On fake lens spaces

Tibor Macko, Christian Wegner

Westfälische Wilhelms Universität Münster
www.math.uni-muenster.de/u/macko

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Outline

- *classical lens spaces vs fake lens spaces*
- *homotopy classification and simple homotopy classification*
- *the surgery exact sequence for a general X*
- *the surgery exact sequence for fake lens spaces*
- *ρ-invariant*
- *calculations when $N = 2^K$*
- *the general case $N = 2^K \cdot M$, M odd*

Classical lens spaces and fake lens spaces

Let $N \in \mathbb{N}$, $N \geq 2$, $G = \mathbb{Z}_N$.

Definition

A *classical lens space* is a space $L^{2d-1}(V)$ obtained as the orbit space $L^{2d-1}(V) = S(V)/G$ where V is a d-dimensional (complex) G-representation, free on $V \setminus 0$.

$L^{2d-1}(V) = L^{2d-1}(N, k_1, \ldots, k_d)$, $(k_i, N) = 1$.

$L^{2d-1}(\alpha_k) := L^{2d-1}(N, k, 1, \ldots, 1)$

Definition

A *fake lens space* is a space $L^{2d-1}(\alpha)$ obtained as the orbit space $L^{2d-1}(\alpha) = S^{2d-1}/\alpha$ of a free action α of G on S^{2d-1}.

Definition

A *polarization* of $L^{2d-1}(\alpha)$ is a pair (T, e), where $\langle T \rangle \in G$ and $e\colon \widetilde{L}^{2d-1}(\alpha) \xrightarrow{\simeq} S^{2d-1}$.

For $i \in \mathbb{N}$ let $L^i(\alpha_1)$ be the i-skeleton.

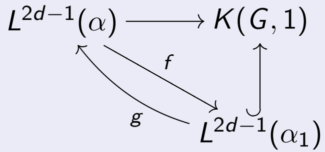
Proposition (Wall)

For any polarized fake lens space $L^{2d-1}(\alpha)$ there exists a polarized simple homotopy equivalence

$$h\colon L^{2d-1}(\alpha) \xrightarrow{\simeq_s} L^{2d-2}(\alpha_1) \cup_\phi e^{2d-1},$$

which is unique in some sense.

Proof.



There exists $g\colon L^{2d-1}(\alpha_1) \to L^{2d-1}(\alpha)$ such that $f \cdot g\colon L^{2d-2}(\alpha_1) \to L^{2d-1}(\alpha_1)$ is the inclusion.

The map g is $(2d-2)$-connected and $\pi_{2d-1}(g) \cong \mathbb{Z}G$.

Attaching a (single) cell kills $\pi_{2d-1}(g)$. □

Homotopy and Simple homotopy classification

Proposition (Wall)

For fixed $N, d \in \mathbb{N}$ there are $\varphi(N) = |\mathbb{Z}_N^\times|$ polarized homotopy types of polarized fake lens spaces $L^{2d-1}(\alpha)$ of dim $2d-1$ and $\pi_1 L(\alpha) = G = \mathbb{Z}_N$. For any polarized fake lens space $L^{2d-1}(\alpha)$ there is a polarized homotopy equivalence

$$L^{2d-1}(\alpha) \longrightarrow L^{2d-1}(\alpha_k).$$

$k_{2d-2}(L^{2d-1}(\alpha)) \in H^{2d}(K(G,1),\mathbb{Z}) \cong \mathbb{Z}_N$

Theorem (Wall)

For a fixed N, $d \in \mathbb{N}$ there is a one-to one correspondence between the polarized simple homotopy types of polarized fake lens spaces $L^{2d-1}(\alpha)$ of dimension $2d-1$ and $\pi_1 L(\alpha) = G = \mathbb{Z}_N$ and the units $u \in R_G^\times/(\pm G)$.

$\Delta(L^{2d-1}(\alpha)) \in \mathbb{Q}R_G^\times/(\pm G)$

Homeomorphism classification

$\mathcal{S}^s(X)$ = the simple structure set of a closed manifold X:

- elements are represented by (M, f), $f : M \xrightarrow{\simeq_s} X$
- $(M, f) \sim (M', f')$ if there exists $h \colon M \xrightarrow{\cong} M'$ such that $f' \cdot h \simeq f$

In TOP version, $\mathcal{S}^s(X)$ can be made into an abelian group.

How to calculate $\mathcal{S}^s(X)$?

The surgery exact sequence for X ($\dim(X) = n \geq 5$)

$$\cdots \mathcal{N}_\partial(X \times I) \xrightarrow{\theta} L^s_{n+1}(G) \xrightarrow{\partial} \mathcal{S}^s(X) \xrightarrow{\eta} \mathcal{N}(X) \xrightarrow{\theta} L^s_n(G)$$

$$\mathcal{S}^s(L(\alpha)) \text{ calculated } \begin{cases} N \text{ odd (Wall)} \\ N = 2 \text{ (Browder, Livesay, López de Medrano, Wall)} \end{cases}$$

In general $\mathcal{S}^s(L(\alpha)) = ?$

The surgery exact sequence

$\mathcal{N}_\partial(X)$ = normal invariants (generalized cohomology theory)
$(f,b)\colon(M,\partial M)\to(X,\partial X)$ of deg 1, $b\colon\nu_M\to\xi$, $(f,b)|_\partial$ is a $\cong$.
$(f,b)\sim(f',b')$ if there is a normal bordism between them

$L_n^s(G)$ = the surgery obstruction group, where $G=\pi_1(X)$ (algebra)

θ: Let $(f,b)\colon(M,\partial M)\to(X,\partial X)$ represent an element in $\mathcal{N}_\partial(X)$
$(f,b)\sim(f',b')$ rel ∂ s.t. f' is a $\simeq_s$ $\iff$ $\theta(f,b)=0\in L_n(G)$, $n\geq 5$

∂: Let $x\in L_{n+1}(G)$. There $\exists$ $(F,B)\colon W\to X\times I$ deg. 1 normal map s.t. $\theta(F,B)=x$, $\partial_0(F,B)=(\mathrm{id},\mathrm{id})$, $\partial_1(F,B)=(f,b)$, with f a $\simeq_s$
$f\colon\partial_1 W\xrightarrow{\simeq_s}X$ represents an element in $\mathcal{S}^s(X)$. Let $\partial(x)=(\partial_1 W,f)$.

