On fake lens spaces

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History

- *1930's Franz, Reidemeister - classical lens spaces*
- *1960's Browder, Livesay, López de Medrano, Petrie, Wall - fake lens spaces N odd, $N = 2$.*

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Outline

- *classical lens spaces*
- *fake lens spaces*
- *homotopy classification*
- *simple homotopy classification*
- *homeomorphism classification*

Classical lens spaces

Let $N \in \mathbb{N}$, $N > 2$, $G = \mathbb{Z}_N$, V f.d. (cplx) G-representation, free on $V \setminus 0$
Denote $LV = SV/G$

We know $V = \oplus_{l=1}^{n} V_l$, $V_l \cong \mathbb{C}$, the G-action $z \mapsto z \cdot e^{\frac{2\pi i k_l}{N}}$, $(k_l, N) = 1$
$L(N, k_1, \ldots, k_n) = LV$

$\pi_1 LV \cong G$, $\pi_i LV \cong \pi_i SV$ for $i \geq 2$

Let $r_l \cdot k_l \equiv 1 \pmod{N}$
The $\mathbb{Z}G$-chain complex $C_*(SV) = C_*(\widetilde{LV})$ is

$$0 \longrightarrow \mathbb{Z}G \xrightarrow{T^{r_1}-1} \mathbb{Z}G \xrightarrow{Z} \mathbb{Z}G \xrightarrow{T^{r_2}-1} \mathbb{Z}G \xrightarrow{Z} \cdots \xrightarrow{T^{r_n}-1} \mathbb{Z}G \longrightarrow 0$$

where $G = \langle T \rangle$, $Z = 1 + T + \cdots + T^{N-1}$

The $\mathbb{Z}$-chain complex $C_*(LV)$ is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{N} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{N} \cdots \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

$H_0(LV, \mathbb{Z}) \cong H_{2n-1}(LV, \mathbb{Z}) \cong \mathbb{Z}$,
$H_{2i-1}(LV, \mathbb{Z}) \cong G$, $H_{2i}(LV, \mathbb{Z}) \cong 0$ for $0 < i < n$

Proposition

The lens spaces $L(N, k_1, \ldots, k_n) \simeq L(N, k'_1, \ldots, k'_n)$ *if and only if*

$$k_1 \cdots k_n \equiv c^n \cdot k'_1 \cdots k'_n \bmod N$$

for some $c \in \mathbb{Z}_N^\times$.

Notice $L(N, k_1, \ldots, k_n) \simeq L(N, k, 1, \ldots, 1)$ for some $k \in \mathbb{Z}$.
$L(5, 1, 1)$ and $L(5, 2, 1)$ are not $\simeq$

Proposition (Franz, Reidemeister)

The lens spaces $L(N, k_1, \ldots, k_n) \cong L(N, k'_1, \ldots, k'_n)$ *if and only if*

$$k_i \equiv \pm c \cdot k'_{\sigma(i)} \bmod N \text{ for all } i$$

for some permutation σ *on* n *letters and some* $c \in \mathbb{Z}_N^\times$.

$L(7, 1, 1) \simeq L(7, 2, 1)$, but they are not $\cong$

Fake lens spaces

Definition

A *fake lens space* is a space $L(\alpha)$ obtained as the orbit space $L(\alpha) = SV/\alpha$ of a free action α of a finite cyclic group $G = \mathbb{Z}_N$ on SV.

Denote $L(0)$ the standard lens space $L(N, 1, \ldots, 1)$. For $i \in \mathbb{N}$ let $L(0)^i$ be its i-skeleton.

Lemma (Wall)

For any fake lens space $L(\alpha)$ there is a homotopy equivalence $h \colon L(\alpha) \longrightarrow L(0)^{2n-2} \cup_\phi e^{2n-1}$

Proof.

$$\begin{array}{ccc} L(\alpha) & \longrightarrow & K(G,1) \\ & \overset{f}{\searrow} & \uparrow \\ & & L(0)^{2n-1} \end{array}$$

There exists $g \colon L(0)^{2n-2} \to L(\alpha)$ s. t. $f \cdot g \colon L(0)^{2n-2} \to L(0)^{2n-1}$ is the inclusion. Also g is $(2n-2)$-connected and s. t. $\pi_{2n-1}(g) \cong \mathbb{Z}G$. Attaching a (single) cell kills $\pi_{2n-1}(g)$. □

Definition

A *polarization* of LV is a pair (T, e), where $\langle T \rangle \in G$ and $e \colon \widetilde{L(\alpha)} \xrightarrow{\simeq} SV$.

