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Distribution of Sequences: A Sampler

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Contents

Preface to the first edition

In the focus of the interest in the present monograph is the set $G(x_n)$ of all distribution functions of a given sequence x_n of real numbers or vectors in unit cubes. We shall identify the notion of the **distribution of a sequence** x_n with the set $G(x_n)$. However, only a relatively small number of sequences x_n are known with a completely described infinite set $G(x_n)$. The majority of sequences x_n for which $G(x_n)$ is completely known is formed by the set of uniformly distributed sequences, i.e. sequences x_n for which $G(x_n)$ is a singleton $\{g(x)\}$ with $g(x) = x$. The importance of the set $G(x_n)$ is reflected in the fact that most properties of a sequence x_n expressed in terms of limiting processes may be characterized using $G(x_n)$. For example, the fundamental Weyl's limit relation[1]

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\,\mathrm{d}x,$$

holding for any continuous function $f(x)$ defined on $[0, 1]$ and any uniformly distributed sequence x_n, can be generalized to the relation

$$\lim_{k\to\infty} \frac{1}{N_k} \sum_{n=1}^{N_k} f(x_n) = \int_0^1 f(x)\,\mathrm{d}g(x)$$

which is true for every $g \in G(x_n)$ if an appropriate index sequence N_k is used.

The description of $G(x_n)$ is of high theoretical importance in the theory of uniform distribution and in the number theory generally. For instance, the detailed knowledge of $G(x_n)$ influences the application of the sequence x_n when calculating some series of arithmetical functions using the generalized Weyl's limit relation. The sequences x_n for which $G(x_n)$ is a singleton (i.e.

[1] Throughout this book we shall use the shorthand notation x_n, $n = 1, 2, \dots$, for sequences, instead of the more common ones (x_n), or $(x_n)_{n\geq 1}$. Consequently, the symbol $G(x_n)$ will stand for $G((x_n))$, while $f(x_n)$ may denote either the value of the function $f(x)$ at $x = x_n$, or the sequence $f(x_n)$, $n = 1, 2, \dots$. The meaning will be clear from the context.

x_n has a limit law) have numerical applications through the so–called Quasi–Monte Carlo method

- in numerical integration,
- when approximating the solutions of differential equations,
- or when approximating the global extremes of continuous functions,
- in searching theory,
- in cryptology,
- or in financial applications,

to mention some areas of applications. For multi–dimensional sequences $\mathbf{x}_n$ the set $G(\mathbf{x}_n)$ can be used in the correlation analysis of co–ordinate sequences of $\mathbf{x}_n$ which yields different results from those obtained by the statistical analysis.

The outline of our conception is as follows. We shall list deterministic – mainly infinite – sequences, including block sequences. A finite sequence will be included if an estimation of its discrepancy is known. We shall not cover

- metric aspects of the theory of distribution,
- integer sequences and sequences from generalized metric spaces,
- distribution problems in finite abstract sets,
- continuously uniform distributions.

In most cases the terms of the listed sequences will be supposed to lie in the unit interval $[0,1]$ or that they are reduced $\bmod 1$. In some special cases we also include unbounded sequences with distribution functions defined on $(-\infty,\infty)$. Infinite sequences will be listed together with their distribution functions, the upper and lower distribution function, discrepancy, diaphony, dispersion, or with their known estimates, of course, depending on our present state of knowledge of all these quantities (often we even do not know anything about their density properties).

The sequences having limits are not listed for the obvious reasons, they have a one–jump asymptotic distribution function and so can be found in other sources. On the other hand, dense statistically convergent sequences which also possess one–jump asymptotic distribution functions are included.

The book itself is divided into four chapters. To make the book more self–contained we repeat the basic definitions or list the fundamental results in Chapter 1. This also will help to unify the exact meaning of the utilized notions which may be in use and to some extend hardly noticeable as to difference in their meaning. Simultaneously we hope thus also to help the non–specialized reader to find the fundamental notions and results of the classical theory on the real line or in multi–dimensional real spaces in one source. Additional theoretical results can be found in Chapter 4. Chapters 2 and 3 contain the promised lists of sequences, which are divided into two main categories:

- one–dimensional sequences (Chapter 2)
- multi–dimensional sequences (Chapter 3).

The sequences are grouped within these two categories according to a dominant (from our point of view) or characteristic feature mainly represented by

- a distribution criteria,
- the distribution as a result of some operations on sequences,
- general functions involved in the definition of the sequence,
- some important special functions appearing in their definitions as sequences involving
  - ∗ logarithmic functions,
  - ∗ trigonometrical functions,
  - ∗ number–theoretic functions,
  - ∗ power function,
  - ∗ exponential sequences, etc.

It is hard to find a unique classification scheme in the labyrinth of the various aspects. From the other classification attributes let us mention

- sequences involving primes,
- sequences of rational numbers or reduced rational numbers,
- the van der Corput sequence and van der Corput – Halton sequence,
- pseudorandom number generators,
- circle sequences.

The so–called completely uniformly distributed sequences can be found in Chapter 3.

Not all of these classification attributes may be immediately clear. Moreover they are neither uniquely determined nor even disjoint, therefore many cross–references should help the reader in orientation amongst other related sequences.

As already mentioned open problems are included not only to complete the picture. These may provide the impetus for further possible research. Having the same aim in mind the reader's attention is also directed to gaps in the presently known results in the theory of the distribution of sequences.

The sections of the book are numbered consecutively, their subsections too. The numbering of the entries starts afresh in each section. The entries are then numbered indicating the chapter by the first number, then the section by the second one, and the final number gives the order within the section. The theorems have the additional fourth number giving their order within the entry number. The notes containing a brief survey of related results together with relevant bibliographies follow immediately the main body of the entry. Here the numbering of the notes corresponds to the numbering in the main part of the entry if any, otherwise the numbering only separates notes from each other (the numbering may also continue if there is no relation to the numbering within the main body of the entry).

The book ends with an extended bibliography with cross–references to the main text, followed by the index of names referred to in the text and the subject index.

It is well–known that the theory of uniform distribution formally began with the pioneering paper *Über die Gleichverteilung von Zahlen mod. Eins* by Hermann Weyl published in 1916. Many important discoveries, the theory of uniform distribution, not excluded often have several forerunners. Results of P. Bohl, W. Sierpiński, S.N. Bernstein, G.H. Hardy and J.E. Littlewood historically paved the road to this theory. Later within some decades several authors, such as J.G. van der Corput, J.F. Koksma, A. Ostrowski, I.M. Vinogradov, and E. Hlawka introduced quantitative methods into the study of the distribution behaviour of sequences. van der Corput defined the discrepancy as a new tool for the quantitative measurement of the distribution behaviour of sequences, a notion which in turn has undergone dramatic development resulting in a variety of modifications and the corresponding avalanche of results.

The prerequisites for using this book are contained in the monographs listed below which are usually recommended as standard references in the general theory of the uniform distribution:

L. Kuipers – H. Niederreiter: *Uniform Distribution of Sequences*,[2] the first comprehensive monograph devoted to uniform distribution published by John Wiley in 1974;

G. Rauzy: *Propriétés statistiques de suites arithmétiques* published by Presses Universitaires de France in 1976;

E. Hlawka: Theorie der Gleichverteilung published in German by Bibliographisches Institut in 1979 and English under the title *The Theory of Uniform Distribution* by A B Academic Publishers in 1984;

N.M. Korobov: *Number–theoretic Methods in Approximate Analysis* published in Russian in 1963;

[2] Hereafter referred to as [KN].

I.M. Sobol': *Multidimensional Quadrature Formulas and Haar Functions* published in Russian in 1969;

Hua Loo Keng – Wang Yuan: *Applications of Number Theory to Numerical Analysis* published by Springer Verlag in 1981;

J. Beck – W.W.L. Chen: *Irregularities of Distribution* published by Cambridge University Press in 1987;

N.M. Korobov: *Trigonometric Sums and its Applications* (mainly Chapter 3) published in Russian in 1989;

H. Niederreiter: *Random Number Generation and Quasi–Monte Carlo Methods* published by SIAM in 1992;

S. Tezuka: *Uniform Random Numbers. Theory and Practice*, published by Kluwer Academic Publishers in 1995;

M. Drmota – R.F. Tichy: *Sequences, Discrepancies and Applications*[3] published by Springer Verlag in 1997. It is mainly devoted to results proved over the two decades 1974 – 1996;

J. Matoušek: *Geometric Discrepancy. An Illustrated Guide, Algorithms and Combinatorics* published by Springer Verlag in 1999; and

J.E. Gentle: *Random Number Generation and Monte Carlo Methods. Statistic and Computing* published by Springer Verlag in 1998 and second edition in 2003.

The interested reader may perhaps also direct his attention to the following expository papers which cover the topic from various points of view:

H. Niederreiter: *Quasi–Monte Carlo methods and pseudo–random numbers*, Bull. Amer. Math. Soc. **84** (1978), no. 6, 957–1040 (MR 80d:65016)

E. Hlawka – Ch. Binder: *Über die Entwicklung der Theorie der Gleichverteilung in den Jahren 1909 bis 1916 (On the development of the theory of uniform distribution in the years 1909 to 1916)*, Arch. Hist. Exact Sci. **36** (1986), no. 3, 197–249 (MR0872356 (88e:01037); Zbl. 0606.10001).

J. Beck – V.T. Sós: *Discrepancy theory* which appeared in Vol. II of the *Handbook of Combinatorics* published by Elsevier in 1995; and

E. Hlawka: *Statistik und Gleichverteilung (Statistics and uniform distribution)*, Grazer Math. Ber. **335** (1998), ii+206 pp (MR1638218 (99g:11093); Zbl. 0901.11027).

The first results from the early period of the development of the theory of uniform distribution can be found in existing classical textbooks:

[3]Hereafter referred to as [DT].

G. PÓLYA AND G. SZEGŐ: *Aufgaben und Lehrsätze aus der Analysis* published by Springer Verlag in several editions, the 3rd one in 1964;

J.F. KOKSMA: *Diophantische Approximationen (Diophantine Approximations)*, (German), published by Springer Verlag previuosly in 1936;

The mosaic of results can be completed using cumulative indices of Mathematical Reviews:

Reviews in Number Theory as printed in Mathematical Reviews 1940 through 1972, Vol. 3, published by AMS in 1974 and edited by W.J. LeVeque;

Reviews in Number Theory 1973–83 as printed in Mathematical Reviews, Vol. 3A, published by AMS in 1984 and edited by R.K. Guy;

Reviews in Number Theory 1984–96 as printed in Mathematical Reviews, Vol. 3B, published by AMS in 1997 and compiled by the Mathematical Reviews staff.

Last but not least, the following proceedings published by the Springer Verlag may be utilized as additional sources:

Monte Carlo and Quasi–Monte Carlo Methods in Scientific Computing (Las Vegas, 1994), Lecture Notes in Statistics, Vol. 106, published in 1995 and edited by H. Niederreiter and P.J. Shiue;

Monte Carlo and Quasi–Monte Carlo Methods 1996 (Salzburg), Lecture Notes in Statistics, Vol. 127, published in 1998 and edited by H. Niederreiter, P. Hellekalek, G. Larcher and P. Zinterhof;

Monte Carlo and Quasi–Monte Carlo Methods 1998 (Clermont), published in 2000 and edited by H. Niederreiter and J. Spanier;

Random and Quasi–Random Point Sets, Lecture Notes in Statistics, Vol. 138, published in 1998 and edited by P. Hellekalek and G. Larcher; and

Monte Carlo and Quasi–Monte Carlo Methods 2000 (Hong Kong), published in 2002 and edited by K.–T. Fang, F.J. Hickernell and H. Niederreiter.

The authors tried to make the presented selection of results as complete as possible in order to reflect the current state of stage. However due to the wealth of material scattered throughout the literature, it is highly probable that some noteworthy results may have been unintentionally omitted or not reproduced completely (or regrettably with errors). We would be grateful to the readers for their remarks, hints and opinion on how to improve or complete the presentation.

During the preparation of the book, valuable advice was provided by Henry Faure, Gérard Rauzy, Michel Mendès France, Robert F. Tichy and in particu-

lar by Pierre Liardet. The authors want to express them and the anonymous referees the deepest gratitude for their useful discussions and helpful comments which were used to improve the presentation of the book. O. Strauch thanks for grants from the the Slovak Academy of Sciences and Grant Agency VEGA in years 1994 – 2003 and presently the grant #2/4138/04. Š. Porubský would like to thank the Grant agency of the Czech Republic for supports on grants #201/93/2122, #201/97/0433, #201/01/0471 and#201/04/0381, and the Slovak Academy of Sciences and the Academy of Sciences of the Czech Republic for their support via the interacademic reciprocity agreement in the final stages of the preparations of the manuscript.

Bratislava and Prague, February 2005 Authors

Preface to the first revised and extended edition

The numbering of the items from the first edition will be kept also in all subsequent editions. It has the form $x.y.z.$ where $x.y.$ denotes the type of the sequences under consideration and z gives its order in the list. The new items added in the second edition are numbered in the form $x.y.z.u.$ where $x.y.z.$ is the label of a sequence from the first edition after which this newly added item is appended.

If a new sequence type is added in the second edition then we imitate the original numbering system in such a way that the added sequence has label $x.y.z.$ where $x.y.$ is the label of this newly added type of sequences and z is again its order in the list. In this case the added numbering obviously does not collide with the original one.

After the first edition of the book in 2005 several new monographs appeared. Let us mention at least the following ones:

E. Novak – H. Woźniakowski: *Tractability of Multivariete Problems*

Volume I: *Linear Information, 2008*,

Volume II: *Standard Information for Functionals, 2010* published by Europen Mathematical Society;

J. Dick – F. Pillichshammer: *Digital Nets and Sequences Discrepancy Theory and Quasi-Monte Carlo Integration* published by Cambridge University Press in 2010;

Monte Carlo and Quasi–Monte Carlo Methods 2006 (Ulm), published in 2008 and edited by A. Keller, S. Heinrich and H. Niederreiter;

Monte Carlo and Quasi–Monte Carlo Methods 2010 (Warsaw), published in 2012 and edited by L. Plaskota and H. Woźniakowski.

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Bratislava and Prague, December 2013 Authors

In December 2016 we added almost 50 new pages to the previous electronic edition. Besides this we made some small revisions of the already published parts to correct misprints or wrongly quoted results we found in the meantime. [4]

Bratislava and Prague, December 2016 Authors

[4]Partially supported by VEGA grant No. 2/0146/14

List of symbols and abbreviations

$\mathbb{N}_0 = \mathbb{Z}_0^+$	the set of non–negative integers
$\mathbb{N} = \mathbb{Z}^+$	the set of positive integers
$\mathbb{Z}$	the set of integers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{Z}_q$	the ring of all integers $\pmod q$
$\mathbb{F}_q$	the finite field of order q
$\mathbb{F}_q(z)$	the rational function field over $\mathbb{F}_q$
$\mathbb{F}_q((z^{-1}))$	the field of formal Laurent series over $\mathbb{F}_q$
$x, y, t, u, v, \dots$	denote real numbers
$x_n, y_n, \dots$	sequences of real numbers, see p. 1 – 1
$\mathbf{x}, \mathbf{y}, \dots$	s–dimensional real vectors
$\mathbf{h} = (h_1, \dots, h_s)$	s–dimensional integral vector, i.e. $h_i \in \mathbb{Z}$
$\mathbf{0} = (0, \dots, 0)$, $\mathbf{1} = (1, \dots, 1)$	
$\mathbf{x}_n, \mathbf{y}_n, \dots$	sequences of s–dimensional real vectors
$a, b, p, q, \dots$	denote positive integers
$m, n, i, j, k, \dots$	denote indices
$c_1, c_2, \dots, c, c', \dots, C, \dots$	will denote constants
$\mathbf{A}$	matrix
$\det(\mathbf{A})$	determinant of $\mathbf{A}$
$\mathrm{rank}(\mathbf{A})$	the number of linearly independent rows (or columns)
X_n, A_n	sequences of blocks, see p. 1 – 31
$\{x\}$	the fractional part of x, see p. 1 – 1
$x \bmod 1$	reduction modulo 1 we identify with $\{x\}$
$[x]$	the integer part of x, or floor of x or greatest integer function, see p. 1 – 1
$\\|x\\| = \min(\{x\}, 1 - \{x\})$	distance of x to the nearest integer
$\Delta x_n = x_{n+1} - x_n$	difference operator
$\Delta^k x_n = \Delta(\Delta^{k-1} x_n)$	difference operator of order k, see p. 2 – 14
$[x, y], [x, y)$	intervals (closed, right open)

$\lvert\mathbf{x}\rvert = \sqrt{\sum_{i=1}^{s} x_i^2}$	Euclidean norm of $\mathbf{x} = (x_1, \dots, x_s)$
$\lvert\mathbf{x}-\mathbf{y}\rvert = \sqrt{\sum_{i=1}^{s}(x_i - y_i)^2}$	Euclidean distance
$\lVert\mathbf{x}\rVert_\infty = \max\limits_{1\le i\le s} \lvert x_i\rvert$	supremum norm of $\mathbf{x} = (x_1, \dots, x_s)$
$\lVert\mathbf{x}-\mathbf{y}\rVert_\infty = \max\limits_{1\le i\le s} \lvert x_i - y_i\rvert$	maximum distance
$r(\mathbf{h}) = \prod_{i=1}^{s} \max(1, \lvert h_i\rvert)$	for $\mathbf{h} = (h_1, \dots, h_s) \in \mathbb{Z}^s$, see p. 1 – 68
$\mathbf{x}\cdot\mathbf{y} = \sum_{i=1}^{s} x_i y_i$	usual inner (scalar) product of $\mathbf{x} = (x_1, \dots, x_s)$ and $\mathbf{y} = (y_1, \dots, y_s)$, see p. 1 – 68
L	s–dimensional integration lattice, see p. 1 – 28
$L^\perp$	the dual lattice of L, see p. 1 – 28
$\rho(L)$	the figure of merit of L, see p. 3 – 83
$r(L)$	the rank of L, see p. 3 – 83
$\mathbf{x}^T$	column vector to the row vector $\mathbf{x}$
$\lvert X\rvert$	the Lebesgue measure of X
$\#X$	the number of terms of the set $X = \{\dots\}$
$\arg z$	argument of the complex number z
$\lvert z\rvert$	norm of the complex number z, see p. 2 – 114
$\Re(z)$	the real part of the complex number z
$n! = 1.2.\dots.n$	factorial, $0! = 1$
$\binom{m}{n} = \frac{m!}{n!(m-n)!}$	binomial coefficient
$a\mid b$	a divides b, $a, b \in \mathbb{Z}$
$a \nmid b$	a does not divide b
$p^\alpha \Vert n$	$p^\alpha\mid n$ and $p^{\alpha+1} \nmid n$
$a \equiv b \pmod m$	means $m\mid(a-b)$
$a \not\equiv b \pmod m$	means $m \nmid (a-b)$
$\prod_{d\mid m}$	the product over the divisors d of m
$\sum_{d\mid m}$	the sum over the divisors d of m
$\left(\frac{a}{p}\right)$	Legendre's symbol
$\gcd(a, b)$	greatest common divisor of a and b
$\mathrm{lcm}[a, b]$	least common multiple
a^*	positive integer $1 \le a^* < n$ satisfying $a.a^* \equiv 1 \pmod n$, see, p. 2 – 257
p_n	unless contrary is stated the nth prime or sequence of weights
$\omega(n) = \#\{p\,;\, p\mid n\}$	the number of distinct prime divisors of n
$\Omega(n)$	the total number of prime factors of n
$\mu(n) = (-1)^{\omega(n)}$	Möbius' function for square-free n

	and $\mu(n) = 0$ otherwise
$d(n)$	the total number of divisors of n (the divisor function)
$v(n)$	the nth Farey fraction, see, p. 2 – 288
$\varphi(n)$	Euler function
$\lambda(n)$	universal exponent of n, see p. 2 – 233
$\pi(n)$	number of all primes $\leq n$
$\mathrm{ord}_p(n) = \alpha$	if $p^\alpha \| n$, see p. 2 – 245
$h(n)$	$\min(\alpha_1, \dots, \alpha_k)$, where $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ is the canonical decomposition of n into primes, see p. 2 – 245
$H(n)$	$\max(\alpha_1, \dots, \alpha_k)$, where $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$, see p. 2 – 245
$\sigma(n)$	the sum of the positive divisors of n
$\sigma_k(n) = \sum_{d \mid n, d>0} d^k$	
$s_q(n)$	sum–of–digits function, see p. 2 – 105
$\gamma_q(n)$	radical inverse function in base q, see p. 2 – 121
$h(-n)$	the class number of $\mathbb{Q}(\sqrt{-n})$, see p. 2 – 261
$\chi(n)$	primitive Dirichlet character modulo q, see p. 2 – 253
$\nu(\lambda)$	the degree of the algebraic number λ, see p. 3 – 51
$\deg p(x)$	the degree of the polynomial $p(x)$
$\alpha = [a_0; a_1, a_2, \dots]$	continued fraction expansion of α with partial quotients $a_0, a_1, \dots$, see p. 2 – 80
$\frac{p_n}{q_n} = [a_0; a_1, a_2, \dots, a_n]$	the nth convergent of a continued fraction
$l(p/q) = i$	the length of $p/q = [a_0; a_1, a_2 \dots a_i]$, see p. 2 – 183
$M(\alpha) = 1/\liminf_{n\to\infty} n\|n\alpha\|$	the Markov constant, see p. 2 – 84
$K = \max_{1\leq i\leq l} a_i$	for $\frac{a}{N} = [a_0; a_1, \dots, a_l]$, see p. 3 – 75
$\rho = \min_{0\leq j\leq j} q_j \lvert q_j a - p_j N\rvert$	for $\frac{a}{N} = [a_0; a_1, \dots, a_l]$, see p. 3 – 75
u.d.	uniform distribution, uniformly distributed, or equi–distributed, see p. 1 – 4
almost u.d.	see p. 1 – 5
u.d. mod Δ	u.d. modulo subdivision, see p. 1 – 5
c.u.d.	continuously uniformly distributed, see p. 2 – 60

completely u.d.	see p. 1 – 21
double u.d.	see p. 1 – 5
u.d.p.	uniform distribution preserving, see p. 2 – 50
w.d.	well distributed, see p. 1 – 5
d.f.	distribution function, see p. 1 – 7
$g(x), \widetilde{g}(x), \dots$	denote d.f., see p. 1 – 7
$F_N(x)$	step d.f. of x_n, see p. 1 – 3
$F_N(\mathbf{x})$	s-dimensional step d.f., see p. 1 – 66
$F(X_n, x)$	step d.f. of block X_n, see p. 1 – 32
a.d.f.	asymptotic distribution function, see p. 1 – 11
$G(x_n)$	the set of the all distribution functions of x_n, see p. 1 – 9
$\underline{g}$, $\overline{g}$	the lower and upper d.f. of x_n, resp., see p. 1 – 11
$\underline{g}_H$, $\overline{g}_H$	the lower and upper d.f. with respect to a set H of d.f.'s, see p. 1 – 63
$c_{[0,x)}(t)$	the indicator of $[0, x)$
$c_\alpha(x)$	one–jump distribution function having a jump of height 1 at $x = \alpha$, see p. 1 – 19
$h_\alpha(x)$	constant distribution function with $h_\alpha(x) = \alpha$ for $x \in (0, 1)$
$r_2(x) = 1 - \left(\frac{\sin \pi x}{\pi x}\right)^2$	the pair correlation function in the GUE, see p. 2 – 250
$t_s(q)$	least t for all (t, s)–sequences in base q, see p. 3 – 99 , Par. (III) and (IX)
$d_s(q)$	least t for all digital (t, s)–sequences over $\mathbb{F}_q$, see p. 3 – 105 , Par. (IV)
$A([x, y); N; x_n)$	counting function, see p. 1 – 2
$A([u_1, v_1) \times \dots \times [u_s, v_s); N; \mathbf{x}_n)$	counting function in the s–dimensional case, see p. 1 – 66
$A([0, x); x_n)$	counting function for integer sequence x_n, see p. 1 – 3
$A([0, x); X_n)$	counting functions for blocks sequence X_n, see p. 1 – 32
$A_q(B_s, N)$	the number of occurrences of the block B_s in q–adic expansion of α, see p. 1 – 34

$\widetilde{A}(X;N;x_n)$	counting function for $X=\cup_{m=1}^{\infty}I_m$, see p. 1 – 40
$A(x;N;(x_n,z_n))$	counting function for diophantine approximations, see p. 1 – 39
$A^*(N,x_n)$	see p. 1 – 39
D_N	extremal discrepancy, see p. 1 – 45
$D_N(\theta)$	extremal discrepancy of $\{n\theta\}$, see p. 2 – 80
D_N^*	star discrepancy, see p. 1 – 45
$D_N^{(2)}$	L^2 discrepancy, see p. 1 – 45 and p. 1 – 80
$DI_N^{(2)}$	diaphony, see p. 1 – 55 and p. 1 – 83
DI_N	Zinterhof's diaphony, see p. 1 – 55
DF_N	diaphony using Walsh or Chrestenson functions, see p. 1 – 84
P_N	polynomial discrepancy, see p. 1 – 58
L_N	logarithmic discrepancy, see p. 1 – 60 and p. 1 – 93
D_r	Abel discrepancy, see p. 1 – 61 and p. 1 – 92
$D_N^{\mathbf{X}}$	discrepancy relative to $\mathbf{X}$, see p. 1 – 85
$D_N^{\mathbf{C}}$, $D_N^{\mathbf{C}(r)}$	discrepancy relative to cubes, see p. 1 – 85
$D_N^{\mathbf{B}(r)}$	discrepancy relative to balls, see p. 1 – 86
D_N^K	discrepancy relative to kernel K, see p. 1 – 89
S_N	spherical–cap discrepancy, see p. 1 – 87
I_N	isotropic discrepancy, see p. 1 – 87
$D_N^{\mathbf{P}}$	partition discrepancy, see p. 1 – 92
$\varphi_\infty(N)$	non–uniformity, see p. 1 – 90
$P_\alpha(L)$	discrepancy for lattice rule L, see p. 3 – 83
$\mathbf{A}$–D_N	matrix discrepancy, see p. 1 – 59
$D_N^{(2)}(x_n,H)$	L^2 discrepancy of x_n with respect to H, see p. 1 – 62
$D_N^{(2)}(x_n,g)$	L^2 discrepancy of x_n with respect to a d.f. g, see p. 1 – 53 and p. 1 – 82
$DS_N^{(2)}((x_n,y_n))$	L^2 discrepancy of statistically independent x_n and y_n, see p. 1 – 57
$D(g)$	discrepancy of a d.f. g, see p. 1 – 63
$U(g,z)$	logarithmic potential of g at z, see p. 1 – 63
$\sigma_N(\mathbf{x}_n)$	spectral test, see p. 1 – 96
$C_N(\theta)=-\frac{N}{2}+\sum_{n=1}^N\{n\theta\}$	see p. 2 – 85

$W_N(x_n)$	well–distribution measure of ± 1 sequence x_n, see p. 2 – 315
$C_N^{(k)}(x_n)$	correlation measure of order k, see p. 2 – 315
$Q_N^{(k)}(x_n)$	combined pseudorandom measure of order k, see p. 2 – 315
$N_N^{(k)}(x_n)$	normality measure of order k, see p.2 – 315
$N_N(x_n)$	normality measure of of ± 1 sequence x_n, see p. 2 – 315
$E_N(x_n)$	mean value of $x_1,\dots,x_n$, see p. 4 – 18
$\mathbf{D}_N^{(2)}(x_n)$	dispersion=variance of $x_1,\dots,x_N$, see p. 4 – 18
$R_N(x_n,y_n)$	correlation coefficient of $x_1,\dots,x_N$ and $y_1,\dots,y_N$, see p. 4 – 18
$D_N=\mathcal{O}(H(N))$	if there exists a number $c>0$ such that $D_N\le cH(N)$ for all sufficiently large N
$D_N \ll H(N)$	the same as $D_N=\mathcal{O}(H(N))$
$D_N \sim H(N)$	the same as $D_N/H(N)\to 1$, $N\to\infty$
$D_N=o(H(N))$	as $N\to\infty$, means $\lim_{N\to\infty}\frac{D_N}{H(N)}=0$
$D_N=\Omega(H(N))$	the same as $D_N\ne o(H(N))$
$\widetilde{d}_N(\theta)$	maximum of distances between consecutive numbers $0,1,\{1\theta\},\dots,\{N\theta\}$, see p. 2 – 84
$d_N^*=\min\limits_{1\le m\ne n\le N}\lvert x_m-x_n\rvert$	see p. 1 – 64
d_N	dispersion of $x_1,\dots,x_N$, see p. 1 – 64 and see p. 1 – 94 in the multi–dimensional case
d_N^∞	dispersion of $\mathbf{x}_1,\dots,\mathbf{x}_N$ with respect to maximum distance, see p. 1 – 94
$d_N(\theta)$	dispersion of $\{1\theta\},\dots,\{N\theta\}$, see p. 2 – 83
$D(\theta)=\limsup_{N\to\infty}Nd_n(\theta)$	see p. 2 – 84
$\overline{d}(a_n)$	upper asymptotic density of a_n, see p. 1 – 3
$\underline{d}(a_n)$	lower asymptotic density of a_n, see p. 1 – 3
$d(a_n)$	asymptotic density of a_n, see p. 1 – 3
$\mathrm{sp}(x_n)$	spectrum of the sequence x_n defined by M. Mendès France, see p. 2 – 45
$\mathrm{Bsp}(x_n)$	Fourier – Bohr spectrum, see p. 2 – 48
$\gamma(k)=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N z_{n+k}\overline{z}_n$	correlation of a complex sequence z_n, see p. 3 – 53

$\psi(k)$	autocorrelation of a real sequence, see p. 2 – 172
$B(q_n)$	normal set associate to q_n, see p. 2 – 95
$\delta_q(\theta) = \max_{1\le j\le s} \Vert q\theta_j\Vert$	see p. 3 – 11 ; here $\theta = (\theta_1, \dots, \theta_s)$ and $q \ge 1$ is an integer
$\lambda_f(t)$	modulus of continuity of f with respect to Euclidean distance, see p. 1 – 74
$\lambda_f^\infty(t)$	modulus of continuity of f with respect to maximum distance, see p. 1 – 74
$V(f)$	variation of f on $[0,1]$ or the Hardy – Krause variation on $[0,1]^s$, see p. 1 – 73
$V^{(k)}(f)$	the Vitali variation on $[0,1]^k$ of the function $f : [0,1]^k \to \mathbb{R}$, see p. 1 – 73
$\sigma^2(f)$	variance of f, see p. 1 – 76
$\Delta(h, J)$	see p. 1 – 73
$\mathrm{d}g(\mathbf{x})$	differential of g at $\mathbf{x}$ see p. 1 – 67
$\Delta_{h_i}^{(i)} g(\mathbf{x})$	difference of g by ith coordinate with increment h_i, see p. 1 – 67
$\log x$	logarithm of x in the base e
$\log^{(k)} n$	kth iterated logarithm, see p. 2 – 139
$\log_y x$	logarithm of x in the base y
$\mathrm{li}(x)$	integral logarithm
$\zeta(z) = \sum_{n=1}^\infty \frac{1}{n^z}$	for $\Re(z) > 1$, Riemann zeta function, see p. 2 – 253
$\rho(n) = \beta(n) + i\gamma(n)$	the sequence of the non–trivial zeros of $\zeta(z)$, see p. 2 – 249
$\mathrm{RH}(\alpha)$	the Riemann hypothesis with $\alpha = 1/2$, see p. 2 – 288
$L(s, \chi)$	Dirichlet L–function, see p. 2 – 253
$w_n(x)$	Walsh function, see p. 2 – 1
$w_{\mathbf{h}}(\mathbf{x})$	Chrestenson function, see p. 1 – 84
$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\,\mathrm{d}t$	complementary error function
$\mathrm{igamc}(u,x) = \frac{1}{\Gamma(u)} \int_x^\infty e^{-t} t^{u-1}\,\mathrm{d}t$	incomplete gamma function
$\Gamma(u) = \int_0^\infty e^{-t} t^{u-1}\,\mathrm{d}t$	gamma function
$\mathrm{sign}(x)$	1 for $x \ge 0$ and -1 others
$\delta(x)$	Dirac δ–function
U	the union of all Hardy fields, see p. 2 – 73

U^+	$f \in U$ satisfying $\lim_{x\to\infty} f(x) = \infty$, see p. 2 – 75
$E_s^\alpha(c)$	the set of periodic functions with bounded Fourier coefficients, see p. 3 – 72
P.V. number	Pisot – Vijayaraghavan number, see p. 2 – 187
F_n	nth Fibonacci number, see p. 2 – 146
B_n	nth Bernoulli number
$B_n(x)$	nth Bernoulli polynomial, see p. 4 – 5
$M_n = 2^n - 1$	nth Mersenne number
$B_n(x; f)$	Bernstein polynomial of degree n, associated with the function $f(x)$, see p. 2 – 4
$e = 2.71828182\ldots$	the base of natural logarithm
$\pi = 3.1415926536\ldots$	the ratio of a circle's circumference to its diameter
$\gamma_0 = 0.57721566490\ldots$	The Euler–Mascheroni constant
Th.	Theorem
Ex.	Example
Exer.	Exercise
Prop.	Proposition
Coroll.	Corollary
Chap.	Chapter
Sect.	Section
Par.	Paragraph
Rem.	Remark
a.e.	almost everywhere
JFM	Jahrbuch über die Fortschritte der Mathematik
MR	Mathematical Reviews
Zbl	Zentralblatt MATH

1. Basic definitions and properties

The main objects of the uniform distribution theory are:
- sequences;
- counting function;
- step distribution function of initial segments of a given sequence;
- distribution function of a given sequence;
- the set of all distribution functions of a given sequence;
- discrepancies.

We shall mainly follow the conventions and the conception used in the monographs *Uniform Distribution of Sequences* by L. Kuipers and H. Niederreiter and *Sequences, Discrepancies and Applications* by M. Drmota and R.F. Tichy which, for the sake of simplicity will be referred to as [KN] and [DT], respectively. Nevertheless, some modifications to the notations used in these books will appear in what follows. In this chapter we shall repeat the fundamentals facilitating a more comfortable reading of the text. Additional technical information can also be found in the Appendix.

1.1 Sequences

• The infinite sequences will be considered as real or complex valued functions defined on the set of positive integers $\mathbb{N}$ and will be denoted by x_n, y_n, z_n etc., with $n = 1, 2, \dots$, or in some explicitly mentioned cases with $n = 0, 1, 2, \dots$. We shall occasionally use the functional notation with the argument appearing in the parentheses instead of in the index position, e.g. $x(n)$, $y(n), \dots$ instead of $x_n, y_n, \dots$. Note, that if $f(x)$ is a function, then $f(x_n)$, $f(y_n), \dots$ also denote sequences.

• The infinite s–dimensional sequences will be considered as sequences of points of the s–dimensional Euclidean space $\mathbb{R}^s$ and denoted by $\mathbf{x}_n$, $\mathbf{y}_n$, $\mathbf{z}_n$ etc., where e.g. $\mathbf{x}_n = (x_{n,1}, \dots, x_{n,s})$ for $n = 1, 2, \dots$.

• We shall also consider finite sequences x_n, y_n, z_n etc., with say $n = 1, \dots, N$. These mostly arise as initial segments of infinite sequences, seldom as finite sequences of single terms.

NOTES: We shall mainly use the name finite sequences for what is often denoted as multisets in the combinatorial sense, i.e. for collection of objects where their multiplicity count. For instance, H. Niederreiter (1992) systematically uses the notion **point set** in such situations.

H. Niederreiter: *Random Number Generation and Quasi–Monte Carlo Methods*, CBMS–NSF Regional Conference Series in Applied Mathematics, 63, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992, vi+241 pp. (MR1172997 (93h:65008); Zbl. 0761.65002).

• Given a real number x, $[x]$ denotes the **integral part** of x, and $x \bmod 1$ stands for the residue of x modulo one, i.e. in other words the **fractional part** $\{x\}$ of x. Almost all sequences x_n, y_n, z_n etc. will be understood to be reduced modulo 1, that is as $x_n \bmod 1, y_n \bmod 1, z_n \bmod 1$ etc. In the multi–dimensional case all the coordinates are reduced modulo 1, i.e. $\mathbf{x}_n \bmod 1 = \big(\{x_{n,1}\}, \dots, \{x_{n,s}\}\big)$.

• We shall also consider the double sequences $x_{m,n}$, $y_{m,n}$, $z_{m,n}$ etc., where $m = 1, 2, \dots$ and $n = 1, 2, \dots$ run independently.

1.2 Counting functions

• Given a sequence x_n of real numbers, a positive integer N and a subset I of the unit interval $[0, 1)$, the **counting function** $A(I; N; x_n \bmod 1)$ is defined as the number of terms of x_n with $1 \le n \le N$, and with x_n taken modulo one, belonging to I, i.e.

$$A(I; N; x_n \bmod 1) = \#\{n \le N\ ;\ \{x_n\} \in I\} = \sum_{n=1}^{N} c_I(\{x_n\}),$$

where $c_I(t)$ is the **characteristic function** of I.

In the previous definition the unit interval and the fractional part have a very close relation. In some situations, however, the distribution property of a sequence x_n with terms belonging to an interval $[\alpha, \beta]$ is studied relative to the interval $[\alpha, \beta]$, in which case the fractional part of x_n in the composition $c_{[\alpha,\beta]}(\{x_n\})$ will not be taken into account, even if written (cf. p. 1 – 11).

Notes: (I) Some other types of counting functions are also used, e.g. O. Strauch (1994, p. 622) uses

$$A(([0, x), y); N; x_n) = \#\{m, n \le N\ ;\ x_m, x_n \in [0, x), |x_m - x_n| < y\}.$$

O. Strauch: *L^2 discrepancy*, Math. Slovaca **44** (1994), 601–632 (MR1338433 (96c:11085); Zbl. 0818.11029).

(II) Another example appears in the definition of the asymptotic density defined in 1.2 bellow which is based on the counting function $A([0, x); x_n)$ defined by (cf. H. Halberstam and K.F. Roth (1966, p. xix))

• Let a_n, $n = 1, 2, \ldots$, be an increasing sequence of positive (non–negative) integers, then

$$A([0,x); a_n) = \#\{n \in \mathbb{N}\,;\, a_n \in [0,x]\} = \sum_{n=1}^{\infty} c_{[0,x)}(a_n)$$

for any real $x > 0$. The **lower asymptotic density** $\underline{d}(a_n)$ and the **upper asymptotic density** $\overline{d}(a_n)$ of the sequence a_n are defined by

$$\underline{d}(a_n) = \liminf_{x\to\infty} \frac{A([0,x); a_n)}{x} = \liminf_{n\to\infty} \frac{n}{a_n},$$

$$\overline{d}(a_n) = \limsup_{x\to\infty} \frac{A([0,x); a_n)}{x} = \limsup_{n\to\infty} \frac{n}{a_n}.$$

If $\underline{d}(a_n) = \overline{d}(a_n)$, we say that the sequence a_n possesses the **asymptotic density** (or the **natural density**) $d(a_n)$, given by this common value. Some further types of densities can be found in G. Tenenbaum (1990, p. 309–314, Sec. III.1).

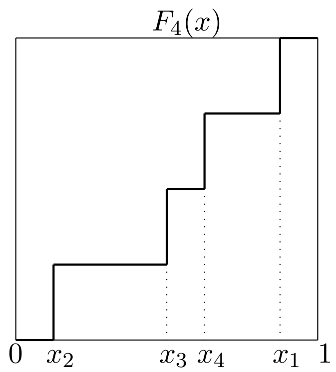
H. Halberstam – K.F. Roth: *Sequences*, Vol. I, Clarendon Press, Oxford, 1966; 2nd ed. 1983 (MR0210679 (**35** #1565); Zbl. 0141.04405).
G. Tenenbaum: *Introduction à la théorie analytique et probabiliste des nombres*, Institut Elie Cartan, Vol. 13, Université de Nancy, Nancy, 1990. (second edition: Société de France, Paris, 1995 (MR1366197 (97e:11005a); Zbl. 0880.11001)). (English translation: Studies in Advanced Mathematics, Vol. 46, Cambridge Univ. Press, Cambridge, 1995 (MR1342300 (97e:11005b); Zbl. 0880.11001)).

1.3 Step distribution function of x_n, $n = 1, 2, \ldots, N$

• For a sequence $x_1, \ldots, x_N$ mod 1 we define the **step distribution function** $F_N(x)$ for $x \in [0,1)$ by

$$F_N(x) = \frac{A([0,x); N; x_n \bmod 1)}{N}$$

while $F_N(1) = 1$, e.g.



Thus if $f : [0,1] \to \mathbb{R}$ is continuous then

$$\frac{1}{N}\sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\,\mathrm{d}F_N(x).$$

NOTES: The notion of the step distribution function was introduced by I.M. Soboľ (1969). The expression via Riemann – Stieljes integral is also valid for sequences $x_n \in [0,1]$ not reduced mod 1. The function $F_N(x)$ is also called the **empirical distribution** of $x_1,\dots,x_N$ mod 1. Weyl limit relation from 1.4 and also its generalization from 1.7 can be derived directly applying the second Helly theorem (cf. 4.1.4.13) to $F_N(x)$.

I.M. SOBOĽ: *Multidimensional Quadrature Formulas and Haar Functions*, (Russian), Library of Applied Analysis and Computational Mathematics, Izd. "Nauka", Moscow, 1969 (MR0422968 (**54** #10952); Zbl. 0195.16903).

1.4 Uniform distribution

• The sequence x_n is said to be **uniformly distributed modulo one** (abbreviated u.d. mod 1) if for every subinterval $[x,y) \subset [0,1]$ we have

$$\lim_{N\to\infty} \frac{A([x,y);N;x_n \bmod 1)}{N} = y - x \ \left(= \lim_{N\to\infty}(F_N(y) - F_N(x))\right).$$

(Note that it suffices to require $\lim_{n\to\infty} F_N(x) = x$ for all $x \in [0,1]$.) Such a sequence x_n is also called **equi–distributed modulo one**.

NOTES: We shall (unless the contrary is stated) firstly reduce the given sequence x_n modulo 1 and only then we proceed to the issue of the uniform distribution.

The next three theorems are of fundamental importance for the theory of u.d.

Theorem 1.4.0.1 (Weyl limit relation). *The sequence* x_n mod 1 *is u.d. if and only if for every continuous* $f : [0,1] \to \mathbb{R}$ *we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(\{x_n\}) = \int_0^1 f(x)\,\mathrm{d}x.$$

Theorem 1.4.0.2 (Weyl criterion). *The sequence* x_n mod 1 *is u.d. if and only if*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h x_n} = 0 \textit{ for all integers } h \neq 0.$$

Theorem 1.4.0.3 (van der Corput's difference theorem). *Let* x_n *be a sequence of real numbers. If for every positive integer* h *the sequence* $x_{n+h} - x_n$ mod 1 *is u.d., then* x_n mod 1 *is u.d.*

NOTES: The formal definition of u.d. was given by H. Weyl (1916) who also proved two of the above mentioned fundamental criteria (cf. 2.1.1, 2.1.2). The difference theorem was proved by van der Corput (1931) (cf. 2.2.1). For the proofs cf. [KN, p. 2, Th. 1.1], [KN, p. 7, Th. 2.1], and [KN, p. 26, Th. 3.1], resp.