Normal invariants

If X is rationally trivial then

$$\mathcal{N}(X) \cong \bigoplus_{i\geq 1} H^{4i}(X;\mathbb{Z}_{(2)}) \oplus \bigoplus_{i\geq 1} H^{4i-2}(X;\mathbb{Z}_2) \oplus KO\left[\frac{1}{2}\right](X)$$

For $N = 2^K \cdot M$, M odd:

$$((\mathbf{t}_{2i})_i, \mathbf{t}_{(odd)}) : \mathcal{N}(L^{2d-1}(\alpha)) \cong \bigoplus_{i=1}^{\lfloor (d-1)/2 \rfloor} \mathbb{Z}_{2^K} \oplus \bigoplus_{i=1}^{\lfloor d/2 \rfloor} \mathbb{Z}_2 \oplus KO\left[\frac{1}{2}\right](L^{2d-1}(\alpha))$$

where $KO\left[\frac{1}{2}\right](L(\alpha))$ is a torsion group of order M^c where $c = \lfloor (d-1)/2 \rfloor$

AHSS: $H^*(L(\alpha); KO\left[\frac{1}{2}\right]) \Rightarrow KO\left[\frac{1}{2}\right](L(\alpha))$

For $G = S^1$:

$$\oplus(\mathbf{s}_{2i})_i : \mathcal{N}(\mathbb{C}P^{d-1}) \cong \bigoplus_{i=1}^{\lfloor (d-1)/2 \rfloor} \mathbb{Z} \oplus \bigoplus_{i=1}^{\lfloor d/2 \rfloor} \mathbb{Z}_2$$

Let $G < S^1$, $\rightsquigarrow$ bundle $p_G : L^{2d-1}(\alpha_1) \longrightarrow \mathbb{C}P^{d-1}$:

$$\begin{array}{ccc} \mathcal{S}^s(\mathbb{C}P^{d-1}) & \xrightarrow{\eta} & \mathcal{N}(\mathbb{C}P^{d-1}) \\ \downarrow{\scriptstyle (p_G)^!} & & \downarrow{\scriptstyle (p_G)^!} \\ \mathcal{S}^s(L_G^{2d-1}(\alpha_1)) & \xrightarrow{\eta} & \mathcal{N}(L_G^{2d-1}(\alpha_1)) \end{array}$$

We have

$$\mathbf{t}_{4i} \circ (p_G)^! = [\mathbf{s}_{4i}]_{2^K}$$
$$\mathbf{t}_{4i-2} \circ (p_G)^! = \mathbf{s}_{4i-2}$$

L-theory

$$L^s_n(G) = L^s_{n+4}(G)$$

$L^s_{2d}(G)$ is a Witt group of non-sing. $(-1)^d$-quadratic forms over $\mathbb{Z}G$

$x \in L^s_{2d}(G) \rightsquigarrow \lambda \colon H \times H \longrightarrow \mathbb{Z}$ a G-invariant $(-1)^d$-sym. bilin. form,
H is a f.g. free based $\mathbb{Z}G$-module, $\lambda(gx, gy) = \lambda(x, y)$, $g \in G$
$\rightsquigarrow \lambda_{\mathbb{C}} \colon H_{\mathbb{C}} \times H_{\mathbb{C}} \longrightarrow \mathbb{C}$
$\rightsquigarrow \mathrm{Sign}(\lambda_{\mathbb{C}}) = [H^+_{\mathbb{C}}] - [H^-_{\mathbb{C}}] \in R(G)$, the complex representation ring.
$\rightsquigarrow$ (± 1)-eigenspaces $R^{(\pm 1)}(G) = \{x \in R(G) \mid x = \pm\bar{x}\}$

$$\text{G-sign} \colon L^s_{2d}(G)[1/2] \xrightarrow{\cong} 4R^{(-1)^d}(G)$$

$$L^s_n(G) = L^s_n(1) \oplus \widetilde{L}^s_n(G)$$

For $G = \mathbb{Z}_N$ we have

$$\widetilde{L}^s_{2d}(G) = 4 \cdot R^{(-1)^d}_{\widehat{G}} =: \Sigma_N(d)$$

where $R^{(-1)^d}_{\widehat{G}} = R(G)^{(-1)^d}/\langle Z \rangle$, Z is the regular representation

$$\operatorname{rank}(\Sigma_N(d)) = \begin{cases} (N-1)/2 & N \text{ odd} \\ (N-2)/2 & N \text{ even, } d \text{ odd} \\ N/2 & N \text{ even, } d \text{ even} \end{cases}$$

If $N = 1$ we have $L^s_n(1) = \mathbb{Z}$, 0, $\mathbb{Z}_2$, 0

If N is odd we have $L^s_n(G) = \mathbb{Z} \oplus \Sigma_N(0)$, 0, $\mathbb{Z}_2 \oplus \Sigma_N(1)$, 0

If N is even we have $L^s_n(G) = \mathbb{Z} \oplus \Sigma_N(0)$, 0, $\mathbb{Z}_2 \oplus \Sigma_N(1)$, $\mathbb{Z}_2$

The surgery exact sequence for fake lens spaces

We have

$$0 \longrightarrow \mathcal{S}^s(\mathbb{C}P^{d-1}) \xrightarrow{\eta} \mathcal{N}(\mathbb{C}P^{d-1}) \xrightarrow{\theta} L_{2d-2}(1) \longrightarrow 0$$

