) = 0$.

Properties of $L_n(R)$.

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Remark

We could have formulated everything in terms of coforms rather than forms.

We would use

$$T\colon \mathrm{Hom}_R(P^*,P) \to \mathrm{Hom}_R(P^*,P)$$
$$\varphi \mapsto \mathrm{ev}^{-1} \circ \varphi^*$$
$$T\colon P \otimes_R P \to P \otimes_R P$$
$$x \otimes y \mapsto \overline{y} \otimes \overline{x}$$

$L(R)$ via chain complexes of f.g. free R-modules (Ranicki)

C- a bounded chain complex of f.g. free R-modules. Recall that $\mathrm{Hom}_R(C, D)$ is a chain cplx with n-chains $\mathrm{Hom}_R(C, D)_n = \mathrm{Hom}_R(C_{*+n}, D_*)$.

Define $T\colon \mathrm{Hom}_R(C^{-*}, C) \to \mathrm{Hom}_R(C^{-*}, C)$ by $T(\varphi) = \mathrm{ev}^{-1} \circ \varphi^*$.

homotopy invariant structures

fixed points $\rightsquigarrow$ homotopy fixed points
orbits $\rightsquigarrow$ homotopy orbits

Let W be a free $\mathbb{Z}[\mathbb{Z}_2]$-resolution of $\mathbb{Z}$, $\mathbb{Z}_2 = \{1, T\}$, $\mathbb{Z}$ is a trivial $\mathbb{Z}[\mathbb{Z}_2]$-module.

An n-dimensional symmetric structure on C is an n-cycle
$\varphi \in Q^n(C) = \mathrm{Hom}_R(C^{-*}, C)_n^{h\mathbb{Z}_2} = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_R(C^{-*}, C))_n$
An n-dimensional quadratic structure on C is an n-cycle
$\psi \in Q_n(C) = (\mathrm{Hom}_R(C^{-*}, C)_n)_{h\mathbb{Z}_2} = (W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}_R(C^{-*}, C))_n$

(C, φ) with $\varphi \in Q^n(C)$ is called a symmetric algebraic Poincaré complex (SAPC) if φ_0 is a chain homotopy equivalence $C^{n-*} \to C$.
(C, ψ), with $\psi \in Q_n(C)$ is called a quadratic algebraic Poincaré complex (QAPC) if $(1 + T) \cdot \psi_0$ is a chain homotopy equivalence $C^{n-*} \to C$.

Examples

1. X - connected finite CW-cplx, $\widetilde{X} \to X$ a universal cover, $\pi = \pi_1 X$. The diagonal map
$$\nabla \colon X \to \widetilde{X} \times_\pi \widetilde{X}$$
is $\mathbb{Z}_2$-equiv, but not cellular in general. There exists a cellular $\mathbb{Z}_2$-equiv
$$\begin{aligned} \nabla^\sharp \colon E\mathbb{Z}_2 \times X &\longrightarrow \widetilde{X} \times_\pi \widetilde{X} \\ \rightsquigarrow \qquad W \otimes C_* X &\longrightarrow C_*\widetilde{X} \otimes_{\mathbb{Z}[\pi]} C_*\widetilde{X} \\ \rightsquigarrow \qquad \varphi \colon C_* X &\longrightarrow \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C_*\widetilde{X} \otimes_{\mathbb{Z}[\pi]} C_*\widetilde{X}) \cong \\ & \qquad \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_{\mathbb{Z}[\pi]}(C^{-*}\widetilde{X}, C_*\widetilde{X})) \end{aligned}$$
An n-cycle $\mu \in C_n X \rightsquigarrow \varphi(\mu) \in Q^n(C_*\widetilde{X})$.
X - n-mfd $\Rightarrow \varphi([X])_0$ non-degenerate, $\Rightarrow (C_*\widetilde{X}, \varphi([X]))$ n-dim'l SAPC.