H. WEYL: *Über die Gleichverteilung von Zahlen mod. Eins*, Math. Ann. **77** (1916), 313–352 (JFM 46.0278.06).
J.G. VAN DER CORPUT: *Diophantische Ungleichungen I. Zur Gleichverteilung modulo Eins*, Acta Math. **56** (1931), 373–456 (MR1555330; JFM 57.0230.05; Zbl. 0001.20102).

1.5 Other types of u.d.

• [KN, p. 53, Def. 7.2]: The sequence x_n is said to be **almost u.d.** mod 1 if a strictly increasing sequence of positive integers $N_1 < N_2 < ...$ exists such that for every subinterval $[x, y) \subset [0, 1]$ we have

$$\lim_{k\to\infty} \frac{A\big([x,y); N_k; x_n \bmod 1\big)}{N_k} = y - x.$$

• [KN, p. 40, Def. 5.1]: The sequence x_n mod 1 is said to be **well distributed** (abbreviated w.d.) if for every subinterval $[x, y) \subset [0, 1]$ we have

$$\lim_{N\to\infty} \frac{A\big([x,y); N; x_{n+k} \bmod 1\big)}{N} = y - x$$

uniformly in $k = 0, 1, 2, \dots$.

NOTES: The well distribution will be considered only occasionally.

• The double sequence $x_{m,n}$ mod 1 is said to be **u.d.** if for every subinterval $[x, y) \subset [0, 1]$ we have

$$\lim_{M,N\to\infty} \frac{A([x,y); M, N; x_{m,n} \bmod 1)}{MN} = y - x,$$

where $A([x, y); M, N; x_{m,n} \bmod 1)$ is the number of $x_{m,n}$, $1 \le m \le M$, $1 \le n \le N$, for which $x \le \{x_{m,n}\} < y$.

The Weyl limit relation and Weyl criterion takes the following form in this case:

Theorem 1.5.0.1 ([KN, p. 18, Th. 2.8]). *The double sequence* $x_{m,n}$ mod 1 *is u.d. if and only if for every Riemann integrable function* f *on* $[0, 1]$ *we have*

$$\lim_{M,N\to\infty} \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} f(\{x_{m,n}\}) = \int_0^1 f(x)\,\mathrm{d}x,$$

Theorem 1.5.0.2 ([KN, p. 18, Th. 2.9]). *The double sequence* $x_{m,n}$ mod 1 *is u.d. if and only if*

$$\lim_{M,N\to\infty}\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}e^{2\pi ihx_{m,n}}=0 \quad \textit{for all integers } h\neq 0.$$

• [KN, p. 4, Def. 1.2]: Let $\Delta=(z_n)_{n=0}^{\infty}$ be a sequence of increasing non–negative real numbers such that $z_0=0$ and z_n tends to infinity with n. We shall call z_n a **subdivision of the interval** $[0,\infty)$. For $z_{k-1}\le x<z_k$ put

$$[x]_\Delta=z_{k-1} \text{ and } \{x\}_\Delta=\frac{x-z_{k-1}}{z_k-z_{k-1}}.$$

The sequence x_n, $n=1,2,\dots$, of non–negative real numbers is said to be **u.d.** mod Δ if the sequence $\{x_n\}_\Delta$ is u.d., and it is said to be **almost u.d.** mod Δ if the sequence $\{x_n\}_\Delta$ is almost u.d. (cf. p. 1 – 5). A necessary condition states:

Theorem 1.5.0.3 (LeVeque (1953, Th. 1, cf. [KN, p. 4, Th. 1.3])). *If an increasing sequence of non–negative reals* x_n *with* $\lim_{n\to\infty}x_n=\infty$ *is u.d.* mod Δ*, then*

$$\lim_{k\to\infty}\frac{\#\{n\in\mathbb{N}\,;\,x_n<z_{k+1}\}}{\#\{n\in\mathbb{N}\,;\,x_n<z_k\}}=1.$$

• Let $I\subset\mathbb{R}$ be an interval of positive length $|I|$. The sequence $x_n\in I$ is said to be **u.d. with respect to** I if

$$\lim_{N\to\infty}\frac{\#\{n\le N\,;\,x_n\in J\}}{N}=\frac{|J|}{|I|}$$

for all subintervals $J\subset I$.

• The sequence $x_n\in\mathbb{R}$, $n=1,2,\dots$, is said to be **u.d. in** $\mathbb{R}$ if the sequence tx_n mod 1 is u.d. for every real number $t\neq 0$.

• Let h be a **measure density** defined on $\mathbb{X}\subset 2^{\mathbb{N}}$, i.e.
$\emptyset\in\mathbb{X}$, $\mathbb{N}\in\mathbb{X}$, and if $X_1,\dots,X_k\in\mathbb{X}$, then $\bigcup_{i=1}^{k}X_i\in\mathbb{X}$,
$h(\emptyset)=0$, $h(\mathbb{N})=1$, $0\le h(X)\le 1$ for $X\in\mathbb{X}$, and
$h(\bigcup_{i=1}^{k}X_i)=\sum_{i=1}^{k}h(X_i)$ for pairwise disjoint $X_i\in\mathbb{X}$.

The sequence $x_n\in[0,1)$ is called h–**u.d.** if for every $x\in[0,1]$ the set

$$A_x=\{n\in\mathbb{N}\,;\,x_n\in[0,x)\}$$

belongs to $\mathbb{X}$ and $h(A_x)=x$.

NOTES: (I) The notion of almost u.d. was introduced by I.I. Pjateckiĭ – Šapiro (1952).
(II) The notion of w.d. was introduced by E. Hlawka (1955) and G.M. Petersen

(1956). For the basic properties of w.d. cf. [KN, pp. 40–47] and [DT, pp. 259–268].
(III) The concept u.d. modulo subdivision goes back to W.J. LeVeque (1953). The case Δ with $z_n = n$ reduces to the ordinary concept of u.d. mod 1.
(IV) The above definition of u.d. in $\mathbb{R}$ is the same as the criterion from [KN, p. 283–284, Ex. 5.4] for u.d. in the locally compact additive group of real numbers. For examples cf. 2.15.1, 2.14.1(V) and 2.3.11.
(V) If $h = d$, the asymptotic density (see p. 1 – 3), then we get the classical u.d. Measure densities may alter: e.g. matrix (cf. 1.8.3), weighted or logarithmic (cf. 1.8.4), Abel (cf. 1.8.6), zeta (cf. 1.8.7), with respect to divisors (cf. 1.8.26), H_∞ (cf. 1.8.5), analytic, uniform, Schnilerman's, Buck's, etc., see G. Tenenbaum (1990), Š. Porubský (1984), M. Paštéka (1992, 1994), A. Fuchs and R. Giuliano Antonini (1990).

A. Fuchs – R. Giuliano Antonini: *Théorie générale des densités*, Rend. Accad. Naz. Sci. XL Mem. Mat. (5) **14** (1990), no. 1, 253–294 (MR1106580 (92f:11018); Zbl. 0726.60004).
E. Hlawka: *Zur formalen Theorie der Gleichverteilung in kompakten Gruppen*, Rend. Circ. Mat. Palermo (2) **4** (1955), 33–47 (MR0074489 (17,594c); Zbl. 0065.26402).
H. Niederreiter – J. Schoissengeier: *Almost periodic functions and uniform distribution* mod 1, J. Reine Angew. Math. **291** (1977), 189–203 (MR0437482 (**55** #10412); Zbl. 0338.10053).
M. Paštéka: *Some properties of Buck's measure density*, Math. Slovaca **42** (1992), no. 1, 15–32 (MR1159488 (93f:11011); Zbl. 0761.11003).
M. Paštéka: *Measure density of some sets*, Math. Slovaca **44** (1994), 515–524 (MR1338425 (96d:11013); Zbl. 0818.11007).
G.M. Petersen: *'Almost convergence' and uniformly distributed sequences*, Quart. J. Math. (2) **7** (1956), 188–191 (MR0095812 (**20** #2313a); Zbl. 0072.28302).
I.I. Pjateckiĭ–Šapiro: *On a generalization of the notion of uniform distribution of fractional parts*, (Russian), Mat. Sb. (N.S.), **30(72)** (1952), 669–676 (MR0056650 (15,106g); Zbl. 0046.04901).
Š. Porubský: *Notes on density and multiplicative structure of sets of generalized integers*, in: *Topics in classical number theory, Vol. I, II (Budapest, 1981)*, (G.Halász ed.), Colloq. Math. Soc. János Bolyai, Vol. 34, North–Holland Publishing Co., Amsterdam, New York, 1984, pp. 1295–1315 (MR0781186 (86e:11011); Zbl. 0553.10038).
G. Tenenbaum: *Introduction à la théorie analytique et probabiliste des nombres*, Institut Elie Cartan, Vol. 13, Université de Nancy, Nancy, 1990. (second edition: Société de France, Paris, 1995 (MR1366197 (97e:11005a); Zbl. 0880.11001)). (English translation: Studies in Advanced Mathematics, Vol. 46, Cambridge Univ. Press, Cambridge, 1995 (MR1342300 (97e:11005b); Zbl. 0880.11001)).
W.J. LeVeque: *On uniform distribution modulo a subdivision*, Pacific J. Math. **3** (1953), 757–771 (MR0059323 (15,511c); Zbl. 0051.28503).

1.6 Distribution functions

• A function $g : [0, 1] \to [0, 1]$ will be called **distribution function** (abbreviated d.f.) if the following two conditions are satisfied:
(i) $g(0) = 0$, $g(1) = 1$,
(ii) g is non–decreasing.

We shall identify any two distribution functions g, $\widetilde{g}$ which coincide at common continuity points, or equivalently, if $g(x) = \widetilde{g}(x)$ a.e.

Notes: (I) **Lebesgue decomposition theorem:** *Any d.f.* $g(x)$ *can be uniquely* *expressed as*

$$g(x) = \alpha_1 g_d(x) + \alpha_2 g_s(x) + \alpha_3 g_{ac}(x),$$

where α_1,α_2,α_3 *are non–negative constants,* $\alpha_1 + \alpha_2 + \alpha_3 = 1$*, and*

- $g_d(x)$ *is a* **discrete d.f.**, *i.e.* $g_d(x) = \sum_{t_n<x} h_n$*, where* t_n *is the sequence of points of discontinuity of* $g(x)$ *with jumps* h_n *at these points,*
- $g_s(x)$ *is a* **singular d.f.**[1]*, i.e. continuous, strictly increasing and having zero derivative a.e.,*
- *and* $g_{ac}(x)$ *is an* **absolutely continuous d.f.**, *i.e.* $g_{ac}(x) = \int_0^x h(t)\,\mathrm{d}t$ *for some non–negative Lebesgue integrable function* $h(t)$ *such that* $\int_0^1 h(t)\,\mathrm{d}t = 1$*. Function* $h(t)$ *is called the* **density of $\boldsymbol{g_{ac}(x)}$**.

(see A.N. Kolmogorov and S.V. Fomin (1972, p. 336)).

(II) The function

$$f(t) = \int_{-\infty}^{\infty} e^{itx}\,\mathrm{d}g(x), \quad t \in \mathbb{R},$$

where $g(x)$ is extended to $(-\infty,\infty)$ by $\mathrm{d}g(x) = 0$ for $x \notin [0,1]$ for integration reasons, is called the **characteristic function of the d.f.** $g(x)$. It has the following properties:

- $f(t)$ is uniformly continuous on $\mathbb{R}$;
- $f(0) = 1$, $|f(t)| \le 1$,$f(-t) = \overline{f}(t)$;
- Let g_1 and g_2 be two d.f. with characteristic functions f_1 and f_2, resp. If $f_1(t) = f_2(t)$ for every $t \in \mathbb{R}$, then $g_1(x) = g_2(x)$ a.e.
- If f is absolutely integrable on $\mathbb{R}$ then the corresponding d.f. g is absolutely continuous. Its density g' exists, is bounded, uniformly continuous and is given by
$$g'(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} f(t)\,\mathrm{d}t \quad \text{for } x \in [0,1],$$
- A d.f. g is continuous if and only if its characteristic function f satisfies
$$\liminf_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(t)|^2\,\mathrm{d}t = 0.$$
- If $F_N(x)$ is the step d.f. of the sequence x_n mod 1, $n = 1, 2, \dots, N$, defined in 1.3, then its characteristic function $f(t)$ is given by
$$f(t) = \int_{-\infty}^{\infty} e^{itx}\,\mathrm{d}F_N(x) = \int_0^1 e^{itx}\,\mathrm{d}F_N(x) = \frac{1}{N}\sum_{n=1}^{N} e^{itx_n}.$$

(e.g. see P.D.T.A. Elliott (1979, pp. 28–29, 112–114; p. 48, Lemma 1.23)).

P.D.T.A. Elliott: *Probabilistic Number Theory I. Mean–value Theorems*, Grundlehren der mathematischen Wissenschaften, Vol. 239, Springer Verlag, New York, Heidelberg, Berlin, 1979 (MR0551361 (82h:10002a); Zbl. 0431.10029).

[1] Also called **singular continuous d.f.**

A.N. Kolmogorov – S.V. Fomin: *Elements of the Theory of Functions and Functional Analysis*, (Russian), 3th ed., Izd. Nauka, Moscow, 1972 (Zbl 0235.46001; 4th ed. MR0435771 (**55** #8728)).

1.7 Distribution functions of a given sequence

• A d.f. g is called a **distribution function** of the sequence $x_n \bmod 1$ if an increasing sequence of positive integers $N_1, N_2, \dots$ exists such that the equality

$$g(x) = \lim_{k\to\infty} \frac{A([0,x);N_k;x_n \bmod 1)}{N_k} \left(= \lim_{k\to\infty} F_{N_k}(x)\right) \qquad (*)$$

holds at every point x, $0 \le x \le 1$, of the continuity of $g(x)$ and thus a.e. on $[0,1]$.

The existence of the above limit for a given sequence N_k is equivalent to the existence of the limit

$$\lim_{k\to\infty} \frac{1}{N_k} \sum_{n=1}^{N_k} f(\{x_n\}) = \int_0^1 f(x)\,\mathrm{d}g(x)$$

for every continuous $f : [0,1] \to \mathbb{R}$. This generalizes the Weyl limit relation 1.4.0.1.[2]

Notes: The above definition differs from that given in [KN, p. 53, Def. 7.2], where it is required that the relation $(*)$ should hold for all $x \in [0,1]$.

• The **set of all distribution functions** of a sequence $x_n \bmod 1$ will be denoted by $G(x_n \bmod 1)$. We shall identify the notion of the **distribution** of a sequence $x_n \bmod 1$ with the set $G(x_n \bmod 1)$, i.e. the distribution of $x_n \bmod 1$ is known if we know the set $G(x_n \bmod 1)$. The set $G(x_n \bmod 1)$ has the following fundamental properties for every sequence $x_n \bmod 1$:

- $G(x_n \bmod 1)$ is non–empty, and it is either a singleton or has infinitely many elements,
- $G(x_n \bmod 1)$ is closed and connected in the topology of the weak convergence, and these properties are characteristic for
- given a non–empty set H of distribution functions, there exists a sequence x_n in $[0,1)$ such that $G(x_n) = H$ if and only if H is closed and connected.

Notes: (I) Proof of non–emptiness can be found in [KN, p. 54, Th. 7.1].

(II) The closedness and connectivity can be derived from the following results proved by van der Corput:

[2] The Riemann – Stieljes integration with limits $\int_{0-0}^{1+0}$ is understood in this case.

Theorem 1.7.0.1 (J.G. van der Corput (1935–36, Satz 10)). *If $g_1(x)$, $g_2(x)$, $g_3(x), \dots$ are d.f.'s of $x_n \bmod 1$ and $\lim_{n\to\infty} g_n(x)$ exists at every common point x of continuity, then the corresponding limit function is also a d.f. of $x_n \bmod 1$.*

Theorem 1.7.0.2 (J.G. van der Corput (1935–36, Satz 5)). *Let H be a non–empty set of d.f.'s. Then there exists a sequence $x_n \in [0,1)$ with $G(x_n) = H$ if and only if there exitst a sequence of d.f.'s $g_n(x)$, $n = 1, 2, \dots$, in H such that*

(i) If $\lim_{k\to\infty} g_{n_k}(x) = g(x)$ at common points x of continuity, then $g \in H$, and conversely, there is such a subsequence for any $g \in H$.

(ii) $\lim_{n\to\infty} g_{n+1}(x) - g_n(x) = 0$ at any common point x of continuity of $g_n(x)$, $n = 1, 2, \dots$.

(III) A purely topological characterization of $G(x_n)$ with a short history can be found in R. Winkler (1997).

(IV) Since the weak topology is metrisable by the metric

$$d(g_1, g_2) = \left(\int_0^1 (g_1(x) - g_2(x))^2 \, \mathrm{d}x \right)^{1/2},$$

a non–empty closed set H is connected if and only if, for any two $g, \widetilde{g} \in H$ and every $\varepsilon > 0$ there exist finitely many $g_1, \dots, g_n \in H$ such that $g_1 = g$, $g_n = \widetilde{g}$ and $d(g_i, g_{i+1}) < \varepsilon$ for $i = 1, \dots, n-1$. Some examples of such H can be found in O. Strauch (1997), cf. 2.2.22.

(V) Instead of $G(x_n)$, the set of all Borel probability measures on $[0,1]$ associated with x_n is occasionally studied (cf. J. Coquet and P. Liardet (1987)).

• The continuity of all d.f.'s of $x_n \bmod 1$ follows from

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} \beta_k = 0, \quad \text{where } \beta_k = \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k x_n} \right|^2.$$

Notes: This is a generalization of the Wiener – Schoenberg theorem (2.1.4(II)) given by P. Kostyrko, M. Mačaj, T. Šalát and O. Strauch (2001).

J. Coquet – P. Liardet: *A metric study involving independent sequences*, J. Analyse Math. **49** (1987), 15–53 (MR0928506 (89e:11043); Zbl. 0645.10044).

P. Kostyrko – M. Mačaj – T. Šalát – O. Strauch: *On statistical limit points*, Proc. Amer. Math. Soc. **129** (2001), no. 9, 2647–2654 (MR1838788 (2002b:40003); Zbl. 0966.40001).

O. Strauch: *On the set of distribution functions of a sequence*, in: *Proceedings of the Conference on Analytic and Elementary Number Theory in Honor of Edmund Hlawka's 80th Birthday, Vienna, July 18–20, 1996*, (W.–G. Nowak, J. Schoißengeier eds.), Universität Wien & Universität für Bodenkultur, Vienna, 1997, pp. 214–229 (Zbl. 0886.11044).

J.G. van der Corput: *Verteilungsfunktionen I – II*, Proc. Akad. Amsterdam **38** (1935), 813–821, 1058–1066 (JFM 61.0202.08, 61.0203.01; Zbl. 0012.34705, 0013.05703).

J.G. van der Corput: *Verteilungsfunktionen III – VIII*, Proc. Akad. Amsterdam **39** (1936), 10–19, 19–26, 149–153, 339–344, 489–494, 579–590 (JFM 61.0204.01, 61.0204.02, 62.0206.06, 62.0207.01, 62.0207.02, 62.0207.03; Zbl. 0013.16001, 0013.20306, 0014.01106, 0014.01107, 0014.20803).

R. Winkler: *On the distribution behaviour of sequences*, Math. Nachr. **186** (1997), 303–312 (MR1461227 (99a:28012); Zbl. 0876.11040).

• Occasionally, if $x_n \in [\alpha, \beta]$, we define a distribution function $g(x)$ of x_n with respect to $[\alpha, \beta]$ as the limit (cf. 2.3.23, 2.14.4)

$$g(x) = \begin{cases} \lim_{k\to\infty} \frac{1}{N_k} \sum_{n=1}^{N_k} c_{[\alpha,x)}(x_n), & \text{at points } \alpha \le x < \beta \text{ of continuity of } g(x), \\ 1, & \text{if } x = \beta, \end{cases}$$

for some increasing sequence N_k. The set of all such d.f.'s will again be denoted by $G(x_n)$.

• Let $x \in [0,1]$. Consider the limits

$$\underline{g}(x) = \liminf_{N\to\infty} \frac{A([0,x);N;x_n \bmod 1)}{N},$$
$$\overline{g}(x) = \limsup_{N\to\infty} \frac{A([0,x);N;x_n \bmod 1)}{N}.$$

The d.f. $\underline{g}$ and $\overline{g}$ will be called the **lower**, and the **upper d.f.** of x_n mod 1, resp. Note, that either the lower or the upper d.f. assigned to a given sequence x_n need not be a d.f. of x_n mod 1 in general, i.e. they do not necessarily belong to $G(x_n \bmod 1)$.

Theorem 1.7.0.3 (O. Strauch (1997)). *The lower and upper d.f.* $\underline{g}$, $\overline{g}$ *of* x_n *belong to* $G(x_n \bmod 1)$ *if and only if*

$$\int_0^1 (\overline{g}(x) - \underline{g}(x))\,\mathrm{d}x = \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \{x_n\} - \liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \{x_n\}.$$

NOTES: The lower and upper d.f.'s were introduced by J.F. Koksma (1933). They can also be defined by relations

$$\underline{g}(x) = \inf_{g\in G(x_n \bmod 1)} g(x), \qquad \overline{g}(x) = \sup_{g\in G(x_n \bmod 1)} g(x).$$

J.F. KOKSMA: *Asymptotische verdeling van reële getallen modulo* 1. *I, II, III*, Mathematica (Leiden) **1** (1933), 245–248, **2** (1933), 1–6, 107–114 (Zbl. 0007.33901).
O. STRAUCH: *On the set of distribution functions of a sequence*, in: *Proceedings of the Conference on Analytic and Elementary Number Theory in Honor of Edmund Hlawka's 80th Birthday, Vienna, July 18–20, 1996*, (W.–G. Nowak, J. Schoißengeier eds.), Universität Wien & Universität für Bodenkultur, Vienna, 1997, pp. 214–229 (Zbl. 0886.11044).

1.8 Various types of distribution of sequences

1.8.1 g–distributed sequences, asymptotic distribution functions

• The sequence x_n mod 1 is said to have the **asymptotic distribution function** (in short a.d.f.) $g(x)$ if the relation

$$g(x) = \lim_{N\to\infty} \frac{A([0,x);N;x_n \bmod 1)}{N}$$

holds at every point x, $0 \le x \le 1$, of continuity of $g(x)$, i.e. if the set $G(x_n \bmod 1)$ of all d.f.'s of $x_n \bmod 1$ reduces to a singleton. The function g is sometimes referred to as the **limit law** or **limiting distribution** of the sequence $x_n \bmod 1$, or that $x_n \bmod 1$ is a g–**distributed sequence**, or $x_n \bmod 1$ is said to have a **distribution**.

NOTES: (I) The notion of the a.d.f. was introduced by I.J. Schoenberg (1928). He required, however at that time, that $g(x)$ is continuous at each $x \in [0,1]$. The above definition was rendered by him in 1939, and again in 1959.

(II) The definition given above differs from that given in [KN, Def. 7.1, p. 53], where it is required that the limit relation should hold for all $x \in [0,1]$. If, in addition, the limit[3]

$$g(x) = \lim_{N\to\infty} \frac{A([0,x];N;x_n \bmod 1)}{N}$$

exists for all $x \in [0,1]$, then g is called the **strong a.d.f.** of x_n. H. Niederreiter (1971, Th. 1) proved that if the sequence x_n has a continuous a.d.f. g (in the sense of [KN, Def. 7.1, p. 53]), then g is also the strong a.d.f. of x_n. He notes that if x_n has a discontinuous strong a.d.f. g, then $\lim_{N\to\infty} A([0,x];N;x_n \bmod 1)/N$ need not be equal to $g(x+0)$. Take for x_n the sequence $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$, then $\lim_{N\to\infty} A([0,0];N;x_n)/N = 0$, but $g(0+0) = 1$.

(III) R. von Mises (1933) proved (cf. 2.6.19) that for every distribution function g there exists a sequence $x_n \in [0,1)$ such that

$$\lim_{N\to\infty} \frac{A([0,x);N;x_n)}{N} = g(x)$$

holds for all $x \in [0,1]$. For sequences in a compact metric space this was generalized by E. Hlawka (1956).

(IV) Niederreiter (1971) assigned to each sequence x_n with elements in $[0,1)$ the partial order $\prec_{x_n}$ on the set of positive integers $\mathbb{N}$ defined by $m \prec_{x_n} n$ if and only if $x_m < x_n$. In terms of this ordering he then characterized sequences in $[0,1)$ having strong or continuous d.f., and sequences dense in $[0,1)$ having continuous d.f. E.g. let $C(n;N)$ denote the number of integers m, $1 \le m \le N$, such that $x_m < x_n$. Then the dense sequence $x_n \in [0,1)$ has a continuous d.f. if and only if the following conditions hold

(i) $\lim_{N\to\infty} \frac{C(n;N)}{N} = \alpha_n$ exists for all $n \ge 1$,

(ii) for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\alpha_n - \alpha_m| < \varepsilon$ whenever $|x_n - x_m| < \delta$.

The Weyl limit relation for a.d.f. becomes the form

Theorem 1.8.1.1. *The sequence* $x_n \bmod 1$ *has the a.d.f.* $g(x)$ *if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{x_n\}) = \int_0^1 f(x)\,\mathrm{d}g(x)$$

[3]Note the closing bracket in $[0,x]$.

for each continuous f defined on $[0,1]$.

NOTES: (I) The existence of the limit on the right–hand side (for every continuous f) is equivalent to the fact that x_n mod 1 has the limiting distribution.
(II) I.J. Schoenberg (1928, Satz I and p. 174) proved Th. 1.8.1.1 for the case of the continuous a.d.f.'s f having bounded variation, and he noted the consequences for $f(x) = x^k$, or $f(x) = e^{2\pi ikx}$, $k = 1, 2, \dots$. Actually he proved these results for block sequences $x_{i1}, \dots, x_{in} \in [0,1]$.

E. HLAWKA: *Folgen auf kompakten Räumen*, Abh. Math. Sem. Univ. Hamburg **20** (1956), 223–241 (MR0081368 (18,390f); Zbl. 0072.05701).
H. NIEDERREITER: *Distribution of sequences and included orders*, Niew Arch. Wisk. **19** (1971), no. 3, 210–219 (MR0364150 (**51** #405); Zbl. 0222.10057).
I.J. SCHOENBERG: *Über die asymptotische Verteilung reeller Zahlen* mod 1, Math. Z. **28** (1928), 171–199 (MR1544950; JFM 54.0212.02).
I.J. SCHOENBERG: *On asymptotic distribution of arithmetical functions*, Trans. Amer. Math. Soc. **39** (1936), 315–330 (MR1501849; Zbl. 0013.39302).
I.J. SCHOENBERG: *The integrability of certain functions and related summability methods*, Amer. Math. Monthly **66** (1959), 361–375 (MR0104946 (**21** #3696); Zbl. 0089.04002).
I.J. SCHOENBERG: *The integrability of certain functions and related summability methods II*, Amer. Math. Monthly **66** (1959), 562–563 (MR0107688 (**21** #6411); Zbl. 0089.04002).
R. VON MISES: *Über Zahlenfolgen, die ein kollektiv–ähnliches Verhalten zeigen*, Math. Ann. **108** (1933), no. 1, 757–772 (MR1512874; Zbl. 0007.21801).

1.8.2 Distribution with respect to a summation method

The next general definition covers the following cases: matrix, weighted, H_∞ and Abel asymptotic distribution.

• A sequence x_n mod 1 has the S–**a.d.f.** $g(x)$, if the sequence $c_{[0,x)}(\{x_n\})$ is S–summable to the d.f. $g(x)$ a.e. on $[0,1]$.

1.8.3 Matrix asymptotic distribution

• cf. [KN, p. 60, Def. 7.3]: Let $\mathbf{A} = (a_{n,k})$, $n = 1, 2, \dots$, $k = 1, 2, \dots$, be a positive Toeplitz matrix[4] and let x_n be a sequence of real numbers. Then d.f. $g(x)$ is the **A–asymptotic distribution function** of x_n mod 1 (abbreviated by **A**–a.d.f.) if

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} c_{[0,x)}(\{x_k\}) = g(x)$$

a.e. on $[0,1]$.

The Weyl limit relation 1.4 for **A**–asymptotic distribution becomes the form

[4] i.e. $a_{n,k} \geq 0$ for all n and k and $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = 1$. To ensure the regularity of $\boldsymbol{A}$ we need $\lim_{n\to\infty} a_{n,k} = 1$ for $k = 1, 2, \dots$, see Silverman – Toeplitz Theorem [KN, p. 62, Th. 7.12].

Theorem 1.8.3.1. *The sequence* $x_n \bmod 1$ *has the* $\mathbf{A}$*–a.d.f.* $g(x)$ *if and only if*

$$\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{n,k} f(\{x_k\}) = \int_0^1 f(x)\,\mathrm{d}g(x)$$

for each continuous f *defined on* $[0,1]$.

The existence of the limit on the right–hand side for every continuous f, is equivalent to the fact that $x_n \bmod 1$ has the $\mathbf{A}$–a.d.f.

Theorem 1.8.3.2 (cf. [KN, p. 62, Ex. 7.1, Th. 7.13]). *The concepts of the a.d.f.* mod 1 *in* $(\mathbf{C},1)$ *(arithmetic means),* $(\mathbf{C},r)$ *(Cesàro means) and* $(\mathbf{H},r)$ *(Hölder means) coincide.*

Notes: (I) Further results can be found in [KN, pp. 207–219].

(II) I.J. Schoenberg (1959) introduced the matrix summation method $\boldsymbol{A} = (a_{n,k})$, where

$$a_{n,k} = \begin{cases} \frac{\varphi(k)}{n}, & \text{if } k|n, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

i.e. the sequence x_n is called φ–**convergent** to α if the sequence $y_n = \frac{1}{n}\sum_{d|n}\varphi(d)x_d$ converges to α. Schoenberg's Theorem 2 (1959) shows that the φ–convergence of x_n implies the classical convergence of x_{n_k} (to the same limit) for every sequence n_k for which $\liminf_{k\to\infty}\frac{\varphi(n_k)}{n_k} > 0$. Since a 0–1 φ–convergent sequence has the φ–limit 0 or 1, no φ–u.d. sequence exists (E. Kováč (2005)).

(III) If $F_N(x) = \sum_{k=1}^{\infty} a_{n,k} c_{[0,x)}(x_k)$, then we can define the set $G(\boldsymbol{A}, x_n)$ of all d.f.'s of the sequence $x_n \in [0,1)$ with respect to a positive Toeplitz matrix $\boldsymbol{A} = (a_{n,k})$ as the set of all possible limits $F_{N_j} \to g(x)$ (a.e.) as $j\to\infty$.

E. Kováč: *On φ–convergence and φ–density*, Math. Slovaca **55** (2005), no. 3, 329–351 (MR2181010 (2007b:40001); Zbl. 1113.40002).

I.J. Schoenberg: *The integrability of certain functions and related summability methods*, Amer. Math. Monthly **66** (1959), 361–375 (MR0104946 (**21** #3696); Zbl. 0089.04002).

I.J. Schoenberg: *The integrability of certain functions and related summability methods II*, Amer. Math. Monthly **66** (1959), 562–563 (MR0107688 (**21** #6411); Zbl. 0089.04002).

1.8.4 Weighted asymptotic distribution

• cf. [KN, p. 250, Def. 2.35]: Let p_n, $n = 1, 2, \dots$, be a sequence of non–negative numbers such that $\sum_{n=1}^{\infty} p_n = \infty$, and set

$$P_N = \sum_{n=1}^{N} p_n.$$

Then the $\mathbf{A}$–a.d.f. $g(x)$ with $\mathbf{A} = (a_{N,n})$ defined by

$$a_{N,n} = \begin{cases} p_n/P_N, & \text{for } n \le N, \\ 0, & \text{for } n > N. \end{cases}$$

is called p_n–**weighted a.d.f.** In the case $p_n = 1/n$ we obtain the so–called **logarithmically weighted a.d.f.**

NOTES: The notion of a p_n–weighted u.d. sequence was introduced by M. Tsuji (1952). He proved

(I) Weyl's criterion 1.4.0.2 and 2.1.2 in the form: The sequence x_n mod 1 is p_n–weighted u.d. if and only if we have $\sum_{n=1}^{N} p_n e^{2\pi i h x_n} = o(\sum_{n=1}^{N} p_n)$ for $h = 1, 2, \dots$.

(II) The van der Corput difference theorem 1.4.0.3 and 2.2.1 in the form: Let the sequence p_n also satisfy the condition that p_n/p_{n+h} is a decreasing function of n for each $h = 1, 2, \dots$. If $(x_{n+h} - x_n)$ mod 1, $n = 1, 2, \dots$, is p_n–weighted u.d. for every $h = 1, 2, \dots$, then also x_n mod 1 is p_n–weighted u.d.

(III) He also proved an analogue to the Fejér's theorem 2.6.1, and that the sequence $\log n$ mod 1 (cf. 2.12.1) is $\frac{1}{n}$–weighted u.d., i.e. logarithmically weighted u.d. Other weights p_n for which $\log_{10} n$ mod 1 is u.d. was found by R. Giuliano Antonini (1991), see 2.12.1(VII).

R. GIULIANO ANTONINI: *On the notion of uniform distribution mod* 1, Fibonacci Quart. **29** (1991), no. 3, 230–234 (MR1114885 (92f:11101); Zbl. 0731.11044).
M. TSUJI: *On the uniform distribution of numbers* mod 1, J. Math. Soc. Japan **4** (1952), 313–322 (MR0059322 (15,511b); Zbl. 0048.03302).

1.8.5 H_∞–uniform distribution

• P. Schatte (1983): Given a sequence t_n, $n = 1, 2, \dots$, of real numbers, the Hölder means $(\mathbf{H}, k)$ are iterated means, i.e. $H_0(t_n) = t_n$ and $H_{k+1}(t_n) = \frac{1}{n}\sum_{j=1}^{n} H_k(t_j)$ for $k = 0, 1, 2, \dots$. If

$$\lim_{k\to\infty} \liminf_{n\to\infty} H_k(t_n) = \lim_{k\to\infty} \limsup_{n\to\infty} H_k(t_n)$$

then the common value is denoted by $H_\infty - \lim t_n$. A sequence x_n in $[0, 1)$ is said to be H_∞–**uniformly distributed** (abbreviated H_∞–u.d.) provided

$$H_\infty - \lim c_{[0,x)}(x_n) = x \quad \text{for every } 0 < x \le 1.$$

The Weyl criterion has in this case the form

Theorem 1.8.5.1. *The sequence x_n mod 1 is H_∞–u.d. if and only if*

$$H_\infty - \lim e^{2\pi i h x_n} = 0$$

for all $h = 1, 2, \dots$.

Theorem 1.8.5.2. *The sequence x_n mod 1 is H_∞–u.d. if and only if for all $h = 1, 2, \dots$*

$$\lim_{k\to\infty} \frac{1}{\log k} \sum_{j=n}^{kn} \frac{e^{2\pi i h x_j}}{j} = 0$$

uniformly in n.

Schatte's (1983) examples are: 2.2.13, 2.6.8.

P. SCHATTE: *On H_∞–summability and the uniform distribution of sequences*, Math. Nachr. **113** (1983), 237–243 (MR0725491 (85f:11057); Zbl. 0526.10043).

1.8.6 Abel asymptotic distribution

• cf. [DT, p. 268]: A sequence x_n mod 1 has the **Abel a.d.f.** $g(x)$ if

$$\lim_{r\to 1-0}(1-r)\sum_{n=0}^{\infty} c_{[0,x)}(\{x_n\})r^n = g(x)$$

a.e. on $[0,1]$.
The concepts of the a.d.f. mod 1 and of Abel a.d.f. coincide.

1.8.7 Zeta asymptotic distribution

• A sequence x_n mod 1 has the **zeta a.d.f.** $g(x)$ if

$$\lim_{\alpha\to 1+0}\frac{1}{\zeta(\alpha)}\sum_{n=1}^{\infty}\frac{c_{[0,x)}(\{x_n\})}{n^\alpha} = g(x)$$

a.e. on $[0,1]$. For an example cf. 2.19.8.

1.8.8 Statistically convergent sequences

The following concept of statistically convergent sequences serves as an example of g–distributed sequences.
• A sequence $x_n \in (-\infty,\infty)$ **statistically converges** to α if

$$\lim_{N\to\infty}\frac{1}{N}\#\{n\le N\,;\,|x_n-\alpha|\ge\varepsilon\} = 0$$

for every $\varepsilon > 0$.
Theorem 1.8.8.1 (I.J. Schoenberg (1959)). *A sequence x_n in $(-\infty,\infty)$ is statistically convergent to α if and only if x_n admits the a.d.f. $c_\alpha(x) := c_{(\alpha,\infty)}(x)$.*
For bounded sequences we have:
Theorem 1.8.8.2. *The sequence x_n in $[a,b]$ is statistically convergent to α if and only if for every real valued continuous function $f(x,y,z,\dots)$, defined on the closed multi–dimensional cube $[a,b]^s$ we have*

$$\lim_{M,N,K,\dots\to\infty}\frac{1}{MNK\dots}\sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{k=1}^{K}\dots f(x_m,x_n,x_k,\dots) = f(\alpha,\alpha,\alpha,\dots).$$

Theorem 1.8.8.3. *The sequence x_n in $[a,b]$ possesses a statistical limit if and only if*

$$\lim_{M,N\to\infty}\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}|x_m-x_n| = 0.$$

Theorem 1.8.8.4. *The sequence x_n in $[a,b]$ is statistically convergent to α if and only if*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}x_n=\alpha,\quad \textit{and}\quad \lim_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^{N}|x_m-x_n|=0.$$

Examples: 2.3.23, 2.20.18, 2.20.19

NOTES: (I) The notion of statistical convergence was independently introduced by H. Fast (1951) and I.J. Schoenberg (1959). H. Fast in his definition, however, assumed that x_n has the a.d.f., which is superfluous.

(II) H. Fast (1951) gave all the known elementary properties of statistically convergent sequences, namely

- The sum, the difference, and the product of statistically convergent sequences is again statistically convergent to the sum, the difference and the product of the corresponding limits, resp.
- A bounded sequence x_n of non–negative real numbers statistically converges to zero if and only if $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}x_n=0$.
- A sequence x_n is statistically convergent to α if and only if there exists a sequence of indices k_n of the asymptotic density $d(k_n)=1$ such that $\lim_{n\to\infty}x_{k_n}=\alpha$ in the standard sense.

(III) The Cauchy condition was introduced by O. Strauch (1995). However, it differs from the concept of the statistically Cauchy sequence defined by J.A. Fridy (1985).

(IV) J.A. Fridy (1993) defined that a real number x is said to be a **statistical limit point** of the given sequence x_n, $n=1,2,\dots$, if there exists a subsequence x_{k_n}, $n=1,2,\dots$, such that $\lim_{n\to\infty}x_{k_n}=x$ and the set of indices k_n has positive upper asymptotic density. P. Kostyrko, M. Mačaj, T. Šalát and O. Strauch (2001) proved that the set of all statistical limit points of the sequence $x_n\in[0,1)$ coincides with the set of all discontinuity points of d.f.'s $g(x)\in G(x_n)$.

H. FAST: *Sur la convergence statistique*, Colloq. Math. **2** (1951/1952), 241–244 (MR0048548 (14,29c); Zbl. 0044.33605).

J.A. FRIDY: *On statistical convergence*, Analysis **5** (1985), 301–313 (MR0816582 (87b:40001); Zbl. 0588.40001).

J.A. FRIDY: *Statistical limit points*, Proc. Amer. Math. Soc. **118** (1993), 1187–1192 (MR1181163 (94e:40008); Zbl. 0776.40001).

P. KOSTYRKO – M. MAČAJ – T. ŠALÁT – O. STRAUCH: *On statistical limit points*, Proc. Amer. Math. Soc. **129** (2001), no. 9, 2647–2654 (MR1838788 (2002b:40003); Zbl. 0966.40001).

I.J. SCHOENBERG: *The integrability of certain functions and related summability methods*, Amer. Math. Monthly **66** (1959), 361–375 (MR0104946 (**21** #3696); Zbl. 0089.04002).

I.J. SCHOENBERG: *The integrability of certain functions and related summability methods II*, Amer. Math. Monthly **66** (1959), 562–563 (MR0107688 (**21** #6411); Zbl. 0089.04002).

O. STRAUCH: *Uniformly maldistributed sequence in a strict sense*, Monatsh. Math. **120** (1995), 153–164 (MR1348367 (96g:11095); Zbl. 0835.11029).

1.8.9 Statistically independent sequences

• G. Rauzy (1976, p. 91, 4.1. Def.): Let x_n and y_n be two infinite sequences from the unit interval $[0,1)$. The pair of sequences (x_n,y_n) is called **statis-**

tically independent if

$$\lim_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N}f(x_n)g(y_n)-\left(\frac{1}{N}\sum_{n=1}^{N}f(x_n)\right)\left(\frac{1}{N}\sum_{n=1}^{N}g(y_n)\right)\right)=0$$

for all continuous real functions f, g defined on $[0,1]$. In other words, the double sequence (x_n, y_n) is called statistically independent if its coordinate sequences x_n and y_n are statistically independent.

The number of continuous functions $f(x)$ and $g(x)$ can be reduced, e.g.

G. Rauzy (1976, pp. 97–98): Two sequences x_n mod 1 and y_n mod 1 are statistically independent if and only if

$$\lim_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i(hx_n+ky_n)}-\left(\frac{1}{N}\sum_{n=1}^{N}e^{2\pi ihx_n}\right)\left(\frac{1}{N}\sum_{n=1}^{N}e^{2\pi iky_n}\right)\right)=0$$

for all integers h and k.

Theorem 1.8.9.1 (G. Rauzy (1976, p. 92, 4.2. par.)). *For every* $(x_n, y_n) \in [0,1)^2$ *we have*

$$(x_n,y_n) \text{ is statistically independent}$$
$$\Updownarrow$$
$$\forall_{g\in G(x_n,y_n)} \quad g(x,y)=g(x,1)g(1,y) \quad \text{a.e. on } [0,1]^2.$$

Notes: (I) This theorem can also be found in P.J. Grabner, O. Strauch and R.F. Tichy (1999) where it is used (p. 109) to give the following s–dimensional generalization of statistical independence: Let $\mathbf{x}_n = (x_{n,1}, \dots, x_{n,s}) \in [0,1)^s$ be an s–dimensional sequence formed from s sequences $x_{n,1}, x_{n,2}, \dots, x_{n,s}$. Then $\mathbf{x}_n$ is called **statistically independent** (or that $\mathbf{x}_n$ has **statistically independent co–ordinates** $x_{n,1}, \dots, x_{n,s}$) if every d.f. $g(\mathbf{x}) \in G(\mathbf{x}_n)$ can be written as a product $g(\mathbf{x}) = g_1(x_1)\dots g_s(x_s)$ of one–dimensional d.f.'s. Here g_i, $i = 1, \dots, s$, can depend on g.

