THE TOTAL SURGERY OBSTRUCTION III

TIBOR MACKO

ABSTRACT. The total surgery obstruction $s(X)$ of a finite n-dimensional Poincaré complex X is an element of a certain abelian group $\mathbb{S}_n(X)$ with the property that $s(X) = 0$ if and only if X is homotopy equivalent to a closed n-dimensional topological manifold.

In a series of three talks we explained the definition of the total surgery obstruction and we gave the proof of the above property. Both the definition and the proof require a substantial amount of technology known as algebraic theory of surgery due to Ranicki. These are informal notes from the last talk.

The material in these notes comes mostly from: [Ran79], [Ran81], [Ran92]. Some background can also be found in [Ran80a], [Ran80b], [Ran02a], [Ran02b] and in the notes from the previous talks[1].

1. Theorem

Theorem 1.1 ([Ran92], Theorem 17.4)**.** *Let X be a finite Poincaré complex of formal dimension $n \geq 5$. Then X is homotopy equivalent to a closed n-dimensional topological manifold if and only if*

$$0 = s(X) \in \mathbb{S}_n(X).$$

By choosing a simplicial complex homotopy equivalent to X we can assume that X is a simplicial complex.

2. Revision of previous talks

Algebraic bordism categories and L-groups. Recall that $\mathbb{S}_n(X)$ is defined as a quadratic L-group of a certain algebraic bordism category. An *algebraic bordism category* $\Lambda = (\mathbb{A}, \mathbb{B}, \mathbb{C}, (T, e))$ consists of an *additive category with chain duality* $(\mathbb{A}, (T, e))$[2] and two full subcategories $\mathbb{C}$, $\mathbb{B} \subseteq \mathbb{B}(\mathbb{A})$ of the category of bounded chain complexes in $\mathbb{A}$ satisfying certain mild assumptions.

In this situation one can define the L-groups of Λ in three different flavors: the *quadratic* $L_n(\Lambda)$, the *symmetric* $L^n(\Lambda)$, and the *normal* $NL^n(\Lambda)$. These are cobordism groups of n-dimensional algebraic complexes in Λ of respective flavors.

For example an element in $L_n(\Lambda)$ is represented by an n-dimensional quadratic algebraic complex (C, ψ) where $C \in \mathbb{B}$, $\psi \in (W_{\%}C)_n$ is a cycle and $S^{-1}C((1 + T)\psi_0 \colon TC^{n-*} \to C_*) \in \mathbb{C}$. This means that (C, ψ) is Poincaré modulo $\mathbb{C}$. We will also often use the notation $L_n(\mathbb{B}, \mathbb{C})$.

In $L^n(\Lambda)$ one replaces ψ by $\phi \in (W^{\%}C)_n$ a cycle. The elements in $NL^n(\Lambda)$ are represented by a more complicated structure which is recalled later in section 3.

Date: July 1, 2009.

[1] I would also like to thank Andrew Ranicki for answering numerous questions

[2] We usually leave out (T, e) from the notation.

Example $\mathbb{Z}[\pi_1(X)]$**.** For any ring with involution R, for example for $\mathbb{Z}[\pi_1(X)]$, we can consider the category of finitely generated free R-modules with the chain duality $T(M) = \mathrm{Hom}_R(M, R)$.

Example $\mathbb{A}_*(X)$**.** Let X be a simplicial complex and let $\mathbb{A}$ be an additive category with chain duality. The category $\mathbb{A}_*(X)$ has as its objects the so-called X-based objects of $\mathbb{A}$, that means objects of $\mathbb{A}$ which come as direct sums

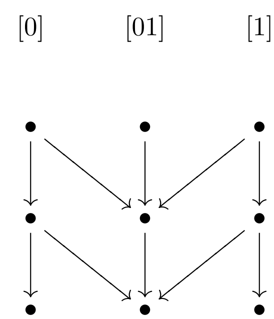
$$M = \sum_{\sigma \in X} M(\sigma).$$

Morphisms are given by

$$\mathrm{Mor}_{\mathbb{A}_*(X)}(M, N) = \{f = \sum_{\sigma,\tau \in X} f(\tau, \sigma) \colon M(\sigma) \to N(\tau) \mid f(\tau, \sigma) = 0 \text{ unless } \sigma \leq \tau\}.$$

An n-dimensional quadratic algebraic complex (C, ψ) in $\mathbb{A}_*(X)$ includes in particular for each $\sigma \in X$ a chain complex $C(\sigma)$ and a duality map $\psi_0(\sigma) \colon \Sigma^n TC(\sigma) \to C(\sigma)$. But it contains more information, there are relations between these data for various simplices and of course chain homotopies ψ_s for $s > 0$.

For example if $X = \Delta^1$ a 1-simplex, then the following is a schematic picture of the underlying chain complex of an object in $\mathbb{A}_*(X)$:



For example, we can take $C = \Delta_*(X')$ the simplicial chain complex of the barycentric subdivision of X and we can define $C(\sigma) = \Delta_*(D(\sigma), \partial D(\sigma))$, the simplicial chain complex of the dual cell relative to its boundary. Then we have $\Sigma^n TC(\sigma) \cong \Delta^{n-|\sigma|-*}(D(\sigma))$. Any n-cycle $[X]$ in C_n defines via the symmetric construction ϕ_X a symmetric structure ϕ on C as a chain complex in $\mathbb{Z}$, whose ϕ_0 is the cap product with $[X]$. When C is viewed as a chain complex over $\mathbb{Z}_*(X)$, there is a refined symmetric construction, so that we obtain an algebraic symmetric structure ϕ in $\mathbb{Z}_*(X)$, which in particular contains duality maps $\phi_0(\sigma) \colon \Sigma^n TC(\sigma) \to C(\sigma)$ which are cap products with certain $(n - |\sigma|)$-dimensional classes $[X](\sigma)$. For more details see [Ran92, Example 6.2]. Quadratic structures over $\mathbb{Z}_*(X)$ can be obtained from a degree one normal map $(f, b) \colon M \to X$ with the target a triangulated manifold. See [Ran92, 9.14].