Homotopy classification

Proposition (Wall)

For fixed $N, n \in \mathbb{N}$ there are $\varphi(N) = |\mathbb{Z}_N^\times|$ polarized homotopy types of polarized fake lens spaces $L(\alpha)$ of dimension $2n-1$. For any polarized fake lens space $L(\alpha)$ there is a polarized homotopy equivalence

$$L(\alpha) \longrightarrow L(N, k, 1, \dots, 1).$$

Use the k-invariant $k_{2n-2}L(\alpha) \in H^{2n}(K(G,1), \mathbb{Z}) \cong \mathbb{Z}_N$.

Simple homotopy classification

D^n, $S^{n-1} = S_-^{n-1} \cup_{S^{n-2}} S_+^{n-1}$, $S_\pm^{n-1} \cong D^{n-1}$
(D^n, S_-^{n-1}), (S_-^{n-1}, S^{n-2}) are CW-pairs, X a CW-complex
$q\colon S_-^{n-1} \longrightarrow X$ a cellular map
$Y = X \cup_q D^n$ the pushout
$j\colon X \hookrightarrow Y$, $\exists\ r\colon Y \to X$, $r \cdot j = \mathrm{id}$, $j \cdot r \simeq \mathrm{id}$
j is called an elementary expansion
r is called an elementary collapse

Definition

A map $h\colon X \to Y$ of CW-complexes is a *simple homotopy equivalence* if it is homotopic to a composition of maps

$$X = X[0] \xrightarrow{h_0} X[1] \xrightarrow{h_1} X[1] \to \cdots \to X[n-1] \xrightarrow{h_{n-1}} X[n] = Y$$

such that each h_i is an elementary collapse or an elementary expansion.

Any homeomorphism is a simple homotopy equivalence.

Whitehead torsion

Theorem

A homotopy equivalence $h\colon X \to Y$ between connected finite CW-complexes X, Y with $G = \pi_1 X = \pi_1 Y$ is a simple homotopy equivalence if and only if $\tau(h) = 0$ in $\mathrm{Wh}(G)$.

Here the Whitehead torsion $\tau(h)$ is a certain well-defined element in the Whitehead group $\mathrm{Wh}(G) = K_1(\mathbb{Z}G)/(\pm G)$.

For $G = \mathbb{Z}_N$ we have $\mathrm{Wh}(G) = \mathbb{Z}G^{\times}/(\pm G)$.

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For $G = \mathbb{Z}_N$ we have $\mathrm{Wh}(G) = \mathbb{Z}G^\times/(\pm G)$.

Proposition (Wall)

For any polarized fake lens space $L(\alpha)$ there exists a polarized simple homotopy equivalence

$$h\colon L(\alpha) \xrightarrow{\simeq_s} L(0)^{2n-2} \cup_\phi e^{2n-1}.$$

Recall from homotopy classification the map $g\colon L(0)^{2n-2} \longrightarrow L(\alpha)$ with $\pi_{2n-1}(g) = \mathbb{Z}G$. One can choose ϕ so that $\tau(h) = 0$ in $\mathrm{Wh}(G)$.

Reidemeister torsion

Definition

Let X be a finite CW -complex with $G = \pi_1 X$ and $a\colon \mathbb{Z}G \to R$ a ring homomorphism for some ring R such that $C' = C_*(\widetilde{X}) \otimes_{\mathbb{Z}G} R$ is acyclic. Then the chain complex C' represents an element $\Delta_a(X)$ in $K_1(R)/(\pm G)$ called the Reidemeister torsion of X.