2. $(f, b) \colon M \to X$ - a degree one normal map of n-dim'l mfds
$$K(f) := Cone\big(f^! \colon C_*\widetilde{X} \simeq C^{n-*}\widetilde{X} \xrightarrow{f^{n-*}} C^{n-*}\widetilde{M} \simeq C_*\widetilde{M}\big)$$
There exists $\psi(f) \in Q_n(K(f))$ such that $(K(f), \psi(f))$ n-dim'l QAPC.

$L(R)$

There exists the notion of a cobordism of SAPCs, QAPCs.

Definition (L-groups)

$$\begin{cases} L^n(R) \\ L_n(R) \end{cases} = \text{cobordism group of } n\text{-dim'l} \begin{cases} SAPCs \\ QAPCs \end{cases} \text{ over } R.$$

There exists the notion of a $(k+2)$-ad of SAPCs, QAPCs.

Definition (L-spaces)

$\mathbb{L}_n(R)$ - a space, k-simplex: $(n+k)$-dimensional $(k+2)$-ad of QAPCs over R.

There are obvious maps $\Sigma\mathbb{L}_n(R) \to \mathbb{L}_{n-1}(R)$ and we have $\mathbb{L}_n(R) \simeq \Omega\mathbb{L}_{n-1}(R)$. Further $\mathbb{L}_n(R) \simeq \mathbb{L}_{n+4}(R)$ by double suspension.

Definition (L-spectra)

$\mathbb{L}_\bullet(R)$ - a spectrum with the n-th space is $\mathbb{L}_{-n}(R)$.

$L(\mathbb{A})$ for $\mathbb{A}$ an additive cat. with chain duality (Ranicki)

$\mathbb{A}$ - an additive category, $\mathbb{B}(\mathbb{A})$ - bounded chain cplxs in $\mathbb{A}$.

Definition

A chain duality on $\mathbb{A}$ is a contravariant functor $T\colon \mathbb{A} \to \mathbb{B}(\mathbb{A})$ with a nat. trans. $e\colon T^2 \to \mathrm{id}$ s. t. for each $M \in \mathbb{A}$

1. $e_{T(M)} \cdot T(e_M) = \mathrm{id}\colon T(M) \to T^3(M) \to T(M)$,
2. $e_M\colon T^2(M) \to M$ is a chain htpy equiv.

$T\colon \mathbb{B}(\mathbb{A}) \to \mathbb{B}(\mathbb{A})$ is given by $C \mapsto \mathrm{Tot}(T(C)_{p,q} = T(C_{-p})_q)$

An n-dimensional symmetric structure on C in $\mathbb{B}(\mathbb{A})$ is an n-cycle $\varphi \in Q^n(C) = \mathrm{Hom}_{\mathbb{A}}(T(C), C)_n^{h\mathbb{Z}_2} = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_{\mathbb{A}}(T(C), C))_n$
An n-dimensional quadratic structure on C in $\mathbb{B}(\mathbb{A})$ is an n-cycle $\psi \in Q_n(C) = (\mathrm{Hom}_{\mathbb{A}}(T(C), C)_n)_{h\mathbb{Z}_2} = (W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}_{\mathbb{A}}(T(C), C))_n$

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(C, ψ), with $\psi \in Q_n(C)$ is called a quadratic algebraic Poincaré complex (QAPC) if $(1 + T) \cdot \psi_0$ is a chain homotopy equivalence $T(C)_{n+*} \to C$.