Hence

$$\oplus_{i=1}^{d-1} \mathbf{s}_{2i} : \mathcal{S}^s(\mathbb{C}P^{d-1}) \longrightarrow \bigoplus \mathbb{Z} \oplus \bigoplus \mathbb{Z}_2$$

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We have

$$\cdots \mathcal{N}_\partial(L(\alpha) \times I) \xrightarrow{\theta} L^s_{2d}(G) \xrightarrow{\partial} \mathcal{S}^s(L(\alpha)) \xrightarrow{\eta} \mathcal{N}(L(\alpha)) \xrightarrow{\theta} L^s_{2d-1}(G)$$

The surgery exact sequence for fake lens spaces

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Denote $\widetilde{\mathcal{N}}(L(\alpha)) = \ker(\mathcal{N}(L(\alpha)) \longrightarrow L^s_{2d-1}(G))$. Then

$$\cdots \mathcal{N}_\partial(L(\alpha) \times I) \xrightarrow{?} L^s_{2d}(G) \xrightarrow{\partial} \mathcal{S}^s(L(\alpha)) \xrightarrow{\eta} \widetilde{\mathcal{N}}(L(\alpha)) \longrightarrow 0$$

Definition of the ρ-invariant

G a finite group, $\beta\colon G \curvearrowright Y^{2d}$, then G-sign$(Y) \in R^{(-1)^d}(G)$.

Proposition (Atiyah-Singer,Wall)

If Y is a closed manifold and β is free, then G-sign$(Y) = m \cdot Z$ *where $m \in \mathbb{Z}$ and Z is the regular representation.*

If Y smooth use ASGIT, if Y top use cobordism theory.

Proposition (Conner-Floyd)

Let $\alpha\colon G \curvearrowright X^{2d-1}$ be free. Then $\exists\, \beta\colon G \curvearrowright Y^{2d}$ free and $k \in \mathbb{N}$, such that $\partial Y = k \cdot X$.

Definition

Let X^{2d-1} have $\pi_1 X = G$. Define

$$\rho(X) = \frac{1}{k}\text{G-sign}(\widetilde{Y}) \in \mathbb{Q}R^{(-1)^d}(G)/\mathbb{Q}\langle Z\rangle =: \mathbb{Q}R_{\widehat{G}}^{(-1)^d}$$

where $X = k \cdot \partial Y$, $\pi_1 Y = G$.

ρ-invariant properties

If $G = \mathbb{Z}_N$ then $R(G) = \mathbb{Z}\widehat{G} = \mathbb{Z}[\chi]/\langle \chi^N - 1 \rangle$,
$\mathbb{Q}R_{\widehat{G}} = \mathbb{Q}R(G)/\mathbb{Q}\langle Z \rangle = \mathbb{Q}[\chi]/\langle 1 + \chi + \cdots + \chi^{N-1} \rangle$.

1. We can show that

$$\widetilde{\rho} \colon \mathcal{S}^s(L(\alpha)) \xrightarrow{\rho(-)-\rho(L(\alpha))} \mathbb{Q}R_{\widehat{G}}^{(-1)^d} \text{ is a homomorphism}$$

2. We have a commutative diagram

$$\begin{array}{ccc} L^s_{2d}(G) & \longrightarrow & \mathcal{S}^s(L(\alpha)) \\ \big\downarrow \text{G-sign} & & \big\downarrow \widetilde{\rho} \\ 4R^{(-1)^d}(G) & \longrightarrow & \mathbb{Q}R_{\widehat{G}}^{(-1)^d} \end{array}$$

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$$\begin{array}{ccccc}
\widetilde{L}^s_{2d}(G) & \longrightarrow & L^s_{2d}(G) & \longrightarrow & \mathcal{S}^s(L(\alpha)) \\
\downarrow \cong & & \downarrow \text{G-sign} & & \downarrow \widetilde{\rho} \\
4R_{\widehat{G}}^{(-1)^d} & \longrightarrow & 4R^{(-1)^d}(G) & \longrightarrow & \mathbb{Q}R_{\widehat{G}}^{(-1)^d}
\end{array}$$

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\end{array}$$

The composition $\widetilde{L}^s_{2d}(G) \longrightarrow \mathbb{Q}R_{\widehat{G}}^{(-1)^d}$ is injective.

The surgery exact sequence revisited

$$? \longrightarrow L^s_{2d}(G) \longrightarrow \mathcal{S}^s(L(\alpha)) \longrightarrow \widetilde{\mathcal{N}}(L(\alpha)) \longrightarrow 0$$

The surgery exact sequence revisited

$$0 \longrightarrow \widetilde{L}^s_{2d}(G) \longrightarrow \mathcal{S}^s(L(\alpha)) \longrightarrow \widetilde{\mathcal{N}}(L(\alpha)) \longrightarrow 0$$

The surgery exact sequence revisited

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0 & \longrightarrow & \widetilde{L}^s_{2d}(G) & \longrightarrow & \mathcal{S}^s(L(\alpha)) & \longrightarrow & \widetilde{\mathcal{N}}(L(\alpha)) & \longrightarrow & 0 \\
 & & \cong \big\downarrow \text{G-sign} & & \big\downarrow \widetilde{\rho} & & & & \\
0 & \longrightarrow & 4R^{(-1)^d}_{\widehat{G}} & \longrightarrow & \mathbb{Q}R^{(-1)^d}_{\widehat{G}} & & & &
\end{array}$$