Theorem

If $h\colon X \to Y$ is a homotopy equivalence between finite connected CW-complexes with $G = \pi_1 X = \pi_1 Y$ and $a\colon \mathbb{Z}G \longrightarrow R$ is a ring homomorphism such that $\Delta_a(X)$ and $\Delta_a(Y)$ are defined then

$$a_*(\tau(h)) = \Delta_a(Y)/\Delta_a(X) \in K_1(R)/(\pm G).$$

Reidemeister torsion of fake lens spaces

Recall $G = \mathbb{Z}_N = \langle T \rangle$. Let $Z = 1 + T + T^2 + \cdots T^{N-1}$.
We have $0 \longrightarrow \langle Z \rangle \longrightarrow \mathbb{Z}G \longrightarrow R_G \longrightarrow 0$.
We have $\mathbb{Q}G \cong \mathbb{Q}\langle Z \rangle \oplus \mathbb{Q}R_G$.
We have $C_*(\widetilde{L(\alpha)}) \otimes_{\mathbb{Z}G} \mathbb{Q}R_G$ is acyclic.
Hence $\Delta(L(\alpha)) \in K_1(\mathbb{Q}R_G)/(\pm G) = \mathbb{Q}{R_G}^{\times}/(\pm G)$
We have $C_*(\widetilde{L(\alpha)})$

$$0 \longrightarrow \mathbb{Z}G \xrightarrow{(T-1)\cdot U} \mathbb{Z}G \xrightarrow{Z} \mathbb{Z}G \xrightarrow{T-1} \cdots \xrightarrow{T-1} \mathbb{Z}G \xrightarrow{Z} \mathbb{Z}G \longrightarrow 0$$

where $U \in \mathbb{Z}G$ maps to $u \in R_G^{\times}$.

We have $C_*(\widetilde{L(\alpha)}) \otimes_{\mathbb{Z}G} \mathbb{Q}R_G$

$$0 \longrightarrow \mathbb{Q}R_G \xrightarrow{(T-1)\cdot u} \mathbb{Q}R_G \xrightarrow{0} \mathbb{Q}R_G \xrightarrow{T-1} \cdots \xrightarrow{T-1} \mathbb{Q}R_G \xrightarrow{0} \mathbb{Q}R_G \longrightarrow 0$$

$\Delta(L(\alpha)) = (T-1)^n \cdot u \in \mathbb{Q}R_G^{\times}/(\pm G)$

Simple homotopy classification of fake lens spaces

Theorem (Wall)

For a fixed N, $n \in \mathbb{N}$ there is a one-to one correspondence between the polarized simple homotopy types of polarized fake lens spaces $L(\alpha)$ of dimension $2n-1$ and $\pi_1 L(\alpha) = G = \mathbb{Z}_N$ and the units $u \in R_G^\times/(\pm G)$.

Homeomorphism classification

$\mathcal{S}^s(X)$ = the simple structure set of a closed manifold X:

- elements are represented by (M, f), $f : M \xrightarrow{\simeq_s} X$
- $(M, f) \sim (M', f')$ if there exists $h \colon M \xrightarrow{\cong} M'$ such that $f \cdot h \simeq f'$

DIFF, PL, TOP version. In TOP version, $\mathcal{S}^s(X)$ is an abelian group.

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Theorem (Wall and others)

- *If N is odd, $\mathcal{S}^s(L(\alpha)) \cong F$, free abelian group of rank $(N-1)/2$.*
- *If $N = 2^K$, $\mathcal{S}^s(L(\alpha)) \cong F \oplus T$, F free ab. group, rank $N/2$ (n even), $(N-2)/2$ (n is odd), and T is a 2-primary torsion group.*

Calculation (Wall and others, M.)

For the group T above we have:

$$T \cong T_N \oplus T_2, \quad T_N \subset \oplus_{k=1}^{a} \mathbb{Z}_N, \quad T_2 \cong \oplus_{k=1}^{a} \mathbb{Z}_2, \quad a = \lfloor (n-1)/2 \rfloor.$$

The order of T_N is N^a/X where $X \leq (N/4)^{(N/4)}$.

The surgery exact sequence

How to calculate $\mathcal{S}^s(X)$?

The surgery exact sequence for X ($\dim(X) = n \geq 5$)

$$\cdots \mathcal{N}_\partial(X \times I) \xrightarrow{\theta} L^s_{n+1}(G) \xrightarrow{\partial} \mathcal{S}^s(X) \xrightarrow{\eta} \mathcal{N}(X) \xrightarrow{\theta} L^s_n(G)$$

$\mathcal{N}_\partial(X)$ = normal invariants (generalized cohomology theory)
$(f,b)\colon(M,\partial M) \to (X,\partial X)$ of deg 1, $b\colon \nu_M \to \nu_X$, $(f,b)|_\partial$ is a $\cong$.
$(f,b) \sim (f',b')$ if there is a bordism between them

$L^s_n(G)$ = the surgery obstruction group, where $G = \pi_1(X)$ (algebra)