(II) Grabner and Tichy (1994) proved that the extremal discrepancy does not characterize statistical independence, but the limit $\lim_{N\to\infty} D_N^{(2)} = 0$ of the L^2 discrepancy (cf. 1.11.4) provides a characterization.

(III) J. Coquet and P. Liardet (1987) call two multi–dimensional sequences $\mathbf{x}_n$ and $\mathbf{y}_n$ **statistically independent** if for every (complex valued) continuous f, g

$$\lim_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N}f(\mathbf{x}_n)g(\mathbf{y}_n)-\left(\frac{1}{N}\sum_{n=1}^{N}f(\mathbf{x}_n)\right)\left(\frac{1}{N}\sum_{n=1}^{N}g(\mathbf{y}_n)\right)\right)=0.$$

If for an integer $s \geq 1$, the s–dimensional sequences $\mathbf{x}_n = (x_{n+1}, \dots, x_{n+s})$ and $\mathbf{y}_n = (y_{n+1}, \dots, y_{n+s})$ are statistically independent, then the one–dimensional sequences x_n and y_n are said to be **statistically independent at rank** s. If they are statistically independent at rank s for all integers s, they are called **completely statistically independent** (for an example cf. 3.10.6).

Coquet and Liardet (1987) defined (following Rauzy (1976)) the **statistical independence for a family of sequences** H using the limit

$$\lim_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} f_1(\mathbf{x}_{n,1})\dots f_k(\mathbf{x}_{n,k}) - \left(\frac{1}{N}\sum_{n=1}^{N} f_1(\mathbf{x}_{n,1})\right)\dots\left(\frac{1}{N}\sum_{n=1}^{N} f_k(\mathbf{x}_{n,k})\right)\right),$$

provided this limit vanishes for any subfamily $\mathbf{x}_{n,1}, \dots, \mathbf{x}_{n,k}$ of H and for every continuous $f_1, \dots, f_k$. The equivalent formulation in terms of the decomposition of any $g \in G(\mathbf{x}_{n,1}, \dots, \mathbf{x}_{n,k})$ into the product of d.f.'s from $G(\mathbf{x}_{n,1}), \dots, G(\mathbf{x}_{n,k})$ they call **independence criterion**. Cf. also Coquet and Liardet (1984).

(IV) Liardet (1990) also defined the statistical independence of a sequence x_n with respect to a set Ψ of mappings $\psi : \mathbb{N} \to \mathbb{N}$ such that $\lim_{n\to\infty} \psi(n) = \infty$ provided the family of sequences $x_{\psi(n)}$, $\psi \in \Psi$, is statistically independent. Along parallel lines to those of the previous note he defined the notion of Ψ–independence at rank s and the complete Ψ–independence.

J. Coquet – P. Liardet: *Répartitions uniformes des suites et indépendance statistique*, Compositio Math. **51** (1984), no. 2, 215–236 (MR0739735 (85d:11072); Zbl. 0537.10030).
J. Coquet – P. Liardet: *A metric study involving independent sequences*, J. Analyse Math. **49** (1987), 15–53 (MR0928506 (89e:11043); Zbl. 0645.10044).
P.J. Grabner – R.F. Tichy: *Remarks on statistical independence of sequences*, Math. Slovaca **44** (1994), 91–94 (MR1290276 (95k:11098); Zbl. 0797.11063).
P.J. Grabner – O. Strauch – R.F. Tichy: *L^p–discrepancy and statistical independence of sequences*, Czechoslovak Math. J. **49(124)** (1999), no. 1, 97–110 (MR1676837 (2000a:11108); Zbl. 1074.11509).
P. Liardet: *Some metric properties of subsequences*, Acta Arith. **55** (1990), no. 2, 119–135 (MR1061633 (91i:11091); Zbl. 0716.11038).
G. Rauzy: *Propriétés statistiques de suites arithmétiques*, Le Mathématicien, Vol. 15, Collection SUP, Presses Universitaires de France, Paris 1976, 133 pp. (MR0409397 (**53** #13152); Zbl. 0337.10036).

1.8.10 Maldistributed sequences

• G. Myerson (1993): The sequence x_n mod 1 is said to be **uniformly maldistributed** if for every non–empty proper subinterval $I \subset [0, 1]$ we have both

$$\liminf_{N\to\infty} \frac{A(I; N; x_n \bmod 1)}{N} = 0, \quad \text{and} \quad \limsup_{N\to\infty} \frac{A(I; N; x_n \bmod 1)}{N} = 1.$$

This distribution can be characterized in terms of d.f.'s as follows (cf. O. Strauch (1995)):

Theorem 1.8.10.1. *A sequence* $x_n \bmod 1$ *is uniformly maldistributed if and only if* $G(x_n \bmod 1) \supset \{c_\alpha(x)\ ;\ \alpha \in [0,1]\}$, *where* $c_\alpha(x)$ *is the* **one–jump d.f.** *defined on* $[0,1]$ *by*

$$c_\alpha(x) = \begin{cases} 0, & \text{if } x \le \alpha, \\ 1, & \text{if } x > \alpha, \end{cases}$$

while always $c_\alpha(1) = 1$.

Examples: 2.12.2, 2.12.4, and in higher dimensions 3.2.2.

NOTES: (I) One–dimensional maldistributed sequences were introduced by G. Myerson (1993). A multi–dimensional analogue was studied by P.J. Grabner, O. Strauch and R.F. Tichy (1997).

R. Winkler (1997) proposed a generalization of the notion of maldistribution of a sequence $x_n \in [0,1)$ in which $G(x_n)$ contains the set all possible d.f.'s.

(II) J.–P. Kahane and R. Salem (1964) called a sequence $x_n \bmod 1$ **badly distributed** if at least one subinterval $I \subset [0,1]$ exists such that

$$\limsup_{N\to\infty} \frac{A(I; N; x_n \bmod 1)}{N} < |I|.$$

P.J. Grabner – O. Strauch – R.F. Tichy: *Maldistribution in higher dimension*, Math. Panon. **8** (1997), no. 2, 215–223 (MR1476099 (99a:11094); Zbl. 0923.11110).

J.–P. Kahane – R. Salem: *Distribution modulo 1 and sets of uniqueness*, Bull. Amer. Math. Soc. **70** (1964), 259–261 (MR0158216 (**28** #1442); Zbl. 0142.29604).

G. Myerson: *A sampler of recent developments in the distribution of sequences*, in: *Number theory with an emphasis on the Markoff spectrum (Provo, UT 1991)*, (A.D. Pollington and W. Moran eds.), Lecture Notes in Pure and App.Math., Vol. 147, Marcel Dekker, New York, Basel, Hong Kong, 1993, pp. 163–190 (MR1219333 (94a:11112); Zbl. 0789.11043).

O. Strauch: *Uniformly maldistributed sequence in a strict sense*, Monatsh. Math. **120** (1995), 153–164 (MR1348367 (96g:11095); Zbl. 0835.11029).

R. Winkler: *On the distribution behaviour of sequences*, Math. Nachr. **186** (1997), 303–312 (MR1461227 (99a:28012); Zbl. 0876.11040).

1.8.11 (λ, λ')–distribution

• J. Chauvineau (1967/68): Let λ and λ' be two real numbers such that $0 < \lambda \le 1 \le \lambda'$. The sequence $x_n \bmod 1$ is said to be (λ, λ')–**distributed** if, for every non–empty proper subinterval $I \subset [0,1]$, we have both

(i) $\liminf_{N\to\infty} \frac{A(I;N;x_n \bmod 1)}{N|I|} \ge \lambda$, and

(ii) $\limsup_{N\to\infty} \frac{A(I;N;x_n \bmod 1)}{N|I|} \le \lambda'$.

For an example cf. 2.12.1(IV). If only (i) is satisfied, the sequence $x_n \bmod 1$ is said to be (λ, ∞)–**distributed** or **positively distributed** (cf. O. Strauch (1982, p. 234)). On the other hand, if only (ii) is true the sequence $x_n \bmod 1$ is said to be $(0, \lambda')$–**distributed**. These distributions can be characterized using d.f.'s as follows (cf. O. Strauch (1997)):

Theorem 1.8.11.1. *A sequence x_n mod 1 is (λ, λ')–distributed if and only if every $g(x) \in G(x_n \bmod 1)$ has the lower derivative $\geq \lambda$ and the upper derivative $\leq \lambda'$ at every point $x \in (0, 1)$.*

J. Chauvineau: *Sur la répartition dans R et dans Q_p*, Acta Arit., **14** (1967/68), 225–313 (MR0245529 (**39** #6835); Zbl. 0176.32902).
O. Strauch: *Duffin – Schaeffer conjecture and some new types of real sequences*, Acta Math. Univ. Comenian. **40–41** (1982), 233–265 (MR0686981 (84f:10065); Zbl. 0505.10026).
O. Strauch: *On the set of distribution functions of a sequence*, in: *Proceedings of the Conference on Analytic and Elementary Number Theory in Honor of Edmund Hlawka's 80th Birthday, Vienna, July 18–20, 1996*, (W.–G. Nowak, J. Schoißengeier eds.), Universität Wien & Universität für Bodenkultur, Vienna, 1997, pp. 214–229 (Zbl. 0886.11044).

1.8.12 Completely u.d. sequences

• N.M. Korobov (1948): The sequence x_n mod 1 is said to be **completely uniformly distributed** (abbreviated completely u.d.) if for any $s \geq 1$ the s–dimensional sequence

$$\mathbf{x}_n = (x_{n+1}, \dots, x_{n+s}) \bmod 1$$

is u.d. in $[0, 1]^s$.

• E. Hlawka (1960): The sequence x_n mod 1 is said to be **u.d. of degree** s if

$$\mathbf{x}_n = (x_{n+1}, \dots, x_{n+s}) \bmod 1$$

is u.d. in $[0, 1]^s$.

• R.F. Tichy (1987): Let $s(N)$ increase monotonically, be unbounded and $s(N) = o(N)$. The sequence x_n mod 1 is said to be $s(N)$–**u.d.** if the discrepancy $D_{N-s(N)}$ of the $s(N)$–dimensional sequence

$$\mathbf{x}_n = (x_{n+1}, \dots, x_{n+s(N)}) \quad \bmod 1, \quad n = 1, 2, \dots, N - s(N),$$

satisfies $\lim_{N\to\infty} D_{N-s(N)} = 0$.

• A.G. Postnikov (1960): The sequence x_n mod 1 is said to be g–**completely distributed** if for any $s \geq 1$ the s–dimensional sequence $(x_{n+1}, \dots, x_{n+s})$ mod 1 has a.d.f.

$$g(\mathbf{x}) = g(x_1)g(x_2)\dots g(x_s).$$

Notes: Completely u.d. sequences were introduced by N.M. Korobov (1948) (see also G. Rauzy (1976, p. 23)). They are often suitable candidates for pseudorandom numbers, cf. D.E. Knuth (1981). For example 3.3.1, 3.6.2, 3.6.3, 3.6.4, 3.10.1, 3.10.2, 3.10.3 are completely u.d. and 3.7.6, 3.7.7 are completely dense. M. Drmota and R. Winkler (1995) proved that almost all sequences are $s(N)$–u.d. if $s(N) = o(\sqrt{N/\log N})$.

M. Drmota – R. Winkler: *$s(N)$–uniform distribution* mod 1, J. Number Theory **50** (1995), 213–225 (MR1316817 (95k:11097); Zbl. 0826.11034).
E. Hlawka: *Erbliche Eigenschaften in der Theorie der Gleichverteilung*, Publ. Math. Debrecen **7** (1960), 181–186 (MR0125103 (**23** #A2410); Zbl. 0109.27501).
D.E. Knuth: *Seminumerical Algorithms*, The Art of Computer Programming, Vol. 2, 2nd ed., Addison Wesley, Reading, MA, 1981 (First ed.: Reading, MA, 1969) (MR0286318 (**44** #3531); Zbl. 0477.65002).
N.M. Korobov: *On functions with uniformly distributed fractional parts*, (Russian), Dokl. Akad. Nauk SSSR **62** (1948), 21–22 (MR0027012 (10,235e); Zbl. 0031.11501).
A.G. Postnikov: *Arithmetic modeling of random processes*, Trudy Math. Inst. Steklov. (Russian), **57** (1960), 1–84 (MR0148639 (**26** #6146); Zbl. 0106.12101).
G. Rauzy: *Propriétés statistiques de suites arithmétiques*, Le Mathématicien, Vol. 15, Collection SUP, Presses Universitaires de France, Paris 1976, 133 pp. (MR0409397 (**53** #13152); Zbl. 0337.10036).
R.F. Tichy: *Ein metrischer Satz über vollständing gleichverteilte Folgen*, Acta Arith. **48** (1987), 197–207 (MR0895440 (88i:11051); Zbl. 0574.10049).

1.8.13 Completely dense sequences

• J. Bukor and J.T. Tóth (1998): A sequence x_n is said to be **completely dense in the interval** $I \subset (-\infty, \infty)$ if for any $s \geq 1$ the s–dimensional sequence

$$(x_{n+1}, \dots, x_{n+s})$$

is dense everywhere in I^s (see 3.7.6, 3.7.7).

J. Bukor – J.T. Tóth: *On completely dense sequences*, Acta Math. Inform. Univ. Ostraviensis **6** (1998), no. 1, 37–40 (MR1822513 (2001k:11147); Zbl. 1024.11052).

1.8.14 Relatively dense universal sequences

• D. Andrica and S. Buzeteanu (1987): A sequence x_n, $n = 1, 2, \dots$, of real numbers is said to be **relatively dense for a function** $f : \mathbb{R} \to \mathbb{R}$ if for every $x, y \in \mathbb{R}$ such that $f(x) < f(y)$ there exists an $n \in \mathbb{N}$ which satisfy $f(x) < f(x_n) < f(y)$. **Relatively dense universal** sequence is such a sequence x_n which is relatively dense for all continuous functions f with an irrational period (cf. 2.6.34, 2.14.8, 2.14.9).

D. Andrica – S. Buzeteanu: *Relatively dense universal sequences for the class of continuous periodical functions of period T*, Math. Rev. Anal. Numér. Théor. Approximation, Anal. Numér. Théor. Approximation **16** (1987), no. 1, 1–9 (MR0938777 (89i:11084); Zbl. 0642.26006).

1.8.15 Low discrepancy sequences

• [DT, p. 369]: an s–dimensional infinite sequence $\mathbf{x}_n$ which discrepancy (for def. of discrepancy D_N see 1.11.2) is bounded from above by

$$D_N(\mathbf{x}_n) = \mathcal{O}\left(\frac{(\log N)^s}{N}\right)$$

is called **low discrepancy sequence** [5]. Here the implied $\mathcal{O}$–constant depends only on dimension s and on the sequence $\mathbf{x}_n$.

NOTES: (I) Well–known conjecture says that this is the optimal order of magnitude of $D_N(\mathbf{x}_n)$ for an infinite sequence x_n in the s–dimensional unit cube. The problem is open for $s \geq 2$. For $s = 1$ the conjecture was proved by W.M. Schmidt (1972). The best general result is due to K.F. Roth (1954), who showed that for an arbitrary s and any s–dimensional infinite sequence $\mathbf{x}_n$ we have $D_N^* \geq c_s \frac{(\log N)^{s/2}}{N}$ for infinitely many N, where the constant c_s depends only on s. A slight improvement was obtained by J. Beck (1989) for $s = 2$ who proved that $D_N^* \geq c_2 \frac{\log N}{N}(\log\log N)^c$ for infinitely many N, where $c > 0$ is an absolute constant.

More generally, the sequence of single blocks $\mathbf{X}_n = (\mathbf{x}_{n,1}, \dots, \mathbf{x}_{n,N_n})$, $\mathbf{x}_{n,i} \in [0,1)^s$, is called **low discrepancy sequence** if

$$D_{N_n}(\mathbf{X}_n) = \mathcal{O}\left(\frac{(\log N_n)^{s-1}}{N_n}\right).$$

NOTES: (II) Cf. Hammersley sequence 3.18.2. It is conjectured (cf. [DT, p. 39] and the conjecture 1.11.2.5 on p. 1 – 72) that for every $s \geq 2$ there is a constant c_s such that for every finite sequence $\mathbf{x}_1, \dots, \mathbf{x}_N$ mod 1 we have $D_N(\mathbf{x}_n) \geq c_s \frac{\log^{s-1} N}{N}$. This conjecture is equivalent to that from (I), cf. 1.11.2.5.

J. BECK: *A two–dimensional van Ardenne-Ehrenfest theorem in irregularities of distribution*, Compositio Math. **72** (1989), no. 3, 269–339 (MR1032337 (91f:11054); Zbl. 0691.10041).
K.F. ROTH: *On irregularities of distribution*, Mathematika **1** (1954), 73–79 (MR0066435 (16,575c); Zbl. 0057.28604).
W.M. SCHMIDT: *Irregularities of distribution. VII*, Acta Arith. **21** (1972), 45–50 (MR0319933 (**47** #8474); Zbl. 0244.10035).

1.8.16 Low dispersion sequences

• An s–dimensional infinite sequence $\mathbf{x}_n$ for which dispersion (for def. see 1.11.17) we have

$$d_N(\mathbf{x}_n) = \mathcal{O}\left(N^{-1/s}\right)$$

is called **low dispersion sequence**. Examples are in 3.19.

1.8.17 (t, m, s)–nets

• [DT, p. 382, Def. 3.11]: Let t and m be integers satisfying $0 \leq t \leq m$ and let $q \geq 2$ be some chosen base. Finite s–dimensional sequence $\mathbf{x}_1, \dots, \mathbf{x}_N$ mod 1, $N = q^m$, is called **(t, m, s)–net in base q**, if

$$A(I; q^m; \mathbf{x}_n \bmod 1) = q^t$$

[5] These sequences are also called **quasirandom sequences**, e.g. H. Niederreiter (1992, p. 23).

for all intervals I of the form

$$I = \prod_{i=1}^{s} \left[\frac{a_i}{q^{d_i}}, \frac{a_i + 1}{q^{d_i}} \right),$$

where $d_i \geq 0$, $0 \leq a_i < q^{d_i}$ for $1 \leq i \leq s$ and $\sum_{i=1}^{s} d_i = m - t$ (i.e. the volume $|I| = q^{t-m}$) and thus $\mathbf{x}_n$ is a (t, m, s)–net if and only if

$$\left| \frac{A(I; N; \mathbf{x}_n \bmod 1)}{N} - |I| \right| = 0$$

for all such intervals I.

1.8.18 (t, s)–sequences

• Cf. [DT, p. 382, Def. 3.12]: Let $t \geq 0$ be an integer. An infinite sequence $\mathbf{x}_n$ mod 1 is called a (t, s)–**sequence in base** q if for all $k \geq 0$ and $m > t$ the finite section

$$\mathbf{x_n}, \qquad kq^m < n \leq (k+1)q^m,$$

is a (t, m, s)–net in base q. Every (t, s)–sequence is a low discrepancy sequence.

Notes: (I) The formal definition of (t, m, s)–nets and (t, s)–sequences in base $q = 2$ together with the method of their construction 3.19.5 and a discrepancy bound was given by I.M. Sobol′ (1966). Full proofs can be found in Sobol′ (1967) or in his monograph (1969, Chap. 3, Part 3, and Chap. 6). An overview of Sobol′'s results is given in Niederreiter (1978, pp. 979–981).
(II) The next contribution to the theory goes back to H. Faure (1982), cf. 3.19.6.
(III) The general theory of (t, m, s)–nets and (t, s)–sequences was developed by H. Niederreiter (1987, 1988).
(IV) H. Niederreiter and C.-P. Xing (1998) gave the following slightly modified definition of the (t, s)–sequence: Let $x \in [0, 1]$ and $x = \sum_{j=1}^{\infty} a_j q^{-j}$ be its infinite q–adic digit expansion (the possibility $a_j = q - 1$ is allowed for all but finitely many j). Given an arbitrary integer $m \geq 1$ define the truncation function

$$[x]_m = \sum_{j=1}^{m} a_j q^{-j}.$$

If $\mathbf{x} = (x_1, \dots, x_s)$ then put $[\mathbf{x}]_m = ([x_1]_m, \dots, [x_s]_m)$. The sequence $\mathbf{x}_n$, $n = 1, 2, \dots,$ of points in $[0, 1]^s$ is called a (t, s)–sequence in base q, if for all integers $k \geq 0$ and $m > t$ the finite section $[\mathbf{x}_n]_m$ for $kq^m < n \leq (k+1)q^m$ forms a (t, m, s)–net in base q.
(V) Surveys can be found in H. Niederreiter (1992, Chap. 4), G. Larcher (1998), H. Niederreiter and C.-P. Xing (1998).
(IV) Examples: 3.19.3, 3.19.4, 3.19.6, 3.19.1, 3.19.2.

H. Faure: *Discrépance de suites associées à un système de numération (en dimension s)*, Acta Arith. **41** (1982), 337–351 (MR0677547 (84m:10050); Zbl. 0442.10035).
G. Larcher: *Digital point sets: Analysis and application*, in: *Random and Quasi–Random Point Sets*, (P. Hellekalek, G. Larcher eds.), Lecture Notes in Statistics, 138, Springer Verlag, New York, Berlin, 1998, pp. 167–222 (MR1662842 (99m:11085); Zbl. 0920.11055).
H. Niederreiter: *Point sets and sequences with small discrepancy*, Monatsh. Math. **104** (1987), 273–337 (MR0918037 (89c:11120); Zbl. 0626.10045).
H. Niederreiter: *Low discrepancy and low–dispersion sequences*, J. Number Theory **30** (1988), 51–70 (MR0960233 (89k:11064); Zbl. 0651.10034).
H. Niederreiter: *Random Number Generation and Quasi–Monte Carlo Methods*, CBMS–NSF Regional Conference Series in Applied Mathematics, 63, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992, vi+241 pp. (MR1172997 (93h:65008); Zbl. 0761.65002).
H. Niederreiter – C.-P. Xing: *Nets, (t,s)–sequences and algebraic geometry*, in: *Random and Quasi–Random Point Sets*, (P. Hellekalek and G. Larcher eds.), Lecture Notes in Statistics, 138, Springer Verlag, New York, Berlin, 1998, pp. 267–302 (MR1662844 (99k:11121); Zbl. 0923.11113).
I.M. Sobol': *Distribution of points in a cube and integration nets*, (Russian), Uspechi Mat. Nauk **21** (1966), no. 5(131), 271–272 (MR0198678 (**33** #6833)).
I.M. Sobol': *Distribution of points in a cube and approximate evaluation of integrals*, (Russian), Ž. Vyčisl. Mat. i Mat. Fiz. **7** (1967), 784–802 (MR0219238 (**36** #2321)).
I.M. Sobol': *Multidimensional Quadrature Formulas and Haar Functions*, (Russian), Library of Applied Analysis and Computational Mathematics, Izd. "Nauka", Moscow, 1969 (MR0422968 (**54** #10952); Zbl. 0195.16903).

1.8.18.1. Digital $(\mathbf{T},s)$-sequence over $\mathbb{F}_q$.

• Let s denote the dimension of the sequence;
• q be a prime;
• Represent $n = n_0 + n_1q + n_2q^2 + \dots$ in base q;
• Let $C_1,\dots,C_s$ be $\mathbb{N}\times\mathbb{N}$-matrices over the finite field $\mathbb{F}_q$;
• $C_i\cdot(n_0,n_1,\dots)^T = (y_0^{(i)},y_1^{(i)},\dots)^T \in \mathbb{F}_q^{\mathbb{N}}$;
• $x_n^{(i)} := \frac{y_0^{(i)}}{q} + \frac{y_1^{(i)}}{q^2} + \dots$;
• The sequence $\mathbf{x}_n = (x_n^{(1)},\dots,x_n^{(s)})$ is said to be a $(\mathbf{T},s)$-sequence if for every $m\in\mathbb{N}$ there exists $\mathbf{T}(m)$ with $0\le\mathbf{T}(m)\le m$ such that for all partitions $d_1+\dots+d_s$ of $m-\mathbf{T}(m)$ the $(m-\mathbf{T}(m))\times m$-matrix which
first d_1 rows are formed by the upper left $d_1\times m$-submatrix of C_1,
the next d_2 rows are formed by the upper left $d_2\times m$-submatrix of C_2,
...
the last block of d_s rows is formed by the upper left $d_s\times m$-submatrix of C_s,
has rank $m-\mathbf{T}(m)$.

If $\mathbf{T}$ is minimal we speak about a **strict digital $(\mathbf{T},s)$-sequence**.

Notes:
(I) These definitions are from R. Hofer and G. Larcher (2010).
(II) An alternative definition is given in 3.19.2.

(III) A strictly digital $(\mathbf{T}, s)$-sequence is u.d. if and only if $\lim_{m\to\infty}(m-\mathbf{T}(m)) = \infty$. If $\mathbf{T}(m) \le t$ for all m, then $(\mathbf{T}, s)$-sequence is a (t, s)-sequence.

R. Hofer – G. Larcher: *On existence and discrepancy of certain digital Niederreiter-Halton sequences*, Acta Arith. **141** (2010), no. 4, 369–394 (MR2587294 (2011b:11108); Zbl. 1219.11112).

1.8.18.2. $(t, \alpha, \beta, n, m, s)$-nets

• Let n, m, s, $\alpha \ge 1$ and $b \ge 2$ be integers;
• $0 \le t \le \beta n$ be an integer, where $0 < \beta \le 1$;
• $\mathbf{k} = (k_1, \dots, k_s) \in \{0, \dots, n\}^s$, $|\mathbf{k}|_1 = \sum_{j=1}^s k_j$;
• $\mathbf{i_k} = (i_{1,1}, \dots, i_{1,k_1}, \dots, i_{s,1}, \dots, i_{s,k_s})$, where $1 \le i_{j,k_j} < \cdots < i_{j,1} \le n$ if $k_j > 0$ and $\{i_{j,1}, \dots, i_{j,k_j}\} = \emptyset$ if $k_j = 0$;
• $\mathbf{a_k} = (a_{1,i_{1,1}}, \dots, a_{1,i_{1,k_1}}, \dots, a_{s,i_{s,1}}, \dots, a_{s,i_{s,k_s}})$ where $\mathbf{a_k} \in \{0, \dots, b-1\}^{|\mathbf{k}|_1}$;
• $J(\mathbf{i_k}, \mathbf{a_k})$ is a generalized elementary interval of volume $b^{|\mathbf{k}|_1}$

$$J(\mathbf{i_k}, \mathbf{a_k}) = \prod_{j=1}^{s} \bigcup_{\substack{a_{j,l}=0 \\ l\in\{1,\dots,n\}\setminus\{i_{j,1},\dots,i_{j,k_j}\}}}^{b-1} \left[\frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n}, \frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n} + \frac{1}{b^n}\right),$$

• A $(t, \alpha, \beta, n, m, s)$-net in base b is a sequence $\mathbf{x}_0, \dots, \mathbf{x}_{b^m-1}$ in $[0, 1)^s$ such that the generalized elementary interval $J(\mathbf{i_k}, \mathbf{a_k})$ contains exactly $b^{m-|\mathbf{k}|_1}$ points of $\mathbf{x}_0, \dots, \mathbf{x}_{b^m-1}$ for each $\mathbf{a_k} \in \{0, \dots, b-1\}^{|\mathbf{k}|_1}$ and for all integers $k_j \ge 0$ and $1 \le i_{j,k_j} < \cdots < i_{j,1}$ satisfying
$\sum_{j=1}^s \sum_{l=1}^{\min(k_j,\alpha)} i_{j,l} \le \beta n - t$,
where if $k_j = 0$ we set the empty sum $\sum_{l=1}^0 i_{j,l} = 0$.

J. Baldeaux – J. Dick – F. Pillichshammer: *A characterization of higher order nets using Weyl sums and its applications*, Unif. Distrib. Theory **5** (2010), no. 1, 133–155 (MR2804667; Zbl. 1249.11071).

1.8.18.3. Niederreiter-Halton (NH) sequence

NH sequence is a combination of different digital $(\mathbf{T}_i, w_i)$-sequences in different prime bases $q_1, \dots, q_r$ with $w_1 + \cdots + w_r = s$ into a single sequence in $[0, 1)^s$.

Finite row NH sequence is a NH sequence in which every generating matrice of the component digital $(\mathbf{T}_i, w_i)$-sequences has in each row only finitely many non-vanishing entries.

Infinite row NH sequence is a (NH) sequence which is not a finite row NH one.

NOTES:
(I) A prototype example is the Halton sequence which is a combination of s digital $(0,1)$-sequences in different prime bases $q_1,\dots,q_s$ generated by the unit matrices in $\mathbb{F}_{q_i}$ for each i.
(II) General NH sequences were first investigated by R. Hofer, P. Kritzer, G. Larcher and F. Pillichshammer (2009). R. Hofer (2009) proved: NH sequence is u.d. if and only if each $(\mathbf{T}_i, w_i)$ is u.d.
(III) R. Hofer and G. Larcher (2010) gave concrete examples of digital $(0,s)$-sequences generated by matrices with finite rows. They noted that all low-discrepancy digital (t,s)-sequences in dimension $s \geq 2$ investigated by I.M. Sobol' (1967), H. Faure (1982), H. Niederreiter (1992), C.-P. Xing and H. Niederreiter (1995) are generated by matrices with infinite rows.

H. FAURE: *Discrépance de suites associées à un système de numération (en dimension s)*, Acta Arith. **41** (1982), 337–351 (MR0677547 (84m:10050); Zbl. 0442.10035).
R. HOFER: *On the distribution properties of Niederreiter-Halton sequences*, J. Number Theory **129** (2009), 451–463 (MR2473892 (2009k:11123); Zbl. 1219.11111).
R. HOFER – P. KRITZER – G. LARCHER – F. PILLICHSHAMMER: *Distribution properties of generalized van der Corput-Halton sequences and their subsequences*, Int. J. Number Theory **5** (2009), 719–746 (MR2532267 (2010d:11082); Zbl. 1188.11038).
R. HOFER – G. LARCHER: *On existence and discrepancy of certain digital Niederreiter-Halton sequences*, Acta Arith. **141** (2010), no. 4, 369–394 (MR2587294 (2011b:11108); Zbl. 1219.11112).
H. NIEDERREITER: *Random Number Generation and Quasi–Monte Carlo Methods*, CBMS–NSF Regional Conference Series in Applied Mathematics, 63, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992, vi+241 pp. (MR1172997 (93h:65008); Zbl. 0761.65002).
I.M. SOBOL': *Distribution of points in a cube and approximate evaluation of integrals*, (Russian), Ž. Vyčisl. Mat. i Mat. Fiz. **7** (1967), 784–802 (MR0219238 (**36** #2321)).
C.-P. XING – H. NIEDERREITER: *A construction of low-discrepancy sequences using global function fields*, Acta Arith. **73** (1995), no. 1, 87–102 (MR1358190 (96g:11096); Zbl. 0848.11038).

1.8.19 Good lattice points sequences

Good lattice points (g.l.p.) are integral vectors $\mathbf{g} = (g_1, g_2, \dots, g_s) \in \mathbb{Z}^s$ (depending on the parameter N) such that the discrepancy of the sequence

$$\mathbf{x}_n = \frac{n}{N}\mathbf{g} = \left(\frac{ng_1}{N}, \frac{ng_2}{N}, \dots, \frac{ng_s}{N}\right) \bmod 1, \quad n = 1, \dots, N,$$

satisfies

$$D_N(\mathbf{x}_n) = \mathcal{O}\left(\frac{(\log N)^s}{N}\right),$$

where the implied constant does not dependent on N.
NOTES: (I) The sequences of this form were first investigated by N.M. Korobov (1959). The existence of g.l.p.'s if N is a prime number was proved by E. Hlawka (1962) and N.M. Korobov (1963); see also [KN, pp. 154–157], H. Niederreiter (1992, Chap. 5) and 3.15.1.
(II) Hlawka (1962) and Korobov (1963, p. 96, Lemma 20) proved that for every

prime p there exists an vector $\mathbf{g} \in \mathbb{Z}^s$ such that

$$\sum_{\substack{0<\|\mathbf{h}\|_\infty<p\\ \mathbf{h}\cdot\mathbf{g}\equiv 0 \pmod p}} \frac{1}{r(\mathbf{h})} < \frac{2}{p}\,(5\log p)^s.$$

Hlawka called such $\mathbf{g}$ as **good lattice point modulo** p (see [KN, p. 156, Ex. 5.4]). Since the left hand side of above expression is directly connected with the discrepancy D_p (see 3.15.1(VII)) of the sequence $\frac{n}{p}\,\mathbf{g} \bmod 1$, $n = 1, \dots, p$, Hlawka definition provides an alternative approach to g.l.p. Korobov (cf. (1963, p. 96)) used a different terminology, he called such a $\mathbf{g}$ an **optimal point**.

E. Hlawka: *Zur angenäherten Berechnung mehrfacher Integrale*, Monatsh. Math. **66** (1962), 140–151 (MR0143329 (**26** #888); Zbl. 0105.04603).
N.M. Korobov: *Approximate evaluation of repeated integrals*, (Russian), Dokl. Akad. Nauk SSSR (N.S.), **124** (1959), 1207–1210 (MR0104086 (**21** #2848); Zbl 0089.04201).
N.M. Korobov: *Number–theoretic Methods in Approximate Analysis*, (Russian), Library of Applicable Analysis and Computable Mathematics, Fizmatgiz, Moscow, 1963 (MR0157483 (**28** #716); Zbl. 0115.11703).
H. Niederreiter: *Random Number Generation and Quasi–Monte Carlo Methods*, CBMS–NSF Regional Conference Series in Applied Mathematics, 63, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992, vi+241 pp. (MR1172997 (93h:65008); Zbl. 0761.65002).

1.8.20 Lattice rules

See H. Niederreiter (1992, pp. 125–146) and 3.17.

• An s–**dimensional lattice** is a discrete additive subgroup

$$L = \left\{ \sum_{i=1}^{s} h_i \mathbf{g}_i \,;\, (h_1, \dots, h_s) \in \mathbb{Z}^s \right\}.$$

of $\mathbb{R}^s$ generated by s linearly independent vectors $\mathbf{g}_1, \dots, \mathbf{g}_s \in \mathbb{R}^s$.

• An s–**dimensional integration lattice** is a lattice containing $\mathbb{Z}^s$.

• The **node set** of a lattice L is determined as the intersection $L \cap [0,1)^s$.

• If $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{N-1}$ is the node set of L then the s–**dimensional lattice rule** L is given by the quasi–Monte Carlo approximation

$$\frac{1}{N}\sum_{n=0}^{N-1} f(\mathbf{x}_n) \quad \text{of} \quad \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

If it is necessary to point out the number N we also speak about the s–**dimensional** N–**point lattice rule**.

• The **dual lattice** $L^\perp$ of the s–dimensional integration lattice L is defined by

$$L^\perp = \{\mathbf{h} \in \mathbb{Z}^s \,;\, \mathbf{h}\cdot\mathbf{x} \in \mathbb{Z} \ \text{ for all } \ \mathbf{x} \in L\}.$$

• The **shifted lattice** with shift $\mathbf{\Delta} \in \mathbb{R}^s$ is the set

$$L + \mathbf{\Delta} = \{\mathbf{x} + \mathbf{\Delta}\,;\ \boldsymbol{x} \in L\}.$$

NOTES: The first steps towards the definition of lattice rules go back to K.K. Frolov (1977), I.H. Sloan (1985), and I.H. Sloan and P. Kachoyan (1987).

K.K. FROLOV: *On the connection between quadrature formulas and sublattices of the lattice of integral vectors*, (Russian), Dokl. Akad. Nauk SSSR **232** (1977), 40–43 (MR0427237 (**55** #272); Zbl. 0368.65016).
H. NIEDERREITER: *Random Number Generation and Quasi–Monte Carlo Methods*, CBMS–NSF Regional Conference Series in Applied Mathematics, 63, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992, vi+241 pp. (MR1172997 (93h:65008); Zbl. 0761.65002).
I.H. SLOAN: *Lattice methods for multiple integration*, J. Comp. Appl. Math. **12/13** (1985), 131–143 (MR0793949 (86f:65045); Zbl. 0597.65014).
I.H. SLOAN – P. KACHOYAN: *Lattice methods for multiple integration: Theory, error analysis and examples*, SIAM J. Numer. Anal. **24** (1987), 116–128 (MR0874739 (88e:65023); Zbl. 0629.65020).

1.8.21 Random numbers

• Three different methods are used in the analysis of random numbers: (i) the structural, (ii) the complexity–theoretic, and (iii) the statistical one, cf. [DT, p. 424]. For example,

(I) D.E. Knuth (1981) proposed a hierarchy of definitions for a sequence x_n of uniform random numbers. In his definition
(i) R1 means that x_n is completely uniformly distributed,
(ii) R4 means that for every effective algorithm that specifies a sequence b_n of distinct positive integers, the sequence x_{b_n} is completely uniformly distributed.

Cf. [DT, pp. 424–430, 3.4.] and J.C. Lagarias (1990, 1992).

(II) A complete collection of tests for random and pseudorandom number generators can be found in A. Rukhin, J. Sotoij ; Soto, J., J. Nechvatal, *et al.* (2001).

(III) Physical random numbers are generated, as the name shows, by physical devices, e.g. coin flipping, roulette wheels, white noise, counts of emitted particles, cf. H. Niederreiter (1978, p. 998).

D.E. KNUTH: *Seminumerical Algorithms*, The Art of Computer Programming, Vol. 2, 2nd ed., Addison Wesley, Reading, MA, 1981 (First ed.: Reading, MA, 1969) (MR0286318 (**44** #3531); Zbl. 0477.65002).
J.C. LAGARIAS: *Pseudorandom number generators in cryptography and number theory*, in: *Cryptology and Computational Number Theory (Boulder, CO, 1989)*, (C. Pomerance ed.), Proc. Sympos. Appl. Math., 42, Amer. Math. Soc., Providence, RI, 1990, pp. 115–143 (MR1095554 (92f:11109); Zbl. 0747.94011).
J.C. LAGARIAS: *Pseudorandom numbers*, in: *Probability and Algorithms*, Nat. Acad. Press, Washington, D.C., 1992, pp. 65–85 (MR1194441; Zbl. 0766.65003).

H. Niederreiter: *Quasi–Monte Carlo methods and pseudo–random numbers*, Bull. Amer. Math. Soc. **84** (1978), no. 6, 957–1041 (MR0508447 (80d:65016); Zbl. 0404.65003).
A. Rukhin – J. Soto – J. Nechvatal – M. Smid – E. Barker – S. Leigh – M. Levenson – M. Vangel – D. Banks – A. Heckert – J. Dray – S. Vo: *A Statistical Test Suite for Random and Pseudorandom Number Generators for Cryptographic Applications*, NIST Special Publication 800-22, (2000 with revision dated May 15, 2001). (http://csrc.nist.gov/rng/SP800-22b.pdf).

1.8.22 Pseudorandom numbers

(I) We distinguish two cases: the uniform and the non–uniform one.

There is no satisfactory formal definition for **uniform pseudorandom numbers** but the basic demands for their generation should meet the following requirements:

- the sequence is generated by a deterministic algorithm,
- the standard generation algorithms are based on recursive procedures and thus yield periodic sequences[6],
- the generated sequences should have a sufficiently large period,
- the generated sequences should be equidistributed within the period in $[0,1]$,
- its successive terms should have reasonable statistical independence properties,
- the generation should possess a reasonably effective computer implementation.

(II) The generation of **non–uniform pseudorandom numbers** usually starts with a sequence of uniform pseudorandom numbers which is then processed by a follow–up transformation to a given distribution using one the following methods (cf. H. Niederreiter (1992, pp. 164–166)):

- the inversion method,
- the rejection method,
- the composition method,
- the ratio–of uniforms method.

(III) The concept of pseudorandom sequences can be interpreted in three different ways as (cf. Ch. Mauduit and A. Sárközy (1997)):

- $[0,1)$ sequences,
- pseudorandom sequences of integers selected from $\{1,2,\dots,N\}$,
- pseudorandom binary, or more generally, q–ary sequences,

(IV) The web site `http://random.mat.sbg.ac.at/` managed by P. Hellekalek is devoted to random numbers and their applications.

Notes: Statistical independence properties are studied in the **correlation analysis**. The correlation analysis of pseudorandom numbers $x_1,\dots,x_M$ should pass

[6] Every sufficiently large initial segment of an infinite u.d. sequence can also be considered as a sequence of u.d. pseudorandom numbers.

the **serial test** (discrepancy) or the **spectral test** (sums of Weyl type) of the **overlapping** s–tuples $\mathbf{x}_n = (x_n, x_{n+1}, \dots, x_{n+s-1})$, $n = 1, \dots, M-(s-1)$, or the **non–overlapping** s–tuples $\mathbf{x}_n = (x_{ns}, x_{ns+1}, \dots, x_{ns+s-1})$, $n = 1, \dots, [M/s]-1$ (cf. P. Hellekalek (1998) and example 2.25.5). H. Niederreiter (1992, pp. 166–168) discusses the following statistical tests: **uniformity test**, **gap test**, **run test**, **permutation test**, and **serial correlation**. Some results illustrating the difficulties of giving a comprehensive and general definition of pseudorandom sequence are discussed by R. Winkler (1993). J. Bass (1957) and J.–P. Bertrandias (1964) defined special types of pseudorandomness on the unit circle (see 3.11). P.J. Grabner, P. Liardet and R.F. Tichy (1995) reformulated these definitions to the case of real sequences x_n under the name

Bertrandias pseudorandomness: If $k \neq 0$ is any integer, then

(i) $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k x_n} = 0$,

(ii) $\gamma(h) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k (x_{n+h}-x_n)}$ exists,

(iii) $\lim_{H\to\infty} \frac{1}{N}\sum_{h=1}^{H} |\gamma(h)|^2 = 0$.

In the case called the **Bass pseudorandomness**, instead of (iii) the following stronger condition is required

(iii′) $\lim_{h\to\infty} \gamma(h) = 0$.

J. Bass: *Sur certaines classes de fonctions admettant une fonction d'autocorrélation continue*, C. R. Acad. Sci. Paris **245** (1957), 1217–1219 (MR0096344 (**20** #2828)); Zbl. 0077.33302).

J.–P. Bertrandias: *Suites pseudo–aléatoires et critères d'équirépartition modulo un*, Compositio Math. **16** (1964), 23–28 (MR0170880 (**30** #1115); Zbl. 0207.05801).

P.J. Grabner – P. Liardet – R.F. Tichy: *Odometres and systems of numeration*, Acta Arith. **70** (1995), no. 2, 103–123 (MR1322556 (96b:11108); Zbl. 0822.11008).

P. Hellekalek: *On correlation analysis of pseudorandom numbers*, in: *Monte Carlo and Quasi–Monte Carlo Methods 1996 (Proceedings of a conference at the University of Salzburg, Austria, July 9–12, 1996)*, (H. Niederreiter, P. Hellekalek, G. Larcher, P. Zinterhof eds.), Lecture Notes in Statistics, 127, Springer Verlag, New York, Berlin, 1998, pp. 251–265 (MR1644524 (99d:65020); Zbl. 0885.65005).