The *assembly* functor $A \colon \mathbb{A}_*(X) \to \mathbb{A}[\pi_1(X)]$ is defined by

$$M \mapsto \sum_{\widetilde{\sigma} \in \widetilde{X}} = M(p(\widetilde{\sigma}))$$

It induces a functor on the chain complexes and it also turns out to 'commute' with the chain duality in a suitable sense so that one obtains maps of the L-groups $A \colon L_n(\mathbb{A}_*(X)) \to L_n(\mathbb{A}[\pi_1(X)])$.

So far we have only presented the underlying additive category $\mathbb{A}_*(X)$ with chain duality. In order to obtain an algebraic bordism category we need to specify interesting subcategories of $\mathbb{B}(\mathbb{A}_*(X))$. We will use three such subcategories, denoted $\mathbb{D} \subset \mathbb{C} \subset \mathbb{B}$. Here are the definitions

$$\begin{aligned} \mathbb{B} &= \mathbb{B}(\mathbb{A}_*(X)) \\ \mathbb{C} &= \{C \in \mathbb{B} \mid A(C) \simeq *\} \\ \mathbb{D} &= \{C \in \mathbb{B} \mid \forall \sigma \in X \;\; C(\sigma) \simeq *\} \end{aligned} \tag{2.1}$$

This gives us three possibilities to construct interesting algebraic bordism categories. In fact the study of the relationship of the L-theories of these three algebraic bordism categories lies in the heart of the arguments presented here.

Connective versions. An important technical aspect is that we need to use connective version of the L-theory spectra. This is related to the difference between topological manifolds and ANR-homology manifolds. We will not discuss this here in detail, we only mention the modifications necessary to obtain the correct result.

Let $q \in \mathbb{Z}$ and let $\Lambda = (\mathbb{A}, \mathbb{B}, \mathbb{C})$ be an algebraic bordism category. Define the subcategory $\mathbb{B}\langle q\rangle \subset \mathbb{B}$ to be the subcategory of chain complexes in $\mathbb{B}$ which are homotopy equivalent to q-connected chain complexes. Further define $\mathbb{C}\langle q\rangle = \mathbb{B}\langle q\rangle \cap \mathbb{C}$. Then $\Lambda\langle q\rangle = (\mathbb{A}, \mathbb{B}\langle q\rangle, \mathbb{C}\langle q\rangle)$ is a new algebraic bordism category. We also define $\Lambda\langle 1/2\rangle = (\mathbb{A}, \mathbb{B}\langle 0\rangle, \mathbb{C}\langle 1\rangle)$. More details are given in [Ran92, chapter 15]

Localization theorem. The following localization theorem is an important tool in comparing the L-groups of various algebraic bordism categories.

Proposition 2.1 ([Ran92], Proposition 3.9)**.** *Let $\mathbb{A}$ be an additive category with chain duality and let $B \subseteq \mathbb{B}(\mathbb{A})$, $\mathbb{C} \subset \mathbb{B}$, $\mathbb{D} \subset \mathbb{C}$ be closed subcategories so that there are the algebraic bordism categories:*

$$\Lambda = (\mathbb{A}, \mathbb{B}, \mathbb{C}), \quad \Lambda' = (\mathbb{A}, \mathbb{B}, \mathbb{D}), \quad \Lambda'' = (\mathbb{A}, \mathbb{C}, \mathbb{D})$$

Then there are the following exact sequences

(1) $\cdots \to L_n(\Lambda'') \to L_n(\Lambda') \to L_n(\Lambda) \to L_{n-1}(\Lambda'') \to \cdots$

(2) $\cdots \to L_n(\Lambda'') \to NL^n(\Lambda') \to NL^n(\Lambda) \to L_{n-1}(\Lambda'') \to \cdots$

We will apply Proposition 2.1 when $\mathbb{A} = \mathbb{A}_*(X)$. In fact we will need three versions of the second exact sequence. The two more versions will be in the cases when $\mathbb{C} = \mathbb{B}$ and $\mathbb{D} = \mathbb{D}$ and when $\mathbb{C} = \mathbb{B}$ and $\mathbb{D} = \mathbb{C}$. Here is the list:

$$\cdots \to L_n(\mathbb{C}, \mathbb{D}) \to L_n(\mathbb{B}, \mathbb{D}) \to L_n(\mathbb{B}, \mathbb{C}) \to L_{n-1}(\mathbb{C}, \mathbb{D}) \to \cdots \tag{2.2}$$

$$\cdots \to L_n(\mathbb{C}, \mathbb{D}) \to NL^n(\mathbb{B}, \mathbb{D}) \to NL^n(\mathbb{B}, \mathbb{C}) \to L_{n-1}(\mathbb{C}, \mathbb{D}) \to \cdots \tag{2.3}$$

$$\cdots \to L_n(\mathbb{B}, \mathbb{C}) \to NL^n(\mathbb{B}, \mathbb{C}) \to NL^n(\mathbb{B}, \mathbb{B}) \to L_{n-1}(\mathbb{B}, \mathbb{C}) \to \cdots \tag{2.4}$$

$$\cdots \to L_n(\mathbb{B}, \mathbb{D}) \to NL^n(\mathbb{B}, \mathbb{D}) \to NL^n(\mathbb{B}, \mathbb{B}) \to L_{n-1}(\mathbb{B}, \mathbb{D}) \to \cdots \tag{2.5}$$