θ: Let $(f,b)\colon(M,\partial M) \to (X,\partial X)$ represent an element in $\mathcal{N}_\partial(X)$
$(f,b) \sim (f',b')$ rel ∂ s.t. f' is a $\simeq_s$ $\iff$ $\theta(f,b) = 0 \in L_n(G)$, $n \geq 5$

∂: Let $x \in L_{n+1}(G)$. There $\exists$ $(F,B)\colon W \to X \times I$ deg. 1 normal map s.t. $\theta(F,B) = x$, $\partial_0(F,B) = (\mathrm{id},\mathrm{id})$, $\partial_1(F,B) = (f,b)$, with f a $\simeq_s$
$f\colon \partial_1 W \xrightarrow{\simeq_s} X$ represents an element in $\mathcal{S}^s(X)$. Let $\partial(x) = (\partial_1 W, f)$.

Normal invariants

We have

$$\mathcal{N}(L(\alpha)) \cong \bigoplus_{k\geq 1} H^{4k}(L(\alpha);\mathbb{Z}_{(2)}) \oplus \bigoplus_{k\geq 1} H^{4k-2}(L(\alpha);\mathbb{Z}_2) \oplus KO[\frac{1}{2}](L(\alpha))$$

Hence if N is odd

$$\mathcal{N}(L(\alpha)) \cong KO[\frac{1}{2}](L(\alpha)) \qquad \text{AHSS: } H^*(L(\alpha);KO[\frac{1}{2}]) \Rightarrow KO[\frac{1}{2}](L(\alpha))$$

It is a torsion group of order N^a where $a = \lfloor (n-1)/2 \rfloor$.

Hence if $N = 2^K$

$$\mathcal{N}(L(\alpha)) \cong \bigoplus_{k\geq 1} H^{4k}(L(\alpha);\mathbb{Z}_{(2)}) \oplus \bigoplus_{k\geq 1} H^{4k-2}(L(\alpha);\mathbb{Z}_2) \cong \bigoplus_{k=1}^{\lfloor (n-1)/2 \rfloor} \mathbb{Z}_N \oplus \bigoplus_{k=1}^{\lfloor n/2 \rfloor} \mathbb{Z}_2$$

L-theory

For any G we have $L^s_n(G) = L^s_{n+4}(G)$

$L^s_{2n}(G)$ is a Witt group of non-sing. $(-1)^n$-quadratic forms over $\mathbb{Z}G$

$x \in L^s_{2n}(G) \rightsquigarrow \lambda \colon H \times H \longrightarrow \mathbb{Z}$ a G-invariant $(-1)^n$-sym. bilin. form,
H is a f.g. free based $\mathbb{Z}G$-module, $\lambda(gx, gy) = \lambda(x, y)$, $g \in G$
$\rightsquigarrow \lambda_{\mathbb{C}} \colon H_{\mathbb{C}} \times H_{\mathbb{C}} \longrightarrow \mathbb{C}$
$\rightsquigarrow \mathrm{Sign}(\lambda_{\mathbb{C}}) = [H^+_{\mathbb{C}}] - [H^-_{\mathbb{C}}] \in R(G)$, the complex representation ring.
$\rightsquigarrow$ G-sign$\colon L^s_{2n}(G) \longrightarrow R(G)$

cplx conjugation $\rightsquigarrow$ (± 1)-eigenspaces $R^{(\pm 1)}(G) = \{x \in R(G) \mid x = \pm \bar{x}\}$

G-sign$\colon L^s_{2n}(G)[1/2] \xrightarrow{\cong} 4R^{(-1)^n}(G)$

For $G = \mathbb{Z}_N$ we have $R(G) = \oplus\mathbb{Z} \oplus \oplus\mathbb{Z}[\mathbb{Z}_2] =$ (type I) $\oplus$ (type II)

and $L^s_{2n}(G) = \Sigma_I \oplus \Sigma_{II} \oplus \Sigma_{III}$

For $G = \mathbb{Z}_N$ we have

$$L^s_{2n}(G) = \Sigma_I \oplus \Sigma_{II} \oplus \Sigma_{III}$$

where

$$\operatorname{rank}(\Sigma_I) = \begin{cases} 0 & n \text{ odd} \\ 1 & n \text{ even}, \ N \text{ odd} \\ 2 & n \text{ even}, \ N \text{ even} \end{cases} \qquad \operatorname{rank}(\Sigma_{II}) = \begin{cases} (N-1)/2 & N \text{ odd} \\ (N-2)/2 & N \text{ even} \end{cases}$$