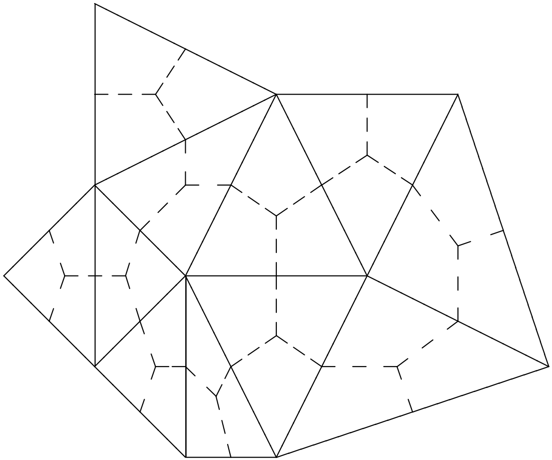
Examples

1. R - ring with involution, cat R-Mod: $T(M) = \mathrm{Hom}_R(M, R)$
2. X - finite simplicial cplx
   $\mathbb{A} = (\mathbb{Z}, X)$ - cat of $(\mathbb{Z}, X)$-modules
   Objects: $M = \sum_{\sigma \in X} M(\sigma)$
   Morph: $f \colon M \to N$, $f = \{f(\sigma,\tau) \colon M(\sigma) \to N(\tau),\ f(\sigma,\tau) = 0$ unless $\sigma \leq \tau\}$.
   Duality: $(TM)_r(\sigma) = T\big(\sum_{\sigma \leq \tau} M\big)_{r+|\sigma|}$
3. X - triangulated mfd of dim n, dual cell dec. $D(X) = \bigcup_{\sigma \in X} D(\sigma, X)$
   $C = C_* X \in \mathbb{B}(\mathbb{Z}, X)$, $C_*(\sigma) = C_*(D(\sigma, X), \partial D(\sigma, X))$
   $T(C)_*(\sigma) = C^{-|\sigma|-*}(D(\sigma, X))$.
   In analogy with the case $\mathbb{Z}[\pi]$ we have

$$\varphi \colon C = C_* X \to \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_{(\mathbb{Z},X)}(T(C), C))$$

$$\varphi([X])_0(\sigma,\sigma) \colon C^{n-|\sigma|-*}(D(\sigma, X)) \to C_*(D(\sigma, X), \partial D(\sigma, X))$$

   defining an n-dim'l SAPC $(C_* X, \varphi([X]))$ in $\mathbb{B}(\mathbb{Z}, X)$.
4. $(f, b) \colon M \to X$ - a degree one normal map of n-dim'l mfds, X-triang.
   Then we have an n-dim'l QAPC $(K(f), \psi(f))$ in $\mathbb{B}(\mathbb{Z}, X)$.



$L(\mathbb{A})$

There exists the notion of a cobordism of SAPCs, QAPCs in $\mathbb{B}(\mathbb{A})$.

Definition (L-groups)

$$\begin{cases} L^n(\mathbb{A}) \\ L_n(\mathbb{A}) \end{cases} = \text{cobordism group of } n\text{-dim'l} \begin{cases} SAPCs \\ QAPCs \end{cases} \text{ in } \mathbb{B}(\mathbb{A}).$$

There exists the notion of a $(k+2)$-ad of SAPCs, QAPCs in $\mathbb{B}(\mathbb{A})$.

Definition (L-spaces)

$\mathbb{L}_n(\mathbb{A})$ - a space, k-simplex: $(n+k)$-dim'l $(k+2)$-ad of QAPCs in $\mathbb{B}(\mathbb{A})$.

There are obvious maps $\Sigma\mathbb{L}_n(\mathbb{A}) \to \mathbb{L}_{n-1}(\mathbb{A})$ and we have $\mathbb{L}_n(\mathbb{A}) \simeq \Omega\mathbb{L}_{n-1}(\mathbb{A})$. Further $\mathbb{L}_n(\mathbb{A}) \simeq \mathbb{L}_{n+4}(\mathbb{A})$ by double suspension.

Definition (L-spectra)

$\mathbb{L}_\bullet(\mathbb{A})$ - a spectrum whose n-th space is $\mathbb{L}_{-n}(\mathbb{A})$.

Theorem (Homology theory)

The functor $X \mapsto \mathbb{L}_n(\mathbb{Z}, X)$ *is a homology theory,* $\mathbb{L}_n(\mathbb{Z}, X) \simeq \Omega^{\infty+n}(X_+ \wedge \mathbb{L}_\bullet)$.