We actually need to use various connective versions. This is indicated below. In addition we have the following isomorphisms (some of which are theorems and some of which are definitions as indicated - the references are to [Ran92]):

$$
\begin{aligned}
L_n(\mathbb{B}\langle 1\rangle, \mathbb{C}\langle 1\rangle) &= L_n(\mathbb{Z}[\pi_1(X)]) \quad \text{Theorems 10.6.,14.4.}\\
L_n(\mathbb{B}\langle 1\rangle, \mathbb{D}\langle 1\rangle) &= H_n(X; \mathbf{L}_\bullet\langle 1\rangle) \quad \text{Theorem 14.5, 15.9}\\
NL^n(\mathbb{B}\langle 0\rangle, \mathbb{D}\langle 0\rangle) &= H_n(X; \mathbf{L}^\bullet\langle 0\rangle)) \quad \text{Theorem 14.5, 15.9}\\
NL^n(\mathbb{B}\langle 0\rangle, \mathbb{B}\langle 1\rangle) &= H_n(X; \mathbf{NL}^\bullet\langle 1/2\rangle)) \quad \text{Theorem 14.5, 15.9}\\
L_n(\mathbb{C}\langle 1\rangle, \mathbb{D}\langle 1\rangle) &= \mathbb{S}_n(X) \quad \text{Definition, chapter 17}\\
NL^n(\mathbb{B}\langle 0\rangle, \mathbb{C}\langle 1\rangle) &= VL^n(X) \quad \text{Definition, chapter 17}
\end{aligned}
$$

2.1. **Flavors of the L-theory.** Let $\mathbb{A}$ be an additive category with chain duality. Recall the exact sequence

$$
\cdots \to L_n(\mathbb{A}) \xrightarrow{1+T} L^n(\mathbb{A}) \to NL^n(\mathbb{A}) \xrightarrow{\partial} L_{n-1}(\mathbb{A}) \to \cdots \tag{2.6}
$$

which can be thought of as induced by the cofibration sequence of spectra ([Ran92, 13.6., 13.9.]):

$$
\mathbf{L}_\bullet \to \mathbf{L}^\bullet \to \mathbf{NL}^\bullet. \tag{2.7}
$$

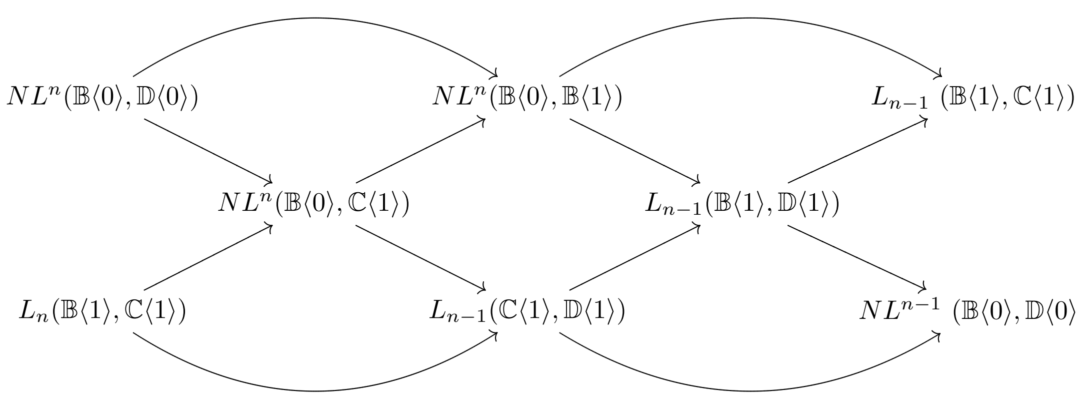
There is a version with connective L-spectra as follows ([Ran92, 15.16]):

$$
\mathbf{L}_\bullet\langle 1\rangle \to \mathbf{L}^\bullet\langle 0\rangle \to \mathbf{NL}^\bullet\langle 1/2\rangle. \tag{2.8}
$$

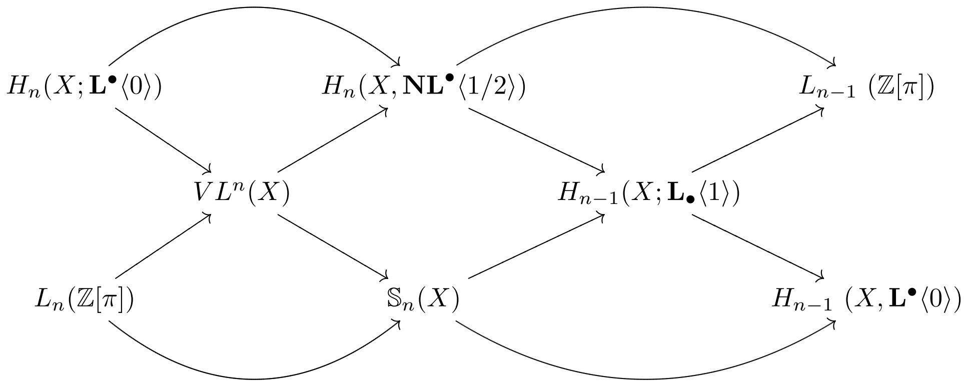
The sequence (2.5) can be thought of as induced by this cofibration sequence ([Ran92, 14.5]). Also the sequence (2.2) can be rewritten and is known as the algebraic surgery exact sequence:

$$
\cdots \to H_n(X, \mathbf{L}_\bullet\langle 1\rangle) \to L_n(\mathbb{Z}[\pi_1(X)]) \to \mathbb{S}_n(X) \to H_{n-1}(X; \mathbf{L}_\bullet\langle 1\rangle) \to \cdots \tag{2.9}
$$

Commutative braids. The whole story can be summarized in the braid diagram ([Ran92, Proposition 15.18]):



where the respective terms are identified as



3. Definition of $s(X)$

Recall first the notions of a geometric normal complex and of an algebraic normal complex over an additive category with chain duality $\mathbb{A}$ ([Ran92, chapter 2]).

Geometric normal complexes. An n-dimensional geometric normal complex is a triple (X,ν,ρ) where X is a finite CW-complex, $\nu\colon X\to BG(k)$ is a spherical fibration and $\rho\colon S^{n+k}\to T(\nu)$ is a map; here $T(\nu)$ denotes the Thom space of ν.

The principal example of a normal complex is of course a geometric Poincaré complex together with its Spivak normal fibration. Note however that in general a geometric normal complex does not have to be Poincaré, in fact it is one of the crucial ideas in this story to recognize the importance of normal complexes which are not Poincaré. They will play a prominent role in Proposition 3.1.