and

$$\Sigma_{III} = \begin{cases} 0 & n \text{ even} \\ \mathbb{Z}_2 & n \text{ odd} \end{cases}$$

$$L^s_n(G) = L^s_n(1) \oplus \widetilde{L}^s_n(G)$$

The surgery exact sequence for fake lens spaces

If $N = 1$ we have $L^s_n(1) = \mathbb{Z}$, 0, $\mathbb{Z}_2$, 0

If N is odd we have $L^s_n(G) = \mathbb{Z} \oplus \Sigma$, 0, $\mathbb{Z}_2 \oplus \Sigma$, 0

If N is even we have $L^s_n(G) = \mathbb{Z} \oplus \mathbb{Z} \oplus \Sigma$, 0, $\mathbb{Z}_2 \oplus \Sigma$, $\mathbb{Z}_2$
where $\Sigma = \Sigma_{II}$.

The surgery exact sequence for fake lens spaces

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If N is even we have $L^s_n(G) = \mathbb{Z} \oplus \mathbb{Z} \oplus \Sigma, 0, \mathbb{Z}_2 \oplus \Sigma, \mathbb{Z}_2$
where $\Sigma = \Sigma_{II}$.

$$\cdots \mathcal{N}_\partial(L(\alpha) \times I) \xrightarrow{\theta} L^s_{2n}(G) \xrightarrow{\partial} \mathcal{S}^s(L(\alpha)) \xrightarrow{\eta} \mathcal{N}(L(\alpha)) \xrightarrow{\theta} L^s_{2n-1}(G)$$

Denote $\widetilde{\mathcal{N}}(L(\alpha)) = \ker(\mathcal{N}(L(\alpha)) \longrightarrow L^s_{2n-1}(G))$
We have $\widetilde{\mathcal{N}}(L(\alpha)) = \mathcal{N}(L(\alpha))$ unless n and N are even.

The surgery exact sequence for fake lens spaces

If $N = 1$ we have $L_n^s(1) = \mathbb{Z}$, 0, $\mathbb{Z}_2$, 0

If N is odd we have $L_n^s(G) = \mathbb{Z} \oplus \Sigma$, 0, $\mathbb{Z}_2 \oplus \Sigma$, 0

If N is even we have $L_n^s(G) = \mathbb{Z} \oplus \mathbb{Z} \oplus \Sigma$, 0, $\mathbb{Z}_2 \oplus \Sigma$, $\mathbb{Z}_2$
where $\Sigma = \Sigma_{II}$.

$$\cdots \mathcal{N}_\partial(L(\alpha) \times I) \xrightarrow{\theta} L_{2n}^s(G) \xrightarrow{\partial} \mathcal{S}^s(L(\alpha)) \xrightarrow{\eta} \mathcal{N}(L(\alpha)) \xrightarrow{\theta} L_{2n-1}^s(G)$$

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We have $\widetilde{\mathcal{N}}(L(\alpha)) = \mathcal{N}(L(\alpha))$ unless n and N are even.

Hence we have

$$\cdots \mathcal{N}_\partial(L(\alpha) \times I) \xrightarrow{?} L_{2n}^s(G) \xrightarrow{\partial} \mathcal{S}^s(L(\alpha)) \xrightarrow{\eta} \widetilde{\mathcal{N}}(L(\alpha)) \longrightarrow 0$$

Definition of the ρ-invariant

G a finite group, $\beta\colon G \curvearrowright Y^{2n}$, then G-sign$(Y) \in R^{(-1)^n}(G)$.

Proposition (Atiyah-Singer,Wall)

If Y is a closed manifold and β is free, then G-sign$(Y) = m \cdot Z$ *where $m \in \mathbb{Z}$ and Z is the regular representation.*

If Y smooth use ASGIT, if Y top use cobordism theory.

Proposition (Conner-Floyd)

Let $\alpha\colon G \curvearrowright X^{2n-1}$ be free. Then $\exists\, \beta\colon G \curvearrowright Y^{2n}$ free and $k \in \mathbb{N}$, such that $\partial Y = k \cdot X$.