Ch. Mauduit – A. Sárközy: *On finite pseudorandom binary sequences, I. Measure of pseudorandomness, the Legendre symbol*, Acta Arith. **82** (1997), no. 4, 365–377 (MR1483689 (99g:11095); Zbl. 0886.11048).

H. Niederreiter: *Random Number Generation and Quasi–Monte Carlo Methods*, CBMS–NSF Regional Conference Series in Applied Mathematics, 63, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992, vi+241 pp. (MR1172997 (93h:65008); Zbl. 0761.65002).

R. Winkler: *Some remarks on pseudorandom sequences*, Math. Slovaca **43** (1993), no. 4, 493–512 (MR1248982 (94g:65009); Zbl. 0813.65001).

1.8.23 Block sequences

Block sequences provide one of the main tools for the construction of sequences with prescribed distribution properties.

• Let a finite sequence

$$X_n = (x_{n,1}, \dots, x_{n,N_n}) \bmod 1$$

be given for every $n \geq 1$. The infinite sequence

$$\omega = (x_{1,1}, \dots, x_{1,N_1}, x_{2,1}, \dots, x_{2,N_2}, \dots) \bmod 1,$$

abbreviated by $\omega = (X_n)_{n=1}^{\infty}$, is called a **block sequence associated with the sequence of single blocks** X_n, $n = 1, 2, \dots$.

• The notion of a d.f. of a block sequence $\omega = (X_n)_{n=1}^{\infty}$ is defined in Sect. 1.7. We shall distinguish between block sequences and sequences of individual blocks:

• For a block X_n we define the **step distribution function** $F(X_n, x)$ by

$$F(X_n, x) = \begin{cases} \dfrac{A([0,x); X_n)}{N_n}, & \text{for } x \in [0,1), \\ 1, & \text{if } x = 1, \end{cases}$$

where

$$A([0,x); X_n) = \#\big\{i \leq N_n \,;\, \{x_{n,i}\} \in [0,x)\big\}.$$

A d.f. $g(x)$ of the sequence of single terms X_n is defined as the limit

$$g(x) = \lim_{n\to\infty} F(X_{k_n}, x)$$

for a suitable sequence of indices $k_1 < k_2 < \dots$ at all continuity points $x \in [0,1]$ of $g(x)$.

- If $k_n = n$, then $g(x)$ is called the a.d.f. of X_n, and if $g(x) = x$, then X_n is called u.d. or **asymptotically u.d.**
- If k_n has the asymptotic density 1, then there exists mostly one such $g(x)$ and it is called the **generalized a.d.f.** of X_n. If $g(x) = x$, then the sequence of single blocks X_n is called **generalized u.d.**
- The set of the all d.f.'s of X_n will be denoted by $G(X_n)$.
- If $N_n = n$, then the sequence of single blocks X_n is called the **triangular array** X_n (cf. R.F. Tichy (1998), E. Hlawka (1979, 1983)).

Notes: In existing literature various types of block sequences have been investigated.

(I) The notion of the a.d.f. of block sequences was actually introduced and studied by I.J. Schoenberg (1928) for X_n with $N_n = n$. He gave some criteria and quotes a result of G. Pólya 2.22.13 that

$$X_n = \left(\frac{n}{1}, \frac{n}{2}, \dots, \frac{n}{n}\right) \bmod 1$$

has a.d.f. $g(x) = \int_0^1 \frac{1-t^x}{1-t}\,\mathrm{d}t$.

(II) In his monograph E. Hlawka (1984, p. 57–60) calls sequences of single blocks X_n

with $N_n = n$ **double sequences**, and with general N_n as N_n–**double sequences**. As an illustration he gives the proof of the u.d. of sequences (cf. 2.23.1)

$$X_n = \left(\frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\right), \quad \text{and} \quad X_n = \left(\frac{1}{n}, \frac{a_2}{n}, \dots, \frac{a_{\varphi(n)}}{n}\right),$$

with $a_1 = 1 < a_2 < \cdots < a_{\varphi(n)}$, $\gcd(a_i, n) = 1$ where $\varphi(n)$ denotes Euler's totient function. For the u.d. of related block sequences $\omega = (X_n)_{n=1}^{\infty}$ see the monograph by L. Kuipers and H. Niederreiter [KN, Lem. 4.1, Ex. 4.1, p. 136].
(III) R.F. Tichy (1998) gave some examples of triangular arrays which are u.d. (cf. also E. Hlawka (1983)).
(IV) G. Myerson (1993, p. 172) calls a sequence of blocks X_n (not taking the ordering of elements of X_n into account) a **sequence of sets**.[7] The same terminology is used by H. Niederreiter in his book (1992). Myerson calls the associated block sequence ω (with X_n endowed with some order) an **underlying sequence** and proved some criteria for the u.d. of such X_n.
(V) Let x_n be an increasing sequence of positive integers. Generalizing a result of S. Knapowski (1958), Š. Porubský, T. Šalát and O. Strauch (1990) have studied a sequence of blocks X_n of the type

$$X_n = \left(\frac{1}{x_n}, \frac{2}{x_n}, \dots, \frac{x_n}{x_n}\right).$$

They completely described the u.d. theory of related block sequences $\omega = (X_n)_{n=1}^{\infty}$, cf. 2.22.1.
(VI) O. Strauch and J.T. Tóth (2000) considered a sequence of blocks X_n, $n = 1, 2, \dots$, where

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right).$$

The associated block sequence $\omega = (X_n)_{n=1}^{\infty}$ denoted as x_m/x_n, $m = 1, \dots, n$, $n = 1, 2, \dots$, is also called the **ratio sequence** of x_n. Everywhere density of x_m/x_n was first investigated by T. Šalát (1969), cf. 2.22.2. The set of limit points of x_m/x_n was described by J. Bukor and J.T. Tóth (1996). O. Strauch and J.T. Tóth (1998) proved that if the lower asymptotic density of x_n is greater than or equal to 1/2, then the ratio sequence x_m/x_n is everywhere dense in $[0, \infty)$. Strauch and Tóth in (2001) studied the set $G(X_n)$ of all d.f.'s of such X_n, cf. 2.22.6, 2.22.7, 2.22.8, 2.19.16.

J. Bukor – J.T. Tóth: *On accumulation points of ratio sets of positive integers*, Amer. Math. Monthly **103** (1996), no. 6, 502–504 (MR1390582 (97c:11009); Zbl. 0857.11004).
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E. Hlawka: *The Theory of Uniform Distribution*, A B Academic Publishers, Berkhamsted, 1984 (translation of the original German edition Hlawka (1979)) (MR0750652 (85f:11056); Zbl. 0563.10001).

[7]Occasionally we shall also use set notation for description of blocks X_n

S. Knapowski: *Über ein Problem der Gleichverteilungstheorie*, Colloq. Math. **5** (1957), 8–10 (MR0092823 (19,1164c); Zbl. 0083.04401).
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R.F. Tichy: *Three examples of triangular arrays with optimal discrepancy and linear recurrences*, in: *Applications of Fibonacci Numbers (The Seventh International Research Conference, Graz, 1996)*, Vol. 7, (G.E. Bergum, A.N. Philippou and A.F. Horadam eds.), 1998, Kluwer Acad. Publ., Dordrecht, Boston, London, pp. 415–423 (MR1638468; Zbl. 0942.11036).

1.8.24 Normal numbers

See also [KN, p. 69, Def. 8.1; p. 71, Def. 8.2], [DT, p. 104–117] and examples in 2.18.

Let $q \geq 2$ be an integer and α be a real number having q–adic digit expansion $\alpha = a_0.a_1a_2\dots a_n\dots$ with digits $a_n, 0 \leq a_n < q$ for $n = 1, 2\dots$. If $B_s = (b_1b_2\dots b_s)$ is a given block of q–adic digits of length $s \geq 1$ put $|B_s| = 1/q^s$. Let $A_q(B_s; N)$ be the number of those n with $1 \leq n \leq N - s + 1$ for which $a_{n+j-1} = b_j$ for $1 \leq j \leq s$, i.e. the number of occurrences of the block B_s in the sequence of blocks

$$(a_1a_2\dots a_s)(a_2a_3\dots a_{s+1})\dots(a_{N-s+1}a_{N-s+2}\dots a_N).$$

The number α is called **normal in the base** q if

$$\lim_{N\to\infty} \frac{A_q(B_s; N)}{N} = \frac{1}{q^s} \ (= |B_s|)$$

for all $s \geq 1$ and all B_s. The number α is called **absolutely normal** if it is normal in all bases $q \geq 2$. The number α is called **simply normal** if the limit holds for $k = 1$, i.e. each digit from 0 to $q - 1$ appears with the asymptotic frequency $1/q$.

The next theorem shows the relation between normal numbers and sequences of the type αq^n mod 1:

Theorem 1.8.24.1 (cf. [KN, p. 70, Th. 8.1]). *The number α is normal in the base q if and only if the sequence $\alpha q^n \bmod 1$, $n = 1, 2, \dots$, is u.d.*

NOTES: (0) A number which is simply normal in any base (called absolutely normal in W. Sierpiński (1964, p. 277)) is absolutely normal in our sense. The existence of such numbers follows from the well–known result proved by E. Borel (1909) saying that almost all real numbers are absolutely normal. The first effective example was given by Sierpiński (1917) and H. Lebesgue (1917).

(I) Another approach (cf. 2.18.19) was discovered by A.G. Postnikov (1952): If there exist two positive constants c and σ such that for every s and B_s

$$\limsup_{N\to\infty} \frac{A_q(B_s;N)}{N} < c|B_s| \left(1 + \log\frac{1}{|B_s|}\right)^{\sigma},$$

then the number α is normal in the base q. Postnikov's result extended a previous result proven by I.I. Šapiro – Pjateckiĭ (1951) in which the right–hand side has the form $c|B_s|$.

(II) If $N = ns$ define a new counting function $\widetilde{A}_q(B_s;N)$ as the number of occurrences of the block B_s in the sequence of blocks

$$(a_1a_2\dots a_s)(a_{s+1}a_{s+2}\dots a_{2s})\dots(a_{(n-1)s+1}a_{(n-1)s+2}\dots a_{ns}).$$

The following theorem of S.S. Pillai (1939, 1940) gives an alternative definition of normality (for a proof cf. Postnikov (1960)): The number α is normal if and only if

$$\lim_{n\to\infty} \frac{\widetilde{A}_q(B_s;ns)}{n} = |B_s|$$

for every s and B_s.

(III) Some elementary properties:

J.E. Maxfield (1953): A non–zero rational number times a normal number in the base q is normal in the same base.

W.M. Schmidt (1960): If there exist positive integers p, q, k, and l such that $p^k = q^l$, then any number normal in the base p is also normal in the base q, and vice versa. If such exponents k, l do not exist, then there exists a real number normal in the base p but non–normal in the base q.

(IV) Theorem 1.8.24.1 provided the impetus for the following general definition: Let $\theta > 1$ be a real number. The number α is called **normal in the real base** θ if the sequence $\alpha\theta^n \bmod 1$, $n = 1, 2, \dots$, is u.d. (for an example cf. 2.18.21). Let $B(\theta)$ denote the set of such numbers α. G. Brown, W. Moran and A.D. Pollington (1993) answered some questions posed by Mendès France:

(a) $B(\theta) = (1/q)B(\theta)$ if and only if for some $j \in \mathbb{N}$ either $\theta^j \pm \theta^{-j} \in \mathbb{N}$ or $\theta^j \in \mathbb{N}$,
(b) $B(\theta_1) = B(\theta_2)$ if and only if there is some $j \in \mathbb{N}$ such that $\theta_1^j, \theta_2^j \in \mathbb{N}$, $\log\theta_1/\log\theta_2 \in \mathbb{Q}$ and $\mathbb{Q}(\theta_1) = \mathbb{Q}(\theta_2)$,
(c) $B(10) \not\subset B(\sqrt{10})$.

(V) For the multi–dimensional case we have the following definitions (cf. [KN, p. 76, Notes]):
J.E. Maxfield (1953): A k–tuple $(\alpha_1, \dots, \alpha_k)$ is called a **normal k–tuple in the base q** if the sequence

$$(q^n\alpha_1, \dots, q^n\alpha_k) \bmod 1, \quad n = 1, 2, \dots,$$

is u.d. in $[0,1]^k$. The Weyl theorem 1.11.1.3 implies that a k–tuple $(\alpha_1, \dots, \alpha_k)$ is normal in the base q if and only if $\sum_{i=1}^k h_i\alpha_i$ is normal in the same base for all integers $(h_1, \dots, h_k) \neq (0, \dots, 0)$.
N.M. Korobov (1952): A k–tuple $(\alpha_1, \dots, \alpha_k)$ is called a **jointly normal in the bases** $q_1, \dots, q_k$ if the sequence

$$(q_1^n\alpha_1, \dots, q_k^n\alpha_k) \bmod 1, \quad n = 1, 2, \dots,$$

is u.d. in $[0,1]^k$ (for an example see 3.2.4).
Matrix normality was considered by L.N. Pushkin (1991): Let $\mathbf{A}$ be a k–dimensional square matrix with real elements. Then a real k–dimensional vector $\boldsymbol{\alpha}$ is said to be **normal with respect to A**, if the sequence $\boldsymbol{\alpha}\mathbf{A}^n \bmod 1$ is u.d. in $[0,1]^k$ and is said to be **absolutely normal** if the sequence $\boldsymbol{\alpha}\mathbf{A}^n \bmod 1$ is u.d. in $[0,1]^k$ for every non–singular matrice which no eigenvalue is a root of unity. Let $\mathbf{A}$ and $\mathbf{B}$ be non–singular matrices with no eigenvalue being a root of unity. If $\mathbf{AB} = \mathbf{BA}$ then the sets of $\mathbf{A}$–normal vectors and $\mathbf{B}$–normal ones coincide if and only if there are integers $i, j \geq 1$ such that $\mathbf{A}^i = \mathbf{B}^j$, cf. G. Brown (1992).
(VI) Some other types of normality:
A.G. Postnikov and I.I. Pjateckiĭ – Šapiro (1957) and A.G. Postnikov (1960): Let p be a number $0 < p < 1$. The number $\alpha = a_0.a_1a_2\dots a_n\dots$ expressed in the base $q = 2$ is called **Bernoulli normal** if

$$\lim_{N\to\infty} \frac{A_q(B_s; N)}{N} = p^j(1-p)^{s-j}$$

for all $s \geq 1$ and all B_s, where j is the number of occurrences of 1 in B_s.
A.G. Postnikov and I.I. Pjateckiĭ – Šapiro ([a]1957): Let
- $\mathbf{P} = (p_{i,j})_{0\leq i,j\leq q-1}$ be an irreducible Markov transition matrix,
- $\mathbf{p} = (p_i)_{0\leq i\leq q-1}$ be the stationary probability vector of $\mathbf{P}$,

The number α is said to be **Markov–normal** if in its q–ary expansion $\alpha = 0.a_1a_2\dots = \sum_{i=1}^\infty a_i/q^i$ each fixed finite block of digits $b_0b_1\dots b_k$ appears with the asymptotic frequency of

$$p_0 p_{b_0,b_1}\cdots p_{b_{k-1},b_k}.$$

Let $g(x)$ be a d.f. defined on $[0,1]$ by

$$g(\gamma_n + 1/q^n) - g(\gamma_n) = p_{c_1}p_{c_1,c_2}\cdots p_{c_{n-1},c_n}$$

for any $\gamma_n = 0.c_1c_2\dots c_n$ where $c_i \in \{0,1,\dots,q-1\}$. Then α is Markov–normal in the base q if and only if the sequence $\alpha q^n \bmod 1$ has a.d.f. $g(x)$.

M.B. Levin (1996) constructed the Markov–normal number α with star discrepancy $D_N^* = \mathcal{O}((\log N)^2)/\sqrt{N}$ where the $\mathcal{O}$–constant depends only on the matrix $\mathbf{P}$.

(VII) Let $\alpha = [0; a_1, a_2, \dots]$ be the continued fraction expansion of $\alpha \in (0,1)$. Given a vector $\mathbf{b} = (b_1, \dots, b_k)$ with positive integer coordinates b_i, put $\Delta_{\mathbf{b}} = \{\alpha \in (0,1)\,;\, a_1 = b_1, \dots, a_k = b_k\}$ (note that this set is an interval). If $T(x) = \{1/x\}$ and $f(x)$ is an L^2 Lebesgue integrable function defined on $[0,1]$, then by ergodic theorem we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^{(n)}(\alpha)) = \frac{1}{\log 2} \int_0^1 \frac{f(x)}{1+x}\, \mathrm{d}x$$

for almost all $\alpha \in (0,1)$. Here $\frac{1}{\log 2}\frac{\mathrm{d}x}{1+x}$ is density of the **Gauss distribution**. If we take the indicator of $\Delta_{\mathbf{b}}$ for $f(x)$, then we get that the frequency of occurrence of $\mathbf{b}$ in $\alpha = [0; a_1, a_2, \dots]$ exists and equals

$$\frac{1}{\log 2} \int_{\Delta_{\mathbf{b}}} \frac{\mathrm{d}x}{1+x}$$

for almost all $\alpha \in (0,1)$.

After R. Adler, M. Keane and M. Smorodinsky (1981) a real number $\alpha \in (0,1)$ is said to be **continued fraction normal** if for every positive integral vector $\mathbf{b}$ the frequency of occurrence of $\mathbf{b}$ in $[0; a_1, a_2, \dots]$ is equal to $\frac{1}{\log 2} \int_{\Delta_{\mathbf{b}}} \frac{\mathrm{d}x}{1+x}$.

The following analogue of Borel's theorem follows from the definition: Almost every $\alpha \in (0,1)$ is continued fraction normal.

B. Volkmann noticed the following characterization in the review (MR 82k:10070): $\alpha = [0; a_1, a_2, \dots]$ is continued fraction normal if and only if the sequence $\alpha_n = [0; a_{n+1}, a_{n+2}, \dots]$, $n = 1, 2, \dots$, has the a.d.f.

$$g(x) = \frac{\log(1+x)}{\log 2}.$$

This a.d.f. is also called **Gaussian a.d.f.** For an example, cf. 2.18.22.

(VIII) The theorem saying that almost every number is normal can be proved using various tools. For instance, M. Kac (1959) proved this theorem for simply normal numbers to base 2 using Rademacher functions and Beppo Levi's Theorem. R. Nillsen (2000), also for binary case, employed series of integrals of step functions without using the measure theory in the proof at the cost of defining the null set in a different way. F. Filip and J. Šustek (2010) gave an elementary proof based on the fact that a bounded monotone function has finite derivative in almost all points. (cf. D. Khoshnevisan (2006), or [KN, p. 74 – 78] for more details.)

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1.8.25 Homogeneously u.d. sequences

• P. Erdős and G.G. Lorentz (1958): The sequence x_n mod 1 is called **homogeneously u.d.** if the sequence

$$\frac{x_{nd}}{d} \bmod 1, \quad n = 1, 2, \dots$$

is u.d. in $[0, 1]$ for every positive integer d.

NOTES: R. Schnabl (1963) gave a generalization based on weighted means.

P. ERDŐS – G.G. LORENTZ: *On the probability that n and $g(n)$ are relatively prime*, Acta Arith. **5** (1958), 35–55 (MR0101224 (**21** #37)).

R. SCHNABL: *Zur Theorie der homogenen Gleichverteilung modulo* 1, Östereich. Akad. Wiss. Math.–Natur. Kl. Sitzungsber. II **172** (1963), 43–77 (MR0164944 (**29** #2235); Zbl. 0121.05101).

1.8.26 u.d. sequences with respect to divisors

Let $d(n)$ denote the number of positive divisors of $n \in \mathbb{N}$.

• The infinite real sequence $x_n \in [0, 1]$ is said to be **u.d. on the divisors** if for some subsequence of indices n with asymptotic density 1 we have

$$\lim_{n\to\infty} \frac{\#\{d \in \mathbb{N}\,;\; d|n, x_d \in [x, y)\}}{d(n)} = y - x$$

for every $[x, y) \subset [0, 1]$.

NOTES: (I) In other words, the sequence of blocks $X_n = (x_d)_{d|n}$ is generalized u.d., cf. 1.8.23.

(II) Y. Dupain, R.R. Hall and G. Tenenbaum (1982) proved that the sequence $n\theta$ mod 1 is u.d. on the divisors if and only if θ is irrational. For other examples cf. 2.20.24.

Y. DUPAIN – R.R. HALL – G. TENENBAUM: *Sur l'équirépartition modulo 1 de certaines fonctions de diviseurs*, J. London Math. Soc. (2) **26** (1982), no. 3, 397–411 (MR0684553 (84m:10047); Zbl. 0504.10029).

1.8.27 Eutaxic sequences

Let $x_n \in [0, 1)$, $z_n \in \mathbb{R}^+$, $n = 1, 2, \dots,$ be two sequences and $x \in [0, 1]$. Strauch (1994) introduced a new counting function

$$A(x; N; (x_n, z_n)) = \#\{n \le N\,;\; |x - x_n| < z_n\}.$$

• The sequence x_n is said to be **eutaxic** if for every non–increasing sequence z_n the divergence of $\sum_{n=1}^{\infty} z_n$ implies that

$$\lim_{N\to\infty} A(x; N; (x_n, z_n)) = \infty$$

for almost all $x \in [0,1]$. If furthermore

$$\lim_{N\to\infty} \frac{A(x;N;(x_n,z_n))}{2\sum_{n=1}^N z_n} = 1$$

then x_n is called **strongly eutaxic**.

NOTES: Eutaxic sequences were introduced by J. Lesca (1968). He proved that if θ is irrational then the sequence $n\theta$ mod 1 is eutaxic if and only if θ has bounded partial quotients. M. Reversat proved the same for the strong eutaxicity of $n\theta$ mod 1, i.e. for sequence $n\theta$ mod 1 both notions coincide. B. de Mathan (1971) defined the counting function

$$A^*(N,x_n) = \#\{0 \le k < N\,;\, \exists n \le N(x_n \in [k/N,(k+1)/N)\}$$

and proved that $\liminf_{N\to\infty} A^*(N,x_n)/N = 0$ implies that x_n is not eutaxic. Since for the sequence $x_n = n\theta$ mod 1 and for θ with unbounded partial quotients we have $\liminf_{N\to\infty} A^*(N,x_n)/N = 0$, de Mathan (1971) recovered half of Lesca's result. A characterization of strong eutaxicity in terms of L^2 discrepancy is an open problem, cf. O. Strauch (1994).

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M. REVERSAT: *Un résult de forte eutaxie*, C. R. Acad. Sci. Paris Sér. A–B **280** (1975), Ai, A53–A55 (MR0366829 (**51** #3075); Zbl. 0296.10032).
O. STRAUCH: *L^2 discrepancy*, Math. Slovaca **44** (1994), 601–632 (MR1338433 (96c:11085); Zbl. 0818.11029).

1.8.28 Uniformly quick sequences

Let $X = \cup_{m=1}^\infty I_m$ be a decomposition of an open set $X \subset [0,1]$ into a sequence I_m, $m = 1,2,\dots$, of pairwise disjoint open subintervals of $[0,1]$ (empty intervals are allowed). Let x_n be an infinite sequence in $[0,1)$. Define a new counting function

$$\widetilde{A}(X;N;x_n) = \#\{m \in \mathbb{N}\,;\, \exists n \le N \text{ such that } x_n \in I_m\} + \\ + \#\{n \le N\,;\, x_n \notin X\},$$

i.e. if $x_n \in X$ for $n = 1,2,\dots$, then $\widetilde{A}(X;N;x_n)$ is the number of intervals I_m containing at least one element of $x_1, x_2, \dots, x_N$.

• The sequence x_n is said to be **uniformly quick** (abbreviated u.q.) if for any open set $X \subset [0,1]$ we have

$$\lim_{N\to\infty} \frac{\widetilde{A}(X;N;x_n)}{N} = 1 - |X|,$$

where $|X|$ denotes Lebesgue measure of X.

• If this limit holds for a special sequence of indices $N_1 < N_2 < \dots$, then x_n is said **almost u.q.**

NOTES: (I) U.q. sequences were introduced and studied by O. Strauch (1982, 1983, 1984, [a]1984, 1986) in connection with the conjecture of Duffin – Schaeffer, cf. R.J. Duffin and A.C. Schaeffer (1941) and G. Harman (1998).

(II) Any u.q. sequence x_n is u.d. in $[0,1]$ and it is also strongly eutaxic.

(III) The sequence $x_n = n\theta \bmod 1$ is u.q. if and only if the simple continued fraction expansion of the irrational θ has bounded partial quotients (cf. O. Strauch ([a]1984)).

(IV) Strauch (1982, Th. 3): The u.d. sequence x_n is u.q. if for infinitely many M there exists c_M, c'_M, and $N_0(M)$ such that $c'_M \to 0$ as $M \to \infty$ and

$$\sum_{\substack{|x_i - x_j| \le t \\ M < i \ne j \le N}} 1 \le c_M t (N-M)^2 + c'_M (N-M)$$

for every $N \ge N_0(M)$ and every $t \ge 0$. For examples see 2.23.6.

(V) The u.q. sequences x_n can be used in the numerical evaluation of integrals $\int_X f(x)\,\mathrm{d}x$ over open subsets X of $[0,1]$. Thus also for Jordan non–measurable sets X, that is sets which boundaries $|\partial X|$ are of positive measure, cf. Strauch (1997).

(VI) Let q_n, $n = 1,2,\dots$, be a one–to–one sequence of positive integers and let $(A_n)_{n=1}^{\infty}$ be a sequence composed from blocks

$$A_n = \left(\frac{1}{q_n}, \frac{a_2}{q_n}, \dots, \frac{a_{\varphi(q_n)}}{q_n} \right),$$

where $1 = a_1 < a_2 < a_3 < \dots < a_{\varphi(q_n)}$ are the integers $< q_n$ coprime to q_n. If $(A_n)_{n=1}^{\infty}$ is almost u.q. $\big($with respect to the set of indices $N_n = \sum_{i=1}^{n} \varphi(q_i)\big)$, then the Duffin – Schaeffer conjecture holds for q_n. In other words, if $f(q_n)$ is non–increasing with $n \to \infty$, then the divergence $\sum_{n=1}^{\infty} \varphi(q_n) f(q_n) = \infty$ implies that for almost all $x \in [0,1]$ the diophantine inequality

$$\left| x - \frac{y}{q_n} \right| < f(q_n)$$

has an integral solution y coprime to q_n for infinitely many n. Examples of such sequences q_n can be found in 2.23.6.

R.J. DUFFIN – A.C. SCHAEFFER: *Khintchine's problem in metric diophantine approximation*, Duke Math. J. **8** (1941), 243–255 (MR0004859 (3,71c); Zbl. 0025.11002).

G. HARMAN: *Metric Number Theory*, London Math. Soc. Monographs, New Series, Vol. 18, Clarendon Press, Oxford, 1998 (MR1672558 (99k:11112); Zbl. 1081.11057).

O. STRAUCH: *Duffin – Schaeffer conjecture and some new types of real sequences*, Acta Math. Univ. Comenian. **40–41** (1982), 233–265 (MR0686981 (84f:10065); Zbl. 0505.10026).

O. STRAUCH: *Some new criterions for sequences which satisfy Duffin – Schaeffer conjecture, I*, Acta Math. Univ. Comenian. **42–43** (1983), 87–95 (MR0740736 (86a:11031); Zbl. 0534.10045).

O. Strauch: *Some new criterions for sequences which satisfy Duffin – Schaeffer conjecture, II*, Acta Math. Univ. Comenian. **44–45** (1984), 55–65 (MR0775006 (86d:11059); Zbl. 0557.10038).
[a] O. Strauch: *Two properties of the sequence $n\alpha$* (mod 1), Acta Math. Univ. Comenian. **44–45** (1984), 67–73 (MR0775007 (86d:11057); Zbl. 0557.10027).
O. Strauch: *Some new criterions for sequences which satisfy Duffin – Schaeffer conjecture, III*, Acta Math. Univ. Comenian. **48–49** (1986), 37–50 (MR0885318 (88h:11053); Zbl. 0626.10046).
O. Strauch: *A numerical integration method employing the Fibonacci numbers*, Grazer Math. Ber. **333** (1997), 19–33 (MR1640470 (99h:65038); Zbl. 0899.11037).

1.8.29 Poissonian sequences

Given an interval $I = [a, b] \subset \mathbb{R}$, let $I/N = [a/N, b/N]$, $I/N \pm 1 = [(a/N) \pm 1, (b/N) \pm 1]$ and $|I| = b - a$. Let x_n, $n = 1, 2, \dots,$ be a sequence of points from the unit interval $[0, 1)$. Define the new counting function by

$$\begin{aligned}&\widetilde{A}(I; N; x_m - x_n) = \\ &\qquad = \#\{1 \le m \ne n \le N \,;\, x_m - x_n \in I/N \cup (I/N + 1) \cup (I/N - 1)\}.\end{aligned}$$

Then the sequence x_n is said to be **Poissonian** if

$$\lim_{N\to\infty} \frac{\widetilde{A}(I; N; x_m - x_n)}{N} = |I|$$

for every interval $I \subset \mathbb{R}$.

Notes: (I) This type of sequences was explicitly introduced by P. Sarnak and Z. Rudnick (1998). For history cf. F.P. Boca and A. Zaharescu (2000).

(II) The sequence $n\theta$ mod 1 is not Poissonian. P. Sarnak and Z. Rudnick (1998) proved that $n^k\theta$ mod 1, $k = 2, 3, \dots,$ is Poissonians for almost all θ. On the other hand, F.P. Boca and A. Zaharescu (2000) showed that for any irrational θ there exist two increasing sequences of positive integers M_j, N_j such that $n^2\theta$ mod 1 with $M_j < n \le M_j + N_j$ is Poissonian as $j \to \infty$.

(III) If the interval I is of the special form $I = [-c, c]$ then we have

$$\widetilde{A}(I; N; x_m - x_n) = A([0, c/N]; N^2; \|x_m - x_n\|) - N,$$

where $A(J; N^2; \|x_m - x_n\|) = \#\big\{1 \le m, n \le N \,;\, \|x_m - x_n\| \in J\big\}$ is the classical counting function and $\|x\| = \min(\{x\}, 1 - \{x\})$.

(IV) There is an alternative definition of Poissonian sequences based on the sequences of differences $|x_m - x_n|$: Let

$$\widetilde{F}_N(x) = \frac{\{1 \le m, n \le N \,;\, |x_m - x_n| \in [0, \frac{x}{N}] \cup [1 - \frac{x}{N}, 1]\}}{2N} - \frac{1}{2}.$$

Then the sequence x_n is Poissonian if and only if $\lim_{N\to\infty} \widetilde{F}_N(x) = x$ for all $x \in [0, \infty)$.

F.P. Boca – A. Zaharescu: *Pair correlation of values of rational functions* (mod q), Duke Math. J. **105** (2000), no. 2, 276–307 (MR1793613 (2001j:11065); Zbl. 1017.11037).
Z. Rudnick – P. Sarnak: *The pair correlation function of fractional parts of polynomials*, Commun. Math. Phys. **194** (1998), no. 1, 61–70 (MR1628282 (99g:11088); Zbl. 0919.11052).

1.8.30 u.d. of matrix sequences

A sequence of $s \times s$ matrices $\mathbf{A}_n$, $n = 1, 2, \dots$, can be considered as a sequence $\mathbf{x}_n$ in $\mathbb{R}^{s^2}$ in a natural way and then we use def. 1.11.1.

NOTES: An exponential sequence $\mathbf{A}^n \bmod 1$, $n = 1, 2, \dots$, is u.d. only if the eigenvalue of $\mathbf{A}$ of largest absolute value is ≥ 1. M. Drmota, R.F. Tichy and R. Winkler (see [DT, p. 202–203]) give an explicit construction of completely u.d. $\mathbf{A}^n \bmod 1$.

1.8.31 u.d. of quadratic forms

• R.F. Tichy (1982): Let $q_A(\mathbf{n}) = q_A(n_1, \dots, n_r) = \mathbf{n}A\mathbf{n}^t$ be a quadratic form associated with the $r \times r$ square matrix A. Order the r–dimensional integral vectors $\mathbf{n}$ lexicographically and put $|\mathbf{N}| = N_1 \dots N_r$. If we define the discrepancy through

$$D_{\mathbf{N}} = D_{\mathbf{N}}(q_A(\mathbf{n})) = \sup_{0 \leq \alpha < \beta \leq 1} \left| \frac{1}{|\mathbf{N}|} \sum_{\mathbf{n} \leq \mathbf{N}} c_{[\alpha,\beta)}(\{q_A(\mathbf{n})\}) - (\beta - \alpha) \right|.$$

then the quadratic form $q_A(\mathbf{n})$ is called u.d. mod 1 provided that

$$\lim_{\mathbf{N} \to \infty} D_{\mathbf{N}}(q_A(\mathbf{n})) = 0,$$

where $\mathbf{N} \to \infty$ means that all its components tend independently towards ∞.

R.F. TICHY: *Gleichverteilung von Mehrfachfolgen und Ketten*, Anz. Österreich. Akad. Wiss. Math.–Natur. Kl. (1978), no. 7, 174–207 (MR0527512 (83a:10087); Zbl. 0401.10061).
R.F. TICHY: *Einige Beiträge zur Gleichverteilung modulo Eins*, Anz. Österreich. Akad. Wiss. Math.–Natur. Kl. **119** (1982), no. 1, 9–13 (MR0688688 (84e:10061); Zbl. 0495.10030).

1.8.32 Hybrid sequences

Let $z_0, z_1, \dots$ be a digital explicit inversive sequence as defined in 2.25.8. Let $q = p^k$ with a prime p and an integer $k \geq 1$. Given an integer t with $1 \leq t \leq q$, choose integers $0 \leq d_1 < d_2 < \cdots < d_t < q$. If $\alpha \in \mathbb{R}^s$ is of finite type η (cf. page 2 – 82), then the discrepenacy D_N of the first N terms of the hybrid sequence

$$\mathbf{x}_n = (\{n\alpha\}, z_{n+d_1}, \dots, z_{n+d_t}) \in [0,1)^{s+t}, \quad n = 0, 1, \dots,$$

satisfies

$$D_N = O_{\alpha,t,\varepsilon}\bigg(\max \big(N^{-1/((\eta-1)s+1)+\varepsilon}, \\ 2^{(k-1)t+k/2} k^{1/2} N^{-1/2} (\log N)^s q^{1/4} (\log q)^t (1 + \log p)^{k/2} \big) \bigg)$$

for all $1 \le N \le q$ and all $\varepsilon > 0$, where the implied constant depends only on α, t, and ε.

NOTES:
(I) H. Niederreiter (2010).
(II) If $\eta = 1$ then

$$D_N = O_{\alpha,t}\left(2^{(k-1)t+k/2}k^{1/2}N^{-1/2}(\log N)^s q^{1/4}(\log q)^t(1+\log p)^{k/2}\right).$$

H. NIEDERREITER: *A discrepancy bound for hybrid sequences involving digital explicit inversive pseudorandom numbers*, Unif. Distrib. Theory **5** (2010), no. 1, 53–63 (MR2804662 (2012f:11143); Zbl. 1249.11074).

1.8.33 Hartmann u.d. sequences

A sequence of integers k_n, $n = 1, 2, \ldots$, is called Hartmann uniformly distributed if for each irrational α the sequence $k_n\alpha$ mod 1 is uniformly distributed and the sequence k_n, $n = 1, 2, \ldots$, is uniformly distributed in $\mathbb{Z}$.

A criterion. A sequence k_n, $n = 0, 1, 2, \ldots$, is Hartman-u.d. if and only if

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i t k_n} = 0$$

for all non-integer t.

Examples.

(i) For irrational α, the sequence $[n\alpha]$, $n = 1, 2, \ldots$, is not Hartman-u.d.

NOTES: (I) See [KN, p. 269, Ex. 5.11]. Also cf. P. Lertchoosakul, A. Jaššová, R. Nair and M. Weber.

P. LERTCHOOSAKUL – A. JAŠŠOVÁ – R. NAIR – M. WEBER: *Distribution functions for subsequences of generalized van der Corput sequences*, Unif. Distrib. Theory (to appear).

1.8.34 L^p good universal sequences

A sequence of integers k_n, $n = 1, 2, \ldots$ is L^p good universal if for each dynamical system (see 4.3.1) $(X, \mathcal{B}, \mu, T)$ and for each L^p function $f : X \to X$ the limit

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(T^{k_n}x) = \tilde{f}(x)$$

exists μ-almost everywhere.

Examples.

(i) $k_n = n$, $n = 1, 2, \ldots$ is L^1 good universal. This is Birkhoff's theorem.

(ii) $k_n = [g(n)]$, where $g(n) = n^\omega$, $\omega > 1$ and $\omega \notin \mathbb{N}$ is L^1 good universal.
(iii) $k_n = [g(n)]$ where $g(n) = e^{\log^\gamma(n)}$ for $\gamma \in (1, 3/2)$.
(iv) $k_n = [g(n)]$, where $g(n)$ is a polynomial with coefficients not all rational multiplies of the same real numbers.

P. Lertchoosakul – A. Jaššová – R. Nair – M. Weber: *Distribution functions for subsequences of generalized van der Corput sequences*, Unif. Distrib. Theory (to appear).

1.9 Classical discrepancies

The notion of discrepancy was introduced to measure the distribution deviation of sequences from the expected ideal one, cf. [KN, Chap. 2], [DT, Chap. 1].

• Let $x_1, \dots, x_N$ be a given sequence of real numbers from the unit interval $[0, 1)$. Then the number

$$D_N = D_N(x_1, \dots, x_N) = \sup_{0 \le \alpha < \beta \le 1} \left| \frac{A([\alpha, \beta); N; x_n)}{N} - (\beta - \alpha) \right|$$

is called the **(extremal) discrepancy** of this sequence. The number

$$D_N^* = \sup_{x \in [0,1]} \left| \frac{A([0, x); N; x_n)}{N} - x \right|$$

is called **star discrepancy**, and the number

$$D_N^{(2)} = \int_0^1 \left(\frac{A([0, x); N; x_n)}{N} - x \right)^2 \mathrm{d}x$$

is called its L^2 **discrepancy**.

Notes: (I) In [KN, p. 97] the L^2 discrepancy is defined as $\sqrt{D_N^{(2)}}$ and is denoted by T_N.
(II) If the sequence x_n of real numbers is infinite or if it has more than N terms, then under the discrepancy $D_N(x_n)$ of x_n we understand the discrepancy of its initial segment of the first N terms.
(III) The extremal discrepancy of the block A_n will be denoted by $D(A_n)$, and similarly $D^*(A_n)$, or $D^{(2)}(A_n)$, resp.

The above discrepancies are mutually related by the following inequalities

$$D_N^* \le D_N \le 2D_N^*, \quad \text{[KN, p. 91]},$$
$$(D_N^*)^3 \le 3D_N^{(2)} \le (D_N^*)^2, \quad \text{[H. Niederreiter (1973)]}.$$

These inequalities hold for the arbitrary sequence $x_1, \dots, x_N$ in $[0,1)$ having N terms.

The following relations can be useful for some computational purposes

$$D_N^* = \frac{1}{2N} + \max_{1\le n\le N} \left| x_n - \frac{2n-1}{2N} \right|, \quad (x_1 \le x_2 \le \dots \le x_N),$$

[H. Niederreiter (1972), cf. [KN, p. 91], H. Niederreiter (1992, p. 15, Th. 2.6)],

$$D_N = \frac{1}{N} + \max_{1\le n\le N} \left(\frac{n}{N} - x_n \right) - \min_{1\le n\le N} \left(\frac{n}{N} - x_n \right), \quad (x_1 \le x_2 \le \dots \le x_N),$$

[L. de Clerck (1981), cf. H. Niederreiter (1992, p. 16, Th. 2.7)],

$$D_N^{(2)} = \frac{1}{3} + \frac{1}{N} \sum_{n=1}^N x_n^2 - \frac{1}{N} \sum_{n=1}^N x_n - \frac{1}{2N^2} \sum_{m,n=1}^N |x_m - x_n|.$$

Some of the important relations and estimates for the discrepancies listed below are valid only under the additional assumption that the sequence $x_1 \le x_2 \le \dots \le x_N$ is ordered according to the non–decreasing magnitude of its terms.

$$D_N^* = \max_{1\le n\le N} \max \left(\left| \frac{n}{N} - x_n \right|, \left| \frac{n}{N} - x_{n+1} \right| \right), \quad (x_1 \le x_2 \le \dots \le x_N),$$

[H. Niederreiter (1992, p. 16)]

$$= \max_{1\le n\le N} \max \left(\left| \frac{n}{N} - x_n \right|, \left| \frac{n-1}{N} - x_n \right| \right), \quad (x_1 \le x_2 \le \dots \le x_N),$$

[H. Niederreiter (1992, p. 16)]

$$D_N^{(2)} = \frac{1}{12N^2} + \frac{1}{N} \sum_{n=1}^N \left(x_n - \frac{2n-1}{2N} \right)^2, \quad (x_1 \le x_2 \le \dots \le x_N),$$

[KN, p. 161, Exer. 5.12]

$$= \frac{1}{N^2} \left(\sum_{n=1}^N \left(x_n - \frac{1}{2} \right) \right)^2 + \frac{1}{2\pi^2 N^2} \sum_{h=1}^\infty \frac{1}{h^2} \left| \sum_{n=1}^N e^{2\pi i h x_n} \right|^2,$$

[KN, p. 110, Lemma 2.8]

$$= \frac{1}{N^2} \left(\sum_{n=1}^N \left(x_n - \frac{1}{2} \right) \right)^2 + \frac{1}{N^2} \int_0^1 \left(\sum_{n=1}^N \left(\{x_n + x\} - \frac{1}{2} \right) \right)^2 \mathrm{d}x,$$

[KN, p. 144, Th. 5.2]

$$= \frac{1}{N^2} \sum_{n=1}^N \left(x_n - \frac{1}{2} \right) + \frac{1}{N} \sum_{n=1}^N \left(x_n - \frac{n}{N} \right)^2 - \frac{1}{6} \qquad (x_1 \le x_2 \le \dots \le x_N),$$

[KN, p. 145, Ex. 5.2]

$$=\frac{1}{3}+\frac{1}{N}\sum_{n=1}^N x_n^2+\frac{1}{N^2}\sum_{n=1}^N x_n-\frac{2}{N^2}\sum_{n=1}^N nx_n \quad (x_1\le x_2\le\dots\le x_N),$$

[KN, p. 145, Ex. 5.2]

$$=\frac{1}{3}+\frac{1}{N}\sum_{n=1}^N x_n^2-\frac{1}{N^2}\sum_{m,n=1}^N \max(x_m,x_n),$$

[KN, p. 145, Th. 5.3], [J.F. Koksma ([a]1942/43)]

$$=\int_0^1\int_0^1 -\frac{|x-y|}{2}\,\mathrm{d}(F_N(x)-x)\,\mathrm{d}(F_N(y)-y), \qquad \text{[O. Strauch (1989)]}$$

$$=\frac{1}{N^2}\sum_{m,n=1}^N F_0(x_m,x_n),\quad F_0(x,y)=\frac{1}{3}+\frac{x^2+y^2}{2}-\frac{x+y}{2}-\frac{|x-y|}{2},$$

[O. Strauch (1994)]

Theorem 1.9.0.1 (H. Niederreiter (1992, p. 15, Lemma 2.5)). *If two finite sequences $x_1,\dots,x_N$ and $y_1,\dots,y_N$ from $[0,1]$ satisfy $|x_n-y_n|\le\varepsilon$ for $1\le n\le N$, then*

$$|D_N^*(x_n)-D_N^*(y_n)|\le\varepsilon, \qquad \text{and} \qquad |D_N(x_n)-D_N(y_n)|\le 2\varepsilon.$$

More precisely,

Theorem 1.9.0.1' ([KN, p. 132, Th. 4.1]). *Let x_n and y_n, $n=1,2,\dots,N$, be two finite sequences in $[0,1)$ such that $|x_n-y_n|\le\varepsilon_n$ for $n=1,2,\dots,N$. Then, for any $\varepsilon\ge 0$, we have*

$$|D_N(x_n)-D_N(y_n)|\le 2\varepsilon+\frac{N_\varepsilon}{N},$$

where $N_\varepsilon=\#\{n\le N;\varepsilon_n>\varepsilon\}$.