Algebraic normal complexes. An n-dimensional algebraic normal complex over $\mathbb{A}$ consists of a four-tuple (C,ϕ,γ,χ), where $C\in\mathbb{B}(\mathbb{A})$, $\phi\in(W_{\%}C)_n$ is a cycle (hence (C,ϕ) is an n-dimensional symmetric complex in $\mathbb{A}$), $\gamma\in(\widehat{W}^{\%}TC)_0$ is a cycle (the pair (C,γ) is called a *chain bundle*) and χ defines compatibility by requiring that $d\chi=J(\phi)-(\widehat{\phi_0^{\%}})(S^n\gamma)$.

Normal signatures over $\mathbb{Z}[\pi_1(X)]$. An n-dimensional geometric normal complex (X,ν,ρ) defines an n-dimensional algebraic normal complex in $\mathbb{Z}[\pi_1(X)]$. Roughly speaking the construction is as follows: the geometric normal complex has a 'want-to-be' fundamental class $[X]=h(\rho)\cap U_\nu\in C_n(X)$, where U_ν is the Thom class of ν. If the normal complex is a Poincaré complex with the Spivak normal fibration then this is a representative the fundamental class of X, in the more general case this is just some cycle. One defines $\phi=\phi_X([X])$, where ϕ_X is the symmetric construction on X. The cycle γ is obtained from U_ν using the symmetric construction ϕ_Y on some Spanier-Whitehead dual Y of the Thom space $T(\nu)$. Again in case the normal space is Poincaré with the Spivak normal fibration one can choose Y to be X, as is well known, and one obtains $J(\phi)=(\widehat{\phi_0^{\%}})(S^n\gamma)$. In the more general case one only obtains a stable map between X and Y and, using it, a weaker compatibility between ϕ and γ as required. Thus we obtain an algebraic normal complex $\widehat{\sigma}^*(X)\in NL^n(\mathbb{Z}[\pi_1(X)])$, called the *normal signature* of (X,ν,ρ).

Normal signatures over $\mathbb{Z}_*(X)$. The following proposition improves these ideas to obtain an n-dimensional algebraic normal complex in $\mathbb{Z}_*(X)$. See [Ran92, 9.12].

Proposition 3.1. *Keep the notation from Example $\mathbb{A}_*(X)$ in section 2.*

1. *An n-dimensional geometric normal complex X defines an n-dimensional algebraic normal complex $\widehat{\sigma}^*(X)$ in the algebraic bordism category $(\mathbb{B}\langle 0\rangle, \mathbb{B}\langle 1\rangle)$ and hence an element in the group $NL^n(\mathbb{B}\langle 0\rangle, \mathbb{B}\langle 1\rangle) \cong H_n(X, \mathbf{NL}^\bullet\langle 1/2\rangle)$.*
2. *If in addition X is Poincaré, then $\partial A(\widehat{\sigma}^*(X)) \simeq *$ and hence $\widehat{\sigma}^*(X)$ defines an element in the algebraic bordism category $(\mathbb{B}\langle 0\rangle, \mathbb{C}\langle 1\rangle)$ and hence an element in the group $NL^n(\mathbb{B}\langle 0\rangle, \mathbb{C}\langle 1\rangle) \cong VL^n(X)$.*

Proof. We think of n-dimensional algebraic normal complexes in $(\mathbb{B}, \mathbb{B})$ as of a collection of normal complexes over $\mathbb{Z}$ indexed by the simplices of X and satisfying certain compatibility conditions. In more detail over a simplex $\sigma \in X$ one has an $(n - |\sigma|)$-dimensional algebraic normal complex.

This 'local' normal structure will be induced for each simplex $\sigma \in X$ by showing that there is a structure of an $(n-|\sigma|)$-dimensional geometric normal complex over the dual cell $D(\sigma)$, which in turn is induced from the global normal space structure on X.

Roughly speaking the idea is to pull-back the global spherical fibration to each dual cell. Then add the normal spherical fibration of each dual cell inside X. The local collapse map is obtained by composing the global collapse map with further collapse.

Notice that although the dual cell $D(\sigma)$ can be complicated it always has a trivial normal bundle $\nu(D(\sigma), X)$ in X which is of dimension $|\sigma|$. Denote the inclusion $i\colon D(\sigma) \hookrightarrow X$ and define

$$\nu(\sigma) = i^*(\nu) \oplus \nu(D(\sigma), X)\colon D(\sigma) \to BG(k + |\sigma|).$$

We also have the obvious map $T(\nu) \to T(\nu(\sigma))$ and can define

$$\rho(\sigma) = S^{(n-|\sigma|)+(k+|\sigma|)} \xrightarrow{\rho} T(\nu) \to T(\nu(\sigma)).$$

Then $(D(\sigma), \nu(\sigma), \rho(\sigma))$ is an $(n - |\sigma|)$-dimensional geometric normal complex and one can apply the normal construction to extract an $(n-|\sigma|)$-dimensional algebraic normal complex over $\mathbb{Z}$ out of it. Some care needs to be taken here, since $D(\sigma)$ has a boundary, but at the end all works fine and the resulting collection of the algebraic normal complexes satisfies certain compatibility and defines an n-dimensional normal complex $\widehat{\sigma}^*(X)$ in $\mathbb{Z}_*(X)$. The required connectivity assumptions are also fulfilled.

If X is Poincaré then one automatically obtains that $\partial A(\widehat{\sigma}^*(X)) \simeq *$ and part (2) follows. □

Definition 3.2. Define (see [Ran92, 17.1]):

$$s(X) := \partial\widehat{\sigma}^*(X) \in \mathbb{S}_n(X). \tag{3.1}$$

4. Scheme of the proof

Classical surgery theory. Recall the classical surgery theory approach to the question whether there exists a closed manifold homotopy equivalent to a finite Poincaré complex X of formal dimension n. It is well known that such an X admits the so-called Spivak normal fibration (SNF): $\nu\colon X \to \mathrm{BG}(k)$ for some large $k \geq 0$. This is a spherical fibration over X which is unique in some sense. If X

is homotopy equivalent to a manifold then the SNF must come from a topological block bundle $\bar{\nu}\colon X \to \mathrm{B}\widetilde{\mathrm{TOP}}(k)$, we say that ν must have a topological block bundle reduction $\bar{\nu}$. Using stable notation this can be described as a lift

$$\begin{array}{ccc} & & \mathrm{BTOP} \\ & \overset{\bar{\nu}}{\nearrow} & \downarrow \\ X & \xrightarrow[\nu]{} & \mathrm{BG} \end{array}$$

Such a lift exists if and only if the composition $J \circ \nu\colon X \to \mathrm{BG} \to \mathrm{B(G/TOP)}$. This gives us the primary obstruction to our question.