Definition

Let X^{2n-1} have $\pi_1 X = G$. Define

$$\rho(X) = \frac{1}{k}\text{G-sign}(\widetilde{Y}) \in \mathbb{Q}R^{(-1)^n}(G)/\mathbb{Q}\langle Z\rangle$$

where $X = k \cdot \partial Y$, $\pi_1 Y = G$.

ρ-invariant properties

If $G = \mathbb{Z}_N$ then $R(G) = \mathbb{Z}\widehat{G}$, $R_{\widehat{G}} = R(G)/\langle Z \rangle$.

1. $\rho(X)$ is an h-cobordism invariant

   $\widetilde{\rho} \colon \mathcal{S}^s(L(\alpha)) \xrightarrow{\rho(-)-\rho(L(\alpha))} \mathbb{Q}R_{\widehat{G}}^{(-1)^n}$ is a homomorphism

2. We have a commutative diagram

$$\begin{array}{ccc} L_{2n}^s(G) & \longrightarrow & \mathcal{S}^s(L(\alpha)) \\ \Big\downarrow{\scriptstyle \text{G-sign}} & & \Big\downarrow{\scriptstyle \widetilde{\rho}} \\ 4R^{(-1)^n}(G) & \longrightarrow & \mathbb{Q}R_{\widehat{G}}^{(-1)^n} \end{array}$$

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$$\begin{array}{ccccc}
\widetilde{L}^s_{2n}(G) & \longrightarrow & L^s_{2n}(G) & \longrightarrow & \mathcal{S}^s(L(\alpha)) \\
\downarrow \cong & & \downarrow \text{G-sign} & & \downarrow \widetilde{\rho} \\
4R_{\widehat{G}}^{(-1)^n} & \longrightarrow & 4R^{(-1)^n}(G) & \longrightarrow & \mathbb{Q}R_{\widehat{G}}^{(-1)^n}
\end{array}$$

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If $G = \mathbb{Z}_N$ then $R(G) = \mathbb{Z}\widehat{G}$, $R_{\widehat{G}} = R(G)/\langle Z \rangle$.

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4R_{\widehat{G}}^{(-1)^n} & \longrightarrow & 4R^{(-1)^n}(G) & \longrightarrow & \mathbb{Q}R_{\widehat{G}}^{(-1)^n}
\end{array}$$

The composition $\widetilde{L}_{2n}^s(G) \longrightarrow \mathbb{Q}R_{\widehat{G}}^{(-1)^n}$ is injective.

The surgery exact sequence revisited

$$? \longrightarrow L^s_{2n}(G) \longrightarrow \mathcal{S}^s(L(\alpha)) \longrightarrow \widetilde{\mathcal{N}}(L(\alpha)) \longrightarrow 0$$

The surgery exact sequence revisited

$$0 \longrightarrow \widetilde{L}^s_{2n}(G) \longrightarrow \mathcal{S}^s(L(\alpha)) \longrightarrow \widetilde{\mathcal{N}}(L(\alpha)) \longrightarrow 0$$

The surgery exact sequence revisited

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{L}^s_{2n}(G) & \longrightarrow & \color{red}{\mathcal{S}^s(L(\alpha))} & \longrightarrow & \widetilde{\mathcal{N}}(L(\alpha)) & \longrightarrow & 0 \\
 & & \cong \big\downarrow \text{G-sign} & & \big\downarrow \widetilde{\rho} & & & & \\
0 & \longrightarrow & 4R^{(-1)^n}_{\widehat{G}} & \longrightarrow & \mathbb{Q}R^{(-1)^n}_{\widehat{G}} & & & &
\end{array}$$

The surgery exact sequence revisited

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{L}^s_{2n}(G) & \longrightarrow & \mathcal{S}^s(L(\alpha)) & \longrightarrow & \widetilde{\mathcal{N}}(L(\alpha)) & \longrightarrow & 0 \\
 & & \cong \big\downarrow \text{G-sign} & & \big\downarrow \widetilde{\rho} & & ? \big\downarrow [\widetilde{\rho}] & & \\
0 & \longrightarrow & 4R^{(-1)^n}_{\widehat{G}} & \longrightarrow & \mathbb{Q}R^{(-1)^n}_{\widehat{G}} & \longrightarrow & \mathbb{Q}R^{(-1)^n}_{\widehat{G}}/4R^{(-1)^n}_{\widehat{G}} & \longrightarrow & 0
\end{array}$$