The surgery exact sequence revisited

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{L}^s_{2d}(G) & \longrightarrow & \mathcal{S}^s(L(\alpha)) & \longrightarrow & \widetilde{\mathcal{N}}(L(\alpha)) & \longrightarrow & 0 \\ & & \cong \downarrow \text{G-sign} & & \downarrow \widetilde{\rho} & & ? \downarrow [\widetilde{\rho}] & & \\ 0 & \longrightarrow & 4R^{(-1)^d}_{\widehat{G}} & \longrightarrow & \mathbb{Q}R^{(-1)^d}_{\widehat{G}} & \longrightarrow & \mathbb{Q}R^{(-1)^d}_{\widehat{G}}/4R^{(-1)^d}_{\widehat{G}} & \longrightarrow & 0 \end{array}$$

Strategies:

The surgery exact sequence revisited

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{L}^s_{2d}(G) & \longrightarrow & \mathcal{S}^s(L(\alpha)) & \longrightarrow & \widetilde{\mathcal{N}}(L(\alpha)) & \longrightarrow & 0 \\ & & \cong \downarrow \text{G-sign} & & \downarrow \widetilde{\rho} & & ? \downarrow [\widetilde{\rho}] & & \\ 0 & \longrightarrow & 4R^{(-1)^d}_{\widehat{G}} & \longrightarrow & \mathbb{Q}R^{(-1)^d}_{\widehat{G}} & \longrightarrow & \mathbb{Q}R^{(-1)^d}_{\widehat{G}}/4R^{(-1)^d}_{\widehat{G}} & \longrightarrow & 0 \end{array}$$

Strategies:

- If $[\widetilde{\rho}]$ is injective, then $\mathcal{S}^s(L(\alpha)) \cong \bar{\Sigma}_N(d) = \mathrm{im}\widetilde{\rho}$
- If $[\widetilde{\rho}] = 0$, then $\mathcal{S}^s(L(\alpha)) = \bar{\Sigma}_N(d) \oplus \widetilde{\mathcal{N}}(L(\alpha)) = \mathrm{im}\widetilde{\rho} \oplus \widetilde{\mathcal{N}}(L(\alpha))$

The surgery exact sequence revisited

$$
\begin{array}{ccccccccc}
 & & & & & & \ker[\widetilde{\rho}] & & \\
 & & & & & \swarrow & \downarrow & & \\
0 & \longrightarrow & \widetilde{L}^s_{2d}(G) & \longrightarrow & \color{red}{\mathcal{S}^s(L(\alpha))} & \longrightarrow & \widetilde{\mathcal{N}}(L(\alpha)) & \longrightarrow & 0 \\
 & & \cong \downarrow \text{G-sign} & & \downarrow \widetilde{\rho} & & \color{red}{?} \downarrow [\widetilde{\rho}] & & \\
0 & \longrightarrow & 4R^{(-1)^d}_{\widehat{G}} & \longrightarrow & \mathbb{Q}R^{(-1)^d}_{\widehat{G}} & \longrightarrow & \mathbb{Q}R^{(-1)^d}_{\widehat{G}}/4R^{(-1)^d}_{\widehat{G}} & \longrightarrow & 0
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The surgery exact sequence revisited

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\begin{array}{ccccccccc}
 & & & & & & \ker[\widetilde{\rho}] & & \\
 & & & \swarrow & & & \downarrow & & \\
0 & \longrightarrow & \widetilde{L}^s_{2d}(G) & \longrightarrow & \mathcal{S}^s(L(\alpha)) & \longrightarrow & \widetilde{\mathcal{N}}(L(\alpha)) & \longrightarrow & 0 \\
 & & \cong \downarrow \text{G-sign} & & \downarrow \widetilde{\rho} & & ? \downarrow [\widetilde{\rho}] & & \\
0 & \longrightarrow & 4R^{(-1)^d}_{\widehat{G}} & \longrightarrow & \mathbb{Q}R^{(-1)^d}_{\widehat{G}} & \longrightarrow & \mathbb{Q}R^{(-1)^d}_{\widehat{G}}/4R^{(-1)^d}_{\widehat{G}} & \longrightarrow & 0
\end{array}
$$

Strategies:

- If $[\widetilde{\rho}]$ is injective, then $\mathcal{S}^s(L(\alpha)) \cong \bar{\Sigma}_N(d) = \mathrm{im}\widetilde{\rho}$
- If $[\widetilde{\rho}] = 0$, then $\mathcal{S}^s(L(\alpha)) = \bar{\Sigma}_N(d) \oplus \widetilde{\mathcal{N}}(L(\alpha)) = \mathrm{im}\widetilde{\rho} \oplus \widetilde{\mathcal{N}}(L(\alpha))$
- In general $\mathcal{S}^s(L(\alpha)) = \bar{\Sigma}_N(d) \oplus \ker[\widetilde{\rho}] \cong \mathrm{im}\widetilde{\rho} \oplus \ker[\widetilde{\rho}]$. Find $\ker[\widetilde{\rho}]$!

Calculating the ρ-invariant

- $\rho(L(\alpha) * L(\alpha')) = \rho(L(\alpha)) \cdot \rho(L(\alpha'))$
- $\rho(L^{2d-1}(\alpha_1)) = f^d = \left(\frac{1+\chi}{1-\chi}\right)^d \in \mathbb{Q}R_{\widehat{G}}$, where $\widehat{G} = \langle \chi \rangle$.

  Note $(1-\chi)^{-1} = (-1/N) \cdot (1 + 2\chi + 3\chi^2 + \cdots + N\chi^{N-1})$.