Theorem 1.9.0.1". *For every $x_1,x_2,\dots,x_N\in[0,1)$ we have*

$$D_N((x_n+y) \bmod 1)=D_N(x_n) \text{ for any } y\in\mathbb{R}, \text{ and}$$
$$D_N(qx_n \bmod 1)\le qD_N(x_n) \text{ for any } q\in\mathbb{N}.$$

Notes:

(I) For an application of (1.10.1) see Ch. Mauduit and A. Sárközy (2000).

Ch. Mauduit – A. Sárközy: *On finite pseudorandom binary sequences. V: On $n\alpha$ and $(n^2\alpha)$ sequences*, Monatsh. Math. **129** (2000), no. 3, 197–216 (MR1746759 (2002c:11088); Zbl. 0973.11076)).

The following theorems demonstrate the role of the discrepancy notions:

Theorem 1.9.0.2 (H. Weyl (1916)). *A sequence $x_n\in[0,1)$ is u.d. if and only if*

$$\lim_{N\to\infty} D_N(x_n)=0.$$

Theorem 1.9.0.3 (J.F. Koksma (1942/43)). *Let $f : [0,1] \to \mathbb{R}$ be a function of bounded variation $V(f)$ on $[0,1]$. Then*

$$\left|\frac{1}{N}\sum_{n=1}^{N} f(x_n) - \int_0^1 f(x)\,\mathrm{d}x\right| \le V(f)D_N^*.$$

Theorem 1.9.0.4 (I.M. Soboľ (1961), S.K. Zaremba (1968)). *If the function $f : [0,1] \to \mathbb{R}$ is a continuously differentiable function then*

$$\left|\frac{1}{N}\sum_{n=1}^{N} f(x_n) - \int_0^1 f(x)\,\mathrm{d}x\right| \le \sqrt{D_N^{(2)}}\sqrt{\int_0^1 (f'(x))^2\,\mathrm{d}x}.$$

Cf. E. Hlawka (1984, p. 107).
Theorem 1.9.0.5 (H. Niederreiter (1972)). *If $f : [0,1] \to \mathbb{R}$ is a continuous function then*

$$\left|\frac{1}{N}\sum_{n=1}^{N} f(x_n) - \int_0^1 f(x)\,\mathrm{d}x\right| \le \lambda_f(D_N^*),$$

where

$$\lambda_f(t) = \sup_{\substack{x,y\in[0,1]\\ |x-y|\le t}} |f(x) - f(y)|$$

is the modulus of continuity of f.
Cf. [KN, p. 146, Th. 5.4] and H. Niederreiter (1992, p. 19, Th. 2.10).
Notes: In [KN, p. 146, Cor. 5.2; Notes, p. 158] the following unpublished Koksma's result is quoted

$$\left|\frac{1}{N}\sum_{n=1}^{N} f(x_n) - \int_0^1 f(x)\,\mathrm{d}x\right| \le 3ND_N^*\lambda_f(1/N).$$

The following trivial estimates [KN, p. 90]

$$\frac{1}{N} \le D_N \le 1 \quad \text{and} \quad \frac{1}{12N^2} \le D_N^{(2)} \le 1$$

hold for every finite sequence in $[0,1)$ and the lower bounds are sharp, see 2.22.15.
van Aardenne – Ehrenfest (1945) showed (cf. note (0) below) that the estimate $D_N(x_n) = \mathcal{O}(1/N)$ cannot hold for an infinite sequence x_n (cf. [KN, p. 109, Th. 2.3], [DT, p. 41, Th. 1.51]). The next result is the best possible.

Theorem 1.9.0.6 (W.M. Schmidt (1972)). *If x_n is an infinite sequence in $[0,1)$ then*

$$D_N > c\frac{\log N}{N}$$

for infinitely many positive integers N.

The best known value of c is (cf. H. Niederreiter (1992, p. 24))

$$c = \max_{a\geq 3} \frac{a-2}{4(a-1)\log a} = 0.120\dots.$$

NOTES: (0) van der Corput (1935, p. 816) conjectured that there is no infinite sequence x_n in a fixed interval I which is **justly distributed over** I, i.e. for which there exists a constant C such that for any pairs of subintervals $I_1, I_2 \subset I$ and for all N we have $|I_1| = |I_2| \Longrightarrow |A(I_1; N; x_n) - A(I_2; N; x_n)| \leq C$. The impossibility of just distribution was proved by van Aardenne – Ehrenfest (1945).

(I) van Aardenne – Ehrenfest (1949) proved that ND_N is never $o\left(\frac{\log\log N}{\log\log\log N}\right)$. Namely she showed that $\limsup_{n\to\infty} \frac{ND_N}{\frac{\log\log N}{\log\log\log N}} \geq \frac{1}{2}$ and noticed far–sightedly "As far as I know for all special infinite sequences, for which ND_N has been calculated, it has been found that $\limsup_{n\to\infty} \frac{ND_N}{\log N} > 0$".

(II) K.F. Roth (1954) improved her result proving that $ND_N > c'\sqrt{\log N}$ for infinitely many N.

(III) W.M. Schmidt (1972) showed that $\limsup_{N\to\infty} ND_N^*/\log N \geq 10^{-2}$.

(IV) R. Béjian (1979) improved (III) to $\limsup_{N\to\infty} ND_N^*/\log N \geq (12\log 4)^{-1}$ and in (1982) he proved that $\limsup_{N\to\infty} ND_N/\log N \geq \max_{a\geq 3}(a-2)/(4(a-1)\log 4)$.

(V) P. Liardet (1979) continued with the inequality $\limsup_{N\to\infty} ND_N/\log N \geq \max_{a\geq 3}(a-2)/(8a\log 4)$, cf. [DT, p. 41, Th. 1.51].

(VI) The fact that the best possible infinite sequence does not exist, i.e. that there does not exit an infinite sequence x_n for which every initial segment $x_1, \dots, x_N$ has minimal ND_N, is called **irregularities of distribution** or **Roth's phenomenon**.

(VII) Given a sequence x_n, $n = 1, 2, \dots$, in $[0,1)$, and a subinterval I of $[0,1]$, define the **local discrepancy function** by $D(N, I) = \big|A(I; N; x_n) - N|I|\big|$. Then for $D(N, I)$ as $N \to \infty$ we have (see the Introduction in W. Steiner (2006)):

(i) For every sequence x_n there exists an interval $I \subset [0,1]$ for which $D(N, I)$ is unbounded (T. van Aardenne-Ehrenfest (1949));

(ii) The set of intervals I for which $D(N, I)$ is bounded is at most countable (W.M. Schmidt (1974));

(iii) If α is irrational and $x_n = n\alpha \bmod 1$, then $D(N, I)$ is bounded if and only if $|I| = k\alpha \bmod 1$ for some integer k (E. Hecke (1921) and H. Kesten (1966));

(iv) If x_n is van der Corput sequence in base q, then $D(N, I)$ is bounded if and only if the length $|I|$ has a finite q-ary expansion (W.M. Schmidt (1974) and L. Shapiro (1978));

(v) If $x_n = \alpha s_q(n)$ with α irrational, and $s_q(n)$ is the sum-of-digit function, then the only intervals I with bounded $D(N, I)$ are the trivial ones $|I| = 0$ and $|I| = 1$ (P. Liardet (1987)).

The upper bounds for discrepancies of finite sequences $x_1, \dots, x_N$ mod 1 are given in the following **LeVeque** and **Erdős − Turán inequalities** (cf. [KN, p. 111, Th. 2.4; p. 112–114,Th. 2.5], [DT, p. 23, Th. 1.27]):

Theorem 1.9.0.7 (W.J. LeVeque (1965)). *The discrepancy D_N of a finite sequence $x_1, \dots, x_N$ reduced* mod 1 *satisfies*

$$D_N \le \left(\frac{6}{\pi^2} \sum_{h=1}^{\infty} \frac{1}{h^2} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} \right|^2 \right)^{1/3}.$$

Theorem 1.9.0.8 (P. Erdős, P. Turán (1948)). *If $x_1, \dots, x_N$* mod 1 *is a finite sequence and m a positive integer, then*

$$D_N \le \frac{6}{m+1} + \frac{4}{\pi} \sum_{h=1}^{m} \left(\frac{1}{h} - \frac{1}{m+1} \right) \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} \right|.$$

In applications the following simpler versions are often useful (cf. [KN, p. 114, relation (2.42)])

$$D_N \le c \left(\frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} \right| \right)$$

or (cf. G. Harman (1998))

$$D_N \le \frac{c_1}{m} + c_2 \sum_{h=1}^{m} \frac{1}{h} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} \right|$$

which hold for all positive integer m. Here c, c_1, c_2 are absolute constants, and their best known values are

$$c_1 = 1, \quad c_2 = 2 + \frac{2}{\pi},$$

cf. R.C. Baker (1986, p. 20) and H.L. Montgomery (1994, p. 8).

The next inequality can be instrumental in lower estimates of the discrepancy: For any N real numbers $x_1, \dots, x_N$ we have (cf. [KN, p. 143, Cor. 5.1])

$$\left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i x_n} \right| \le 4 D_N^*(x_n \bmod 1).$$

A weaker inequality with 4 replaced by 2π can be found in van der Corput and Pisot (1939).

Notes: (I) The star discrepancy was introduced by Weyl (1916), but the notion of the discrepancy probably goes back to van der Corput and the first systematic

investigation of this important notion can be found in his joint paper with C. Pisot (1939). Consult [KN, p. 97–99, Notes] for further details.
(II) A weaker form of Erdős – Turán inequality was proved by van der Corput in 1935 but never published (cf. J.F. Koksma (1936, Kapitel IX, Satz 4) or Koksma (1950)).
(III) I.Z. Ruzsa (1994) investigated how bad the Erdős – Turán estimate can be. Let

$$B_N = \min_{m\in\mathbb{N}} \left(\frac{1}{m} + \sum_{h=1}^{m-1} \frac{1}{h} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} \right| \right).$$

He showed that given N, D with $N \geq 2$ and $\frac{2}{N} \leq D \leq 1$, there exist N real numbers $x_1, \dots, x_N$ such that $D_N \leq D$ and $B_N \geq cD^{\frac{2}{3}}$, where the positive constant c is absolute. Since $12D_N \geq B_N^{3/2}$, this result is the best possible.
(IV) A multi–dimensional variant of the Erdős – Turán inequality was proved by Koksma (1950), cf. 1.11.2.1.
(V) J.D. Vaaler (1985) using Beurling and Selberg majorising and minorising functions proved modified forms of the Berry – Esseen and of the Erdős – Turán inequalities.
(VI) A modified form of Erdős – Turán theorem was also proved by Y. Ohkubo (1999) (cf. also Th. 1.10.7.2):
Theorem 1.9.0.9. *For any* $0 < \delta \leq 1$ *there exists a constant* $c(\delta)$ *such that for every finite sequence* $x_1, \dots, x_N \bmod 1$ *we have*

$$D_N \leq F(N) + \frac{c(\delta)}{N} \sum_{1\leq h\leq N^\delta} \frac{1}{h} \sup_{h^{1/\delta}<b\leq N} \left| \sum_{h^{1/\delta}\leq n\leq B} e^{2\pi i h x_n} \right|,$$

where

$$F(N) = \begin{cases} \left(\dfrac{1}{2^{1-\delta}-1} + 1 \right) \dfrac{1}{N^\delta}, & \textit{if } 0 < \delta < 1, \\ \left(\dfrac{1}{\log 2} + 1 \right) \dfrac{1 + \log N}{N}, & \textit{if } \delta = 1. \end{cases}$$

(VII) An unsolved problem asks for the exact value of $\sup \frac{1}{ND_N^*} \left| \sum_{n=1}^{N} e^{2\pi i x_n} \right|$, where the supremum is extended over all finite sequence $x_1, \dots, x_N$, cf. [KN, p. 160, Exer. 5.7].
(VIII) K. Goto and Y. Ohkubo (2004) proved that

$$\left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} \right| \leq 4hD_N^*(x_n \bmod 1) \qquad \text{for } h = 1, 2, \dots.$$

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1.10 Other discrepancies

1.10.1 Discrepancy for g–distributed sequences

The L^2 **discrepancy** of a sequence $x_n \in [0,1)$ with a.d.f. $g(x)$, denoted by $D_N^{(2)}(x_n, g)$ or in the abbreviated form by $D_N^{(2)}$, is defined through

$$D_N^{(2)}(x_n, g) = D_N^{(2)} = \int_0^1 \left(\frac{A([0,x);N;x_n)}{N} - g(x) \right)^2 \mathrm{d}x.$$

It can be expressed in the form

$$D_N^{(2)} = \frac{1}{N^2} \sum_{m,n=1}^{N} F(x_m, x_n),$$

where

$$F(x,y) = \int_0^1 g^2(t)\,\mathrm{d}t - \int_x^1 g(t)\,\mathrm{d}t - \int_y^1 g(t)\,\mathrm{d}t + 1 - \max(x,y).$$

Similarly, the **extremal discrepancy** $D_N(x_n, g)$ and the **star discrepancy** $D_N^*(x_n, g)$ of a sequence $x_n \in [0,1)$ with respect to a.d.f $g(x)$ is defined by

$$D_N = \sup_{0\le\alpha<\beta\le1} \left| \frac{A([\alpha,\beta);N;x_n)}{N} - (g(\beta) - g(\alpha)) \right|,$$

and

$$D_N^* = \sup_{x\in[0,1]} \left| \frac{A([0,x);N;x_n)}{N} - g(x) \right|,$$

resp. Note that it is necessary to assume here that the d.f. g(x) is continuous for every $x \in [0,1]$. We again have:
Theorem 1.10.1.1. *A sequence x_n in $[0,1]$ has a.d.f $g(x)$ if and only if*

$$\lim_{N\to\infty} D_N^{(2)}(x_n, g) = 0.$$

If g is continuous then also the limit $\lim_{N\to\infty} D_N = 0$ or $\lim_{N\to\infty} D_N^* = 0$ characterizes the g–distribution.
NOTES: (I) For the proof of this expression of the L^2 discrepancy cf. O. Strauch (1994, p. 618). Taking $g(x) = c_\alpha(x)$ he found (1994, p. 619) that a sequence x_n statistically convergent to α (cf. 1.8.8) can be characterized by its L^2 discrepancy $D_N^{(2)} \to 0$ which can be given in the form

$$D_N^{(2)} = \frac{1}{N}\sum_{n=1}^{N} |x_n - \alpha| - \frac{1}{2N^2}\sum_{m,n=1}^{N} |x_m - x_n|.$$

(II) If $g(x)$ is continuous and x_n is g–distributed then the sequence $g(x_n)$ is u.d. and the Erdős – Turán inequality takes the form (cf. K. Goto and T. Kano (1993))

$$D_N \le \frac{6}{m+1} + \frac{4}{\pi}\sum_{h=1}^{m}\left(\frac{1}{h} - \frac{1}{m+1}\right)\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h g(x_n)}\right|.$$

(III) Let ψ be an increasing function on $[0,1]$ such that $\psi(0) = 0$. P.D. Proinov (1985) defined the ψ–**discrepancy** $D_N^{(\psi)}$ by

$$D_N^{(\psi)} = \int_0^1 \psi\left(\left|\frac{A([0,x);N;x_n)}{N} - g(x)\right|\right) \mathrm{d}x,$$

and he proved that

$$\Psi(D_N^*) \le D_N^{(\psi)} \le \psi(D_N^*),$$

where $\Psi(x) = \int_0^x \psi(t)\,\mathrm{d}t$.

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1.10.2 Diaphony

The following modification of the L^2 discrepancy is also used to characterize the g–distributed sequences $x_n \in [0,1)$

$$DI_N^{(2)} = \iint\limits_{0\le x\le y\le 1} \left(\frac{A([x,y);N;x_n)}{N} - (g(y)-g(x))\right)^2 \mathrm{d}x\,\mathrm{d}y =$$
$$= \int_0^1 \left(\frac{A([0,x);N;x_n)}{N} - g(x)\right)^2 \mathrm{d}x -$$
$$- \left(\int_0^1 \left(\frac{A([0,x);N;x_n)}{N} - g(x)\right)\mathrm{d}x\right)^2.$$

It can be expressed as the classical L^2 discrepancy in the form

$$DI_N^{(2)} = \frac{1}{N^2}\sum_{m,n=1}^{N} F(x_m,x_n),$$

where (cf. O. Strauch (1994, p. 621))

$$F(x_m,x_n) = \int_0^1 g^2(x)\,\mathrm{d}x - \left(\int_0^1 g(x)\,\mathrm{d}x\right)^2 - (x_m+x_n)\int_0^1 g(x)\,\mathrm{d}x +$$
$$+ \int_0^{x_m} g(x)\,\mathrm{d}x + \int_0^{x_n} g(x)\,\mathrm{d}x + \min(x_m,x_n) - x_m x_n.$$

The case in which $g(x)=x$ was investigated by P. Zinterhof (1976). More precisely, Zinterhof defined **diaphony** through

$$DI_N = \left(\frac{1}{N^2}\sum_{m,n=1}^{N}\frac{\pi^2}{2}\left((1-2\{x_m-x_n\})^2 - \frac{1}{3}\right)\right)^{1/2}$$

which is equal to (cf. O. Strauch (1999, p. 80))

$$\left(4\pi^2 \iint\limits_{0\le x\le y\le 1}\left(\frac{A([x,y);N;x_n)}{N} - (y-x)\right)^2 \mathrm{d}x\,\mathrm{d}y\right)^{1/2},$$

i.e.

$$DI_N = \left(4\pi^2 DI_N^{(2)}\right)^{1/2}.$$

Theorem 1.10.2.1. *Assume that d.f. $g(x)$ is continuous at 0 and 1. Then the sequence x_n in $[0,1]$ has a.d.f. $g(x)$ if and only if*

$$\lim_{N\to\infty} DI_N^{(2)} = 0.$$

NOTES: (I) The following general expression was proved by O. Strauch (1994, p. 620):

$$\iint\limits_{0\le x\le y\le 1} \left(\frac{A([x,y);N;x_n)}{N} - g(x,y) \right)^2 \mathrm{d}x\,\mathrm{d}y = \frac{1}{N^2} \sum_{m,n=1}^{N} F(x_m, x_n),$$

where

$$F(x_m,x_n) = \iint\limits_{0\le x\le y\le 1} g^2(x,y)\,\mathrm{d}x\,\mathrm{d}y - \int_0^{x_m} \mathrm{d}x \int_{x_m}^1 g(x,y)\,\mathrm{d}y - \\ - \int_0^{x_n} \mathrm{d}x \int_{x_n}^1 g(x,y)\,\mathrm{d}y + \min(x_m,x_n) - x_m x_n.$$

Applying this to $g(x,y) = g(y) - g(x)$ he found the expression for $DI_N^{(2)}$.
(II) For the classical L^2 discrepancy with respect to $g(x) = x$ we get the Koksma formula

$$D_N^{(2)} = \frac{1}{N^2} \left(\sum_{n=1}^{N} \left(x_n - \frac{1}{2} \right) \right)^2 + DI_N^{(2)}.$$

(III) The following expression was proved by L. Kuipers (1968)

$$DI_N^{(2)} = \frac{1}{2\pi^2} \sum_{h=1}^{\infty} \frac{1}{h^2} \left| \frac{1}{N} \sum_{n=1}^{N} e^{-2\pi i h x_n} - \int_0^1 e^{-2\pi i h x}\,\mathrm{d}g(x) \right|^2,$$

which, in the case $g(x) = x$ can be found in W.J. LeVeque (1965).
(IV) Another expression can be found in C. Amstler (1997) and two alternative definitions of the diaphony subject to some restrictions can be found in the monograph [DT, pp. 24–26].

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1.10.3 L^2 discrepancy of statistically independent sequences

Given two sequences x_n and y_n, $(x_n, y_n) \in [0,1)^2$, write (cf. 1.11)

$$F_N(x,y) = \frac{1}{N}\sum_{n=1}^{N} c_{[0,x)}(x_n)c_{[0,y)}(y_n),$$

where for $x = 1$ we take the interval $[0,x]$ and similarly for $y = 1$. The L^2 discrepancy characterizing the statistical independence of x_n and y_n (cf. 1.8.9) can be expressed in the forms

$$\begin{aligned}
DS_N^{(2)}((x_n,y_n)) = &\int_0^1\int_0^1 \big(F_N(x,y) - F_N(x,1)F_N(1,y)\big)^2\,\mathrm{d}x\,\mathrm{d}y = \\
=&\frac{1}{16\pi^4}\sum_{\substack{k,l=-\infty\\ k,l\neq 0}}^{\infty}\frac{1}{k^2l^2}\Bigg|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i(kx_n+ly_n)} - \\
&\qquad\qquad - \frac{1}{N^2}\sum_{m,n=1}^{N} e^{2\pi i(kx_n+ly_m)}\Bigg|^2 \\
=&\frac{1}{N^2}\sum_{m,n=1}^{N}(1-\max(x_m,x_n))(1-\max(y_m,y_n)) + \\
&+\frac{1}{N^4}\sum_{m,n,k,l=1}^{N}(1-\max(x_m,x_k))(1-\max(y_n,y_l)) - \\
&-\frac{2}{N^3}\sum_{m,k,l=1}^{N}(1-\max(x_m,x_k))(1-\max(y_m,y_l)).
\end{aligned}$$

NOTES: The first expression was proved by P.J. Grabner and R.F. Tichy (1994) and the second one by O. Strauch (1994). Grabner and Tichy (1994) proved that the analogue of the standard extremal discrepancy

$$DS_N^* = \sup_{x,y\in[0,1]}\left|\frac{A([0,x)\times[0,y));N;(x_n,y_n)}{N} - \frac{A([0,x);N;x_n)}{N}\frac{A([0,y);N;y_n)}{N}\right|$$

is not suitable for the characterization of statistical independence, but that the L^2 discrepancy $DS_N^{(2)}$ is. The so–called **Wiener L^2 discrepancy of statistical independence** of x_n and y_n can be defined by

$$WS_N^{(2)} = \int_{\mathbf{X}}\int_{\mathbf{X}}\left(\frac{1}{N}\sum_{n=1}^{N} f(x_n)g(y_n) - \frac{1}{N}\sum_{n=1}^{N} f(x_n)\frac{1}{N}\sum_{n=1}^{N} g(x_n)\right)^2 \mathrm{d}f\,\mathrm{d}g,$$

where the set $\mathbf{X} = \{f\,;\,[0,1] \to \mathbb{R}, f(0) = 0, f$ is continuous$\}$ is equipped with the classical Wiener sheet measure $\mathrm{d}f$ normed by $\int_{\mathbf{X}} f(x)f(y)\,\mathrm{d}f = \min(x,y)$.
O. Strauch (1994) found an expression for $WS_N^{(2)}$ which coincides with that for $DS_N^{(2)}$ given above, if all occurrences of the function $1 - \max(x,y)$ are replaced by the function $\min(x,y)$. Since $DS_N^{(2)}$ is invariant under the transformation $(x_n, y_n) \to (1-x_n, 1-y_n)$, this implies that $WS_N^{(2)} = DS_N^{(2)}$; see also P.J. Grabner, O. Strauch and R.F. Tichy (1996).

P.J. Grabner – O. Strauch – R.F. Tichy: *L^p–discrepancy and statistical independence of sequences*, Czechoslovak Math. J. **49(124)** (1999), no. 1, 97–110 (MR1676837 (2000a:11108); Zbl. 1074.11509).
P.J. Grabner – R.F. Tichy: *Remarks on statistical independence of sequences*, Math. Slovaca **44** (1994), 91–94 (MR1290276 (95k:11098); Zbl. 0797.11063).
O. Strauch: *L^2 discrepancy*, Math. Slovaca **44** (1994), 601–632 (MR1338433 (96c:11085); Zbl. 0818.11029).

1.10.4 Polynomial discrepancy

E. Hlawka ([a]1975, [b]1975, [c]1975) defined the so–called **polynomial discrepancy**

$$P_N(x_n) = \sup_{k=1,2,\dots} \left| \frac{1}{N} \sum_{n=1}^{N} x_n^k - \frac{1}{k+1} \right|$$

for $x_n \in [0,1)$ and he proved that

$$P_N \le D_N \le c \frac{1}{|\log P_N|}.$$

W.M. Schmidt (1993) showed that $D_N > e^{-1} |\log P_N|^{-1}$.
Notes: Let γ_k, $k = 1, 2, \dots$, be an increasing sequence of positive real numbers. Hlawka (1986) introduced the discrepancy

$$P_N(x_n, \gamma_k) = \sup_{k=1,2,\dots} \left| \frac{1}{N} \sum_{n=1}^{N} x_n^{\gamma_k} - \frac{1}{\gamma_k + 1} \right|.$$

We again have $P_N(x_n, \gamma_k) \le D_N(x_n)$. Hlawka proved that if $\lim_{k\to\infty} \gamma_k = \infty$ and $\sum_{k=1}^{\infty} 1/\gamma_k = \infty$, then $\lim_{N\to\infty} P_N(x_n, \gamma_k) = 0$ implies that x_n is u.d. His proof uses the known Müntz (1914) theorem: A continuous $f : [0,1] \to \mathbb{R}$ can be uniformly approximated by polynomials in $1, x^{\gamma_1}, x^{\gamma_2}, \dots$, with limit $\lim_{k\to\infty} \gamma_k = \infty$, if and only if $\sum_{k=1}^{\infty} 1/\gamma_k = \infty$.

[a] E. Hlawka: *Zur quantitativen Theorie der Gleichverteilung*, Österreich. Akad. Wiss. Math.–Natur. Kl. Sitzungsber. II **184** (1975), 355–365 (MR0422183 (**54** #10175); Zbl. 0336.10049).
[b] E. Hlawka: *Zur Theorie der Gleichverteilung*, Anz. Österreich. Akad. Wiss. Math.–Natur. Kl., (1975), no. 2, 13–14 (MR0387223 (**52** #8066); Zbl. 0315.10030).

[c] E. Hlawka: *Zur Theorie der Gleichverteilung*, Anz. Österreich. Akad. Wiss. Math.–Natur. Kl., (1975), no. 3, 23–24 (MR0387223 (**52** #8066); Zbl. 0319.10043).
E. Hlawka: *Gleichverteilung und ein Satz von Műntz*, J. Number Theory **24** (1986), no. 1, 35–46 (MR0852188 (88b:11050); Zbl. 0588.10057).
C.H. Müntz: *Über den Approximationssatz von Weierstraß*, in: *Mathematische Abhandlungen Hermann Amandus Schwarz*, (C. Carathéodory, G. Hessenberg, E. Landau, L. Lichtenstein eds.), Springer Berlin Heidelberg, Berlin, Heidelberg, 1914, 303–312 (JFM 45.0633.02).
W.M. Schmidt: *Bemerkungen zur Polynomdiskrepanz*, Österreich. Akad. Wiss. Math.–Natur. Kl. Abt. Sitzungsber. II **202** (1993), no. 1–10, 173–177 (MR1268810 (95d:11095); Zbl. 0790.11056).

1.10.5 A–discrepancy

• [DT, p. 251, Def. 2.36]: Let $\mathbf{A} = (a_{N,n})$ be a positive Toeplitz matrix. The **A–discrepancy** $\mathbf{A}\text{–}D_N(x_n)$ of the sequence $x_n \in [0,1]$ is defined by

$$\mathbf{A}\text{–}D_N(x_n) = \sup_{[x,y)\subset[0,1]} \left| \sum_{n=1}^{\infty} a_{N,n} c_{[x,y)}(x_n) - (y-x) \right|.$$

Similarly, the **star discrepancy** $\mathbf{A}\text{–}D_N^*(x_n)$ is defined with the supremum which runs over the all $[0,x) \subset [0,1]$.

1.10.6 Weighted discrepancies

Let the matrix $\mathbf{A} = (a_{N,n})$ be defined by

$$a_{N,n} = \begin{cases} \frac{p_n}{P_N}, & \text{if } n \le N, \\ 0, & \text{if } n > N, \end{cases}$$

where p_n, $n = 1, 2, \dots$, is a sequence (the so–called **weight sequence**) of positive real numbers such that $P_N = \sum_{n=1}^{N} p_n \to \infty$ for $N \to \infty$. The extremal and the star **A**–discrepancies are also called **weighted extremal** and **weighted star discrepancies** of the given sequence $x_n \in [0,1]$; cf. 2.12.12, 2.6.3, 2.8.11.

Given a real $p > 0$, the **weighted L^p discrepancy** $D_N^{(p)}$ of x_n is defined by

$$D_N^{(p)}(x_n) = \int_0^1 \left| \sum_{n=1}^{N} \frac{p_n}{P_N} c_{[0,x)}(x_n) - x \right|^p \mathrm{d}x.$$

If p is an even positive integer and $x_1 \le x_2 \le \dots \le x_N$ then

$$D_N^{(p)}(x_n) = \frac{1}{p+1} \sum_{n=1}^{N} \left(\left(x_n - \frac{P_{n-1}}{P_N} \right)^{p+1} - \left(x_n - \frac{P_n}{P_N} \right)^{p+1} \right).$$

NOTES: A similar expression for $D_N^{(p)}$ for $p = 2, 4$ was proved by P.D. Proinov and V.A. Andreeva (1986), and for the general even p by M. Paštéka (1988). The above expression was given by G. Turnwald in his review MR 90c:11047.

M. PAŠTÉKA: *Solution of one problem from the theory of uniform distribution*, C. R. Acad. Bulgare Sci. **41** (1988), no. 11, 29–31 (MR0985877 (90c:11047); Zbl. 0659.10060).
P.D. PROINOV – V.A. ANDREEVA: *Note on theorem of Koksma on uniform distribution*, C. R. Acad. Bulgare Sci. **39** (1986), no. 7, 41–44 (MR0868698 (88d:11070); Zbl. 0598.10055).

1.10.7 Logarithmic discrepancy

The logarithmic discrepancy can be viewed as a special case of weighted discrepancy.

• Given a sequence x_n in $[0, 1]$, the **logarithmic discrepancy** is defined by

$$L_N(x_n) = \sup_{0\le x\le 1} \left| \frac{1}{\sum_{n=1}^N n^{-1}} \sum_{n=1}^N \frac{1}{n}\, c_{[0,x)}(x_n) - x \right|.$$

It characterizes the u.d. with respect to the **logarithmically weighted means** (cf. [DT, p. 252], and 2.12.1, 2.12.31). Note that it coincides with the star discrepancy for the related matrix. A logarithmically weighted version of Erdős – Turán inequality was proved by R.F. Tichy and G. Turnwald (1986):

Theorem 1.10.7.1. *For any finite sequence $x_1, \dots, x_N$ and any positive integer m, we have*

$$L_N \le \frac{1}{m+1} + 3\left(\sum_{n=1}^N \frac{1}{n}\right)^{-1} \sum_{h=1}^m \left| \sum_{n=1}^N \frac{1}{n} e^{2\pi i h x_n} \right|.$$

NOTES: A more manageable version was given by R.C. Baker and G. Harman (1990):

Theorem 1.10.7.2. *For any $0 < \delta \le 1$ there exists a constant $c(\delta) > 0$ such that the inequality*

$$\left(\sum_{n=1}^N \frac{1}{n}\right) L_N < c(\delta) + 24 \sum_{1\le h\le N^\delta} \frac{1}{h} \max_{A\ge h^{1/\delta}} \left| \sum_{n=1}^A \frac{1}{n}\, e^{2\pi i h x_n} \right|$$

holds for every finite sequence $x_1, \dots, x_N$.

(I) J. Rivat and G. Tenenbaum (2005) proved the following form of Erdős-Turán inequality for weighted discrepancies: Let x_n, $n = 1, 2, \dots,$ be a sequence in $[0, 1)$, $w_n > 0$ be a sequence of weights and $W_N = \sum_{n=1}^N w_n$. Define the weighted discrepancy with respect to weights w_n by

$$D_N = \sup_{0\le x\le 1} \left| \frac{1}{W_N} \sum_{n=1}^N w_n c_{[0,x)}(x_n) - x \right|.$$

If the weights are $w_n = 1/n$ or $w_n = n \log n$ for all n then for every natural number H we have

$$D_N \leq \frac{3}{2}\left(\frac{2}{H+1} + \sum_{h=1}^{H} \frac{1}{W_N}\left|\sum_{n=1}^{N} w_n e^{2\pi i h x_n}\right|\right).$$

R.C. Baker – G. Harman: *Sequences with bounded logarithmic discrepancy*, Math. Proc. Cambridge Philos. Soc. **107** (1990), no. 2, 213–225 (MR1027775 (91d:11091); Zbl. 0705.11040).
J. Rivat – G. Tenenbaum: *Constantes d'Erdős – Turán*, (French), Ramanujan J. **9** (2005), no. 1–2, 111–121 (MR2166382 (2006g:11158); Zbl. 1145.11318).
R.F. Tichy – G. Turnwald: *Logarithmic uniform distribution of* $(\alpha n + \beta \log n)$, Tsukuba J. Math. **10** (1986), no. 2, 351–366 (MR0868660 (88f:11069); Zbl. 0619.10031).

1.10.8 Abel discrepancy

Let x_n, $n = 0, 1, 2, \dots$, be an infinite sequence in $[0, 1]$.

• Let $0 < r < 1$. Then the **Abel discrepancy** $D_r(x_n)$ is defined by (cf. E. Hlawka (1973), [DT, pp. 268-275])

$$D_r(x_n) = \sup_{[x,y)\subset[0,1]} \left|(1-r)\sum_{n=0}^{\infty} c_{[x,y)}(x_n) r^n - (y-x)\right|,$$

and similarly the star discrepancy

$$D_r^*(x_n) = \sup_{[0,x)\subset[0,1]} \left|(1-r)\sum_{n=0}^{\infty} c_{[0,x)}(x_n) r^n - x\right|.$$

Notes: (I) The theory of u.d. with respect to Abel's summation method was initiated by E. Hlawka (1973) and was further developed by Niederreiter (1975).
(II) The real sequence x_n, $n = 0, 1, 2, \dots$, is said to be **Abel limitable** to x if

$$\lim_{r\to 1-0}(1-r)\sum_{n=0}^{\infty} x_n r^n = x.$$

Though the Abel summation method is not a matrix method, it is regular and therefore the bounded sequences are Abel limitable if and only if they are Cesàro limitable. Thus $\lim_{r\to 1-0} D_r(x_n) = 0$ characterizes the usual u.d..
(III) For any r such that $0 < r < 1$ and any sequence x_n, $n = 0, 1, 2, \dots$, in $[0, 1]$ we have

- $D_r^*(x_n) \leq D_r(x_n) \leq 2D_r^*(x_n)$
- $D_r^*(x_n) \geq \frac{1-r}{2}$ (Hlawka (1973)
- there exists a y_n, $n = 0, 1, 2, \dots$, such that $D_r^*(y_n) = \frac{1-r}{2}$
- $D_r(x_n) \leq 4 \sup_{N \geq (1-r)^{-1/2}} D_N(x_n)$ $\big($[DT, p. 269, Th. 2.61]$\big)$

- $D_N(x_n) \le \left(-\log(D_{r(N)}(x_n))\right)^{-1}$, where $r(N) = N^{-1/N}$ and $c > 0$ is an absolute constant.

(IV) For the analogs to the Koksma's inequality, Erdős – Turán's inequality and LeVeque's inequality consult [DT, pp. 271-272, Th. 2.64-65], or Niederreiter (1975, Th. 5).

Given an infinite sequence x_n, $n = 0, 1, 2, \dots$, in $[0, 1]$ and $0 < r < 1$, then:
- if $f : [0, 1] \to \mathbb{R}$ is of bounded variation $V(f)$ then

$$\left|(1-r)\sum_{n=0}^{\infty} f(x_n)r^n - \int_0^1 f(x)\,\mathrm{d}x\right| \le V(f)D_r(x_n),$$

- if m is an arbitrary positive integer then

$$D_r(x_n) \le \left(\frac{3}{2}\right)\cdot\left(\frac{2}{m+1} + (1-r)\sum_{h=1}^{m}\frac{1}{h}\left|\sum_{n=0}^{\infty} e^{2\pi i h x_n} r^n\right|\right),$$

- if r is any positive number with $0 < r < 1$ and m is an arbitrary positive integer m then

$$D_r(x_n) \le \frac{4}{m+1} + \frac{4(1-r)}{\pi}\sum_{h=1}^{m}\left(\frac{1}{h} - \frac{1}{m+1}\right)\left|\sum_{n=0}^{\infty} e^{2\pi i h x_n} r^n\right|,$$

- and

$$D_r(x_n) \le \left(\frac{6}{\pi^2}\sum_{h=1}^{\infty}\frac{1}{h^2}\left|(1-r)\sum_{n=0}^{\infty} e^{2\pi i h x_n} r^n\right|^2\right)^{1/3}.$$

E. Hlawka: *Über eine Methode von E. Hecke in der Theorie der Gleichverteilung*, Acta Arith. **24** (1973), 11–31 (MR0417092 (**54** #5153); Zbl. 0231.10029).
H. Niederreiter: *Quantitative versions of a result of Hecke in the theory of uniform distribution mod 1*, Acta Arith. **28** (1975/76), no. 3, 321–339 (MR0389778 (**52** #10609); Zbl. 0318.10037).

1.10.9 Discrepancy with respect to a set of distribution functions

Let H be a non–empty closed set of d.f.'s and $x_n \in [0, 1]$ be an arbitrary sequence.
- The L^2 **discrepancy of** x_n **with respect to** H is defined by

$$D_N^{(2)}(x_n, H) = \min_{g\in H}\int_0^1 (F_N(x) - g(x))^2\,\mathrm{d}x.$$

In this notation the L^2 discrepancy of a g–distributed sequence x_n may be written as $D_N^{(2)}(x_n, g)$. The following generalization of Theorem 1.9.0.2 can be proved:

Theorem 1.10.9.1. *For every sequence* $x_n \in [0,1)$ *we have*

$$G(x_n) \subset H \qquad \Longleftrightarrow \qquad \lim_{N\to\infty} D_N^{(2)}(x_n, H) = 0.$$

NOTES: O. Strauch (1997). He used this discrepancy notion in the following result:
Theorem 1.10.9.2. *Let* H *be non–empty, closed, and connected set of d.f.'s. Denote* $\underline{g}_H(x) = \inf_{g\in H} g(x)$ *and* $\overline{g}_H(x) = \sup_{g\in H} g(x)$. *Further, if* $g \in H$ *let* $\mathrm{Graph}(g)$ *be the continuous curve formed by all the points* $(x, g(x))$ *for* $x \in [0,1]$, *and the all line segments connecting the points of discontinuity* $(x, \liminf_{x'\to x} g(x'))$ *and* $(x, \limsup_{x'\to x} g(x'))$. *Assume that for every* $g \in H$ *there exists a point* $(x,y) \in \mathrm{Graph}(g)$ *such that* $(x,y) \notin \mathrm{Graph}(\widetilde{g})$ *for any* $\widetilde{g} \in H$ *with* $\widetilde{g} \neq g$. *If moreover* $\underline{g} = \underline{g}_H$ *and* $\overline{g} = \overline{g}_H$ *for the lower d.f.* $\underline{g}$ *and the upper d.f.* $\overline{g}$ *of the sequence* $x_n \in [0,1)$ *(cf. p. 1 – 11) and* $G(x_n) \subset H$, *i.e. if* $\lim_{N\to\infty} D_N^{(2)}(x_n, H) = 0$, *then* $G(x_n) = H$.

O. Strauch: *On the set of distribution functions of a sequence*, in: *Proceedings of the Conference on Analytic and Elementary Number Theory in Honor of Edmund Hlawka's 80th Birthday, Vienna, July 18–20, 1996*, (W.–G. Nowak, J. Schoißengeier eds.), Universität Wien & Universität für Bodenkultur, Vienna, 1997, pp. 214–229 (Zbl. 0886.11044).

1.10.10 Discrepancy of distribution functions

Let $g_1(x)$, $g_2(x)$ be two d.f.'s defined on the interval $[-1,1]$ (i.e. g_1, g_2 are non–decreasing, $g_1(-1) = g_2(-1) = 0$ and $g_1(1) = g_2(1) = 1$). The difference $g(x) = g_1(x) - g_2(x)$ is called the **signed Borel measure**. The **discrepancy of** $g(x)$ is defined by

$$D(g) = \sup_{[x,y)\subset[-1,1]} |g(y) - g(x)|$$

and the **logarithmic potential** of $g(x)$ with respect to the complex number z is given by

$$U(g,z) = \int_{-1}^{1} \log\left(\frac{1}{|z-x|}\right) \mathrm{d}g(x).$$

If E_a is the ellipse with foci ± 1 and the major axis $a + \frac{1}{a}$, let

$$u(a) = \max_{z\in E_a} |U(g,z)|.$$

Theorem 1.10.10.1. *Let* $M > 0$, $0 < \gamma \le 1$ *be constants such that*

$$g_1(y) - g_1(x) \le M(y-x)^\gamma$$

holds for all subintervals $[x, y) \subset [-1, 1]$. *Then there exists a constant* $c = c(M, \gamma)$ *such that*

$$D(g) \leq c\, u(a) \log\left(\frac{1}{u(a)}\right)$$

for all $a \leq 1 + u(a)^{1+1/\gamma}$ *and* $u(a) < 1/e$.

NOTES: H.–P. Blatt and H.N. Mhaskar (1993, Th. 2.1). For details concerning logarithmic potential see M. Tsuji (1950), and for applications of the above Theorem consult V.V. Andrievskii, H.–P. Blatt and H.N. Mhaskar (2001).