If there is such a reduction then the Pontrjagin-Thom construction can be used to produce a degree one normal map $(f,b)\colon M \to X$ from an n-dimensional closed manifold M to X. If X is homotopy equivalent to a manifold N via $h\colon N \xrightarrow{\simeq} X$ then the homotopy equivalence h defines such a degree one normal and it can be obtained from a suitable topological reduction of ν. On the other hand surgery theory can answer the question whether the degree one normal map $(f,b)\colon M \to X$ is normally cobordant to a homotopy equivalence. This is if and only if the quadratic signature $\sigma_*(f,b) \in L_n(\mathbb{Z}[\pi_1(X)])$ is zero. In this way we obtain the secondary obstruction: there must exist a degree one normal map (f,b) with $\sigma_*(f,b) = 0$.

The above approach has been successful in many applications, but has certain disadvantages, namely the fact that it is a two-stage theory. Therefore there is some need for another theory where this two-stage obstruction theory is replaced by a single element in some group. This is what the total surgery obstruction yields. The new theory is not completely independent from the old one. In fact the old theory is used in the proof of the main theorem of the new theory.

Scheme of the proof. Recall the algebraic surgery exact sequence:

$$\cdots \to H_n(X, \mathbf{L}_\bullet\langle 1\rangle) \to L_n(\mathbb{Z}[\pi_1(X)]) \to \mathbb{S}_n(X) \to H_{n-1}(X; \mathbf{L}_\bullet\langle 1\rangle) \to \cdots$$

Denote by $t(X) \in H_{n-1}(X, \mathbf{L}_\bullet\langle 1\rangle)$ the image of $s(X)$. We sketch the proof of the following

Proposition 4.1. *Let X be a finite Poincaré complex of formal dimension $n \geq 5$. Then we have*

(1) $t(x) = 0$ *if and only if there exists a topological block bundle reduction of the SNF* ν
(2) *If* $t(X) = 0$ *then we have*

$$\partial^{-1} s(X) = \{ -\sigma_*(f,b) \in L_n(\mathbb{Z}[\pi_1(X)]) \mid \\ (f,b) : M \to X \text{ degree one normal map, } M \text{ manifold}\}.$$

Proof of the main theorem assuming Proposition 4.1.

If X is homotopy equivalent to a manifold then $t(X) = 0$ and by (2) the set $\partial^{-1}s(X)$ contains 0, hence $s(X) = 0$.

If $s(X) = 0$ then $t(X) = 0$ and hence by (1) the SNF of X has a topological block bundle reduction. Also $\partial^{-1}s(X)$ must contain 0 and hence by (2) there exists a degree one normal map with the target X and with the surgery obstruction 0. □

Remark 4.2. The condition (1) above might actually be quite puzzling for the following reason. As recalled earlier, the classical surgery gives an obstruction to the reduction of the SNF in the group $[X, \mathrm{B(G/TOP)}] = H^1(X; \mathrm{G/TOP})$. It is important to note that here the Ω^∞-space structure is used which corresponds to the Whitney sum and hence not the one that is compatible with the homotopy equivalence $\mathrm{G/TOP} \simeq \mathbf{L}\langle 1 \rangle_0$. On the other hand $t(X) \in H_{n-1}(X; \mathbf{L}_\bullet\langle 1\rangle)$. We note that the claim of (1) is NOT that the two groups are isomorphic, it merely says that one obstruction is zero if and only if the other is zero.

5. Sketch of the proof of Proposition 4.1.

Proof of (1). The main idea is to translate the statement about the lift of ν into a statement about orientations with respect to L-theory spectra. The main references for this part are [Ran79] and [Ran92, chapter 16].

Orientations. An orientation of a spherical fibration $\nu \colon X \to \mathrm{BG}(k)$ with respect to a ring spectrum $\mathbf{E}$, shortly we will talk about an $\mathbf{E}$-*orientation*, is a homotopy class of maps $U_\nu \colon \mathbf{T}(\nu) \to \mathbf{E}$, where $\mathbf{T}(\nu)$ denotes the Thom spectrum of ν, such that for each $x \in X$, the restriction $(U_\nu)_x \colon \mathbf{T}(\nu_x) \to \mathbf{E}$ represents a generator of $\mathbf{E}^*(\mathbf{T}(\nu_x)) \cong \mathbf{E}^*(S^k)$, here of course ν_x denotes the fiber of ν over x.

Canonical orientations. Denote by $\mathbf{MG}$ the Thom spectrum of the universal stable spherical fibrations over the classifying space BG. Similarly denote by $\mathbf{MTOP}$ the Thom spectrum of the universal stable topological block bundles over the classifying space BTOP.

Proposition 5.1 ([Ran79], pages 280-283)**.**

(1) *Any spherical fibration* $\alpha \colon X \to \mathrm{BG}(k)$ *has a canonical orientation* $U_\alpha \in H^k(T(\alpha); \mathbf{MG})$.

(2) *Any topological block bundle* $\beta \colon X \to \mathrm{B}\widetilde{\mathrm{TOP}}(k)$ *has a canonical orientation* $U_\beta \in H^k(T(\beta); \mathbf{MTOP})$.

This follows since any spherical fibration (or a topological block bundle) is a pullback of the universal via the classifying map.

Proposition 5.2 ([Ran79], pages 284-289)**.**

(1) *Any spherical fibration* $\alpha \colon X \to \mathrm{BG}(k)$ *has a canonical orientation* $\widehat{\sigma}^*(U_\alpha) \in H^k(T(\alpha); \mathbf{NL}^\bullet\langle 1/2\rangle)$.