Functor of additive categories with chain duality

$$\alpha\colon (\mathbb{Z}, X) \to \mathbb{Z}[\pi]$$
$$M \mapsto \sum_{p(\widetilde{\sigma})=\sigma} M(\widetilde{\sigma})$$

$p\colon \widetilde{X} \to X$ - the universal cover, $M(\widetilde{\sigma}) := M(p(\widetilde{\sigma}))$.

Definition (The assembly map)

$$\alpha\colon \mathbb{L}_n(\mathbb{Z}, X) \to \mathbb{L}(\mathbb{Z}[\pi]).$$

$\mathbb{B}(\mathbb{Z}, X)^\alpha$ - chain cplxes $C \in \mathbb{B}(\mathbb{Z}, X)$ with $\alpha(C) \simeq *$. $\mathbb{S}(X) := \mathbb{L}_n((\mathbb{Z}, X)^\alpha)$.

Theorem (The algebraic surgery fibration sequence - a local. seq.)

$$\mathbb{S}(X) \longrightarrow \mathbb{L}_n(\mathbb{Z}, X) \xrightarrow{\alpha} \mathbb{L}_n(\mathbb{Z}[\pi]).$$

Properties of ASFS

- All dimensions
- $\pi_0 \mathbb{S}(X)$ is a group. ASES is a LES of groups.
- 4-periodicity (double suspension)
  $\mathbb{S}(X) = \mathbb{L}_n((\mathbb{Z}, X)^\alpha) = \mathbb{L}_{n+4}((\mathbb{Z}, X)^\alpha) = \Omega^4 \mathbb{L}_n((\mathbb{Z}, X)^\alpha) = \Omega^4 \mathbb{S}(X)$.
- Comparison GSFS $\rightsquigarrow$ ASFS (assembly map = surgery obstruction map)

$$\begin{array}{ccccc}
\widetilde{\mathcal{S}}(X) & \longrightarrow & \mathcal{N}(X) & \xrightarrow{\theta} & \mathcal{L}(X) \\
\downarrow & & \downarrow & & \downarrow \simeq \\
\mathbb{S}(X) & \longrightarrow & \mathbb{L}_n(\mathbb{Z}, X) & \xrightarrow{\alpha} & \mathbb{L}_n(\mathbb{Z}[\pi]) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & *
\end{array}$$

- Algebraic model
  $\mathbb{S}(X)$ - k-simplex: $(n+k)$-dim'l $(k+2)$-ad of QAPCs in $\mathbb{B}(\mathbb{Z}, X)$

$$(C, \psi) = \sum_{\sigma \in X} (C(\sigma), \psi(\sigma)) \text{ with } \alpha(C) \simeq *.$$

Reference: A. Ranicki: Algebraic L-theory and top. manifolds, CUP, 1992.

Manifolds of the form SV/Γ with $\Gamma \leq S^1$

$$S(V \oplus W) = SV * SW \rightsquigarrow \mathrm{AUT}_\Gamma(SV) \times \mathrm{AUT}_\Gamma(SW) \to \mathrm{AUT}_\Gamma(S(V \oplus W))$$

Functors

Fix a standard $\Gamma \to \mathrm{TOP}(S\mathfrak{k})$, $\mathfrak{k} = \mathbb{R}$ or $\mathbb{C}$
$\rightsquigarrow$ a continuous functor $V \mapsto \mathrm{AUT}_\Gamma(SV)$, f.d. $\mathfrak{k}$-inner product spaces to spaces

$$\widetilde{\mathcal{S}}_\Gamma(SV) \simeq \widetilde{\mathcal{S}}(SV/\Gamma)$$

Theorem (Browder,Wall,Madsen-Rothenberg,'68-'88)

If $\Gamma = \mathbb{Z}_p$, p odd, then $\widetilde{\mathcal{S}}^{\mathrm{TOP}}(SV/\Gamma) \xrightarrow{\simeq} \widetilde{\mathcal{S}}^{\mathrm{TOP}}(S(V \oplus W)/\Gamma)$.
If $\Gamma = \mathbb{Z}_2$ or S^1 then these maps are not $\simeq$ in general.