Calculating the ρ-invariant

- $\rho(L(\alpha) * L(\alpha')) = \rho(L(\alpha)) \cdot \rho(L(\alpha'))$
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  Note $(1-\chi)^{-1} = (-1/N) \cdot (1 + 2\chi + 3\chi^2 + \cdots + N\chi^{N-1})$.
- Let G be a cpct Lie group (e.g. S^1), $\alpha \colon G \curvearrowright X^{2d-1}$ free, $\beta \colon G \curvearrowright Y^{2d}$, $\partial Y = X$, $\alpha = \beta|$ can define

$$\rho(X) \colon g \mapsto \rho(g, X) = \text{G-sign}(g, Y) - L(g, Y) \in \mathbb{C}$$

  where $L(g, Y)$ is the "right-hand side" in the ASGIT.
- Whenever both definitions apply they agree.
- ρ is natural in G

- Let $h\colon Q \xrightarrow{\simeq} \mathbb{C}P^{d-1} \in \mathcal{S}^s(\mathbb{C}P^{d-1})$ with $s_{2i} = \mathbf{s}_{2i}(h)$. Wall:

$$\widetilde{\rho}(g,h) = \rho(g,\widetilde{Q}) - \rho(g,S^{2d-1}) = \sum_i 8s_{4i}(f^{d-2i} - f^{d-2i-2}) \in \mathbb{C}$$

where $f = \frac{1+g}{1-g}$, $g \in S^1$.

- $\mathcal{S}^s(\mathbb{C}P^{d-1}) \xrightarrow{\eta} \widetilde{\mathcal{N}}(\mathbb{C}P^{d-1}) \xrightarrow{p^!} \widetilde{\mathcal{N}}(L^{2d-1}(\alpha_1))$ is (almost) surjective.

Let $N = 2^K$. Then $[\widetilde{\rho}]\colon \widetilde{\mathcal{N}}(L^{2d-1}(\alpha_1)) \longrightarrow \mathbb{Q}R_{\widehat{G}}^{(-1)^d}/4R_{\widehat{G}}^{(-1)^d}$:

$$d = 2e \rightsquigarrow [\widetilde{\rho}]\colon t = (t_{2i})_i \mapsto \sum_i 8\overline{t}_{4i}(f^{d-2i} - f^{d-2i-2})$$

$$d = 2e+1 \rightsquigarrow [\widetilde{\rho}]\colon t = (t_{2i})_i \mapsto \sum_i 8\overline{t}_{4i}(f^{d-2i} - f^{d-2i-2}) + 8\overline{t}_{4e}f$$

where $f = \frac{1+\chi}{1-\chi}$, $t = (t_{2i})_i$, $\overline{t}_{4i} \in \mathbb{Z}$, $\overline{t}_{4i} \equiv t_{4i} \bmod 2^K$.

Calculation of $\ker[\widetilde{\rho}]$ for $N = 2^K$

Our aim is to calculate $\ker\big([\widetilde{\rho}] : \widetilde{\mathcal{N}}(L^{2d-1}(\alpha)) \to \mathbb{Q}R_{\widehat{G}}^{(-1)^d}/4 \cdot R_{\widehat{G}}^{(-1)^d}\big)$ for $G = \mathbb{Z}_N$ with $N = 2^K$.

In the sequel we will focus on the case $\alpha = \alpha_1$ and $d = 2e + 1$.

Formula (for $\alpha = \alpha_1$, $d = 2e + 1$)

$$[\widetilde{\rho}](t) = 8 \cdot f \cdot q_{\bar{t}}(f^2)$$

where

$$q_{\bar{t}}(x) := \sum_{i=1}^{e-1} \bar{t}_{4i} \cdot x^{e-i-1} \cdot (x-1) + \bar{t}_{4e} \quad \in \mathbb{Z}[x]$$

with $\bar{t}_{4i} \in \mathbb{Z}$ such that $\bar{t}_{4i} \equiv t_{4i} \bmod 2^K$.

Calculation of $\ker[\widetilde{\rho}]$ for $N = 2^K$

The isomorphism

$$\begin{aligned} \widetilde{\mathcal{N}}(L^{4e+1}(\alpha_1)) &\cong \bigoplus_{i=1}^{e} \mathbb{Z}_2 \oplus \left\{ q \in \mathbb{Z}[x] \,\middle|\, \deg(q) \le e-1 \right\} / \; 2^K \cdot \mathbb{Z}[x] \\ t = (t_{2i})_i &\mapsto \left((t_{4i-2})_i \, , q_{\bar{t}} \right) \end{aligned}$$

restricts to an isomorphism

$$\ker[\widetilde{\rho}] \cong \bigoplus_{i=1}^{e} \mathbb{Z}_2 \oplus \left\{ q \in \mathbb{Z}[x] \,\middle|\, \deg(q) \le e-1, 8 \cdot f \cdot q(f^2) \in 4R_{\widehat{G}} \right\} / \; 2^K \mathbb{Z}[x].$$

Which polynomials satisfy $8 \cdot f \cdot q(f^2) \in 4 \cdot R_{\widehat{G}}$?

Which polynomials satisfy $8 \cdot f \cdot q(f^2) \in 4 \cdot R_{\widehat{G}}$?

$N = 2$:
all polynomials because $f = 0$,
hence $\ker[\widetilde{\rho}] \cong \bigoplus_{i=1}^{e} \mathbb{Z}_2 \oplus \bigoplus_{i=1}^{e} \mathbb{Z}_2$.

$N = 4$:
all polynomials because $f^{2n+1} = \frac{(-1)^n}{2} \cdot (\chi - \chi^3)$,
hence $\ker[\widetilde{\rho}] \cong \bigoplus_{i=1}^{e} \mathbb{Z}_2 \oplus \bigoplus_{i=1}^{e} \mathbb{Z}_4$.

$N = 8$:
$f = \frac{3}{4} \cdot (\chi - \chi^7) + \frac{1}{2} \cdot (\chi^2 - \chi^6) + \frac{1}{4} \cdot (\chi^3 - \chi^5) \Longrightarrow 8 \cdot f \notin 4 \cdot R_{\widehat{G}}$,
$f^3 = -\frac{11}{4} \cdot (\chi - \chi^7) - \frac{7}{2} \cdot (\chi^2 - \chi^6) - \frac{9}{4} \cdot (\chi^3 - \chi^5), \ \ldots$
A long calculation shows $\ker[\widetilde{\rho}] \cong \bigoplus_{i=1}^{e} \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \bigoplus_{i=2}^{e} \mathbb{Z}_8$.