(2) *Any topological block bundle* $\beta \colon X \to \mathrm{B}\widetilde{\mathrm{TOP}}(k)$ *has a canonical orientation* $\sigma^*(U_\beta) \in H^k(T(\beta); \mathbf{L}^\bullet\langle 0\rangle)$.

Proof. As suggested by the notation these orientations are obtained from maps between spectra. The precise claim is that there exists the following commutative diagram of spectra:

$$\begin{array}{ccc} \mathbf{MTOP} & \xrightarrow{\sigma^*} & \mathbf{L}^\bullet\langle 0\rangle \\ \downarrow & & \downarrow \\ \mathbf{MG} & \xrightarrow[\widehat{\sigma}^*]{} & \mathbf{NL}^\bullet\langle 1/2\rangle \end{array}$$

Recall that the spectra $\mathbf{NL}^\bullet\langle 1/2\rangle$, $\mathbf{L}^\bullet\langle 0\rangle$ are constructed using m-ads (by this we mean objects in $\mathbb{Z}^*(\Delta^m)$). We note that the one can construct cobordism spectra $\mathbf{\Omega}^N$, and $\mathbf{\Omega}^{STOP}$ using m-ads (normal spaces, resp. manifolds with boundary decomposed into pieces as the boundary of an m-simplex Δ^m). The normal construction yields the map $\mathbf{\Omega}^N \to \mathbf{NL}^\bullet\langle 1/2\rangle$ and the symmetric construction yields the map $\mathbf{\Omega}^{STOP} \to \mathbf{L}^\bullet\langle 0\rangle$. The normal transversality yields a homotopy equivalence $\mathbf{\Omega}^N \simeq \mathbf{MG}$ (perhaps the definition of "normal transversality" is not clear. We suggest that the reader takes as a definition the fact that these two spectra are homotopy equivalent, which can be elementary verified). The topological transversality yields a homotopy equivalence $\mathbf{\Omega}^{STOP} \simeq \mathbf{MTOP}$. □

Lifts. Notice: if X is a Poincaré complex with the SNF $\nu\colon X \to BG$ then we have $\mathrm{SW}(T(\nu)) \simeq X$, where SW denotes the Spanier-Whitehead dual. Also if $\bar{\nu}\colon X \to \mathrm{BTOP}$ is such that $J\bar{\nu} \simeq \nu$ then $\mathrm{SW}(T(\bar{\nu})) \simeq X$. Therefore in this situation we have

$$H^k(T(\nu);\mathbf{NL}^\bullet\langle 1/2\rangle) \cong H_n(X;\mathbf{NL}^\bullet\langle 1/2\rangle)$$
$$H^k(T(\bar{\nu});\mathbf{L}^\bullet\langle 0\rangle) \cong H_n(X;\mathbf{L}^\bullet\langle 0\rangle).$$

Furthermore

Proposition 5.3 ([Ran92], Proposition 16.1.)**.** *If X is an n-dimensional geometric Poincaré complex with the Spivak normal fibration $\nu\colon X \to \mathrm{BG}(k)$ then we have*

$$\mathrm{SW}(\widehat{\sigma}^*(U_\nu)) = \widehat{\sigma}^*(X) \in H_n(X;\mathbf{NL}^\bullet\langle 1/2\rangle)$$

In addition we need the following crucial result

Theorem 5.4 ([Ran79], pages 290-292)**.** *There is a one-to-one correspondence between*

(1) *stable oriented topological block bundles over X*
(2) *stable oriented spherical fibrations over X with an $\mathbf{L}^\bullet\langle 0\rangle$-lift of the canonical $\mathbf{NL}^\bullet\langle 1/2\rangle$-orientation*

Proof. The above remarks yield a map from (1) to (2). We give arguments for the inverse map. It boils down to the existence of the following diagram:

$$\begin{array}{ccc} \mathrm{BTOP} & \longrightarrow & \mathrm{B}\mathbf{L}^\bullet\langle 0\rangle\mathrm{G} \\ \downarrow & & \downarrow \\ \mathrm{BG} & \longrightarrow & \mathrm{B}\mathbf{NL}^\bullet\langle 1/2\rangle\mathrm{G} \end{array}$$

Here the entries in the right hand column are: the classifying space for spherical fibrations with an $\mathbf{NL}^\bullet\langle 1/2\rangle$-orientation $\mathrm{B}\mathbf{NL}^\bullet\langle 1/2\rangle\mathrm{G}$ and the classifying space for spherical fibrations with an $\mathbf{L}^\bullet\langle 0\rangle$-orientation $\mathrm{B}\mathbf{L}^\bullet\langle 0\rangle\mathrm{G}$.

We need to show that this diagram is homotopy cartesian. For any Omega ring spectrum $\mathbf{E}$ with $\pi_0(\mathbf{E}) = \mathbb{Z}$ let $\mathbf{E}_\otimes$ be the component of $1 \in \mathbb{Z}$. Then it can be shown that there is a homotopy fibration sequence

$$\mathbf{E}_\otimes \to \mathrm{B}\mathbf{E}\mathrm{G} \to \mathrm{BG}$$

Using this one sees that the homotopy fiber of the right hand column map is $(\mathbf{L}_\bullet)_\otimes$. The homotopy fiber of the left hand column is G/TOP. It is well-known that these two spaces are homotopy equivalent and in fact the induced map of the homotopy fibers is that well-known homotopy equivalence. □

Consider the exact sequence:

$$\begin{array}{ccccc} H^k(T(\nu);\mathbf{L}^\bullet\langle 0\rangle) & \longrightarrow & H^k(T(\nu);\mathbf{NL}^\bullet\langle 1/2\rangle) & \longrightarrow & H^{k+1}(T(\nu);\mathbf{L}_\bullet\langle 1\rangle) \\ {\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle =} \\ H_n(X;\mathbf{L}^\bullet\langle 0\rangle) & \longrightarrow & H_n(X;\mathbf{NL}^\bullet\langle 1/2\rangle) & \longrightarrow & H_{n-1}(X;\mathbf{L}_\bullet\langle 1\rangle) \end{array}$$