Rem: They also calculated $\pi_* \widetilde{\mathcal{S}}^{\mathrm{TOP}}(SV/\Gamma)$ and the htpy type for all $\Gamma < S^1$.
A deficiency:

$$\begin{array}{ccccc}
\widetilde{\mathcal{S}}^{\mathrm{TOP}}(SV/\Gamma) & \longrightarrow & \mathcal{N}^{\mathrm{TOP}}(SV/\Gamma) & \longrightarrow & \mathcal{L}(SV/\Gamma) \\
\downarrow & & \downarrow \nexists!!! & & \downarrow \nexists!!! \\
\widetilde{\mathcal{S}}^{\mathrm{TOP}}(S(V \oplus W)/\Gamma) & \longrightarrow & \mathcal{N}^{\mathrm{TOP}}(S(V \oplus W)/\Gamma) & \longrightarrow & \mathcal{L}(S(V \oplus W)/\Gamma)
\end{array}$$

A description of $\widetilde{\mathcal{S}}^{\mathrm{TOP}}(SV/\Gamma)$ natural in V?

Orthogonal calculus

For E a continuous functor from $\mathfrak{k}$-inner product spaces to spaces

$$\Omega^\infty((S(V)_+ \wedge E^{(1)})_{h\mathrm{Aut}(1)}) \to T_1E(0) \to T_1E(V),$$

where T_1E - polynomial approx of degree ≤ 1, $E^{(1)}$ - derivative of E at infinity.

Let $F_\Gamma(V) = \widetilde{\mathcal{S}}^{\mathrm{TOP}}(SV/\Gamma)$.

Theorem (M.)

If $\Gamma = \mathbb{Z}_2$ *or* S^1, $\dim(V) \geq 5$, *then*

$$\begin{array}{ccccc}
\Omega^\infty((S(V)_+ \wedge F_\Gamma^{(1)})_{h\mathrm{Aut}(1)}) & \longrightarrow & T_1F_\Gamma(0) & \longrightarrow & T_1F_\Gamma(V) \\
\simeq \uparrow & & \uparrow & & \simeq \uparrow \\
\Omega^\infty((S(V)_+ \wedge \mathbb{L}_\bullet)_{h\mathrm{Aut}(1)}) & \longrightarrow & ? & \longrightarrow & F_\Gamma(V)
\end{array}$$

Question

What is $T_1 F_{\mathbb{Z}_2}(0)$?
What is $T_1 F_{\mathbb{Z}_2}(V)$ when $\dim(V) < 5$?

(joint work with M. Weiss)

Guess

$T_1 F_{\mathbb{Z}_2}(V) \simeq \mathrm{hofiber}(\mathcal{N}^{\mathrm{TOP}}(SV/\mathbb{Z}_2) \to \mathcal{L}(SV/\mathbb{Z}_2))$.

Idea: work with $\mathbb{S}(SV/\mathbb{Z}_2)$ rather than with $\widetilde{\mathcal{S}}^{\mathrm{TOP}}(SV/\mathbb{Z}_2)$.

Theorem (M.-Weiss)

The assignment $F^a_\Gamma \colon V \mapsto \mathbb{S}(SV/\Gamma)$ *is a continuous functor for* $\Gamma < S^1$.

Theorem (M.-Weiss)

We have $F^a_{\mathbb{Z}_2}(V) \xrightarrow{\simeq} T_1 F^a_{\mathbb{Z}_2}(V)$ *for all* V.

Corollary

We have $T_1 F^a(0) \simeq F^a(0) \simeq \mathbb{L}_0(\mathbb{Z}[\mathbb{Z}_2])$.