Strategies:

The surgery exact sequence revisited

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{L}^s_{2n}(G) & \longrightarrow & \mathcal{S}^s(L(\alpha)) & \longrightarrow & \widetilde{\mathcal{N}}(L(\alpha)) & \longrightarrow & 0 \\ & & \cong \big\downarrow \text{G-sign} & & \big\downarrow \widetilde{\rho} & & ? \big\downarrow [\widetilde{\rho}] & & \\ 0 & \longrightarrow & 4R^{(-1)^n}_{\widehat{G}} & \longrightarrow & \mathbb{Q}R^{(-1)^n}_{\widehat{G}} & \longrightarrow & \mathbb{Q}R^{(-1)^n}_{\widehat{G}}/4R^{(-1)^n}_{\widehat{G}} & \longrightarrow & 0 \end{array}$$

Strategies:

- If $[\widetilde{\rho}]$ is injective, then $\mathcal{S}^s(L(\alpha)) \cong F \cong \widetilde{L}^s_{2n}(G)$
- If $[\widetilde{\rho}] = 0$, then $\mathcal{S}^s(L(\alpha)) = F \oplus T \cong \widetilde{L}^s_{2n}(G) \oplus \widetilde{\mathcal{N}}(L(\alpha))$

The surgery exact sequence revisited

$$
\begin{array}{ccccccccc}
 & & & & & & T & & \\
 & & & & & \swarrow & \downarrow & & \\
0 & \longrightarrow & \widetilde{L}^s_{2n}(G) & \longrightarrow & \color{red}{\mathcal{S}^s(L(\alpha))} & \longrightarrow & \widetilde{\mathcal{N}}(L(\alpha)) & \longrightarrow & 0 \\
 & & \cong \downarrow \text{G-sign} & & \downarrow \widetilde{\rho} & & \color{red}{?} \downarrow [\widetilde{\rho}] & & \\
0 & \longrightarrow & 4R^{(-1)^n}_{\widehat{G}} & \longrightarrow & \mathbb{Q}R^{(-1)^n}_{\widehat{G}} & \longrightarrow & \mathbb{Q}R^{(-1)^n}_{\widehat{G}}/4R^{(-1)^n}_{\widehat{G}} & \longrightarrow & 0
\end{array}
$$

Strategies:

- If $[\widetilde{\rho}]$ is injective, then $\mathcal{S}^s(L(\alpha)) \cong F \cong \widetilde{L}^s_{2n}(G)$
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The surgery exact sequence revisited

$$
\begin{array}{ccccccccc}
 & & & & & & T & & \\
 & & & & & \swarrow & \downarrow & & \\
0 & \longrightarrow & \widetilde{L}^s_{2n}(G) & \longrightarrow & \mathcal{S}^s(L(\alpha)) & \longrightarrow & \widetilde{\mathcal{N}}(L(\alpha)) & \longrightarrow & 0 \\
 & & \cong \downarrow \text{G-sign} & & \downarrow \widetilde{\rho} & & ? \downarrow [\widetilde{\rho}] & & \\
0 & \longrightarrow & 4R^{(-1)^n}_{\widehat{G}} & \longrightarrow & \mathbb{Q}R^{(-1)^n}_{\widehat{G}} & \longrightarrow & \mathbb{Q}R^{(-1)^n}_{\widehat{G}}/4R^{(-1)^n}_{\widehat{G}} & \longrightarrow & 0
\end{array}
$$

Strategies:

- If $[\widetilde{\rho}]$ is injective, then $\mathcal{S}^s(L(\alpha)) \cong F \cong \widetilde{L}^s_{2n}(G)$
- If $[\widetilde{\rho}] = 0$, then $\mathcal{S}^s(L(\alpha)) = F \oplus T \cong \widetilde{L}^s_{2n}(G) \oplus \widetilde{\mathcal{N}}(L(\alpha))$
- $T = \ker([\widetilde{\rho}])$, then $\mathcal{S}^s(L(\alpha)) = F \oplus T$, $F \cong \widetilde{L}^s_{2n}(G)$. Find T!

Results

Theorem (Wall)

If N is odd then the map $[\widetilde{\rho}]$ is injective, hence $\mathcal{S}^s(L(\alpha)) \cong \widetilde{L}^s_{2n}(G)$.