$N = 2^K$ with $K \geq 4$:
difficult

The ring $\mathbb{Q}R_{\widehat{G}}$ and the Chinese Remainder Theorem

Recall that $\mathbb{Q}R_{\widehat{G}} = \mathbb{Q}[\chi]/I\langle K\rangle$ with

$$\begin{aligned} I\langle K\rangle &:= \left\langle 1+\chi+\chi^2+\ldots+\chi^{2^K-1} \right\rangle \\ &= \left\langle (1+\chi)\cdot(1+\chi^2)\cdot\ldots\cdot(1+\chi^{2^{K-1}}) \right\rangle. \end{aligned}$$

Consider the projections $pr_l : \mathbb{Q}R_{\widehat{G}} \to \mathbb{Q}[\chi]/\langle 1+\chi^{2^l}\rangle$ for $0 \leq l \leq K-1$.

The Chinese Remainder Theorem tells us that the ring homomorphism

$$\mathbb{Q}R_{\widehat{G}} \to \mathbb{Q}[\chi]/\langle 1+\chi\rangle \times \cdots \times \mathbb{Q}[\chi]/\langle 1+\chi^{2^{K-1}}\rangle$$

given by the projections pr_l is an isomorphism.

Usually, it is easier to make computations in $\mathbb{Q}[\chi]/\langle 1+\chi^{2^l}\rangle$ instead of $\mathbb{Q}R_{\widehat{G}}$.

When does an element $r \in \mathbb{Q}R_{\widehat{G}}$ lie in $4 \cdot R_{\widehat{G}}$?

Lemma

Let $r \in \mathbb{Q}R_{\widehat{G}}$. Let $r_l \in \mathbb{Q}R_{\widehat{G}}$ $(0 \leq l \leq K-1)$ such that $pr_l(r_l) = pr_l(r)$. Then

$$r = \sum_{l=0}^{K-1} 2^{l-K} \cdot r_l \cdot (1-\chi) \cdot \prod_{\substack{0 \leq n \leq K-1 \\ n \neq l}} (1+\chi^{2^n}).$$

Proof: Because of the Chinese Remainder Theorem it suffices to check

$$pr_{l'}(r) = pr_{l'}\Big(\sum_{l=0}^{K-1} 2^{l-K} \cdot r_l \cdot (1-\chi) \cdot \prod_{\substack{0 \leq n \leq K-1 \\ n \neq l}} (1+\chi^{2^n})\Big).$$

for all $0 \leq l' \leq K-1$. This is an easy calculation.

When does an element $r \in \mathbb{Q}R_{\widehat{G}}$ lie in $4 \cdot R_{\widehat{G}}$?

Corollary

Let $r \in \mathbb{Q}R_{\widehat{G}}$. If $pr_l(r) \in 2^{K+2-l} \cdot \mathbb{Z}[\chi]/\langle 1 + \chi^{2^l} \rangle$ for all $0 \leq l \leq K-1$ then $r \in 4 \cdot R_{\widehat{G}}$.

We are interested in how often the factor 2 is contained in $pr_l(r)$. We denote this number by $w_l(r)$.

Example: $pr_0(r) \in \mathbb{Q}[\chi]/\langle 1 + \chi \rangle = \mathbb{Q}$. We have $w_0(r) = z \in \mathbb{Z}$ if $pr_0(r) = 2^z \cdot \frac{a}{b}$ with odd integers a, b. If $pr_0(r) = 0$ then $w_0(r) = \infty$.

w_l-functions

Definition

Consider the ring of algebraic integers $\mathbb{Z}[\zeta_{2^{l+1}}]$ in the cyclotomic field $\mathbb{Q}(\zeta_{2^{l+1}})$ with $\zeta_{2^{l+1}} := \exp\left(2^{-l} \cdot \pi \cdot i\right)$. The ideal $\mathcal{P} := (2, 1 - \zeta_{2^{l+1}})$ in $\mathbb{Z}[\zeta_{2^{l+1}}]$ is a prime ideal satisfying $\mathcal{P}^{2^l} = (2)$. We define

$$w_l(r) := \frac{1}{2^l} \cdot \nu_{\mathcal{P}}\left(\alpha(pr_l(r))\right) \in \frac{1}{2^l} \cdot \mathbb{Z}$$

with $\nu_{\mathcal{P}}$ the valuation and $\alpha : \mathbb{Q}[\chi]/\langle 1 + \chi^{2^l} \rangle \cong \mathbb{Q}(\zeta_{2^{l+1}}), \chi \mapsto \zeta_{2^{l+1}}$.

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Lemma

- $w_l(g_1 \cdot g_2) = w_l(g_1) + w_l(g_2)$.
- *If* $w_l(g_1) \neq w_l(g_2)$ *then* $w_l(g_1 + g_2) = \min\left\{w_l(g_1), w_l(g_2)\right\}$.
- *If* $w_l(g_1) = w_l(g_2)$ *then* $w_l(g_1 + g_2) > w_l(g_1) = w_l(g_2)$.

w_l-functions

Proposition

Let $r \in \mathbb{Q}R_{\widehat{G}}$ such that $pr_l(r) \in 4 \cdot \mathbb{Z}[\chi]/\langle 1 + \chi^{2^l} \rangle$ for all $0 \leq l \leq K - 1$.

- *If $w_l(r) \geq 2 + K - l - 2^{-l}$ for all $0 \leq l \leq K - 1$ then $r \in 4 \cdot R_{\widehat{G}}$.*
- *If there exist $s \in R_{\widehat{G}}$ and $0 \leq l' \leq K - 1$ such that*

$$w_l(r) + w_l(s) \geq 2 + K - l - 2^{-l} \text{ for all } l \neq l' \text{ and}$$
$$w_{l'}(r) + w_{l'}(s) < 2 + K - l' - 2^{-l'}$$

then $r \notin 4 \cdot R_{\widehat{G}}$.