Putting this and the previous results together we obtain

Corollary 5.5. *Let X be an n-dimensional geometric Poincaré complex with the Spivak normal firbation $\nu\colon X \to \mathrm{BG}$. Then the following are equivalent*

1. *There exists a lift $\bar{\nu}\colon X \to \mathrm{BTOP}$ of ν*
2. *There exists a lift of the normal signature $\widehat{\sigma}^*(X) \in H_n(X;\mathbf{NL}^\bullet\langle 1/2\rangle)$ in the group $H_n(X;\mathbf{L}^\bullet\langle 0\rangle)$.*
3. *$t(X) = 0$.*

Proof of (2). We suppose $t(X) = 0$ and hence ν has a topological block bundle reduction and hence there exists a degree one normal map $(f,b)\colon M \to X$ from an n-dimensional topological manifold M. To prove the desired equality we need some preparation. The main references for this part are [Ran92, chapter 17]., [Ran81, chapeter 7.3].

Quadratic signatures revisited. Note that a degree one normal map $(f,b)\colon M \to X$ from an n-dimensional manifold M to an n-dimensional Poincaré complex X has a quadratic signature $\sigma_*(f,b) \in L_n(\mathbb{Z}[\pi_1(X)])$, which comes as an n-dimensional quadratic algebraic Poincaré complex over the category $\mathbb{Z}[\pi_1(X)]$. However, the map $\partial\colon L_n(\mathbb{Z}[\pi_1(X)]) \to \mathbb{S}_n(X)$ is obtained when working over the category $\mathbb{Z}_*(X)$. So we need an improvement on the quadratic signature. We need to have $\sigma_*(f,b)$ as an n-dimensional quadratic algebraic complex over the category $\mathbb{Z}_*(X)$ which is globally Poincaré.

We start with the description of the map ∂. Recall that we think of an element in $L_n(\mathbb{Z}[\pi_1(X)]) \cong L_n(\mathbb{B},\mathbb{C})$ as represented by an n-dimensional chain complex (C,ψ) in $\mathbb{B}(\mathbb{A}_*(X))$ such that $A(C)$ is Poincaré. Then the image $\partial(C,\psi) = (\partial C, \partial\psi)$ is an $(n-1)$-dimensional quadratic chain complex in $\mathbb{B}(\mathbb{A}_*(X))$ such that $(\partial C)(\sigma) = S^{-1}C(\Sigma^n TC(\sigma) \to C(\sigma))$ and respective $\psi(\sigma)$. In addition we automatically have that ∂C is globally contractible and locally Poincaré.

Now let us proceed with the desired improvement of the quadratic construction. We need to produce an n-dimensional quadratic chain complex in $\mathbb{B}(\mathbb{Z}_*(X))$ which is globally Poincaré. In particular we need for each $\sigma \in X$ a chain complex $C(\sigma)$ with a duality map $\psi(\sigma)\colon \Sigma^n TC(\sigma) \to C(\sigma)$.

The idea in obtaining this is to make f transverse to the dual cells of X and consider the restrictions

$$(f,b)(\sigma)\colon (M(\sigma), \partial M(\sigma)) \to (D(\sigma), \partial D(\sigma)) \tag{5.1}$$

These are degree one normal maps, but the target $(D(\sigma),\partial D(\sigma))$ is only a normal space which can be non Poincaré. We need to associate to $(f,b)(\sigma)$ an $(n-|\sigma|)$-dimensional quadratic chain complex $C(\sigma)$ over $\mathbb{Z}$ so that $A(C) \simeq C(f^!)$.

We review how to do this. We start with an explanation of [Ran81, Proposition 7.3.4]. Consider the following situation: $(g,c)\colon N \to Y$ is a degree one normal map from an n-dimensional manifold to an n-dimensional normal space.

In this case we have defined the normal signature $\widehat{\sigma}^*(X)$ which is an n-dimensional normal complex which is not Poincaré. Its boundary $\partial\widehat{\sigma}^*(X)$ is an $(n-1)$-dimensional quadratic complex which is Poincaré.

In case Y is Poincaré there is defined the algebraic Umkehr map $g^!\colon C_*(\widetilde{Y}) \to C_*(\widetilde{N})$ and one obtains the symmetric signature $\sigma^*(g,c)$ with the underlying chain complex the algebraic mapping cone $C(g^!)$. This can be further refined to a quadratic structure. In fact one has

$$\sigma^*(g,c) \oplus \sigma^*(Y) = \sigma^*(N) \qquad \text{and} \qquad C(g^!) \oplus C_*(\widetilde{Y}) \simeq C_*(\widetilde{N})$$

In case Y is not Poincaré there is defined the generalized algebraic Umkehr map $g^!\colon C^{n-*}(\widetilde{Y}) \to C_*(\widetilde{N})$ and $C(g^!)$ turns out to be an underlying chain complex of the n-dimensional quadratic complex $\sigma_*(g,c)$ which is not necessarily Poincaré. Hence it has a potentially interesting $(n-1)$-dimensional quadratic boundary $\partial\sigma_*(g,c)$ The above statements generalize as follows:

Proposition 5.6 ([Ran81], Proposition 7.3.4)**.** *Let $(g,c)\colon N \to Y$ be a degree one normal map from an n-dimensional manifold to an n-dimensional normal space. Then there are chain homotopy equivalences of symmetric complexes*

$$h\colon \partial\sigma^*(g,c) \xrightarrow{\simeq} -\partial\sigma^*(Y)$$
$$\sigma^*(M) \simeq \sigma^*(g,c) \cup_h \sigma^*(Y)$$

and a homotopy equivalence of quadratic refinements

$$h\colon \partial\sigma_*(g,c) \xrightarrow{\simeq} -\partial\widehat{\sigma}^*(Y).$$

So now starting with a degree one normal map $(f,b)\colon M \to X$ from an n-dimensional manifold to an n-dimensional Poincaré complex X we can make it transverse to the dual cells and consider a collection of degree one normal maps (5.1). Then use the relative version of the generalized quadratic construction above to obtain a collection of chain complexes $C(\sigma)$ with duality maps $\psi(\sigma)$ which together build an n-dimensional quadratic chain complex over the category $\mathbb{Z}_*(X)$ which is globally Poincaré, and denote it $\sigma_*(f,b)$.