V.V. ANDRIEVSKII – H.–P. BLATT – H.N. MHASKAR: *A local discrepancy theorem*, Indag. Mathem., N.S. **12** (2001), no. 1, 23–39 (MR1908137 (2003g:11084); Zbl. 1013.42017).
H.–P. BLATT – H.N. MHASKAR: *A general discrepancy theorem*, Ark. Mat. **31** (1993), no. 2, 219–246 (MR1263553 (95h:31002); Zbl. 0797.30032).
M. TSUJI: *Potential Theory in Modern Function Theory*, Maruzen Co., Ltd., Tokyo, 1959 (MR0114894 (**22** #5712); Zbl. 0087.28401); Reprinted: Chelsea Publ. Co., New York, 1975 (MR0414898 (**54** #2990); Zbl. 0322.30001).

1.10.11 Dispersion

Dispersion serves as a means for the quantitative measurement of the density of a sequence.

• Let $x_1, x_2, \dots, x_N$ belong to $[0, 1]$. Then the **dispersion** d_N of x_n's in $[0, 1]$ is defined as

$$d_N = d_N(x_1, \dots, x_N) = \sup_{x \in [0,1]} \min_{1 \leq n \leq N} |x - x_n|.$$

An alternative definition requires the reordering of $x_1, \dots, x_N$ into a non–decreasing sequence $x_{i_1} \leq x_{i_2} \leq \dots \leq x_{i_N}$. Then

$$d_N = \max\left(\frac{1}{2} \max_{1 \leq j \leq N-1} (x_{i_{j+1}} - x_{i_j}), x_{i_1}, 1 - x_{i_N}\right).$$

Evidently

$$d_N \leq D_N$$

and the infinite sequence $x_n \in [0, 1]$, $n = 1, 2, \dots$, is dense everywhere in $[0, 1]$ if and only if

$$\lim_{N \to \infty} d_N = 0.$$

If we define the quantities

$$d_N^* = \min_{1 \leq m \neq n \leq N} |x_m - x_n|, \qquad d_N^{**} = \max_{1 \leq j \leq N-1} (x_{i_{j+1}} - x_{i_j}).$$

then for every one–to–one infinite sequence x_n in $[0,1]$, $n=0,1,2,\dots$, with $x_0=1$, $x_1=0$, we have

$$\liminf_{N\to\infty} Nd_N^* \le \frac{1}{\log 4} \le \limsup_{N\to\infty} Nd_N$$

and the bounds are attained, cf. 2.12.3. For the dispersion of multidimensional sequences, see 1.11.17.

Notes: (I) The inequality between dispersion and discrepancy in the multi–dimensional case (cf. 1.11.17) was proved by H. Niederreiter (1983).

(II) The constant $1/\log 4$ has been discovered independently by several authors: N.G. de Bruijn and P. Erdős (1949), A. Ostrowski (1957, [a]1957), A. Schönhage (1957) and G.H. Toulmin (1957). More precisely:

- Motivated by the T. van Aardenne – Ehrenfest results (1945, 1949), (cf. p. 1 – 49 (0)) the quantities d_N^* and d_N^{**} were first studied by de Bruijn and Erdős (1949) for sequences x_n lying on the circle of unit length. They found the exact values
$$\inf_{(x_n)_{n=1}^\infty} \limsup_{N\to\infty} Nd_N^{**} = 1/\log 2, \qquad \sup_{(x_n)_{n=1}^\infty} \liminf_{N\to\infty} Nd_N^* = 1/\log 4$$
and proved that these values are attained for the sequence 2.12.3.
- Ostrowski (1957) independently studied the quantity d_N^*, and he proved that if x_n is an infinite sequence in $[0,1)$, then $\frac{1}{2} \le \liminf_{N\to\infty} Nd_N^* \le \frac{1}{4-2\sqrt{2}} = 0.853\dots$ and that the lower bound is attained for the sequence $x_1=0$, $x_2=\frac12$, $x_3=\frac14$, $x_4=\frac34,\dots$, $x_{2^k+i}=\frac{i-1}{2^k}+\frac{1}{2^{k+1}}$. Here $d_{2^k+i}^*=\frac{1}{2^{k+1}}$. Later ([a]1957) he improved the upper bound to $\frac{1}{\log 4}=0.7213\dots$.
- Toulmin (1957) reproved de Bruijn's and Erdős's result that the upper bound is exactly $1/\log 4$, and that this bound is attained by sequence 2.12.3 and he also proved that $\limsup_{N\to\infty} Nd_N^{**} \ge 1/\log 2$ and that this bound is attained for the same sequence 2.12.3.
- Schönhage (1957) also reproved the upper bound by means of sequences x_n for which $\lim_{N\to\infty} Nd_N^* = \alpha_m$ and $\lim_{m\to\infty}\alpha_m = 1/\log 4$.
- A multi–dimensional generalization was proved by H. Groemer (1960).

(III) H. Niederreiter (1984, Th. 1) proved that the dispersion d_N of the sequence $x_1\le x_2\le\dots\le x_N$ in $[0,1)$ satisfies

$$d_N \le c\left(\frac{1}{m+1} + \sum_{h=1}^m \left(\frac1h - \frac{1}{m+1}\right)\cdot\left|\sum_{n=0}^N (x_{n+1}-x_n)e^{2\pi i h x_n}\right|\right)$$

for all $m\in\mathbb{N}$, where $x_0=0$, $x_{N+1}=1$, and c is an absolute constant.

(IV) O. Strauch (1995) proved that for any sequence $x_1,\dots,x_N$ in $[0,1]$ we have

$$d_N \le \max(A_N, 1-B_N, 2C_N),$$

where

$$A_N = \min_{1\le M\le N} \frac{1}{M}\sum_{n=1}^{M} x_n, \quad B_N = \max_{1\le M\le N} \frac{1}{M}\sum_{n=1}^{M} x_n,$$

and if the min and max are attained at $M = M_1$ and $M = M_2$, resp., then

$$C_N = \min_{\min(M_1,M_2)\le M\le \max(M_1,M_2)} \frac{1}{M^2}\sum_{m,n=1}^{M} |x_m - x_n|.$$

(V) Dispersion $d_N(\theta)$ of the sequence $n\theta$ mod 1 can be found in 2.8.1(VIII).

N.G. de Bruijn – P. Erdős: *Sequences of points on a circle*, Nederl. Akad. Wetensch., Proc. **52** (1949), 14–17 (MR0033331 (11,423i); Zbl. 0031.34803). (=Indag. Math. **11** (1949), 46–49).
H. Groemer: *Über den Minimalabstand der ersten N Glieder einer unendlichen Punktfolge*, Monatsh. Math. **64** (1960), 330–334 (MR0117466 (**22** #8245); Zbl. 0094.02804).
H. Niederreiter: *A quasi–Monte Carlo method for the approximate computation of the extreme values of a function*, (P.Erdős – L.Álpár – G.Halász – A.Sárkőzy eds.), in: *To the memory of Paul Turán*, Studies in pure mathematics, Birkäuser Verlag & Akadémiai Kiadó, Basel, Boston, Stuttgart & Budapest, 1983, pp. 523–529 (MR0820248 (86m:11055); Zbl. 0527.65041).
H. Niederreiter: *On a measure of denseness for sequences*, in: *Topics in classical number theory, Vol. I, II (Budapest 1981)*, (G.Halász ed.), Colloq. Math. Soc. János Bolyai, Vol. 34, North–Holland Publishing Co., Amsterdam, New York, 1984, pp. 1163–1208 (MR0781180 (86h:11058); Zbl. 0547.10045).
A. Ostrowski: *Zum Schubfächerprinzip in einem linearen Intervall*, Jber. Deutsch. Math. Verein. **60** (1957), Abt. 1, 33–39 (MR0089232 (19,638a); Zbl. 0077.26703).
[a] A. Ostrowski: *Eine Verschärfung des Schubfächerprinzips in einem linearen Intervall*, Arch. Math. **8** (1957), 1–10 (MR0089233 (19,638b); Zbl. 0079.07302).
A. Schönhage: *Zum Schubfächerprinzip im linearen Intervall*, Arch. Math. **8** (1957), 327–329 (MR0093511 (**20** #35); Zbl. 0079.07303).
O. Strauch: *Uniformly maldistributed sequence in a strict sense*, Monatsh. Math. **120** (1995), 153–164 (MR1348367 (96g:11095); Zbl. 0835.11029).
G.H. Toulmin: *Subdivision of an interval by a sequence of points*, Arch. Math **8** (1957), 158–161 (MR0093513 (**20** #37); Zbl. 0086.03801).

1.11 The multi–dimensional case

In the multi–dimensioanl case we can proceed in a manner similar to the one–dimensional one.

First of all, if $\mathbf{x} = (x_1, \dots, x_s) \in \mathbb{R}^s$ is given, then $\mathbf{x}$ mod 1 denotes the sequence $(\{x_1\}, \dots, \{x_s\})$. If $\mathbf{x}_n = (x_{n,1}, \dots, x_{n,s})$ is the sequence of points in $\mathbb{R}^s$ then define:

• the s**–dimensional counting function** by

$$A([u_1, v_1) \times \cdots \times [u_s, v_s); N; \mathbf{x}_n \bmod 1) = \\ \#\{n \le N\ ;\ \{x_{n,1}\} \in [u_1, v_1), \dots, \{x_{n,s}\} \in [u_s, v_s)\}.$$

• the s**–dimensional step d.f.** also called the **empirical distribution** by

(i) $F_N(\mathbf{x}) = \frac{1}{N}A([0,x_1)\times\cdots\times[0,x_s);N;\mathbf{x}_n \bmod 1)$ if $\mathbf{x}\in[0,1)^s$,
(ii) $F_N(\mathbf{x}) = 0$ for every $\mathbf{x}$ having a vanishing coordinate,
(iii) $F_N(\mathbf{1}) = \mathbf{1}$,
(iv) $F_N(1,\dots,1,x_{i_1},1,\dots,1,x_{i_2},1,\dots,1,x_{i_l},1\dots,1) = F_N(x_{i_1},x_{i_2},\dots,x_{i_l})$ for every restricted l–dimensional face sequence $(x_{n,i_1},x_{n,i_2},\dots,x_{n,i_l})$ of $\mathbf{x}_n$ for $l=1,2,\dots,s$.

Then

• If $f:[0,1]^s\to\mathbb{R}$ is continuous, again

$$\frac{1}{N}\sum_{n=1}^{N} f(\mathbf{x}_n \bmod 1) = \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}F_N(\mathbf{x}).$$

• A function $g:[0,1]^s\to[0,1]$ is called a **d.f.** if
(i) $g(\mathbf{1}) = 1$,
(ii) $g(\mathbf{0}) = 0$, and moreover $g(\mathbf{x}) = 0$ for any $\mathbf{x}$ with a vanishing coordinate,
(iii) $g(\mathbf{x})$ is non–decreasing, i.e. $\Delta^{(s)}_{h_s}(\dots(\Delta^{(1)}_{h_1}g(x_1,\dots,x_s)))\geq 0$ for any $h_i\geq 0$, $x_i+h_i\leq 1$, where $\Delta^{(i)}_{h_i}g(x_1,\dots,x_s) = g(x_1,\dots,x_i+h_i,\dots,x_s)-g(x_1,\dots,x_i,\dots,x_s)$.

• If g is such d.f. then $\int_{[0,1]^2}\mathrm{d}g(\mathbf{x}) = 1$.

• If $\mathrm{d}g(\mathbf{x}) = \Delta^{(s)}_{\mathrm{d}x_s}\dots\Delta^{(1)}_{\mathrm{d}x_1}g(x_1,\dots,x_s)$ is the differential of $g(\mathbf{x})$ at the point $\mathbf{x}=(x_1,\dots,x_s)$, then also $\mathrm{d}g(\mathbf{x}) = \Delta(g,J)$, where $J=[x_1,x_1+\mathrm{d}x_1]\times\cdots\times[x_s,x_s+\mathrm{d}x_s]$, see 1.11.3. Moreover, $g(\mathbf{x})$ is non–decreasing if and only if $\mathrm{d}g(\mathbf{x})\geq 0$ for every $\mathbf{x}\in[0,1]^s$.

• The d.f. $g(1,\dots,1,x_{i_1},1,\dots,1,x_{i_2},1,\dots,1,x_{i_l},1\dots,1)$ is called an **l–dimensional face d.f. of g in variables** $(x_{i_1},x_{i_2},\dots,x_{i_l})\in(0,1)^l$, $0\leq l\leq s$.

• We shall identify two d.f.'s $g(\mathbf{x})$ and $\widetilde{g}(\mathbf{x})$ if:
(i) $g(\mathbf{x}) = \widetilde{g}(\mathbf{x})$ at every common point $\mathbf{x}\in(0,1)^s$ of continuity, and
(ii) $g(1,\dots,1,x_{i_1},1,\dots,1,x_{i_2},1,\dots,1,x_{i_l},1\dots,1) =$
$= \widetilde{g}(1,\dots,1,x_{i_1},1,\dots,1,x_{i_2},1,\dots,1,x_{i_l},1\dots,1)$
at every common point $(x_{i_1},x_{i_2},\dots,x_{i_l})\in(0,1)^l$ of continuity in every l–dimensional face d.f. of g and $\widetilde{g}$, $l=1,2,\dots,s$.

• The s–dimensional d.f. $g(\mathbf{x})$ is a d.f. of the sequence $\mathbf{x}_n \bmod 1$ if
(i) $g(\mathbf{x}) = \lim_{k\to\infty}F_{N_k}(\mathbf{x})$ for all continuity points $\mathbf{x}\in(0,1)^s$ of g (the so–called **weak limit**) and,

(ii) $g(1,\dots,1,x_{i_1},1,\dots,1,x_{i_2},1,\dots,1,x_{i_l},1\dots,1) = \lim_{k\to\infty} F_{N_k}(x_{i_1},x_{i_2},\dots,x_{i_l})$
weakly over $(0,1)^l$ and every l–dimensional face sequence of $\mathbf{x}_n$ for $l = 1,2,\dots,s$, and for a suitable sequence of indices $N_1 < N_2 < \dots$.

• The Second Helly theorem (see 4.1.4.15) shows that the weak limit[8] $F_{N_k}(\mathbf{x}) \to g(\mathbf{x})$ implies

$$\int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}F_{N_k}(\mathbf{x}) \to \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}g(\mathbf{x})$$

for every continuous $f : [0,1]^s \to \mathbb{R}$.

• $G(\mathbf{x}_n \bmod 1)$ is the set of all d.f.'s of $\mathbf{x}_n$ mod 1. It is again a non–empty, closed and connected set, and either it is a singleton or it has infinitely many elements.

1.11.1 u.d. sequences

NOTES: Write $\mathbf{x}\cdot\mathbf{y} = \sum_{i=1^s} x_i y_i$ for the standard inner product and $r(\mathbf{x}) = \prod_{i=1}^s \max(1,|x_i|)$.

The sequence $\mathbf{x}_n$ mod 1 is u.d. in $[0,1)^s$ if

$$\lim_{N\to\infty} \frac{A([u_1,v_1)\times\dots\times[u_s,v_s);N;\mathbf{x}_n \bmod 1)}{N} = (v_1-u_1)\dots(v_s-u_s)$$

for every subintervals $[u_1,v_1)\times\dots\times[u_s,v_s) \subset [0,1)^s$.

Theorem 1.11.1.1 (Weyl's limit relation). *A sequence* $\mathbf{x}_n$ mod 1 *is u.d. if and only if*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(\{\mathbf{x}_n\}) = \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

holds for all continuous $f : [0,1]^s \to \mathbb{R}$.

Theorem 1.11.1.2 (Weyl's criterion). *A sequence* $\mathbf{x}_n$ mod 1 *is u.d. if and only if*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i\mathbf{h}\cdot\mathbf{x}_n} = 0$$

holds for all $\mathbf{h} \in \mathbb{Z}^s$, $\mathbf{h} \neq \mathbf{0}$.

The concept of multi–dimensional u.d. sequences can be reduced to the concept of a one–dimensional u.d. as the following result demonstrates.

[8] that is (i), and (ii) above are fulfilled

Theorem 1.11.1.3 (H. Weyl (1916)). *An s–dimensional sequence $\mathbf{x}_n \bmod 1$ is u.d. if and only if for every integral vector $(h_1, \dots, h_s) \neq (0, \dots, 0)$ the one–dimensional sequence*

$$h_1 x_{n,1} + \cdots + h_s x_{n,s} \bmod 1, \quad n = 1, 2, \dots,$$

is u.d.

H. Weyl: *Über die Gleichverteilung von Zahlen mod. Eins*, Math. Ann. **77** (1916), 313–352 (JFM 46.0278.06).

1.11.2 Extremal and star discrepancy

The **extremal discrepancy** of $\mathbf{x}_n \bmod 1$ is defined by

$$D_N(\mathbf{x}_n \bmod 1) = \\ = \sup_{[u_1,v_1)\times\cdots\times[u_s,v_s)\subset[0,1]^s} \left| \frac{A([u_1, v_1) \times \cdots \times [u_s, v_s); N; \mathbf{x}_n \bmod 1)}{N} - \right. \\ \left. - (v_1 - u_1) \dots (v_s - u_s) \right|$$

and the **star discrepancy** by

$$D_N^*(\mathbf{x}_n \bmod 1) = \\ = \sup_{[0,v_1)\times\cdots\times[0,v_s)\subset[0,1]^s} \left| \frac{A([0, v_1) \times \cdots \times [0, v_s); N; \mathbf{x}_n \bmod 1)}{N} - \right. \\ \left. - v_1 \dots v_s \right|.$$

Both are connected by the relations

$$D_N^* \le D_N \le 2^s D_N^* \quad \text{[KN, p. 93]}.$$

Theorem 1.11.2.1 (Erdős – Turán – Koksma's inequality). *Let $\mathbf{x}_1$, $\mathbf{x}_2, \dots, \mathbf{x}_N$ be points in the s–dimensional unit cube $[0, 1)^s$ and H be an arbitrary positive integer. Then*

$$D_N(\mathbf{x}_n) \le \left(\frac{3}{2}\right)^s \left(\frac{2}{H+1} + \sum_{0<\|\mathbf{h}\|_\infty \le H} \frac{1}{r(\mathbf{h})} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \mathbf{h}\cdot\mathbf{x}_n} \right| \right),$$

where $r(\mathbf{h}) = \prod_{i=1}^{s} \max(1, |h_i|)$.

Theorem 1.11.2.2 (H. Niederreiter (1992, p. 43, Coroll. 37). *For arbitrary s–dimensional sequence $\mathbf{x}_1, \dots, \mathbf{x}_N$ in $[0,1)^s$ and for any non–zero lattice point $\mathbf{h} = (h_1, \dots, h_s)$ we have*

$$D_N(\mathbf{x}_n) \geq \frac{\pi}{2((\pi+1)^m - 1)} \cdot \frac{1}{r(\mathbf{h})} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \mathbf{h}\cdot\mathbf{x}_n} \right|,$$

where m is the number of non–zero coordinates of $\mathbf{h}$.

NOTES: (I) Though a multi–dimensional generalization of Erdős – Turán inequality to several dimensions had already been known to van der Corput in (1935) (cf. J.F. Koksma (1936, Kapitel X, Satz 2)) it was not published. In 1950 Koksma (1950) published a version thereof, and later, independently, also P. Szüsz (1952) (we refer the reader, for instance, to the corresponding reviews in MR for more exact forms of their results). Multi–dimensional generalizations are usually referred to as the Erdős – Turán – Koksma inequality. ([KN, p. 116])
(II) The above version, with the given constant, is due to P.J. Grabner (1989), cf. [DT, p. 15, Th. 1.21]. Grabner (1989) and Grabner and R.F. Tichy (1990) generalized an argument used by J.D. Vaaler (1985) (cf. p. 1 – 50).
(III) A similar result using a related technique was also proved by T. Cochrane (1988).
(IV) Generalizations of the inequality are also given by H. Niederreiter and W. Philipp (1972, 1973).
(V) If $\mathbf{x}_1, \dots, \mathbf{x}_N$ is a finite sequence, then the trivial lower bound is

$$D_N(\mathbf{x}_n) \geq \frac{1}{N},$$

but finite sequences satisfying the equality can only exist in the one–dimensional case, see 2.22.15.
(VI) The star discrepancy D_N^* is also known as the **two–sided Kolmogorov – Smirnov statistic test** in the goodness–of fit testing.
(VII) H. Niederreiter and I.H. Sloan (1990) proved: Let $\mathbf{x}_1, \dots, \mathbf{x}_N$ be a finite sequence in $\mathbb{R}^s$ and suppose that there exists an $\mathbf{h} = (h_1, \dots, h_s) \in \mathbb{Z}^s$ with $\sum_{i=1}^{s} |h_i| \geq 2$ and $\theta \in [0,1)$ such that $\{\mathbf{h} \cdot \mathbf{x}_n\} = \theta$ for $n = 1, 2, \dots, N$. Then the discrepancy D_N of $\mathbf{x}_n$ mod 1 satisfies

$$D_N \geq \frac{1}{m^m r(\mathbf{h})},$$

where m is the number of non–zero coordinates of $\mathbf{h}$ (also cf. Niederreiter (1992, p. 137, Lemma 5.36)).

T. COCHRANE: *Trigonometric approximation and uniform distribution modulo one*, Proc. Amer. Math. Soc. **103** (1988), no. 3, 695–702 (MR0947641 (89j:11071); Zbl. 0667.10031).
P.J. GRABNER: *Harmonische Analyse, Gleichverteilung und Ziffernentwicklungen*, TU Vienna, Ph.D. Thesis, Vienna, 1989.
P.J. GRABNER – R.F. TICHY: *Remark on an inequality of Erdős – Turán – Koksma*, Anz. Österreich. Akad. Wiss. Math.–Natur. Kl. **127** (1990), 15–22 (1991) (MR1112638 (92h:11065); Zbl. 0715.11037).

J.F. Koksma: *Diophantische Approximationen*, Ergebnisse der Mathematik und Ihrer Grenzgebiete, Vol. 4, Julius Springer, Berlin, 1936 (Zbl. 0012.39602; JFM 62.0173.01).
J.F. Koksma: *Some theorems on Diophantine inequalities*, Math. Centrum, (Scriptum no. **5**), Amsterdam, (1950) (i+51 pp.), (MR0038379 (12,394c); Zbl. 0038.02803).
H. Niederreiter: *Random Number Generation and Quasi–Monte Carlo Methods*, CBMS–NSF Regional Conference Series in Applied Mathematics, 63, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992, vi+241 pp. (MR1172997 (93h:65008); Zbl. 0761.65002).
H. Niederreiter – W. Philipp: *On a theorem of Erdős and Turán on uniform distribution*, in: *Proc. Number Theory Conference (Univ. Colorado, Boulder, Colo., 1972*, Univ. Colorado, Boulder, Colo. 1972, pp. 180–182 (MR0389821 (**52** #10651); Zbl. 0323.10039).
H. Niederreiter – W. Philipp: *Berry – Esseen bounds and a theorem of Erdős and Turán on uniform distribution* mod 1, Duke Math. J. **40** (1973), 633–649 (MR0337873 (**49** #2642); Zbl. 0273.10043).
H. Niederreiter – I.H. Sloan: *Lattice rules for multiple integration and discrepancy*, Math. Comp. **54** (1990), 303–312 (MR0995212 (90f:65036); Zbl. 0689.65006).
P. Szüsz: *On a problem in the theory of uniform distribution* (Hungarian. Russian and German summary), in: *Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août – 2 September 1950*, Akadémiai Kiadó, Budapest, 1952, pp. 461–472 (MR0056036 (15,15c); Zbl. 0048.28001).
J.D. Vaaler: *Some extremal functions in Fourier analysis*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), no. 2, 183–216 (MR0776471 (86g:42005); Zbl. 0575.42003).

Theorem 1.11.2.3 (LeVeque's inequality). *Let $\mathbf{x}_1, \dots, \mathbf{x}_N$ be points in the s–dimensional unit cube $[0,1)^s$. Then*

$$D_N(\mathbf{x}_n) \le 6\left(\frac{3}{2}\right)^s \left(\sum_{0\neq\mathbf{h}\in\mathbb{Z}^s} \frac{1}{r(\mathbf{h})^2}\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i\mathbf{h}\cdot\mathbf{x}_n}\right|^2\right)^{\frac{1}{s+2}}.$$

Notes: This result was proved by H. Stegbuchner (1979), cf. [DT, p. 23, Th. 1.28].

H. Stegbuchner: *Eine mehrdimensionale Version der Ungleichung von LeVeque*, Monatsh. Math. **87** (1979), 167–169 (MR0530461 (80i:10048); Zbl. 0369.10022).

The computation of the star discrepancy in multi–dimensional cases is much more difficult than that for dimension $s = 1$ (cf. 1.9), e.g. for $s = 2$ we have:
Theorem 1.11.2.4. *Let (x_n, y_n), $n = 1, 2, \dots, N$, be a finite sequence in $[0,1)^2$ such that $x_1 \le x_2 \le \dots \le x_N$. Let $(x_0, y_0) = (0,0)$ and $(x_{N+1}, y_{N+1}) = (1,1)$. For every $k = 0, 1, 2, \dots, N$ rearrange y_i, $i = 0, 1, 2, \dots, k, n+1$, in a non–decreasing order and rewrite them as $0 = t_{k,0} \le t_{k,1} \le \dots \le t_{k,k} < t_{k,k+1} = 1$. Then*

$$D_N^* = \max_{0\le k\le N}\max_{0\le n\le k}\max\left(\left|\frac{n}{N} - x_k t_{k,n}\right|, \left|\frac{n}{N} - x_{k+1} t_{k,n+1}\right|\right)$$

Notes: P. Bundschuh and Y. Zhu (1993) proved this theorem (and also for the case $s = 3$) motivated by a result proved by L. de Clerck (1984) and (1986). She considered only two–dimensional sequences such that $x_i < x_j$ and $y_i \neq y_j$ for any $i < j$. A general formula was proved by L. Achan, cf. [DT, p. 377, Th. 3.6], but it seems practically intractable for large dimensions.

L. ACHAN: *Discrepancy in* $[0,1]^s$, (Preprint).
P. BUNDSCHUH – Y. ZHU: *A method for exact calculation of the discrepancy of low–dimensional finite point set. I.*, Abh. Math. Sem. Univ. Hamburg **63** (1993), 115–133 (MR1227869 (94h:11070); Zbl. 0789.11041).
L. DE CLERCK: *De exacte berekening van de sterdiscrepantie van de rijen van Hammersley in 2 dimensies*, (Dutch), Ph.D. Thesis, Leuven, 1984.
L. DE CLERCK: *A method for exact calculation of the stardiscrepancy of plane sets applied to the sequences of Hammersley*, Monatsh. Math. **101** (1986), no. 4, 261–278 (MR0851948 (87i:11096); Zbl. 0588.10059).

For the extremal (and also for the star) discrepancy we have:

Conjecture 1.11.2.5. *For every dimension s, $s \geq 2$, there is a constant c_s depending only on the dimension s such that for every finite sequence $\mathbf{x}_1, \dots,$ $\mathbf{x}_N$ in $[0,1)^s$, $s \geq 2$, we have*

$$D_N(\mathbf{x}_n) \geq c_s \frac{\log^{s-1} N}{N}.$$

Everything indicates that this lower bound is possibly the best one. It was only proved for $s = 2$ by W. Schmidt (1972). The conjecture can be reformulated for infinite sequences: the s–dimensional conjecture for finite sequences gives the $(s-1)$–dimensional conjecture for infinite sequences, and vice–versa, cf. [DT, p. 40, Th. 1.49].

Conjecture 1.11.2.6. *There are constants c_s such that for any infinite sequence $\mathbf{x}_n$ in $[0,1)^s$, $s \geq 1$, we have*

$$D_N(\mathbf{x}_n) \geq c_s \frac{\log^{s} N}{N}$$

for infinitely many N.

Theorem 1.11.2.7 (K.F. Roth (1954)). *For any infinite sequence $\mathbf{x}_n$ in $[0,1)^s$ with $s \geq 1$ we have*

$$D_N(\mathbf{x}_n) \geq \frac{1}{2^5} \cdot \frac{1}{2^{4s}} \cdot \frac{1}{(s \log 2)^{s/2}} \cdot \frac{\log^{s/2} N}{N}$$

for infinitely many positive integers N .

Cf. [KN, p. 105, Th. 2.2] and [DT, p. 40, Th. 1.50].

NOTES: (I) The situation that in the s–dimensional unit cube $[0,1)^s$ the optimal discrepancy of finite sequences is better than the optimal discrepancy of infinite sequences, is known under the name **irregularities of distribution** (or **Roth's phenomenon**), cf. the monograph J. Beck and W.W.L. Chen (1987).

(II) Roth's theorem for finite sequences has the form 1.11.4.1.

J. Beck – W.W.L. Chen: *Irregularities of Distribution*, Cambridge Tracts in Mathematics, Vol. 89, Cambridge University Press, Cambridge, New York, 1987 (MR0906524 (89c:11117); Zbl. 0631.10034).
K.F. Roth: *On irregularities of distribution*, Mathematika **1** (1954), 73–79 (MR0066435 (16,575c); Zbl. 0057.28604).
W.M. Schmidt: *Irregularities of distribution. VII*, Acta Arith. **21** (1972), 45–50 (MR0319933 (**47** #8474); Zbl. 0244.10035).

1.11.3 The multi–dimensional numerical integration

Multivariate quadrature formulas.
Theorem 1.11.3.1 (Koksma – Hlawka's inequality). *Let $f : [0,1]^s \to \mathbb{R}$ be of the bounded variation $V(f)$ in the sense of Hardy and Krause. Then for any sequence $\mathbf{x}_1, \dots, \mathbf{x}_N$ in $[0,1)^s$ we have*

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \right| \le V(f) D_N^*(\mathbf{x}_n).$$

Notes: (I) A multi–dimensional analogue to the Koksma's inequality 1.9.0.3 was proved by E. Hlawka (1961), cf. [KN, p. 151, Th. 5.5.], H. Niederreiter (1978, p. 966, Th. 2.9), H. Niederreiter (1992, p. 20, Th. 2.11) and [DT, p. 10, Th. 1.14.].
(II) The **Hardy – Krause variation $\boldsymbol{V(f)}$** is defined by

$$V(f) = \sum_{k=1}^{s} \sum_{1 \le i_1 < i_2 < \dots < i_k \le s} V^{(k)}(f_{i_1,\dots,i_k})$$

where $f_{i_1,\dots,i_k} = f(1,\dots,1,x_{i_1},1,\dots,1,x_{i_k},1,\dots)$ is the restriction of f to the k–dimensional face

$$\{(x_1,\dots,x_s) \in [0,1]^s ; x_j = 1 \text{ for } j \ne i_1,\dots,i_k\}.$$

On the other hand, the **Vitali variation $\boldsymbol{V^{(k)}(h)}$** of an $h : [0,1]^k \to \mathbb{R}$ is defined by

$$V^{(k)}(h) = \sup_{\mathbf{P}} \sum_{J \in \mathbf{P}} |\Delta(h,J)|,$$

where the supremum is extended over all partitions $\mathbf{P}$ of $[0,1]^k$ into subintervals J, and $\Delta(h,J)$ is an alternating sum of the values of h at the vertices of J (function values at the adjacent vertices have opposite signs), i.e.

$$\Delta(h,J) = \sum_{\varepsilon_1=1}^{2} \cdots \sum_{\varepsilon_k=1}^{2} (-1)^{\varepsilon_1+\dots+\varepsilon_k} h(x_{\varepsilon_1}^{(1)},\dots,x_{\varepsilon_k}^{(k)})$$

for an interval $J = [x_1^{(1)}, x_2^{(1)}] \times [x_1^{(2)}, x_2^{(2)}] \times \dots \times [x_1^{(k)}, x_2^{(k)}] \subset [0,1]^k$. Vitali variation can be written in a more convenient form

$$V^{(k)}(h) = \int_0^1 \dots \int_0^1 \left| \frac{\partial^k h}{\partial x_1 \dots \partial x_k} \right| \mathrm{d}x_1 \dots \mathrm{d}x_k$$

provided the partial derivative is continuous on $[0,1]^k$.
Let $g(\mathbf{x})$ be an s–dimensional d.f. with density $h(\mathbf{x})$. E. Hlawka and R. Mück (1972) constructed $g^{-1}(\mathbf{x})$ such that for the u.d. sequence $\mathbf{x}_n$, the sequence $\mathbf{y}_n = g^{-1}(\mathbf{x}_n)$ has a.d.f. $g(x)$. J. Spanier and E. Maize (1994) continued with the following generalization of the Koksma – Hlawka inequality: For every sequence $\mathbf{x}_1, \dots, \mathbf{x}_N$ in $[0,1)^s$ and every $f : [0,1]^s \to \mathbb{R}$ with bounded Hardy – Krause variation $V(f/h)$ we have

$$\left| \frac{1}{N} \sum_{n=1}^{N} \frac{f(\mathbf{y}_n)}{h(\mathbf{y}_n)} - \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \right| \le V(f/h) D_N^*(\mathbf{x}_n),$$

cf. also A. Keller (1998).
(III) E. Hlawka (1971) proved (cf. [KN, p. 158, Notes])

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \right| \le (2^{2s-1}+1)\lambda_f^{\infty}\left(\left[(D_N^*(\mathbf{x}_n))^{-1} \right]^{-1/s} \right).$$

Here λ_f^{∞} is the **modulus of continuity of** f

$$\lambda_f^{\infty}(t) = \sup_{\substack{\mathbf{x},\mathbf{y}\in[0,1]^s \\ \|\mathbf{x}-\mathbf{y}\|_\infty \le t}} |f(\mathbf{x}) - f(\mathbf{y})|$$

with respect to the maximum distance $\|\mathbf{x}-\mathbf{y}\|_\infty = \max_{1\le i\le s} |x_i - y_i|$ for $\mathbf{x} = (x_1, \dots, x_s)$ and $\mathbf{y} = (y_1, \dots, y_s)$. If we replace $\|\mathbf{x}-\mathbf{y}\|_\infty$ by the Euclidean distance $|\mathbf{x}-\mathbf{y}| = \sqrt{\sum_{i=1}^{s}(x_i - y_i)^2}$ the corresponding modulus of continuity of f will be denoted by $\lambda_f(t)$.
(IV) P.D. Proinov (1988) proved the following multi–dimensional variant of Niederreiter's Theorem 1.9.0.5:

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \right| \le 4\lambda_f^{\infty}\left((D_N^*(\mathbf{x}_n))^{1/s} \right).$$

(V) An analogue to the Koksma – Hlawka's inequality for **A**–discrepancy can be found in [DT, p. 251, Th. 2.38].
(Va) More precisely, the Koksma – Hlawka inequality has the form (cf. H. Niederreiter (1978, p. 966, Th. 2.9))

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \right| \le \sum_{k=1}^{s} \sum_{1\le i_1 < i_2 < \dots i_k \le s} V^{(k)}(f_{i_1,\dots,i_k}) D_N^*(\mathbf{x}_n^{(i_1,\dots,i_k)}),$$

where $\mathbf{x}_n^{(i_1,\dots,i_k)} = (x_{n,i_1}, \dots, x_{n,i_s})$.

(Vb) I.M. Sobol̃ (1969, p. 268) found the expression

$$\int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x} - \frac{1}{N}\sum_{n=1}^{N} f(\mathbf{x}_n) = \sum_{k=1}^{s}\sum_{1\le i_1<i_2<\dots i_k\le s}(-1)^k\int_{[0,1]^k}\Bigg(x_{i_1}\dots x_{i_k} - \frac{A\big([0,x_{i_1})\times\dots\times[0,x_{i_k});N;\mathbf{x}_n^{(i_1,\dots,i_k)}\big)}{N}\Bigg)\frac{\partial^k f_{i_1,\dots,i_k}}{\partial x_{i_1}\dots\partial x_{i_k}}\,\mathrm{d}x_{i_1}\dots\mathrm{d}x_{i_k}$$

for f having continuous partial derivatives. This gives the following L^2 discrepancy variant of Koksma – Hlawka inequality

$$\left|\frac{1}{N}\sum_{n=1}^{N} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x}\right| \le$$
$$\le \sum_{k=1}^{s}\sum_{1\le i_1<i_2<\dots i_k\le s}\sqrt{D_N^{(2)}(\mathbf{x}_n^{(i_1,\dots,i_k)})}\,\sqrt{\int_{[0,1]^k}\left(\frac{\partial^k f_{i_1,\dots,i_k}}{\partial x_{i_1}\dots\partial x_{i_k}}\right)^2\mathrm{d}x_{i_1}\dots\mathrm{d}x_{i_k}},$$

see S.K. Zaremba (1968, Prop. 3) and I.M. Sobol̃ (1969, p. 271, Th. 2) for discrepancy $D_N^{(q)} = \int_{[0,1]^s}|F_N(\mathbf{x}) - x_1\dots x_s|^q\,\mathrm{d}\mathbf{x}$, where $\mathbf{x} = (x_1,\dots,x_s)\in[0,1]^s$.
(Vc) S.K. Zaremba (1970) proved that (cf. H. Niederreiter (1978, p. 970))

$$\left|\frac{1}{N}\sum_{\substack{n=1\\ \mathbf{x}_n\in\mathbf{E}}}^{N} f(\mathbf{x}_n) - \int_{\mathbf{E}} f(\mathbf{x})\,\mathrm{d}\mathbf{x}\right| \le (V(f) + |f(1,\dots,1)|)I_N$$

for every convex subset $\mathbf{E}\subset[0,1]^s$ and every function $f(\mathbf{x})$ of bounded Hardy – Krause variation $V(f)$, where I_N is the isotropic discrepancy (cf. 1.11.9) of $\mathbf{x}_1,\dots,\mathbf{x}_N$.
H. Niederreiter (1973) extended this result to every Jordan–measurable set $\mathbf{E}\subset[0,1]^s$, however with discrepancy $D_N^{\mathbf{X}}$ (cf. 1.11.6) instead of I_N, and with $\mathbf{X}$ being a family of subsets of $[0,1]^s$ which approximate $\mathbf{E}$ in some sense. O. Strauch (1997) found a more complicated formula on how to approximate $\int_{\mathbf{E}} f(x)\,\mathrm{d}x$ by

$$\frac{1}{N}\sum_{\substack{n=1\\ x_n\in\mathbf{E}}}^{N} f(x_n) - \frac{1}{N}\cdot\sum_{\substack{m=1\\ A(J_m;N;x_n)>0}}^{\infty}\frac{1}{A(J_m;N;x_n)}\cdot\sum_{\substack{n=1\\ x_n\in J_m}}^{N} f(x_n)$$

for every open subset $\mathbf{E} = \bigcup_{m=1}^{\infty} J_m\subset[0,1]$ (here J_m are pairwise disjoint open one–dimensional intervals in $[0,1]$).
Other quadrature formulas are in: 1.11.12, 3.15(XII), 3.17(II).
(Vd) N.N. Čencov (1961) proved: If $\mathbf{x}_n\in[0,1)^s$ is an **infinite** sequence and f runs over the class of all analytic functions defined on $[0,1]^s$ then the best possible error

term is

$$\max_f \left| \frac{1}{N} \sum_{n=1}^{N} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \right| = \mathcal{O}\left(\frac{1}{N}\right),$$

see N.M. Korobov (1963, p. 51, Th. 4). Thus the result mentioned in 3.4.1 Note (X) is the best possible. Korobov (1963, p. 45, Th. 1) also noted: Given a **finite** sequence $\mathbf{x}_1, \dots, \mathbf{x}_N$ in $[0,1)^s$ there exists an $f \in E_s^\alpha(c)$ (for def. of $E_s^\alpha(c)$ see p. 3 – 72) such that

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \right| > \frac{c.c_1}{N^\alpha},$$

where c_1 depends only on α and s.
(VI) The **Monte Carlo method** may be described as a numerical method based on random sampling and the **quasi–Monte Carlo method** as the deterministic version of the Monte Carlo method. The effectiveness of the Monte Carlo versus quasi–Monte Carlo methods for numerical multiple integration over the s–dimensional unit cube with very large values of s is an open problem. For instance, when a classical financial problem which requires the evaluation of the mortage backed security portfolio is expressed as an integral then this problem is nominally 360–dimensional (cf. [DT, pp. 389–390]).
(VIa) The Monte Carlo method for numerical integration yields a probabilistic error bound of the form $\mathcal{O}(N^{-1/2})$ depending on the number N of nodes and this order does not depend on the dimension s, but we need to repeat the computation sufficiently enough times (we do not know exactly how many times). Precisely (cf. H. Niederreiter (1992, p. 5))

$$\lim_{N\to\infty} \mathrm{Prob}\left(\frac{c_1\sigma(f)}{\sqrt{N}} \le \frac{1}{N} \sum_{n=1}^{N} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \le \frac{c_2\sigma(f)}{\sqrt{N}} \right) = \\ = \frac{1}{\sqrt{2\pi}} \int_{c_1}^{c_2} e^{-t^2/2}\,\mathrm{d}t$$

which implies

$$\mathrm{Prob}\left(\left| \frac{1}{N} \sum_{n=1}^{N} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \right| \le \frac{3\sigma(f)}{\sqrt{N}} \right) = 0.997\dots$$

for large N. Here the variance

$$\sigma^2(f) = \int_{[0,1]^s} \left(f(\mathbf{x}) - \int_{[0,1]^s} f(\mathbf{u})\,\mathrm{d}\mathbf{u} \right)^2 \mathrm{d}\mathbf{x}$$

and Prob is the Borel measure on the space of all sequences in $[0,1]^s$.
Koksma – Hlawka formula yields for the quasi–Monte Carlo method a much better deterministic error bound $\mathcal{O}(N^{-1}(\log N)^{s-1})$ for a suitably chosen sets of N nodes, but for a big s we need a very big N (cf. H. Niederreiter (1992, Chap. 1)).
(VIb) Given dimension s and $N \in \mathbb{N}$ define

- $D_N^{(2)}(s) = \inf_{\mathbf{x}_1,\dots,\mathbf{x}_N\in[0,1)^s} D_N^{(2)}(\mathbf{x}_n)$,
- $D_N^*(s) = \inf_{\mathbf{x}_1,\dots,\mathbf{x}_N\in[0,1)^s} D_N^*(\mathbf{x}_n)$,

and their inverses

- $N^{(2)}(s,\varepsilon) = \min\{N; D_N^{(2)}(s) \le \varepsilon\}$,
- $N^*(s,\varepsilon) = \min\{N; D_N^*(s) \le \varepsilon\}$.

I.H. Sloan and H. Woźniakowski (1998) proved that

$$\lim_{s\to\infty} \frac{D_N^{(2)}(s)}{D_0^{(2)}(s)} = 1,$$

with $N = [1.0463^s]$. Using the expression for $D_N^{(2)}$ given in 1.11.4 with $N = 0$ we get $D_0^{(2)}(s) = 3^{-s}$ (a similar result also holds for the weighted L^2 discrepancy). Furthermore, the minimal number N of points in $[0,1]^s$ with L^2 discrepancy $D_N^{(2)} \le \varepsilon 3^{-s}$ must satisfy $N = N^{(2)}(s, \varepsilon 3^{-s}) \ge (1-\varepsilon^2)(9/8)^s$ and thus the L^2 version of the Koksma – Hlawka inequality is intractable.