If $N = 2^K$ then $\widetilde{\mathcal{N}}(L(\alpha)) = \oplus_{k=1}^{a}\mathbb{Z}_N \oplus \oplus_{k=1}^{a}\mathbb{Z}_2$, where $a = \lfloor (n-1)/2 \rfloor$. We have $\oplus_{k=1}^{a}\mathbb{Z}_2 \subset T$. Further

$$[\widetilde{\rho}] \colon \oplus_{k=1}^{a} \mathbb{Z}_N \longrightarrow \oplus_{k=1}^{A}\mathbb{Z}_N \subset \mathbb{Q}R^{(-1)^n}_{\widehat{G}} / 4R^{(-1)^n}_{\widehat{G}}$$

where $A = (N-1)/2$ or $N/2$. Clearly T_N has order at least N^a/N^A.

Calculation

If $N = 2^K$ then the order of T_N is at least N^a/X, where $X = (N/4)^{N/4}$.

If $N = 4$ then the order of T_N is N^a.
If $N = 8$ then the order of T_N is $N^a/2$.

Tools - more properties of the ρ-invariant

1. $\rho(L(\alpha) * L(\alpha')) = \rho(L(\alpha)) \cdot \rho(L(\alpha'))$
2. $\rho(L(N,k)) = \left(\frac{1+t^k}{1-t^k}\right) \in \mathbb{Q}R_{\widehat{G}}$, where $\widehat{G} = \langle t \rangle$.
Note $(1-t)^{-1} = (-1/N) \cdot (1 + 2t + 3t^2 + \cdots + Nt^{N-1})$.
3. Suppose $\pi_1 X^{2n-1} = G$, and $\exists\, \beta \colon G \curvearrowright Y^{2n}$, $\partial Y = \widetilde{X}$. Then

$$\rho(X) \colon g \mapsto \rho(g,X) = \text{G-sign}(g,Y) - L(g,Y) \in \mathbb{C}$$

where $L(g,Y)$ is the "right-hand side" in the ASGIT. When $X = L(\alpha)$ can choose $Y = DV$.

When N is odd, the proof is by induction. One can calculate the ρ-invariant for classical lens spaces, which exhaust the normal invariants in a low-dimensional case. Then one uses the action of the $L^s_{2n}(G)$ and the join to produce enough fake lens spaces in higher dimensions.

4. The ρ-invariant can be defined for $\alpha\colon G \curvearrowright X^{2n-1}$ free, where G is a cpct Lie group ($G = S^1$).

$$\rho(X)\colon t \mapsto \rho(t, X) \in \mathbb{C} \text{ for } t \in S^1$$

Let $h\colon Q \xrightarrow{\simeq} \mathbb{P}V \in \mathcal{S}^s(\mathbb{P}V) \rightsquigarrow \alpha\colon \curvearrowright \widetilde{Q} \cong SV$ free. Wall gives the formula

$$\widetilde{\rho}(h) = \rho(t, \widetilde{Q}) - \rho(t, SV) = \sum_k 8 s_{4k}(f^{n-2k} - f^{n-2k-2})$$

where $f = \frac{1+t}{1-t}$ adn $s_{4k} \in \mathbb{Z}$ are the normal invariants of $h\colon Q \xrightarrow{\simeq} \mathbb{P}V \in \widetilde{\mathcal{N}}(\mathbb{P}V) \cong \oplus_{k=1}^{a} \mathbb{Z} \oplus \oplus_{k=1}^{a} \mathbb{Z}_2$.

Further $\mathcal{S}^s(\mathbb{P}V) \to \widetilde{\mathcal{N}}(\mathbb{P}V) \to \widetilde{\mathcal{N}}(L(0))$ is (almost) surjective.

The formula above is natural for $G < S^1$. Hence we obtain a formula for $[\widetilde{\rho}]$ in terms of $r_{4k} \in \widetilde{\mathcal{N}}(L(0)) \cong \oplus_{k=1}^{a} \mathbb{Z}_N \oplus \oplus_{k=1}^{a} \mathbb{Z}_2$.

The formula can be used to test when does $[\widetilde{\rho}]$ equal 0.

Summary

The invarinats of fake lens spaces are: polarization, Reidemeister torsion, ρ-invariant if N is odd. If $N = 2^K$ then we also need normal invariants.

More problems: study the join map, fibering over complex projective spaces, the structure sets $\mathcal{S}^s_\partial(L(\alpha) \times D^k)$.