Sketch of proof

- Agreeement with geometry

  $F_{\mathbb{Z}_2}(V) \to F^a_{\mathbb{Z}_2}(V) \to \mathbb{Z}$ natural in V, $\dim(V) \geq 5$

  $\rightsquigarrow F^a_{\mathbb{Z}_2}(V) \xrightarrow{\simeq} T_1 F^a_{\mathbb{Z}_2}(V)$ for $\dim(V) \geq 5$

  $\rightsquigarrow F^a_{\mathbb{Z}_2}(V \oplus W) \xrightarrow{\simeq} T_1 F^a_{\mathbb{Z}_2}(V \oplus W)$ for all V if $\dim(W) \geq 5$
- 4-periodicity

  $F^a_{\mathbb{Z}_2}(V) \xrightarrow{\simeq} \Omega^{4W} F^a_{\mathbb{Z}_2}(V)$ is natural in V

  $\rightsquigarrow T_1 F^a_{\mathbb{Z}_2}(V) \xrightarrow{\simeq} T_1 \Omega^{4W} F^a_{\mathbb{Z}_2}(V)$ for all V
- Thom isomorphism, $n = \dim(V)$, $m = \dim(W)$

$$
\begin{array}{ccccc}
\Omega^{4W} F^a_{\mathbb{Z}_2}(V) & \longrightarrow & F^a_{\mathbb{Z}_2}(V \oplus 4W) & \longrightarrow & \mathbb{L}_{n+4m-1}(\mathbb{Z}, SW/\mathbb{Z}_2) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
\rightsquigarrow T_1 \Omega^{4W} F^a_{\mathbb{Z}_2}(V) & \longrightarrow & T_1 F^a_{\mathbb{Z}_2}(V \oplus 4W) & \longrightarrow & \mathbb{L}_{n+4m-1}(\mathbb{Z}, SW/\mathbb{Z}_2)
\end{array}
$$

- Conclusion $T_1 F^a_{\mathbb{Z}_2}(V) \xrightarrow{\simeq} T_1 \Omega^{4W} F^a_{\mathbb{Z}_2}(V) \xleftarrow{\simeq} \Omega^{4W} F^a_{\mathbb{Z}_2}(V) \xleftarrow{\simeq} F^a_{\mathbb{Z}_2}(V)$

$T_1 F^a_{\mathbb{Z}_2}(0) = F^a_{\mathbb{Z}_2}(0) = \Omega^{4W} F^a_{\mathbb{Z}_2}(0) = ?$

$$\begin{array}{ccccccc}
\mathbb{L}_{4m}(\mathbb{Z}[\mathbb{Z}_2]) & \longrightarrow & F^a_{\mathbb{Z}_2}(4W) & \longrightarrow & \mathbb{L}_{4m-1}(\mathbb{Z}, SW/\mathbb{Z}_2) & \longrightarrow & \mathbb{L}_{4m-1}(\mathbb{Z}[\mathbb{Z}_2]) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
\Omega^{4W} F^a_{\mathbb{Z}_2}(0) & \longrightarrow & F^a_{\mathbb{Z}_2}(4W) & \longrightarrow & \mathbb{L}_{4m-1}(\mathbb{Z}, SW/\mathbb{Z}_2) & \longrightarrow & \mathbb{L}_{4m-1}(\mathbb{Z}[\mathbb{Z}_2])
\end{array}$$

Corollary

There is a homotopy equivalence of homotopy fibration sequences for all V

$$\begin{array}{ccccc}
\Omega^\infty((S(V)_+ \wedge F^a_{\mathbb{Z}_2}{}^{(1)})_{hO(1)}) & \longrightarrow & T_1 F^a_{\mathbb{Z}_2}(0) & \longrightarrow & T_1 F^a_{\mathbb{Z}_2}(V) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
\Omega^\infty((S(V)_+ \wedge \mathbb{L}_\bullet)_{hO(1)}) & \longrightarrow & \mathbb{L}_0(\mathbb{Z}[\mathbb{Z}_2]) & \longrightarrow & \mathbb{S}(SV/\mathbb{Z}_2)
\end{array}$$