(VIc) S. Heinrich, *et al.* (2001) proved the estimate $D_N^*(s) \le c\frac{\sqrt{s}}{\sqrt{N}}$ with an unknown absolute constant c. They showed that the dependence on s cannot be improved. This yields that $N^*(s,\varepsilon) = \mathcal{O}(s\varepsilon^{-2})$ with an unknown $\mathcal{O}$–constant. They also proved another estimate not containing unknown constants (Th. 1)

$$D_N^*(s) \le 2\sqrt{2}\, N^{-1/2} \left(s \log\left(\left\lceil \frac{sN^{1/2}}{2(\log 2)^{1/2}} \right\rceil + 1 \right) + \log 2 \right)^{1/2}$$

which yields $N^*(s,\varepsilon) = \mathcal{O}(s\varepsilon^{-2}(\ln s + \ln\varepsilon^{-1}))$ with a known $\mathcal{O}$–constant. A bound (Lem. 2 & Th. 6) implying $N^*(s,\varepsilon) \le C_k s^2 \varepsilon^{-2-1/k}$ with an explicitly given C_k is also given (Th. 7). Some lower bounds for the inverse of the star discrepancy are also known. The best one says (Th. 8) that positive numbers c and ε_0 exist such that $N^*(s,\varepsilon) \ge cs\ln\varepsilon^{-1}$ for all s and all $\varepsilon \in (0,\varepsilon_0]$. In particular, $N^*(s, \frac{1}{64}) \ge 0.18s$ for all positive integers s. Thus the classical star discrepancy version of the Koksma – Hlawka inequality 1.11.3.1 is tractable, but, in practice the L^2 discrepancy $D_N^{(2)}$ can be easily evaluated by reasonably fast algorithms (see [DT, pp. 372–377]), while the computation of the star discrepancy D_N^* seems practically intractable for large dimensions (see [DT, p. 377, Th. 3.6]).

(VId) See also J. Matoušek (1999).

(VII) **Quasi-Monte Carlo integration in Hilbert space with reproducing kernel.** Denote

- $x = \frac{x_0}{b} + \frac{x_1}{b^2} + \dots$ is a b-adic representation of $x \in [0,1)$.
- $\sigma = \frac{\sigma_0}{b} + \frac{\sigma_1}{b^2} + \dots$
- $k = k_0 + k_1 b + k_2 b^2 + \dots + k_n b^n$ is a b-adic expression of the integer k, $k_n \ne 0$.
- $\mathrm{wal}_k(x) = e^{\frac{2\pi i}{b}(k_0x_0 + k_1x_1 + \dots + k_nx_n)}$ is the k-th Walsh function $\mathrm{wal}_k : [0,1] \to \mathbb{C}$ in base b.
- $\mathrm{wal}_{\mathbf{k}}(\mathbf{x}) = \prod_{i=1}^s \mathrm{wal}_{k_i}(x_i)$.
- $x \oplus \sigma = \frac{x_0+\sigma_0 \pmod b}{b} + \frac{x_1+\sigma_1 \pmod b}{b^2} + \dots$.

• $\mathbf{x} \oplus \boldsymbol{\sigma} = (x_1 \oplus \sigma_1, x_2 \oplus \sigma_2, \dots, x_s \oplus \sigma_s)$.
• $\mathbf{x} + \boldsymbol{\sigma} \bmod 1 = (\{x_1 + \sigma_1\}, \{x_2 + \sigma_2\}, \dots, \{x_s + \sigma_s\})$.
• Let H be a Hilbert space of functions $f, g, \dots : [0,1]^s \to \mathbb{R}$ with a scalar product $f(\mathbf{x}) \odot g(\mathbf{x})$ and a norm $||f|| = \sqrt{f(\mathbf{x}) \odot f(\mathbf{x})}$. The reproducing kernel $K(\mathbf{x}, \mathbf{y})$ of H is a function $K : [0,1]^{2s} \to \mathbb{R}$ satisfying (also see 1.11.12)

(i) $K(\mathbf{x}, \mathbf{y}) \in H$ for each fixed $\mathbf{y} \in [0,1]^s$.
(ii) $f(\mathbf{x}) \odot K(\mathbf{x}, \mathbf{y}) = f(\mathbf{y})$ for each fixed $\mathbf{y} \in [0,1]^s$ and for all $f(\mathbf{x}) \in H$.
(iii) $K(\mathbf{x}, \mathbf{y}) = K(\mathbf{y}, \mathbf{x})$ for $\mathbf{x}, \mathbf{y} \in [0,1]^s$.
(iv) $K(\mathbf{u}, \mathbf{x}) \odot K(\mathbf{x}, \mathbf{v}) = K(\mathbf{u}, \mathbf{v})$.
(v) H is the closure of the linear envelope of $K(\mathbf{x}, \mathbf{y})$, $\mathbf{y} \in [0,1]^s$.
(vi) $K(\mathbf{x}, \mathbf{y})$ is determined uniquely by (i)-(v).
(vii) $K(\mathbf{x}, \mathbf{y})$ is positive semi-definite, i.e. $\sum_{m,n=0}^{N-1} t_m t_n K(\mathbf{x}_m, \mathbf{x}_n) \geq 0$ for all choices of $t_0, \dots, t_{N-1} \in \mathbb{R}$ and $\mathbf{x}_0, \dots, \mathbf{x}_{N-1} \in [0,1]^s$.
(viii) For every symmetric positive semi-definite $K(\mathbf{x}, \mathbf{y})$ there is a unique Hilbert space H with reproducing kernel $K(\mathbf{x}, \mathbf{y})$.

I.H. Sloan and H. Woźniakowski (1998) found the following form for *the square worst-case quasi-Monte Carlo error*

$$\sup_{\substack{f\in H\\ ||f||\leq 1}} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \right|^2$$
$$= \int_{[0,1]^{2s}} K(\mathbf{x}, \mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} - \frac{2}{N} \sum_{n=0}^{N-1} \int_{[0,1]^s} K(\mathbf{x}_n, \mathbf{y})\,\mathrm{d}\mathbf{y} + \frac{1}{N^2} \sum_{n,m=0}^{N-1} K(\mathbf{x}_m, \mathbf{x}_n).$$

V. Baláž, J. Fialová, V.S. Grozdanov, S. Stoilova and O. Strauch (2013) replaced the sequence $\mathbf{x}_0, \dots, \mathbf{x}_{N-1}$ by $\Phi(\mathbf{x}_0 \oplus \boldsymbol{\sigma}), \dots, \Phi(\mathbf{x}_{N-1} \oplus \boldsymbol{\sigma})$ and expressed *the mean square worst-case error* in the form

Theorem 1.11.3.2. *For every sequence* $\mathbf{x}_0, \dots, \mathbf{x}_{N-1}$ *in the unit cube* $[0,1)^s$ *and every u.d.p. map* $\Phi(\mathbf{x})$ *and an arbitrary kernel* $K(\mathbf{x}, \mathbf{y})$ *with Fourier-Walsh expansion we have*

$$\int_{[0,1]^s} \sup_{\substack{f\in H\\ ||f||\leq 1}} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(\Phi(\mathbf{x}_n \oplus \boldsymbol{\sigma})) - \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \right|^2 \mathrm{d}\boldsymbol{\sigma}$$
$$= \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^s\\ \mathbf{k}\neq\mathbf{0}}} \widehat{K}_1(\mathbf{k}, \mathbf{k}) \left| \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{wal}_{\mathbf{k}}(\mathbf{x}_n) \right|^2,$$

where $\widehat{K}_1(\mathbf{k}, \mathbf{k}) = \int_{[0,1]^{2s}} K(\Phi(\mathbf{x}), \Phi(\mathbf{y}))\mathrm{wal}_{\mathbf{k}}(\mathbf{x})\overline{\mathrm{wal}_{\mathbf{k}}(\mathbf{y})}\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}$.

This is an extension of L.L. Cristea, J. Dick, G. Leobacher and F. Pillichshammer (2007, Th. 4), and of J. Dick, and F. Pillichshammer (2010, Th. 12.7).
An application: Let b, $\alpha < \beta$ be integers and consider a Sobolev space with kernel
(i) $K(x,y) = 1 + \gamma B_1(x)B_1(y) + \frac{\gamma^2}{4}B_2(x)B_2(y) - \frac{\gamma^2}{24}B_4(|x-y|)$,
(ii) $b = 2$, $N = 2^\beta$,
(iii) $x_0, x_1, \dots, x_{N-1}$ be the van der Corput sequence in base $b = 2$.
Then

$$\int_0^1 \sup_{\substack{f\in H\\ ||f||\le 1}} \left|\frac{1}{N}\sum_{n=0}^{N-1} f(x_n \oplus \sigma) - \int_0^1 f(x)\,\mathrm{d}x\right|^2 \mathrm{d}\sigma = \frac{\gamma}{12\cdot 2^{2\beta}} + \frac{\gamma^2}{360\cdot 2^{4\beta}};$$

$$\int_0^1 \sup_{\substack{f\in H\\ ||f||\le 1}} \left|\frac{1}{N}\sum_{n=0}^{N-1} f(\Phi(x_n \oplus \sigma)) - \int_0^1 f(x)\,\mathrm{d}x\right|^2 \mathrm{d}\sigma = \frac{\gamma^2}{30\cdot 2^{4\beta}}$$

where $\Phi(x)$ denotes the tent transformation $\Phi(x) = 1 - |2x-1|$.

The application of the tent transformation leads to an improvement of the estimation $O(1/N^2)$ for the mean square worst-case error to $O(1/N^4)$, see F.J. Hickernell (2002), L.L. Cristea, J. Dick, G. Leobacher and F. Pillichshammer (2007). V. Baláž, J. Fialová, V. Grozdanov, S. Stoilova and O. Strauch (2013) considered also other u.d.p. transformations $\Phi(x)$ and proved the given numerical values in the expressins above. Note that the idea of using Walsh functions goes back to G. Larcher (1993).

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S.K. Zaremba: *Some applications of multidimensional integration by parts*, Ann. Polon. Math. **21** (1968), 85–96 (MR0235731 (**38** #4034); Zbl. 0174.08402).
S.K. Zaremba: *La discrépance isotrope et l'intégration numérique*, Ann. Mat. Pura Appl. (4) **87** (1970), 125–136 (MR0281349 (**43** #7067); Zbl. 0212.17601).

1.11.4 L^2 discrepancy

The s–dimensional L^2 discrepancy of $\mathbf{x}_1, \dots, \mathbf{x}_N$ in $[0,1)^s$, where $\mathbf{x}_n = (x_{n,1}, \dots, x_{n,s})$ can be expressed in the form

$$D_N^{(2)} = \int_{[0,1]^s} \left(\frac{A([0,v_1) \times \cdots \times [0,v_s); N; \mathbf{x}_n)}{N} - v_1 \dots v_s \right)^2 \mathrm{d}v_1 \dots \mathrm{d}v_s =$$
$$= \frac{1}{3^s} + \frac{1}{N^2} \sum_{m,n=1}^{N} \prod_{j=1}^{s} (1 - \max(x_{m,j}, x_{n,j})) - \frac{1}{2^{s-1}N} \sum_{n=1}^{N} \prod_{j=1}^{s} (1 - x_{n,j}^2).$$

Notes: (I) This formula can be found in T.T. Warnock (1972). S. Heinrich (1996) found an efficient algorithm for computing L^2 discrepancy of the worst case complexity $\mathcal{O}(N(\log N)^s)$, cf. [DT, p. 372–377]. Previously known algorithms required $\mathcal{O}(N^2)$ operations. The quantity $D_N^{(2)}$ in the associated goodness-of fit test in statistics is known as the **Cramér − von Mises statistic test**.

(II) As in 1.10.3, we can define the so–called **Wiener L^2 discrepancy** of $\mathbf{x}_1,\dots,\mathbf{x}_N$ by

$$W_N^{(2)} = \int_{\mathbf{X}} \left(\frac{1}{N}\sum_{n=1}^{N} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \right)^2 \mathrm{d}f,$$

where the set $\mathbf{X} = \{f : [0,1]^s \to \mathbb{R}\,;\, f(\mathbf{0}) = 0, f \text{ is continuous}\}$ is equipped with the Wiener measure $\mathrm{d}f$ normed by $\int_{\mathbf{X}} f(\mathbf{x})f(\mathbf{y})\,\mathrm{d}f = \min(\mathbf{x},\mathbf{y}) = \prod_{i=1}^{s}\min(x_i,y_i)$.
H. Woźniakovski (1991) proved that $W_N^{(2)}(\mathbf{x}_n) = D_N^{(2)}(\mathbf{1}-\mathbf{x}_n)$, where $\mathbf{1}-\mathbf{x}_n = (1-x_{n,1},\dots,1-x_{n,s})$. (For $s=1$ we have $W_N^{(2)}(x_n) = D_N^{(2)}(x_n)$, a result independently proved also by O. Strauch (1994)). This means that on average the integration error depends only on the L^2 discrepancy.
For integration over Wiener measure $\mathrm{d}f$ cf. I.M. Gel′fand and A.M. Jaglom (1956).

I.M. Gel′fand – A.M. Jaglom: *Integration in functional spaces and its applications in quantum physic* (Russian), Uspekhi Mat. Nauk **11** (1956), no. 1, 77–114 (English translation: J. Math. Phys. **1** (1960), 48–69 (MR0112604 (**22** #3455); Zbl. 0092.45105)).
S. Heinrich: *Efficient algorithms for computing the L_2 discrepancy*, Math. Comp. **65** (1996), no. 216, 1621–1633 (MR1351202 (97a:65024); Zbl. 0853.65004).
O. Strauch: *L^2 discrepancy*, Math. Slovaca **44** (1994), 601–632 (MR1338433 (96c:11085); Zbl. 0818.11029).
T.T. Warnock: *Computational investigations of law discrepancy point sets*, in: *Applications of Number Theory to Numerical Analysis (Proc. Sympos. Univ. Montreal, Montreal, Que., 1971)*, (S.K. Zaremba ed.), Academic Press, New York, London, 1972, pp. 319–343 (MR0351035 (**50** #3526); Zbl. 0248.65018).
H. Woźniakowski: *Average case complexity of multivariate integration*, Bull. Amer. Math. Soc. (N.S.) **24** (1991), no. 1, 185–194 (MR1072015 (91i:65224); Zbl. 0729.65010).

Theorem 1.11.4.1 (K.F. Roth (1954)). [9] *For any finite sequence $\mathbf{x}_1,\dots,\mathbf{x}_N$ in $[0,1)^s$ with $s\ge 2$ we have*

$$D_N(\mathbf{x}_n) \ge D_N^*(\mathbf{x}_n) \ge \sqrt{D_N^{(2)}(\mathbf{x}_n)} \ge \frac{1}{2^{4s}}\cdot\frac{1}{((s-1)\log 2)^{\frac{s-1}{2}}}\cdot\frac{\log^{\frac{s-1}{2}} N}{N}.$$

Theorem 1.11.4.2 (K.F. Roth (1980)). *There are constants c_s such that for every $N = 1,2,\dots$ there exits a finite sequence $\mathbf{x}_1,\dots,\mathbf{x}_N$ in $[0,1)^s$ such that*

$$\sqrt{D_N^{(2)}(\mathbf{x}_n)} \le c_s\frac{\log^{\frac{s-1}{2}} N}{N}.$$

Notes: (I) The first constructions of sequences satisfying Th. 1.11.4.2 were given by H. Davenport (1956)[10] and Roth (1979) for dimensions $s=2$ and $s=3$ resp., and for an arbitrary dimension by Roth (1980). Another proof can be found in N.M. Dobrovol′skiĭ (1984). For the early history of this topic consult J. Beck and

[9] cf. [KN, p.105, Th. 2.1] and [DT, p. 29, Th. 1.40]
[10] His sequence is basically of the form $\left(\frac{n}{N},\{n\alpha\}\right)$, $n = 0,1,2,\dots,N-1$, with an irrational α having bounded partial quotients.

W.W.L. Chen (1978) and for $s = 2$ see H. Niederreiter (1978, p. 977).
(II) The Roth's bound for the L^2 discrepancy is optimal. Roth's bound for the extremal and star discrepancies is the best known one for $s > 3$. For $s = 3$ it was sharpened by J. Beck (1989) (cf. [DT, p. 44, Th. 1.58]):
Theorem 1.11.4.3. *For any finite sequence* $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N$ *in* $[0,1)^3$ *and for any* $\varepsilon > 0$ *we have*

$$D_N^* \geq \frac{\log N}{N} (\log\log N)^{\frac{1}{8}-\varepsilon}$$

for sufficiently large N.

(III) W.W.L. Chen and M.M. Skriganov (2002) proved:
Theorem 1.11.4.4. *Let* $p \geq 2s^2$ *be a prime. Then given any* $N > 1$, *a sequence* $\mathbf{x}_1, \dots, \mathbf{x}_N$ *of* N *points in the unit cube* $[0,1)^s$ *can be explicitly constructed for which*

$$N\sqrt{D_N^{(2)}} < 2^{s+1} p^{2s} (\log N + 2s + 1)^{\frac{s-1}{2}}.$$

J. Beck: *A two–dimensional van Ardenne-Ehrenfest theorem in irregularities of distribution*, Compositio Math. **72** (1989), no. 3, 269–339 (MR1032337 (91f:11054); Zbl. 0691.10041).
J. Beck – W.W.L. Chen: *Irregularities of Distribution*, Cambridge Tracts in Mathematics, Vol. 89, Cambridge University Press, Cambridge, New York, 1987 (MR0906524 (89c:11117); Zbl. 0631.10034).
W.W.L. Chen – M.M. Skriganov: *Explicit constructions in the classical mean square problem in irregularities of point distribution*, J. Reine Angew. Math., **545** (2002), 67–95 (MR1896098 (2003g:11083); Zbl. 1083.11049).
H. Davenport: *Note on irregularities of distribution*, Mathematika **3** (1956), 131–135 (MR0082531 (18,566a); Zbl. 0073.03402).
N.M. Dobrovol'skiĭ: *An effective proof of Roth's theorem on quadratic deviation*, (Russian), Uspekhi. Mat. Nauk **39** (1984), 155–156 (MR0753777 (86c:11055); Zbl. 0554.10030).
H. Niederreiter: *Quasi–Monte Carlo methods and pseudo–random numbers*, Bull. Amer. Math. Soc. **84** (1978), no. 6, 957–1041 (MR0508447 (80d:65016); Zbl. 0404.65003).
K.F. Roth: *On irregularities of distribution*, Mathematika **1** (1954), 73–79 (MR0066435 (16,575c); Zbl. 0057.28604).
K.F. Roth: *On irregularities of distribution. III.*, Acta Arith. **35** (1979), no. 4, 373–384 (MR0553291 (81a:10065); Zbl. 0425.10056).
K.F. Roth: *On irregularities of distribution. IV.*, Acta Arith. **37** (1980), 67–75 (MR0598865 (82f:10063); Zbl. 0425.10057).

The discrepancies D_N, D_N^* and $D_N^{(2)}$ are linked together by inequalities

$$c_s D_N^{s+2} \leq D_N^{(2)} \leq (D_N^*)^2 \quad \text{H. Niederreiter (1973, Th. 4.2)},$$

where the constant $c_s > 0$ depends only on s.

If $\mathbf{x}_n \in [0,1)^s$ is a g–distributed s–dimensional sequence then (cf. O. Strauch

(1994))

$$
\begin{aligned}
&D_N^{(2)}(\mathbf{x}_n, g) = \\
&= \int_{[0,1]^s} \left(\frac{A\big([0,v_1)\times\cdots\times[0,v_s); N; \mathbf{x}_n\big)}{N} - g(v_1,\dots,v_s) \right)^2 \mathrm{d}v_1 \dots \mathrm{d}v_s = \\
&\qquad\qquad = \frac{1}{N^2} \sum_{m,n=1}^{N} F\big((x_{m,1},\dots,x_{m,s}),(x_{n,1},\dots,x_{n,s})\big),
\end{aligned}
$$

where

$$
\begin{aligned}
&F\left((x_{m,1},\dots,x_{m,s}),(x_{n,1},\dots,x_{n,s})\right) = \\
&= \int_{[0,1]^s} g^2(v_1,\dots,v_s)\,\mathrm{d}v_1\dots\mathrm{d}v_s - \int_{x_{m,1}}^1 \mathrm{d}v_1 \dots \int_{x_{m,s}}^1 g(v_1,\dots,v_s)\,\mathrm{d}v_s - \\
&\qquad - \int_{x_{n,1}}^1 \mathrm{d}v_1 \dots \int_{x_{n,s}}^1 g(v_1,\dots,v_s)\,\mathrm{d}v_s + \prod_{j=1}^{s} \left(1 - \max(x_{m,j}, x_{n,j})\right).
\end{aligned}
$$

Notes: If in the multi–dimensional cases $g(\mathbf{x})$ is continuous then the limit $\lim_{k\to\infty} D_{N_k}^{(2)}(\mathbf{x}_n, g) = 0$ implies $g \in G(\mathbf{x}_n)$ (see def. 1.11). If $g(\mathbf{x})$ is discontinuous then to obtain the inclusion $g \in G(\mathbf{x}_n)$, the existence of the limits

$$\lim_{k\to\infty} D_{N_k}^{(2)}\big((x_{n,i_1},\dots,x_{n,i_l}), g(1,\dots,1,x_{i_1},1,\dots,1,x_{i_2},1,\dots,1,x_{i_l},1\dots,1)\big) = 0$$

is necessary for every face sequence $(x_{n,i_1},\dots,x_{n,i_l})$ of $\mathbf{x}_n$, $l = 1,2,\dots,s$.

H. Niederreiter: *Application of diophantine approximations to numerical integration*, in: *Diophantine Approximation and Its Applications (Washington, D.C., 1972)*, (C.F. Osgood ed.), Academic Press, New York, 1973, pp. 129–199 (MR0357357 (**50** #9825); Zbl. 0268.65014).
O. Strauch: L^2 *discrepancy*, Math. Slovaca **44** (1994), 601–632 (MR1338433 (96c:11085); Zbl. 0818.11029).

1.11.5 Diaphony

For the multi–dimensional finite sequence $\mathbf{x}_n = (x_{n,1},\dots,x_{n,s}) \in [0,1)^s$, $n = 1,2,\dots,N$, the **diaphony** is defined by (cf. W. Morokoff and R.E. Caflisch (1994))

$$DI_N^{(2)} = \int\limits_{\substack{[0,1]^{2s} \\ 0\le u_i<v_i\le 1, i=1,\dots,s}} \left(\frac{A([u_1,v_1)\times\dots[u_s,v_s);N;\mathbf{x}_n)}{N} - \right.$$
$$\left. - (v_1-u_1)\dots(v_s-u_s)\right)^2 \mathrm{d}u_1\dots\mathrm{d}u_s\,\mathrm{d}v_1\dots\mathrm{d}v_s =$$
$$= \frac{1}{(12)^s} + \frac{1}{N^2}\sum_{m,n=1}^{N}\prod_{j=1}^{s}\big(1-\max(x_{m,j},x_{n,j})\big)\min(x_{m,j},x_{n,j}) -$$
$$- \frac{1}{2^{s-1}N}\sum_{n=1}^{N}\prod_{j=1}^{s}(1-x_{n,j})x_{n,j}.$$

Another definition of the diaphony says

$$DI_N(\mathbf{x}_n) = \left(\sum_{\mathbf{0}\neq\mathbf{h}\in\mathbb{Z}^s}\frac{1}{r(\mathbf{h})^2}\left|\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i\mathbf{h}\cdot\mathbf{x}_n}\right|^2\right)^{\frac{1}{2}}$$

(for ex. cf. 2.11.1, 2.11.2). W. Fleischer and H. Stegbuchner (1982) proved that

$$DI_N \le (5\pi+1)^s D_N^*$$

for any sequence $\mathbf{x}_n \in [0,1]^s$. This result is the best possible in exception of the constant involved.

A.V. Bikovsky (1985) (cf. V.S. Grozdanov and S.S. Stoilova (2003)) proved that

$$DI_N > c(s)\frac{(\log N)^{\frac{s-1}{2}}}{N}$$

for every $\mathbf{x}_n \in [0,1)^s$, $n=1,2,\dots,N$.

NOTES: P. Hellekalek and H. Leeb (1997), using the Walsh functions, and V.S. Grozdanov and S.S. Stoilova (2001, 2003), using the Chrestenson functions (see below), introduced a new version of diaphony called q–**adic diaphony**:

$$DF_N(\mathbf{x}_n) = \left(\frac{1}{(q+1)^s-1}\sum_{\mathbf{0}\neq\mathbf{h}\in\mathbb{N}_0^s}\frac{1}{\rho(\mathbf{h})^2}\left|\frac{1}{N}\sum_{n=1}^{N}w_{\mathbf{h}}(\mathbf{x}_n)\right|^2\right)^{\frac{1}{2}}.$$

Here the **Chrestenson function** $w_{\mathbf{h}}(\mathbf{x})$ of order q is defined by:

$w_{\mathbf{h}}(\mathbf{x}) = \prod_{i=1}^{s} w_{h_i}(x_i)$, where $\mathbf{h} = (h_1, \dots, h_s)$ and $\mathbf{x} = (x_1, \dots, x_s)$,
$w_n(x) = \prod_{i=0}^{k(n)} (r_i(x))^{a_i}$, if $n = \sum_{i=0}^{k(n)} a_i q^i$ is the q–adic digit expansion of n, $x \in [0, 1)$,
$r_i(x) = r_0(q^i x)$, $i = 1, \dots, k(n)$, while
$r_0(x) = e^{2\pi i(k/q)}$ provided $x \in [k/q, (k+1)/q)$ for $k = 0, 1, \dots q-1$.
Furthermore
$\rho(\mathbf{h}) = \prod_{i=1}^{s} \rho(h_i)$ if $\mathbf{h} = (h_1, \dots, h_s)$, and
$\rho(0) = 1$ and $\rho(h) = q^{-2k}$ if $q^k \le h < q^{k+1}$, $k \in \mathbb{N}_0$.
If $q = 2$ then $w_n(x)$ reduces to Walsh function, cf. 2.1.1(II).

A.V. Bikovski: *On the exact order of the error of the cubature formulas in space with dominating derivative and quadratic discrepancy of sets*, (Russian), Computing center, DVNC AS SSSR, Vladivostok, 1985 (preprint).
H.E. Chrestenson: *A class of generalized Walsh functions*, Pac. J. Math. **5** (1955), 17–31 (MR0068659 (16,920c); Zbl. 0065.05302).
W. Fleischer – H. Stegbuchner: *Über eine Ungleichung in der Theorie Gleichverteilung* mod 1, Österreich. Akad. Wiss. Math.–Natur. Kl. Sitzungsber. II **191** (1982), no. 4–7, 133–139 (MR0705432 (85e:11050); Zbl. 0511.10037).
V.S. Grozdanov – S.S. Stoilova: *On the theory of b–adic diaphony*, C. R. Acad. Bulgare Sci. **54** (2001), no. 3, 31–34 (MR1829550 (2002e:11101); Zbl. 0974.60002).
V.S. Grozdanov – S.S. Stoilova: *On the b–adic diaphony of the Roth net and generalized Zaremba net*, Math. Balkanica (N.S.) **17** (2003), no. 1–2, 103–112 (MR2096244 (2005f:11144); Zbl. 1053.11066).
P. Hellekalek – H. Leeb: *Dyadic diaphony*, Acta Arith. **80** (1997), no. 2, 187–196 (MR1450924 (98g:11090); Zbl. 0868.11034).
W. Morokoff – R.E. Caflisch: *Quasi–random sequences and their discrepancies*, SIAM J. Sci. Comput. **15** (1994), no. 6, 1251–1279 (MR1298614 (95e:65009); Zbl. 0815.65002).

1.11.6 Discrepancy relative to sets systems X

Let $\mathbf{X}$ be a system of bounded measurable subsets X of $\mathbb{R}^s$. The discrepancy $D_N^{\mathbf{X}}(\mathbf{x}_n)$ of the sequence $\mathbf{x}_1, \dots, \mathbf{x}_N$ of points in $[0, 1)^s$ is defined by

$$D_N^{\mathbf{X}} = \sup_{X \in \mathbf{X}} \left| \frac{A(X \bmod 1; N; \mathbf{x}_n)}{N} - |X| \right|,$$

where $|X|$ denotes the Lebesgue measure of X and $X \bmod 1$ is defined by considering the **multiplicity** $\mathbf{x} \bmod 1$ of $\mathbf{x} \in X$. The basic prototypes of $\mathbf{X}$ are set boxes, cubes, balls, convex sets, etc. The classical discrepancies D_N, D_N^* are defined relative to rectangular parallelepipeds aligned with the axes.

1.11.7 Discrepancy relative to cubes (cube-discrepancy)

Denote

$$D_N^{\mathbf{C}} = \sup_{C} \left| \frac{A(C \bmod 1; N; \mathbf{x}_n)}{N} - |C| \right|$$

if $\mathbf{x}_n \in [0, 1)^s$, where the supremum is taken over all cubes $C \subset \mathbb{R}^s$ aligned with the axes. Similarly define discrepancy $D_N^{\mathbf{C}(r)}$, where $\mathbf{C}(r)$ is the class of all s–dimensional cubes of edge length not exceeding r (again aligned with axes). If $s = 2$ then

$$D_N^{\mathbf{C}} \le D_N \le 11 D_N^{\mathbf{C}}$$

for any sequence $\mathbf{x}_n \in [0,1)^2$.

NOTES: These boundaries were proved by I.Z. Ruzsa (1993), see also [DT, p. 35]. G. Larcher (1991) found boundaries of $D_N^{\mathbf{C}(r)}$ for Kronecker sequences, cf. 3.4.1(IIIb).

G. LARCHER: *On the cube–discrepancy of Kronecker–sequences*, Arch. Math. (Basel) **57** (1991), no. 4, 362–369 (MR1124499 (93a:11064); Zbl. 0725.11036).

I.Z. RUZSA: *The discrepancy of rectangles and squares*, in: *Österreichisch – Ungarisch – Slowakisches Kolloquium über Zahlentheorie (Maria Trost, 1992)*, (F. Halter–Koch, R.F. Tichy eds.), Grazer Math. Ber., Vol. 318, Karl–Franzes – Univ. Graz, 1993, pp. 135–140 (MR1227410 (94j:11070); Zbl. 0784.11038).

1.11.8 Discrepancy relative to balls

• Let $\mathbf{B}(r)$ be the family of balls $B = \{\mathbf{x} \in \mathbb{R}^s\,;\, |\mathbf{x} - \mathbf{c}| \le r\}$ with radius r and centered at $\mathbf{c}$, where $\mathbf{c}$ is taken over all $\mathbf{c} \in \mathbb{R}^s$. The **ball–discrepancy** of $\mathbf{x}_1, \dots, \mathbf{x}_N$ in $[0,1)^s$ is

$$D_N^{\mathbf{B}(r)} = \sup_{B \in \mathbf{B}(r)} \left| \frac{A(B \bmod 1; N; \mathbf{x}_n)}{N} - |B| \right|.$$

NOTES: J.J. Holt (1996) proved a variant of the Erdős – Turán inequality 1.9.0.8 for $D_N^{\mathbf{B}(r)}$ involving Bessel functions on the right–hand side. G. Harman (1998) proved a stronger version which holds for all $t > 0$

$$D_N^{\mathbf{B}(r)} \le c_1(s)\left(\frac{r^{s-1}}{t} + \frac{1}{t^s}\right) + \\ + c_2(s) \sum_{0<|\mathbf{h}|<t, \mathbf{h}\in\mathbb{Z}^s} \left(\frac{1}{t^s} + \min\left(r^s, \frac{r^{(s-1)/2}}{|\mathbf{h}|^{(s+1)/2}}\right)\right) \left|\frac{1}{N}\sum_{n=1}^N e^{2\pi i \mathbf{h}\cdot\mathbf{x}_n}\right|$$

with $|\mathbf{h}|$ denoting the Euclidean metric. If $\mathbf{B} = \bigcup_{0<r\le\frac12} \mathbf{B}(r)$ then this gives

$$D_N^{\mathbf{B}} \le c_3(s)\frac{1}{t} + c_4(s) \sum_{\substack{0<|\mathbf{h}|<t,\\ \mathbf{h}\in\mathbb{Z}^s}} \frac{1}{|\mathbf{h}|^{(s+1)/2}} \left|\frac{1}{N}\sum_{n=1}^N e^{2\pi i \mathbf{h}\cdot\mathbf{x}_n}\right|.$$

J. Beck and W.W.L. Chen (1986) proved that $D_N^{\mathbf{X}}$ does not satisfy Roth's phenomenon (for def. cf. 1.9 Note (VI)).

J. BECK – W.W.L. CHEN: *Note on irregularities of distribution*, Mathematika **33** (1986), 148–163 (MR0859507 (88a:11071); Zbl. 0601.10039).

G. Harman: *On the Erdős–Turán inequality for balls*, Acta Arith. **85** (1998), no. 4, 389–396 (MR1640987 (99h:11086); Zbl. 0918.11044).
J.J. Holt: *On a form of the Erdős – Turán inequality*, Acta Arith. **74** (1996), no. 1, 61–66 (MR1367578 (96k:11098); Zbl. 0851.11042).

1.11.9 Isotropic discrepancy

• The **isotropic discrepancy** I_N of the sequence $\mathbf{x}_n$ in $[0,1)^s$ is defined by

$$I_N = \sup_C \left| \frac{A(C;N;\mathbf{x}_n)}{N} - |C| \right|,$$

where the supremum is taken over all convex subset C of $[0,1]^s$.
For any $\mathbf{x}_1,\dots,\mathbf{x}_N$ in $[0,1)^s$ we have

$$D_N(\mathbf{x}_n) \le I_N(\mathbf{x}_n) \le 4sD_N(\mathbf{x}_n)^{1/s}.$$

Notes: (I) This was proved by H. Niederreiter (1992, p. 17) using the bound $I_N \le s\left(\frac{4cs}{s-1}\right)^{(s-1)/s} D_N^{1/s}$ with an absolute constant c (cf. also the paper H. Niedereiter and J.M. Wills (1975)). In [KN, p. 95, Th. 1.6.] the form $I_N \le (4s\sqrt{s}+1)D_N^{1/s}$ can be found.
(II) G. Larcher (1986, 1988) achieved some improvements for special sequences (see 3.4.1, 3.18.2, 3.18.1).
(III) E. Hlawka (1971) originally proved that $I_N \le 72^s D_N^{1/s}$.

E. Hlawka: *Zur Definition der Diskrepanz*, Acta Arith. **18** (1971), 233–241 (MR0286757 (**44** #3966); Zbl. 0218.10063).
G. Larcher: *Über die isotrope Discrepanz von Folgen*, Arch. Math. (Basel) **46** (1986), no. 3, 240–249 (MR0834843 (87e:11091); Zbl. 0568.10029).
G. Larcher: *On the distribution of s–dimensional Kronecker sequences*, Acta Arith. **51** (1988), no. 4, 335–347 (MR0971085 (90f:11065); Zbl. 0611.10033).
H. Niederreiter: *Random Number Generation and Quasi–Monte Carlo Methods*, CBMS–NSF Regional Conference Series in Applied Mathematics, 63, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992, vi+241 pp. (MR1172997 (93h:65008); Zbl. 0761.65002).
H. Niederreiter – J.M. Wills: *Diskrepanz und Distanz von Maßen bezüglich konvexer und Jordanschen Mengen*, Math. Z. **144** (1975), no. 2, 125–134 (MR0376588 (**51** #12763); Zbl. 0295.28028).

1.11.10 Spherical–cap discrepancy

• [DT, p. 231]: Let $\mathbf{S} = \{\mathbf{x} \in \mathbb{R}^{s+1} : |\mathbf{x}| = 1\}$ be the s–dimensional sphere and $C = C(\mathbf{x}, r) = \{\mathbf{y} \in \mathbf{S}\,;\, \mathbf{x}\cdot\mathbf{y} \ge r\}$, $-1 \le r \le 1$, be a **spherical cap** with **normalized surface measure** $\sigma(C)$. The **spherical–cap discrepancy** is defined by

$$S_N = \sup_C \left| \frac{A(C;N;\mathbf{x}_n)}{N} - \sigma(C) \right|,$$

where $\mathbf{x}_n$ is a sequence on $\mathbf{S}$, and the supremum is taken over all spherical caps. Moreover, $S_N \gg N^{-1/2-1/(2s)}$ for every sequence $\mathbf{x}_n \in S$.

NOTES: The analogue to the Erdős – Turán – Koksma's inequality for the spherical–cap dicrepancy was proved by P.J. Grabner (1991).

P.J. GRABNER: *Erdős – Turán type discrepancy bounds*, Monatsh. Math. **111** (1991), no. 2, 127–135 (MR1100852 (92f:11108); Zbl. 0719.11046).

1.11.11 L^2 discrepancy relative to a counting function

Let $\mathbf{X} = \{X(\mathbf{t})\,;\,\mathbf{t} \in [0,1]^s\}$ be a system of subsets of $[0,1]^s$. Let $A\big(X(\mathbf{t});\mathbf{x}_1,\dots,\mathbf{x}_N\big)$ be the **generalized counting function** defined for $\mathbf{t},\mathbf{x}_1,\dots,\mathbf{x}_N$ from $[0,1]^s$ by the conditions

(i) $A\big(X(\mathbf{t});\mathbf{x}_1,\dots,\mathbf{x}_N\big) = \sum_{n=1}^N A\big(X(\mathbf{t});\mathbf{x}_n\big)$,

(ii) $A\big(X(\mathbf{t});\mathbf{x}\big) = 0 \vee 1$,

(iii) $\mathbf{t} \le \mathbf{t}' \Rightarrow A\big(X(\mathbf{t});\mathbf{x}\big) \le A\big(X(\mathbf{t});\mathbf{x}\big)$, where $\mathbf{t} = (t_1,\dots,t_s) \le \mathbf{t}' = (t'_1,\dots,t'_s)$ if $t_i \le t'_i$ for $i = 1,2,\dots,s$,

(iv) $T(\mathbf{x}) = \big\{\mathbf{t} \in [0,1]^s\,;\,A\big(X(\mathbf{t});\mathbf{x}\big) = 1\big\}$ is measurable in the Lebesque sense.

In what follows we assume that (i)–(iv) holds for every $\mathbf{t},\mathbf{t}'$, $\mathbf{x}_1,\dots,\mathbf{x}_N$, $\mathbf{x}$ in $[0,1]^s$. Then if $g(\mathbf{x})$ is a d.f. defined on $[0,1]^s$ then the L^2 discrepancy

$$D_N^{(2)}(A,g) = \int_{[0,1]^s} \left(\frac{A(X(\mathbf{t});\mathbf{x}_1,\dots,\mathbf{x}_N)}{N} - g(\mathbf{x})\right)^2 \mathrm{d}\mathbf{t}$$

can be expressed in the form

$$D_N^{(2)}(A,g) = \frac{1}{N^2}\sum_{m,n=1}^N F(\mathbf{x},\mathbf{y}),$$

where

$$F(\mathbf{x},\mathbf{y}) = \int_{[0,1]^s} g^2(\mathbf{t})\,\mathrm{d}\mathbf{t} - \int_{T(\mathbf{x})} g(\mathbf{t})\,\mathrm{d}\mathbf{t} - \int_{T(\mathbf{y})} g(\mathbf{t})\,\mathrm{d}\mathbf{t} + \int_{T(\mathbf{x})\cap T(\mathbf{y})} 1\cdot \mathrm{d}\mathbf{t},$$

and $\lim_{N\to\infty} D_N^{(2)}(A,g) = 0$ if and only if

$$\frac{A(X(\mathbf{t});\mathbf{x}_1,\dots,\mathbf{x}_N)}{N} \to g(\mathbf{t})$$

for every point $\mathbf{t} \in [0,1]^s$ of continuity of $g(\mathbf{t})$.

NOTES: For a more general form which can be applied to 1.10.1, 1.10.2, 1.10.3, 1.11.4, 1.11.5 consult O. Strauch (1994, p. 608–609, Th. 2,3). The above expression of the L^2 discrepancy $D_N^{(2)}(A,g)$ gives impetus to the following generalization:

$$D_N^{(2)}(F,\mathbf{x}_n) = \frac{1}{N^2}\sum_{m,n=1}^{N} F(\mathbf{x}_m,\mathbf{x}_n)$$

where $F(\mathbf{x},\mathbf{y})$ is continuous on $[0,1]^{2s}$. For such discrepancy Strauch (1994, p. 612, Th. 4) proved that

$$G(\mathbf{x}_n) \subset G(F) \Longleftrightarrow \lim_{N\to\infty} D_N^{(2)}(F,\mathbf{x}_n) = 0,$$

where

$$G(F) = \left\{\text{d.f. } g(\mathbf{x})\ ;\ \int_{[0,1]^{2s}} F(\mathbf{x},\mathbf{y})\,\mathrm{d}g(\mathbf{x})\,\mathrm{d}g(\mathbf{y}) = 0\right\}.$$

O. Strauch: L^2 *discrepancy*, Math. Slovaca **44** (1994), 601–632 (MR1338433 (96c:11085); Zbl. 0818.11029).

1.11.12 Discrepancy relative to reproducing kernel

Reproducing kernel $K(\mathbf{x},\mathbf{y})$ (see also 1.11.3, Note (VIII)) is a function on $[0,1)^s \times [0,1)^s$ which satisfies
(i) $K(\mathbf{x},\mathbf{y}) = K(\mathbf{y},\mathbf{x})$ for all $\mathbf{x},\mathbf{y} \in [0,1)^s$,
(ii) $\sum_{m,n=1}^{N} t_m t_n K(\mathbf{x}_m,\mathbf{x}_n) \geq 0$ for all $t_n \in \mathbb{R}$, $\mathbf{x}_n \in [0,1)^s$, $N = 1,2,\dots$,
i.e. $K(\mathbf{x},\mathbf{y})$ is symmetric and positive definite. Then the **discrepancy involving the reproducing kernel** K is defined as

$$D_N^K = \int_{[0,1]^{2s}} K(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} - \frac{2}{N}\sum_{n=1}^{N}\int_{[0,1]^s} K(\mathbf{x}_n,\mathbf{y})\,\mathrm{d}\mathbf{y} + \frac{1}{N^2}\sum_{m,n=1}^{N} K(\mathbf{x}_m,\mathbf{x}_n).$$

The definition of this discrepancy is motivated by the following quadrature error formula: Let W be a Hilbert space of all real valued function $f(\mathbf{x})$ on $[0,1)^s$ endowed with an inner product $f(\mathbf{x})\cdot g(\mathbf{x})$ which satisfies
(j) $K(\mathbf{x},\mathbf{y}) \in W$ for every fixed $\mathbf{y} \in [0,1)^s$,
(jj) $f(\mathbf{y}) = K(\mathbf{x},\mathbf{y})\cdot f(\mathbf{x})$ for all $f \in W$ and $\mathbf{y} \in [0,1)^s$.
Then for every $f \in W$ and $\mathbf{x}_1,\dots,\mathbf{x}_N$ in $[0,1)^s$ we have

$$\left|\int_{[0,1]^s} f(\mathbf{x}) - \frac{1}{N}\sum_{n=1}^{N} f(\mathbf{x}_n)\right| \leq \sqrt{D_N^K}\,V_K(f),$$

where $V_K(f) = \|f(\mathbf{x}) - (f(\mathbf{x})\cdot\mathbf{1})/(\mathbf{1}\cdot\mathbf{1})\|$ with $\|h(x)\| = \sqrt{h(\mathbf{x})\cdot h(\mathbf{x})}$. Here the error bound is attained for constant functions.