Furthermore, the above proposition generalizes as (see [Ran92, chapter 17]):

Proposition 5.7. *Let $(f,b)\colon M \to X$ be a degree one normal map from an n-dimensional manifold to an n-dimensional Poincaré complex. Then there are chain homotopy equivalences of symmetric complexes*

$$h\colon \partial\sigma^*(f,b) \xrightarrow{\simeq} -\partial\sigma^*(X) \qquad \text{in } \mathbb{Z}_*(X)$$
$$\sigma^*(M) \simeq \sigma^*(f,b) \cup_h \sigma^*(X) \qquad \text{in } \mathbb{Z}_*(X)$$

and a homotopy equivalence of quadratic refinements

$$h\colon \partial\sigma_*(f,b) \xrightarrow{\simeq} -\partial\widehat{\sigma}^*(X) \qquad \text{in } \mathbb{Z}_*(X).$$

The last statement shows the inclusion $\supseteq$.

It remains to show $\subseteq$. The idea is to show that the right-hand side of the desired equation is a coset of

$$\ker\,(\mathbb{S}_n(X) \to H_{n-1}(X;\mathbf{L}_\bullet\langle 1\rangle) = \operatorname{im}\,(H_n(X;\mathbf{L}_\bullet\langle 1\rangle) \to L_n(\mathbb{Z}[\pi_1(X)])).$$

Since the left hand side is also a coset of the same subgroup and they have nonempty intersection the desired claim follows.

Fix first one degree one normal map $(g,c)\colon N \to X$. Note that then degree one normal maps into X relative to (g,c) can be classified by $[X;\mathrm{G/TOP}]$. In more detail, the degree one normal maps are in 1-1 correspondence with the reductions of the SNF ν. If we choose the reduction which corresponds to (g,c), say $\bar{\nu}$, then the differences $(\bar{\nu})' - \bar{\nu}$ are in 1-1 correspondence with $[X;\mathrm{G/TOP}] \cong H^0(X;\mathbf{L}_\bullet\langle 1\rangle)$.

We have the Poincaré duality $H^0(X;\mathbf{L}_\bullet\langle 1\rangle) \cong H_n(X;\mathbf{L}_\bullet\langle 1\rangle)$ obtained using the fundamental class $[X]_{\mathbf{L}^\bullet\langle 0\rangle} = U_{\bar{\nu}} \in H_n(X;\mathbf{L}^\bullet\langle 0\rangle) = H^k(T(\bar{\nu});\mathbf{L}^\bullet\langle 0\rangle)$[3]

Given the degree one normal maps (f,b) and (g,c) which give rise to n-dimensional quadratic algebraic complexes (C,ψ_C) and (D,ψ_D) in $\mathbb{Z}_*(X)$ which are globally Poincaré, their difference $(C,\psi_C) - (D,\psi_D)$ turns out to be also locally Poincaré and hence gives an element in $H_n(X;\mathbf{L}_\bullet\langle 1\rangle)$. This maps via the assembly to the difference of the quadratic signatures $\sigma_*(f,b) - \sigma_*(g,c)$. It follows that the right-hand side of the desired equation is a coset of $\mathrm{im}(H_n(X;\mathbf{L}_\bullet\langle 1\rangle) \to L_n(\mathbb{Z}[\pi_1(X)]))$ and the claim follows.

6. Concluding remarks

These notes contain just a sketch of the proof of the main theorem. The topic itself is larger and interesting generalizations and applications can be found in Part II of [Ran92]. We would just like to mention some of these briefly.

One important generalization is the theory when one works with the spectrum $\mathbf{L}_\bullet\langle 0\rangle$ rather than with $\mathbf{L}_\bullet\langle 1\rangle$. This yields an analogous theory for the ANR-homology manifolds rather than for topological manifolds. The Quinn resolution obstruction also fits nicely into this theory. For details see [Ran92, chapter 25].

Another important application is that the total surgery obstruction can be used to identify the geometric structure set of an n-dimensional manifold M with $\mathbb{S}_{n+1}(M)$. The geometric surgery exact sequence can be identified with the algebraic surgery exact sequence. For more details see [Ran92, chapter 18].

Interesting examples of geometric Poincaré complexes with non-trivial total surgery obstruction can be found in [Ran92, chapter 19].

References

[Ran79] Andrew Ranicki. The total surgery obstruction. In *Algebraic topology, Aarhus 1978 (Proc. Sympos., Univ. Aarhus, Aarhus, 1978)*, volume 763 of *Lecture Notes in Math.*, pages 275–316. Springer, Berlin, 1979.

[Ran80a] Andrew Ranicki. The algebraic theory of surgery. I. Foundations. *Proc. London Math. Soc. (3)*, 40(1):87–192, 1980.

[Ran80b] Andrew Ranicki. The algebraic theory of surgery. II. Applications to topology. *Proc. London Math. Soc. (3)*, 40(2):193–283, 1980.

[Ran81] Andrew Ranicki. *Exact sequences in the algebraic theory of surgery*, volume 26 of *Mathematical Notes*. Princeton University Press, Princeton, N.J., 1981.

[Ran92] A. A. Ranicki. *Algebraic L-theory and topological manifolds*, volume 102 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1992.

[Ran02a] Andrew Ranicki. Foundations of algebraic surgery. In *Topology of high-dimensional manifolds, No. 1, 2 (Trieste, 2001)*, volume 9 of *ICTP Lect. Notes*, pages 491–514. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2002.

[Ran02b] Andrew Ranicki. The structure set of an arbitrary space, the algebraic surgery exact sequence and the total surgery obstruction. In *Topology of high-dimensional manifolds, No. 1, 2 (Trieste, 2001)*, volume 9 of *ICTP Lect. Notes*, pages 515–538. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2002.

[3] Note that $\mathbf{L}_\bullet\langle 1\rangle$ is a module over $\mathbf{L}^\bullet\langle 0\rangle$.