The proof uses the lemma which says that

$$r = \sum_{l=0}^{K-1} 2^{l-K} \cdot r_l \cdot (1 - \chi) \cdot \prod_{\substack{0 \leq n \leq K-1 \\ n \neq l}} (1 + \chi^{2^n}) \text{ if } pr_l(r_l) = pr_l(r).$$

Good polynomials

Our strategy is to determine polynomials with leading coefficient 1 such that

$$8 \cdot f \cdot q(f^2) \in 4 \cdot \mathbb{Z}[\chi]/I\langle K \rangle$$

for a largest possible K in comparison with other polynomials of the same degree.
Good candidates are the following polynomials $q_n(x)$.

Definition

Let $n \geq 0$. Define $a(n), b(n) \geq 0$ as the integers satisfying $n + 1 = 2^{a(n)} + b(n)$ and $0 \leq b(n) \leq 2^{a(n)} - 1$. We define

$$q_n(x) := \prod_{r=1}^{a(n)} p_r(x) \cdot (x-1)^{b(n)}$$

with $p_1(x) := x + 1$ and $p_{k+1}(x) := p_k\left(\frac{(x+1)^2}{4x}\right) \cdot (4x)^{2^{k-1}} \in \mathbb{Z}[x]$.

The polynomials $q_n(x)$

The polynomials $q_n(x)$ have leading coefficient 1 and degree n.

We calculate

$$w_l(8 \cdot f \cdot q_n(f^2)) = \begin{cases} \infty & l \leq a(n) \\ 2n + 3 - a(n) - \frac{b(n)}{2^{l-1}} & l \geq a(n) + 1 \end{cases}$$

and conclude

$$\begin{aligned} 8 \cdot f \cdot q_n(f^2) &\in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+1 \rangle, \\ 8 \cdot f \cdot q_n(f^2) &\in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2 \rangle \Longleftrightarrow b(n) = 0, \\ 8 \cdot f \cdot q_n(f^2) &\notin 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+3 \rangle. \end{aligned}$$

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The polynomials $q_n(x)$ have leading coefficient 1 and degree n.

We calculate

$$w_l(8 \cdot f \cdot q_n(f^2)) = \begin{cases} \infty & l \leq a(n) \\ 2n + 3 - a(n) - \frac{b(n)}{2^{l-1}} & l \geq a(n) + 1 \end{cases}$$

and conclude

$$\begin{aligned} 8 \cdot f \cdot q_n(f^2) &\in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+1 \rangle, \\ 8 \cdot f \cdot q_n(f^2) &\in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2 \rangle \iff b(n) = 0, \\ 8 \cdot f \cdot q_n(f^2) &\notin 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+3 \rangle. \end{aligned}$$

Are the polynomials $q_n(x)$ best possible? Or does there exist a polynomial $q(x)$ with leading coefficient 1 and of degree n such that

- $8 \cdot f \cdot q(f^2) \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2 \rangle$ if $b(n) \neq 0$?
- $8 \cdot f \cdot q(f^2) \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+3 \rangle$ if $b(n) = 0$?

The polynomials $\tilde{q}_n(x)$

It turns out that the polynomials $q_0(x)$ and $q_1(x)$ are best possible. We set $\tilde{q}_n(x) := q_n(x)$ for $n = 0, 1$.

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We can show that there exist a unique $a_0 \in \{0, 1\}$ such that

$$\tilde{q}_2(x) := q_2(x) + a_0 \cdot 2^3 \cdot \tilde{q}_0(x)$$

satisfies $8 \cdot f \cdot \tilde{q}_2(f^2) \in 4 \cdot \mathbb{Z}[\chi]/I\langle 6\rangle$. (Here, $a_0 = 1$.)

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satisfies $8 \cdot f \cdot \tilde{q}_2(f^2) \in 4 \cdot \mathbb{Z}[\chi]/I\langle 6\rangle$. (Here, $a_0 = 1$.)

We can proceed in this way. For all $n \in \mathbb{N}$ there exist unique $a_l \in \{0, 1\}$ $(0 \leq l \leq \lfloor \frac{n}{2} \rfloor - 1)$ such that

$$\tilde{q}_n(x) := q_n(x) + \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor - 1} a_l \cdot 2^{2(n-l)-1} \cdot \tilde{q}_l(x)$$

satisfies $8 \cdot f \cdot \tilde{q}_n(f^2) \in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n + 2\rangle$.

The polynomials $\tilde{q}_n(x)$

The polynomials $\tilde{q}_n(x)$ have the properties

$$\begin{aligned} 8 \cdot f \cdot \tilde{q}_n(f^2) &\in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2\rangle, \\ 8 \cdot f \cdot \tilde{q}_n(f^2) \cdot (1-\chi)^{2n} &\notin 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+3\rangle. \end{aligned}$$

The polynomials $\tilde{q}_n(x)$

The polynomials $\tilde{q}_n(x)$ have the properties

$$\begin{aligned} 8 \cdot f \cdot \tilde{q}_n(f^2) &\in 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+2\rangle, \\ 8 \cdot f \cdot \tilde{q}_n(f^2) \cdot (1-\chi)^{2n} &\notin 4 \cdot \mathbb{Z}[\chi]/I\langle 2n+3\rangle. \end{aligned}$$

Using these properties we can show

$$\left\{ q \in \mathbb{Z}[x] \,\middle|\, \deg(q) \leq e-1, 8 \cdot f \cdot q(f^2) \in 4 \cdot \mathbb{Z}[\chi]/I\langle K\rangle \right\} =$$
$$\left\{ \sum_{n=0}^{e-1} c_n \cdot 2^{\max\{K-2n-2,0\}} \cdot \tilde{q}_n \,\middle|\, c_n \in \mathbb{Z} \right\}.$$

Calculation of $\ker[\widetilde{\rho}]$ for $N = 2^K$

We obtain

$$\ker[\widetilde{\rho}] \cong$$

$$\bigoplus_{i=1}^{e} \mathbb{Z}_2 \oplus \left\{ q \in \mathbb{Z}[x] \,\middle|\, \deg(q) \leq e-1, 8 \cdot f \cdot q(f^2) \in 4R_{\widehat{G}} \right\} / \; 2^K \mathbb{Z}[x] \cong$$

$$\bigoplus_{i=1}^{e} \mathbb{Z}_2 \oplus \left\{ \sum_{n=0}^{e-1} c_n \cdot 2^{\max\{K-2n-2,0\}} \cdot \tilde{q}_n \,\middle|\, c_n \in \mathbb{Z} \right\} / \; 2^K \cdot \mathbb{Z}[x] \cong$$

$$\bigoplus_{i=1}^{e} \mathbb{Z}_2 \oplus \bigoplus_{n=0}^{e-1} 2^{\max\{K-2n-2,0\}} \cdot \mathbb{Z} \; / \; 2^K \cdot \mathbb{Z} \cong$$

$$\bigoplus_{i=1}^{e} \mathbb{Z}_2 \oplus \bigoplus_{i=1}^{e} \mathbb{Z}_{2^{\min\{K,2i\}}}.$$

Calculation of $\ker[\widetilde{\rho}]$ for $N = 2^K$

So far, we have proven

$$\ker\big([\widetilde{\rho}] : \widetilde{\mathcal{N}}(L^{2d-1}(\alpha)) \to \mathbb{Q}R_{\widehat{G}}^{(-1)^d}/4 \cdot R_{\widehat{G}}^{(-1)^d}\big) \;\cong\; \bigoplus_{i=1}^{c} \mathbb{Z}_2 \oplus \bigoplus_{i=1}^{c} \mathbb{Z}_{2^{\min\{K,2i\}}}$$

with $c := \lfloor (d-1)/2 \rfloor$ for $\alpha = \alpha_1$ and d odd.

There is a similar proof for $\alpha = \alpha_k$ and for d even.

Calculation of $\ker[\widetilde{\rho}]$ for $N = 2^K$

So far, we have proven

$$\ker\big([\widetilde{\rho}] : \widetilde{\mathcal{N}}(L^{2d-1}(\alpha)) \to \mathbb{Q}R_{\widehat{G}}^{(-1)^d}/4 \cdot R_{\widehat{G}}^{(-1)^d}\big) \;\cong\; \bigoplus_{i=1}^{c} \mathbb{Z}_2 \oplus \bigoplus_{i=1}^{c} \mathbb{Z}_{2^{\min\{K,2i\}}}$$

with $c := \lfloor (d-1)/2 \rfloor$ for $\alpha = \alpha_1$ and d odd.

There is a similar proof for $\alpha = \alpha_k$ and for d even.

As explained in the first half of the talk, we conclude

$$\mathcal{S}^s(L^{2d-1}(\alpha_k)) \cong \bar{\Sigma}_N(d) \oplus \bigoplus_{i=1}^{c} \mathbb{Z}_2 \oplus \bigoplus_{i=1}^{c} \mathbb{Z}_{2^{\min\{K,2i\}}}$$

where $\bar{\Sigma}_N(d)$ is a free abelian group of rank $N/2 - 1$ if $d = 2e + 1$ and $N/2$ if $d = 2e$.

Classification of fake lens spaces for $N = 2^K$

For any fake lens space $L^{2d-1}(\alpha)$ there exists $k \in \mathbb{N}$ and a homotopy equivalence $h\colon L^{2d-1}(\alpha) \longrightarrow L^{2d-1}(\alpha_k)$. It induces an isomorphism $h_*\colon \mathcal{S}^s(L^{2d-1}(\alpha)) \to \mathcal{S}^s(L^{2d-1}(\alpha_k))$.

Hence we obtain

Theorem (M., W.)

Let $L^{2d-1}(\alpha)$ be a fake lens space with $\pi_1(L^{2d-1}(\alpha)) \cong \mathbb{Z}_N$ where $N = 2^K$ and $d \geq 3$. Then we have

$$\mathcal{S}^s(L^{2d-1}(\alpha)) \cong \bar{\Sigma}_N(d) \oplus \bigoplus_{i=1}^{c} \mathbb{Z}_2 \oplus \bigoplus_{i=1}^{c} \mathbb{Z}_{2^{\min\{K,2i\}}}$$

where $\bar{\Sigma}_N(d)$ is a free abelian group of rank $N/2 - 1$ if $d = 2e + 1$ and $N/2$ if $d = 2e$ and $c := \lfloor (d-1)/2 \rfloor$.

Classification of fake lens spaces (general case)

Using the results for the cases N odd and $N = 2^K$ we can prove

Theorem (M., W.)

Let $L^{2d-1}(\alpha)$ be a fake lens space with $\pi_1(L^{2d-1}(\alpha)) \cong \mathbb{Z}_N$ where $N = 2^K \cdot M$ with $K \geq 0$, M odd and $d \geq 3$. Then we have

$$\mathcal{S}^s(L^{2d-1}(\alpha)) \cong \bar{\Sigma}_N(d) \oplus \bigoplus_{i=1}^{c} \mathbb{Z}_{2^{\min\{K,1\}}} \oplus \bigoplus_{i=1}^{c} \mathbb{Z}_{2^{\min\{K,2i\}}}$$

where $\bar{\Sigma}_N(d)$ is a free abelian group. If N is odd then its rank is $(N-1)/2$. If N is even then its rank is $N/2 - 1$ if $d = 2e + 1$ and $N/2$ if $d = 2e$. In the torsion summand we have $c := \lfloor (d-1)/2 \rfloor$.