NOTES: F.J. Hickernell (1998). He also noted that the choice of $K(\mathbf{x},\mathbf{y})$ which satisfies (i) and (ii) uniquely determines W and the accompanying inner product (see G. Wahba (1990)). In (2002) he gave the following two examples of Hilbert spaces W_1 and W_2 of integrands:

- If $u \subset S = \{1, 2, \dots, s\}$ then $|u|$ denotes the cardinality of u,
- $\mathbf{x}_u$ denotes the vector of elements of $\mathbf{x} = (x_1, \dots, x_s)$ indexed by elements of u,
- $[0,1]^u$ denotes the $|u|$–dimensional unit cube,
- γ_i, $i = 1, 2, \dots, s$, are arbitrary positive numbers,
- $\gamma_u = \prod_{i\in u} \gamma_i$,
- $||f||_1 = \sum_{u\subset S} \gamma_u^{-2} \int_{[0,1]^u} \left(\int_{[0,1]^{S\setminus u}} \frac{\partial^{|u|} f}{\partial \mathbf{x}_u} \,\mathrm{d}\mathbf{x}_{S\setminus u}\right)^2 \mathrm{d}\mathbf{x}_u$,
- $||f||_2 = \sum_{u\subset S} \sum_{v\subset u} \gamma_u^{-2}\gamma_v^{-2} \int_{[0,1]^v} \left(\int_{[0,1]^{S\setminus v}} \frac{\partial^{|u|+|v|} f}{\partial \mathbf{x}_u \partial \mathbf{x}_v} \,\mathrm{d}\mathbf{x}_{S\setminus v}\right)^2 \mathrm{d}\mathbf{x}_v$,
- $W_j = \{f \,;\, ||f||_j < \infty\}$,
- $K_j(\mathbf{x},\mathbf{y}) = \prod_{i=1}^s \left(-\frac{(-\gamma_i^2)^j}{(2j)!} B_{2j}\big(\{x_i - y_i\}\big) + \sum_{k=0}^j \frac{\gamma_i^{2k}}{(k!)^2} B_k(x_i) B_k(y_i)\right)$,

where $j = 1, 2$ and $B_k(x)$ denotes the kth Bernoulli polynomial (the so-called Sobolev weighted space).

F.J. HICKERNELL: *Lattice rules: How well do they measure up?* (P. Hellekalek and G. Larcher eds.), in: *Random and quasi-random point sets*, Lecture Notes in Statistics **138**, pp. 109–166, Springer, New York, NY, 1998 (MR1662841 (2000b:65007); Zbl. 0920.65010).
F.J. HICKERNELL: *Obtaining $O(N^{-2+\varepsilon})$ convergence for lattice quadrature rules*, (K.–T. Fang, F.J. Hickernell, H. Niederreiter eds.), in: *Monte Carlo and quasi-Monte Carlo methods 2000. Proceedings of a conference, held at Hong Kong Baptist Univ., Hong Kong SAR, China, November 27 – December 1, 2000*, pp. 274–289, Springer, Berlin, 2002 (MR1958860; Zbl. 1002.65009).
G. WAHBA: *Spline Models for Observational Data*, CBMS–NSF Regional Conference Series in Applied Mathematics, 59, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1990 (MR1045442 (91g:62028); Zbl. 0813.62001).

1.11.13 Non–uniformity

We shall use the following notations:

- a **dyadic interval** is an interval of the $\left[\frac{j}{2^m}, \frac{j+1}{2^m}\right)$,
- a **dyadic box** $B \subset [0,1]^s$ is a Cartesian product of s dyadic intervals,
- $t_1, \dots, t_s$ denote the (new) coordinates in the coordinate system with the origin moved to the center of B,
- B^+ denotes the union of those "quadrants" of B for which $\mathrm{sign}(t_1 \dots t_s) > 0$,
- B^- denotes the union of the elements of the partition of B not belonging to B^+,
- $C_N^{(s)} = \sup_B |A(B^+; N; \mathbf{x}_n) - A(B^-; N; \mathbf{x}_n)|$ is called the s–**dimensional non–uniformity** of the given collection of points $\mathbf{x}_n \in [0,1)^s$, $n = 1, 2, \dots, N$, where the supremum is extended over all possible dyadic boxes $B \subset [0,1]^s$.

(I) Project the given points $\mathbf{x}_1, \dots, \mathbf{x}_N$ orthogonally onto the various k–dimensional faces of $[0,1]^s$ and calculate the k–dimensional non–uniformity $C_N^{(k)}$ of the projected points in the respective face. The **non–uniformity** of $\mathbf{x}_1, \dots, \mathbf{x}_N$ is then defined as

$$\varphi_\infty(N) = \max_{1\le k\le s} C_N^{(k)}.$$

NOTES: (I) This discrepancy was introduced by I.M. Soboľ (1960) (cf. H. Niederreiter (1978, p. 967)). He proved that $\varphi_\infty(N) = o(N)$ characterizes the u.d. of $\mathbf{x}_n$.
(II) Niederreiter (1978, p. 968) noted that $\varphi_\infty(N) \le 2^s N D_N$ for any N points in $[0,1]^s$.
(III) Soboľ (1960) (cf. Soboľ (1969, Ch. 4)) proved that if $f(\mathbf{t}) = f(t_1, \dots, t_s)$ is a function which possesses continuous mixed partial derivative $\frac{\partial^k f}{\partial t_{i_1} \dots \partial t_{i_k}}$ for all $1 \le i_1 < i_2 < \dots < i_k \le s$, and all $1 \le k \le s$, then

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{t})\,\mathrm{d}\mathbf{t} \right| \le c(f) \varphi_\infty(N) \frac{(\log N)^s}{N}.$$

(IV) For a fixed integer q, Soboľ (1957) introduced the quantity

$$\varphi_q(N) = \sup_{(m_1,\dots,m_s)} \left(\sum_{(j_1,\dots,j_s)} |A(B^+; N; \mathbf{x}_n) - A(B^-; N; \mathbf{x}_n)|^q \right)^{1/q},$$

where

$$B = \left[\frac{j_1}{2^{m_1}}, \frac{j_1+1}{2^{m_1}} \right) \times \dots \times \left[\frac{j_s}{2^{m_s}}, \frac{j_s+1}{2^{m_s}} \right).$$

He claims that $\varphi_q(N) = o(N)$ characterizes the u.d. of $\mathbf{x}_n$, and that $\lim_{q\to\infty} \varphi_q(N)$ is hard to compute.
(V) Soboľ (1969, p. 114, Chap. 3) mentioned that the non–uniformity φ_∞ does not satisfy Roth's phenomenon (for the def. cf. 1.9, Note (VI)). In other words there exist infinitely many sequences $x_n \in [0,1)$ such that their every initial segment $x_1, \dots, x_N$ attains the absolute minimum of $\varphi_\infty(N)$, e.g. every $(0,1)$–sequence (cf. 1.8.18).

H. NIEDERREITER: *Quasi–Monte Carlo methods and pseudo–random numbers*, Bull. Amer. Math. Soc. **84** (1978), no. 6, 957–1041 (MR0508447 (80d:65016); Zbl. 0404.65003).
I.M. SOBOĽ: *Multidimensional integrals and the Monte–Carlo method*, (Russian), Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), no. 4, 706–709 (MR0092205 (19,1079b); Zbl. 0091.14601).
I.M. SOBOĽ: *Accurate estimate of the error of multidimensional quadrature formulas for functions of class S_p*, (Russian), Dokl. Akad. Nauk SSSR **132** (1960), 1041–1044cpa (English translation: Soviet Math. Dokl. **1** (1960), 726–729 (MR0138198 (**25** #1645); Zbl. 0122.30702)).
I.M. SOBOĽ: *Multidimensional Quadrature Formulas and Haar Functions*, (Russian), Library of Applied Analysis and Computational Mathematics, Izd. "Nauka", Moscow, 1969 (MR0422968 (**54** #10952); Zbl. 0195.16903).

1.11.14 Partition discrepancy

The notion of non–uniformity from the previous item can be generalized as follows:

• Let $\mathbf{P} = \{\mathbf{X}_t\,;\, t \in T\}$ be a partition of $\mathbf{X} = \bigcup_{t\in T} \mathbf{X}_t$ into disjoint classes $\mathbf{X}_t$ of sets of equal measure in $[0,1]^s$. For the sequence $\mathbf{x}_1,\dots,\mathbf{x}_N$ of points in $[0,1)^s$ define the **partition discrepancy** by

$$D_N^{\mathbf{P}} = \frac{1}{N}\max_{t\in T}\left(\max_{X\in\mathbf{X}_t}\sum_{n=1}^{N} c_X(\mathbf{x}_n) - \min_{X\in\mathbf{X}_t}\sum_{n=1}^{N} c_X(\mathbf{x}_n)\right).$$

We immediately get

$$D_N^{\mathbf{X}} \le D_N^{\mathbf{P}} \le 2D_N^{\mathbf{X}}.$$

Notes: I.M. Soboľ and O.V. Nuzhdin (1991) gave this definition based on the dyadic boxes

$$\mathbf{X}_t = \left\{\prod_{i=1,\dots,s}[(u_i-1)2^{-m_i}, u_i 2^{-mi}]\,;\, u_i = 1,\dots,2^{m_i}, i = 1,\dots,s\right\},$$

where $t = (m_1,\dots,m_s) \in \mathbb{N}_0^s - \{\mathbf{0}\}$ (cf. also Soboľ and B.V. Shukhman (1992)). P.J. Grabner (1992) modified it for general $\mathbf{X}$.

P.J. Grabner: *Metric results on a new notion of discrepancy*, Math. Slovaca **42** (1992), no. 5, 615–619 (MR1202177 (93m:11071); Zbl. 0765.11031).
I.M. Soboľ – O.V. Nuzhdin: *A new measure of irregularity of distribution*, J. Number Theory **39** (1991), no. 3, 367–373 (MR1133562 (93a:11065); Zbl. 0743.11039).
I.M. Soboľ – B.V. Shukhman: *On computational experiments in uniform distribution*, Österreich. Akad. Wiss. Math.–Natur. Kl. Abt. Sitzungsber. II **201** (1992), no. 1–10, 161–167 (MR1237371 (95d:11097); Zbl. 0784.11039).

1.11.15 Abel discrepancy

• [DT, p. 268, 2.2.3.]: Let $\mathbf{x}_n$, $n = 0,1,\dots,$ be a sequence in the s–dimensional unit cube $[0,1)^s$ and put $[\mathbf{x},\mathbf{y}) = [x_1,y_1)\times\cdots\times[x_s,y_s)$. If $0 < r < 1$ then define the s**–dimensional Abel's discrepancy** $D_r(\mathbf{x}_n)$ by

$$D_r(\mathbf{x}_n) = \sup_{[\mathbf{x},\mathbf{y})\subset[0,1]^s}\left|(1-r)\sum_{n=0}^{\infty} c_{[\mathbf{x},\mathbf{y})}(\mathbf{x}_n)r^n - (y_1-x_1)\dots(y_s-x_s)\right|.$$

The Erdős – Turán – Koksma's inequality for Abel discrepancy has the form (cf. [DT, p. 272, Th. 2.65.] and H. Niederreiter (1975, Th. 4)):

Theorem 1.11.15.1. *For an arbitrary positive integer H*

$$D_r(\mathbf{x}_n) \le \left(\frac{3}{2}\right)^s \left(\frac{2}{H+1} + (1-r) \sum_{0<\|\mathbf{h}\|_\infty \le H} \frac{1}{r(\mathbf{h})} \left|\sum_{n=0}^{\infty} e^{2\pi i \mathbf{h}\cdot\mathbf{x}_n} r^n\right|\right).$$

The Koksma – Hlawka inequality has the form (cf. [DT, p. 271, Th. 2.64]):
Theorem 1.11.15.2. *Let $f : [0,1]^s \to \mathbb{R}$ be of bounded variation $V(f)$ in the sense of Hardy and Krause. Then for any sequence $\mathbf{x}_n$, $n = 0,1,2,\dots,$ in $[0,1)^s$ we have*

$$\left|(1-r)\sum_{n=0}^{\infty} f(\mathbf{x}_n) r^n - \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x}\right| \le V(f) D_r(\mathbf{x}_n).$$

H. Niederreiter: *Quantitative versions of a result of Hecke in the theory of uniform distribution mod 1*, Acta Arith. **28** (1975/76), no. 3, 321–339 (MR0389778 (**52** #10609); Zbl. 0318.10037).

1.11.16 Polynomial discrepancy

• If $\mathbf{x}_1,\dots,\mathbf{x}_N$ (where $\mathbf{x}_n = (x_{n,1},\dots,x_{n,s})$) is a finite sequence in the s–dimensional unit cube then the **polynomial discrepancy** P_N is defined by

$$P_N = \sup_{(m_1,\dots,m_s)\in\mathbb{N}^s} \left|\frac{1}{N}\sum_{n=1}^{N} x_{n,1}^{m_1}\dots x_{n,s}^{m_s} - \prod_{j=1}^{s}\frac{1}{m_j+1}\right|.$$

Its relation to the standard discrepancy D_N is given by the inequalities

$$P_N \le D_N \le c_s \frac{1}{|\log P_N|}.$$

Given an $\varepsilon > 0$, there exists an integer N and a set $\mathbf{x}_1,\dots,\mathbf{x}_N$ in $[0,1]^s$ such that $P_N < \varepsilon$ and

$$D_N > c_s^* \frac{1}{|\log P_N|^s}$$

where c_s^* depends only on the dimension s.

Notes: The notion of the multi–dimensional polynomial discrepancy P_N was also introduced by E. Hlawka (1975) (for the one–dimensional case cf. 1.10.4). The double-sided inequality for the extremal discrepancy was proved by R.F. Tichy (1984) and the above lower bound by B. Klinger and R.F. Tichy (1997).

E. Hlawka: *Zur quantitativen Theorie der Gleichverteilung*, Österreich. Akad. Wiss. Math.–Natur. Kl. Sitzungsber. II **184** (1975), 355–365 (MR0422183 (**54** #10175); Zbl. 0336.10049).
B. Klinger – R.F. Tichy: *Polynomial discrepancy of sequences*, J. Comput. Appl. Math. **84** (1997), no. 1, 107–117 (MR1474405 (98j:11058); Zbl. 0916.11045).
R.F. Tichy: *Beiträge zur Polynomdiskrepanz*, Österreich. Akad. Wiss. Math.–Natur. Kl. Sitzungsber. II **193** (1984), no. 8–10, 513–519 (MR0817922 (87g:11091); Zbl. 0564.10052).

1.11.17 Dispersion

See H. Niederreiter (1992, p. 147–159, Chap. 6) and [DT, p. 11–13]:

• If $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N$ belong to $[0,1]^s$ then the **dispersion** d_N of $\mathbf{x}_n$'s in $[0,1]^s$ is defined by

$$d_N = d_N(\mathbf{x}_1, \dots, \mathbf{x}_N) = \sup_{\mathbf{x}\in[0,1]^s} \min_{1\le n\le N} |\mathbf{x} - \mathbf{x}_n|,$$

where $|\mathbf{x} - \mathbf{x}_n|$ is the Euclidean distance. The dispersion based on the maximum distance $\|\mathbf{x} - \mathbf{x}_n\|_\infty$ will be denoted by d_N^∞. We immediately have

$$d_N^\infty \le d_N \le \sqrt{s}\, d_N^\infty,$$

and for an arbitrary finite sequence $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N$ of points in $[0,1]^s$ with the extremal discrepancy D_N we have

- $d_N \le \sqrt{s}\, D_N^{1/s}$ (Niederreiter (1983, Th. 3)),
- $d_N^\infty \le \frac{1}{2}\, D_N^{1/s}$ (see Niederreiter (1992, p. 152, Th. 6.6) and [DT, p. 12, Th. 1.17]),
- $d_N^\infty \ge \frac{1}{2[N^{1/s}]}$, and this bound is sharp, because for every N and s there exists a sequence $\mathbf{x}_n$, $n = 1, 2, \dots, N$, in $[0,1]^s$ such that $d_N^\infty = \frac{1}{2[N^{1/s}]}$ (see Niederreiter (1985, Th. 1; 1992, p. 154, Th. 6.8) and [DT, p. 12, Remark 5]).
- For every dimension s, there is an infinite sequence $\mathbf{x}_n$, $n = 1, 2, \dots$, in $[0,1]^s$ such that
$$d_N^\infty = \mathcal{O}(N^{-1/s})$$
and the order of magnitude of the error is the best possible. Sequences $\mathbf{x}_n$ fulfilling this condition are called **low–dispersion sequences** (see Niederreiter (1984; 1985; 1986; 1992, p. 154, Th. 6.8), also cf. 3.19).
- [DT, p. 12, Th. 1.16]: The infinite sequence $\mathbf{x}_n$, $n = 1, 2, \dots$, in $[0,1]^s$ is dense in $[0,1]^s$ if and only if
$$\lim_{N\to\infty} d_N = 0.$$
- G. Larcher and H. Niederreiter (1993) showed that the dispersion of infinite sequences $\mathbf{x}_n$, $n = 1, 2, \dots$, satisfies the lower estimate
$$\limsup_{N\to\infty} N^{\frac{1}{s}} d_N \ge \frac{1}{2}\left(\frac{s-1}{s(2^{(s-1)/s} - 1)}\right)^{1/s}.$$

• Niederreiter (1984; 1992, p. 153, Th. 6.7) proved that for an infinite sequence $\mathbf{x}_n$ in $[0,1]^s$ we have

$$\limsup_{N\to\infty} N d_N \geq \frac{1}{\log 4}$$

and that (1985; 1992, p. 155, Th. 6.9) there exists a sequence $\mathbf{x}_n$ such that

$$\lim_{N\to\infty} N^{1/s} d_N^{\infty} = \frac{1}{\log 4}.$$

Notes: For the one-dimensional variant of the dispersion see 1.10.11.
(I) Niederreiter (1983; 1992, p. 148) proposed a quasi–Monte Carlo method for the approximate evaluation of the extremes of continuous functions (called **quasirandom search** or **crude search**) and showed that

$$m_N \leq \sup_{\mathbf{x}\in[0,1]^s} f(\mathbf{x}) \leq m_N + \lambda_f(d_N),$$

and

$$m_N \leq \sup_{\mathbf{x}\in[0,1]^s} f(\mathbf{x}) \leq m_N + \lambda_f^{\infty}(d_N^{\infty}),$$

where $m_1 = f(\mathbf{x}_1)$ and

$$m_{n+1} = \begin{cases} m_n, & \text{if } f(\mathbf{x}_{n+1}) \leq m_n, \\ f(\mathbf{x}_{n+1}), & \text{if } f(\mathbf{x}_{n+1}) > m_n. \end{cases}$$

Here $\lambda_f(t)$ and $\lambda_f^{\infty}(t)$ denote the moduli of the continuity of f (cf. p. 1 – 74). A refinement of the crude search was proposed by Niederreiter and P. Peart (1986).
(II) J.P. Lambert (1988) describes a recursive method for the generation of points of a low–dispersion sequence in the unit square.
(III) An explicit formula for the dispersion of two–dimensional g.l.p. sequences is given in G. Larcher (1986), see 3.15.2(V).
(IV) Niederreiter (1992, p. 155) notes that the problem of determining the minimal value of d_N for a fixed N is equivalent to a difficult geometric problem of finding the most economical covering of $\mathbb{R}^s$ by balls of equal radius in the Euclidean metric. This problem has been solved only for $s = 1, 2$.

J.P. Lambert: *A sequence well dispersed in the unit square*, Proc. Amer. Math. Soc. **103** (1988), no. 2, 383–388 (MR0943050 (89i:11090); Zbl. 0655.10055).
G. Larcher: *The dispersion of a special sequence*, Arch. Math. (Basel) **47** (1986), no. 4, 347–352 (MR 88k:11044; Zbl. 584.10031).
G. Larcher – H. Niederreiter: *A lower bound for the dispersion of multidimensional sequences*, in: *Analytic number theory and related topics (Tokyo, 1991)*, World Sci. Publishing, River Edge, NJ, 1993, pp. 81–85 (MR1342309 (96e:11097); Zbl. 0978.11035).

H. Niederreiter: *A quasi–Monte Carlo method for the approximate computation of the extreme values of a function*, (P.Erdős – L.Álpár – G.Halász – A.Sárkőzy eds.), in: *To the memory of Paul Turán*, Studies in pure mathematics, Birkäuser Verlag & Akadémiai Kiadó, Basel, Boston, Stuttgart & Budapest, 1983, pp. 523–529 (MR0820248 (86m:11055); Zbl. 0527.65041).
H. Niederreiter: *On a measure of denseness for sequences*, in: *Topics in classical number theory, Vol. I, II (Budapest 1981)*, (G.Halász ed.), Colloq. Math. Soc. János Bolyai, Vol. 34, North–Holland Publishing Co., Amsterdam, New York, 1984, pp. 1163–1208 (MR0781180 (86h:11058); Zbl. 0547.10045).
H. Niederreiter: *Quasi–Monte Carlo methods for global optimization*, in: *Proc. Fourth Pannonian Symp. on Math. Statistics (Bad Tatzmannsdorf, 1983)*, Riedel, Dordrecht, 1985, pp. 251–267 (MR0851058 (87m:90092); Zbl. 0603.65043).
H. Niederreiter: *Good lattice points for quasirandom search methods*, in: *System Modelling and Optimization*, (A. Prékopa, J. Szelezsán, and B. Strazicky eds.), Lecture Notes in Control and Information Sciences, Vol. 84, Springer, Berlin, 1986, pp. 647–654 (MR0903508 (89f:11112); Zbl. 0619.90066).
H. Niederreiter: *Random Number Generation and Quasi–Monte Carlo Methods*, CBMS–NSF Regional Conference Series in Applied Mathematics, 63, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992, vi+241 pp. (MR1172997 (93h:65008); Zbl. 0761.65002).
H. Niederreiter – P. Peart: *Localization of search in quasi–Monte Carlo methods for global optimization*, SIAM J. Sci. Statist. Comput.) **7** (1986), no. 2, 660–664 (MR0833928 (87h:65017); Zbl. 0613.65067).

1.11.18 Spectral test

• Let $\mathbf{x}_n$ be a sequence of points from $[0,1)^s$. The **spectral test** $\sigma_N(\mathbf{x}_n)$ of its first N elements is given by the quantity

$$\sigma_N(\mathbf{x}_n) = \sup_{\mathbf{k}\in\mathbb{Z}^s,\mathbf{k}\neq\mathbf{0}} \frac{1}{|\mathbf{k}|}\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i\mathbf{k}\cdot\mathbf{x}_n}\right|,$$

where $|\mathbf{k}| = \sqrt{k_1^2+\cdots+k_s^2}$ denotes the Euclidean norm of $\mathbf{k} = (k_1,\dots,k_s)$. Then $\mathbf{x}_n$ is u.d. in $[0,1)^s$ if and only if

$$\lim_{N\to\infty} \sigma_N(\mathbf{x}_n) = 0.$$

P. Hellekalek: *On the assessment of random and quasi–random point sets*, in: *Random and Quasi–Random Point Sets*, (P. Hellekalek, G. Larcher eds.), Lecture Notes in Statistics, 138, Springer Verlag, New York, Berlin, 1998, pp. 49–108 (MR1662840 (2000c:11127); Zbl. 0937.65004).

1.12 Quasi–Monte Carlo applications

• E. Hlawka (1998) discusses the following modelling problems: Mendel's laws from genetics; entropy; Bell's inequality (in quantum physics); Bayesian statistics; regression; random flight; a model for light; the Coulomb gas; average length of a molecule; model of turbulence.

• M. Drmota and R.F. Tichy in [DT, pp. 368–432, Chap. 3] discusses: numerical integration in mathematical finance; average case analysis; spherical designs and Chebyshev quadrature; slice dispersion and polygonal approximation of curves; the heat equation; the Boltzmann equation.

• In [Monte Carlo and Quasi–Monte Carlo Methods 2000] (2002) are discussed: finance and insurance; experimental design; control variates; simulation of diffusion; Markov chain simulation in statistical physics; evaluation of the Asian basket option; transport problems; computing extremal eigenvalues; American option pricing; non–linear time series; etc.

• In [Quasi–Monte Carlo Methods in Finance and Insurance] (2002) are discussed: strategies for pricing Asian options; value at risk; simulation of generalized ruin model; differential equations with multiple delayed arguments.

• N.M. Korobov (1963, p. 190–213) applied the theory of good lattice points (see 3.15) to approximations of solutions of Fredholm integral equations of the second type.

• Hua Loo Keng and Wang Yuan (1981, pp. 159–224, Chap. 8–10) show the applications to: estimations of numerical errors for quadrature formulas; interpolation of functions by polynomials; approximate solutions of Fredholm integral equations of the second type; Volterra equation; eigenvalues; Cauchy and Dirichlet problem of partial differential equations.

• R. F. Tichy (1990) used quasi–Monte Carlo method to compare three different types of sequences (good lattice points sequences, cf. 3.15.1, practical lattice points. see 3.15.1 Note(X), and lattice rules, see 3.17) in order to find an approximate solution of a special class of partial differential equation.

• S. Tezuka (1998) surveys applications in financial mathematics.

• A. Keller (1998) used a low discrepancy sequence for quasi–Monte Carlo integration of the Neumann series, where it is applied to global illumination problem.

• O. Strauch (2003) computed the a.d.f. for scalar product $\mathbf{x}_n \cdot \mathbf{y}_n$ and applied it to a modified one–time pad cipher, see 2.3.24.

• M. Drmota (1988) investigated a robust control system by using the so–called practical lattice points (cf. 3.15.1 Note (X)).

M. Drmota: *Such– und Prüfprozesse mit praktischen Gitterpunkten*, Anz. Österreich. Akad. Wiss. Math.–Natur. Kl. **125** (1988), 23–28 (MR1003653 (90m:11113); Zbl. 0825.11005).

E. Hlawka: *Statistik und Gleichverteilung*, Grazer Math. Ber. **335** (1998), ii+206 pp (MR1638218 (99g:11093); Zbl. 0901.11027).

L.–K. Hua – Y. Wang: *Applications of Number Theory to Numerical Analysis*, Springer Verlag & Science Press, Berlin, Heidelberg, New York, Beijing, 1981 (MR0617192 (83g:10034); Zbl. 0465.10045). (Chinese edition: Science Press, Beijing, 1978 (MR0617192 (83g:10034); Zbl. 0451.10001)).

A. Keller: *The quasi–random walk*, in: *Monte Carlo and Quasi–Monte Carlo Methods 1996 (Proceedings of a conference at the University of Salzburg, Austria, July 9-12, 1996)*, (H. Niederreiter, P. Hellekalek, G. Larcher, P. Zinterhof eds.), Lecture Notes in Statistics, 127, Springer Verlag, New York, Berlin, 1998, pp. 277–299 (MR1644526 (99d:65368); Zbl 0885.65150).

N.M. Korobov: *Number–theoretic Methods in Approximate Analysis*, (Russian), Library of Applicable Analysis and Computable Mathematics, Fizmatgiz, Moscow, 1963 (MR0157483 (**28** #716); Zbl. 0115.11703).
Monte Carlo and Quasi–Monte Carlo Methods 2000 (Proceedings of a Conference held at Hong Kong Baptist University, Hong Kong SAR, China, Nov. 27–Dec. 1, 2000, (Kai–Tai Fang, F.J. Hickernell, H. Niederreiter eds.), Springer Verlag, New York, Berlin, 2002 (MR1958842 (2003i:65006); Zbl. 0980.00040).
Quasi–Monte Carlo Methods in Finance and Insurance, (R. Tichy ed.), Grazer Math. Ber., 345, 2002, 129 pp. (MR1985927 (2004a:62012); Zbl. 1006.00021).
O. Strauch: *On distribution functions of sequences generated by scalar and mixed product*, Math. Slovaca **53** (2003), no. 5, 467–478 (MR2038514 (2005d:11108); Zbl. 1061.11042).
R.F. Tichy: *Random points in the cube and on the sphere with applications to numerical analysis*, J. Comput. Appl. Math. **31** (1990), no. 1, 191–197 (MR1068159 (91j:65009); Zbl. 0705.65003).
S. Tezuka: *Financial applications of Monte–Carlo and quasi–Monte Carlo methods*, in: *Random and Quasi–Random Point Sets*, (P. Hellekalek, G. Larcher eds.), Lecture Notes in Statistics, 138, Springer Verlag, New York, Berlin, 1998, pp. 303–332 (MR1662845; Zbl. 0928.91023).

2. One–dimensional sequences

2.1 Criteria for asymptotic distribution functions

2.1.1. Weyl's limit relation. The sequence $x_n \bmod 1$ is u.d. if and only if for every continuous function $f : [0,1] \to \mathbb{R}$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{x_n\}) = \int_0^1 f(x)\,\mathrm{d}x.$$

NOTES: (I) (H. Weyl (1916), cf. [KN, p. 2, Th. 1.1]): The maximal class of the applicable functions f is the class of the Riemann integrable functions, cf. J.F. Koksma and R. Salem (1950), N.G. de Bruijn and K.A. Post (1968), or Ch. Binder (1971). On the other hand, the same conclusion follows if f is restricted to some proper subclasses of the class of continuous functions as

(a) the set of all polynomials, or even the set of polynomials of the form x^h with $h = 1, 2, \dots$,

(b) the set of the all trigonometric polynomials, or simply the set of exponentials $e^{2\pi i h x}$ with $h = \pm 1, \pm 2, \dots$ (cf. 2.1.2),

(c) the set of periodic Bernoulli polynomials $B_h(x)$, $h = 1, 2, \dots$.

(II) B.G. Sloss and W.F. Blyth (1993) replaced the class of continuous functions by the class of **Walsh's functions** $w_h(x)$, $h = 1, 2, \dots$, which are orthogonal in $[0,1]$. These are defined for $h = \sum_{k=0}^{\infty} a_k 2^k$ by $w_h(x) = \prod_{k=0}^{\infty} (r_k(x))^{a_k}$, where $r_k(x) = r(2^k x)$ and

$$r(x) = \begin{cases} 1, & \text{if } x \in [0, 1/2), \\ -1, & \text{if } x \in [1/2, 1]. \end{cases}$$

Generally, let

- $k = k_0 + k_1 b + k_2 b^2 + \dots + k_n b^n$, $k_n \neq 0$, be a b-adic expression of an integer k,
- $x = \frac{x_0}{b} + \frac{x_1}{b^2} + \dots$ be a b-adic representation of $x \in [0,1)$. Then
- $\mathrm{wal}_k(x) = e^{\frac{2\pi i}{b}(k_0 x_0 + k_1 x_1 + \dots + k_n x_n)}$, $\mathrm{wal}_k : [0,1] \to \mathbb{C}$, is the k-th Walsh function in the base b, and
- $\mathrm{wal}_{\mathbf{k}}(\mathbf{x}) = \prod_{i=1}^{s} \mathrm{wal}_{k_i}(x_i)$, where
- $\mathbf{k} = (k_1, k_2, \dots, k_s)$ is a vector with nonnegative integer coordinates.

(III) J. Horbowicz (1981) reduced the length of interval of the integration: Let f be Riemann integrable and assume that the set $\{x \in [0,1]; f(x) = 0\}$ has zero Lebesgue measure. Then the sequence x_n mod 1 is u.d. if and only if for every subinterval $[\alpha,\beta)\subset[0,1]$ we have

$$\lim_{n\to\infty} N^{-1}\sum_{n=1}^{N} f(x_n)c_{[\alpha,\beta)}(x_n)=\int_\alpha^\beta f(x)\,\mathrm{d}x.$$

If $f(x)=x$ this gives the criterion proved by Pólya and Szegő (1964, Aufg. 163). The condition "zero Lebesgue measure" can be replaced by "zero Jordan measure", cf. T. Šalát (1987).
(IV) Let $f:[0,1]\to\mathbb{R}$ be a bounded function and X be the set of all limit points of the integral sums $\sum_{n=1}^{N} f(t_n)(y_n-y_{n-1})$, where $t_n\in[y_{n-1},y_n]$ and the diameter of the partition $0=y_0<y_1<\cdots<y_N=1$ tends to 0 (i.e. X is the set of all Riemann integrals of the function f over $[0,1]$). S. Salvati and A. Volčič (2001) proved that for every non–empty compact and connected set $C\subset X$ there exists a u.d. sequence x_n in $[0,1)$ such that the set of all limit points of

$$\frac{1}{N}\sum_{n=1}^{N} f(x_n),\quad N=1,2,\dots,$$

coincides with C.

Ch. Binder: *Über einen Satz von de Bruijn und Post*, Österreich. Akad. Wiss. Nath.–Natur. Kl. Sitzungsber. II **179** (1971), 233–251 (MR0296224 (**45** #5285); Zbl. 0262.26010).
N.G. de Bruijn – K.A. Post: *A remark on uniformly distributed sequences and Riemann integrability*, Nederl. Akad. Wetensch. Proc. Ser. A 71 **30** (1968), 149–150 (MR0225946 (**37** #1536); Zbl. 0169.38401). (=Indag. Math. **30** (1968), 149–150).
J. Horbowicz: *Criteria for uniform distribution*, Nederl. Akad. Wetensch. Indag. Math. **43** (1981), no. 3, 301–307 (MR0632169 (82k:10068); Zbl. 0465.10039).
J.F. Koksma – R. Salem: *Uniform distribution and Lebesgue integration*, Acta Sci. Math. (Szeged) **12B** (1950), 87–96 (MR0032000 (11,239b); Zbl. 0036.03101).
G. Pólya – G. Szegő: *Aufgaben und Lehrsätze aus der Analysis*, Vol. 1 & 2, 3rd corr. ed., Grundlehren d. math. Wiss., Band 19, Springer Verlag, Berlin, Göttingen, Heidelberg, New York, 1964 (MR0170985 (**30** #1219a); MR0170986 (**30** #1219b); Zbl. 0122.29704).
T. Šalát: *Criterion for uniform distribution of sequences and a class of Riemann integrable functions*, Math. Slovaca **37** (1987), no. 2, 199–203 (MR0899436 (88j:11042); Zbl. 0673.10038).
S. Salvati – A. Volčič: *A quantitative version of a de Bruijn – Post theorem*, Math. Nachr. **229** (2001), 161–173 (MR1855160 (2002g:11108); Zbl. 0991.11045).
B.G. Sloss – W.F. Blyth: *Walsh functions and uniform distribution* mod 1, Tôhoku Math. J. (2) **45** (1993), no. 4, 555–563 (MR1245722 (94k:11087); Zbl. 0799.11020).
H. Weyl: *Über die Gleichverteilung von Zahlen mod. Eins*, Math. Ann. **77** (1916), 313–352 (JFM 46.0278.06).

2.1.2. Weyl's criterion. The sequence x_n mod 1 is u.d. if and only if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h x_n}=0\quad\text{for all integers } h\neq 0.$$

NOTES: H. Weyl (1916), cf. [KN, p. 7, Th. 2.1]. It is sufficient to consider only the values $h = 1, 2, \dots$.

H. WEYL: *Über die Gleichverteilung von Zahlen mod. Eins*, Math. Ann. **77** (1916), 313–352 (JFM 46.0278.06).

2.1.3. $\mathbf{L^2}$ discrepancy criterion. The sequence x_n in $[0, 1)$ is u.d. if and only if

$$\lim_{N\to\infty}\left(\frac{1}{3}+\frac{1}{N}\sum_{n=1}^{N}x_n^2-\frac{1}{N}\sum_{n=1}^{N}x_n-\frac{1}{2N^2}\sum_{m,n=1}^{N}|x_m-x_n|\right)=0,$$

or equivalently, if and only if
(i) $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}x_n=\frac{1}{2}$, and
(ii) $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}x_n^2=\frac{1}{3}$, and
(iii) $\lim_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^{N}|x_m-x_n|=\frac{1}{3}$.
NOTES: [KN, p. 145, Th. 5.3]

2.1.4.
(I) The sequence x_n mod 1 has the given a.d.f. $g(x)$ if and only if for every continuous function $f : [0, 1] \to \mathbb{R}$ we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}f(\{x_n\})=\int_0^1 f(x)\,\mathrm{d}g(x),$$

and x_n mod 1 has the a.d.f. if and only if the limit on the left hand side exists for every continuous f. Note that it is sufficient to take the polynomials $x, x^2, x^3, \dots$ for $f(x)$.
(II) In order that x_n mod 1 has the a.d.f., it is both necessary and sufficient that (1) the limit

$$\beta_k=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i k x_n}$$

exists for every integer k. This a.d.f. will then be continuous if and only if (2)

$$\liminf_{N\to\infty}\frac{1}{2N+1}\sum_{k=-N}^{N}|\beta_k|^2=0,$$

and absolutely continuous with the derivative belonging to $L^2(0,1)$ if and only if (3)

$$\sum_{k=-\infty}^{\infty} |\beta_k|^2 < \infty.$$

(III) Let $g(x)$ be continuous at $x = 0$ and $x = 1$. Then the sequence x_n mod 1 has a.d.f. $g(x)$ if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} = \int_0^1 e^{2\pi i h x} \, \mathrm{d}g(x) \quad \text{for all integers } h \neq 0.$$

(IV) Given a d.f. $g(x)$, the sequence x_n mod 1 has a.d.f. $g(x)$ if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \frac{h}{2}\{x_n\}} = \int_0^1 e^{2\pi i \frac{h}{2} x} \, \mathrm{d}g(x) \quad \text{for all integers } h \neq 0.$$

Notes: (I) This is a modification to the Weyl's limit relation. The second Helly theorem 4.1.4.13 $\left(\text{saying that } \frac{1}{N}\sum_{n=1}^{N} f(\{x_n\}) = \int_0^1 f(x)\,\mathrm{d}F_N(x) \to \int_0^1 f(x)\,\mathrm{d}g(x)\right)$ implies the necessary condition in (I). The sufficiency follows from the first Helly theorem 4.1.4.12.
The reduction to $f(x) = x, x^2, x^3, \dots$ is clear, it follows, for instance, from the approximation of $f(x)$ by Bernstein polynomial of degree n

$$B_n(x; f) = \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right) \tbinom{n}{k} x^k (1-x)^{n-k}.$$

Let us mention the Hausdorff moment problem here: Let $s_0 = 1, s_2, s_3, \dots$ be a given sequence in $[0,1]$. Then there exists a d.f. $g(x)$ such that

$$s_n = \int_0^1 x^n \, \mathrm{d}g(x), \quad n = 0, 1, 2, \dots,$$

if and only if

$$\sum_{i=0}^{m} (-1)^i \tbinom{m}{i} s_{i+k} \geq 0 \quad \text{for } m, k = 0, 1, 2, \dots,$$

and the solution function $g(x)$ is unique (cf. N.I. Achyeser (1961), J.A. Shohat and J.D. Tamarkin (1943)).
(II) The assertion (2) involving the continuity of the a.d.f. is due to N. Wiener (1924) and I.J. Schoenberg (1928), and is called **Wiener – Schoenberg theorem** (cf. [KN, p. 55, Th. 7.5]). The case (3) involving the absolute continuity is due to R.E. Edwards (1967), cf. P.D.T.A. Elliott (1979, Vol. 1, p. 67, Lemma 1.46). Note that the conditions in (II) are equivalent to the following ones: (1) the coefficients β_k exist for $k = 1, 2, \dots$, (2) $\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} |\beta_k|^2 = 0$, and (3) $\sum_{k=1}^{\infty} |\beta_k|^2 < \infty$.
(III) The Weyl criterion cannot be modified immediately, for instance because $c_0(x)$,

$c_1(x)$, $h_\beta(x)$ cannot be distinguished by $e^{2\pi i h x}$ with $h = 0, \pm1, \pm2, \dots$.
(IV) Note that every continuous $f : [0,1] \to \mathbb{R}$ can be approximated by polynomials in $e^{2\pi i \frac{h}{2} x}$ with $h = 0, \pm1, \pm2, \dots$ (cf. O. Strauch (1999, p. 34, Th. 1,2)).

N.I. Achyeser (Achieser): *The Classical Problem of Moments*, (Russian), Gos. Izd. Fiz. – Mat. Literatury, Moscow, 1961.
S. Bernstein: *On the Best Approximation of Continuous Functions by Polynomials of a Given Degree*, (Russian), Charkov, 1912 (JFM 43.0493.01).
R.E. Edwards: *Fourier Series. A Modern Introduction*, Vol. I, Holt, Rinehart and Winston, Inc., New York, Toronto, London, 1967 (MR0216227 (**35** #7062); Zbl. 0152.25902).
P.D.T.A. Elliott: *Probabilistic Number Theory I. Mean–value Theorems*, Grundlehren der mathematischen Wissenschaften, Vol. 239, Springer Verlag, New York, Heidelberg, Berlin, 1979 (MR0551361 (82h:10002a); Zbl. 0431.10029).
F. Hausdorff: *Momentprobleme für ein endliches Interval*, Math. Zeitschr. **16** (1923), 220–248 (MR1544592; JFM 49.0193.01).
I.J. Schoenberg: *Über die asymptotische Verteilung reeller Zahlen* mod 1, Math. Z. **28** (1928), 171–199 (MR1544950; JFM 54.0212.02).
J.A. Shohat – J.D. Tamarkin: *The Problem of Moments*, Mathematical Surveys, Vol. 1, Amer. Math. Soc., Providence, Rhode Island, 1943 (MR0008438 (5,5c) ; Zbl. 0063.06973).
O. Strauch: *Distribution of Sequences* (in Slovak), Mathematical Institute of the Slovak Academy of Sciences, DSc Thesis, Bratislava, Slovakia, 1999.
N. Wiener: *The quadratic variation of a function and its Fourier coefficients*, J. Math. Phys. **3** (1924), 72–94 (JFM 50.0203.01).

2.1.5. $\mathbf{L^2}$ discrepancy criterion. The sequence x_n in $[0,1)$ has the a.d.f. $g(x)$ if and only if

$$\lim_{N\to\infty}\left(1+\int_0^1 g^2(x)\,\mathrm{d}x - 2\int_0^1 g(x)\,\mathrm{d}x + \frac{2}{N}\sum_{n=1}^N\int_0^{x_n} g(x)\,\mathrm{d}x - \right.$$
$$\left. - \frac{1}{N}\sum_{n=1}^N x_n - \frac{1}{2N^2}\sum_{m,n=1}^N |x_m - x_n|\right) = 0,$$

or equivalently, if and only if
(i) $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N x_n = \int_0^1 x\,\mathrm{d}g(x)$,
(ii) $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N\int_0^{x_n} g(x)\,\mathrm{d}x = \int_0^1\left(\int_0^x g(t)\,\mathrm{d}t\right)\mathrm{d}g(x)$,
(iii) $\lim_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^N |x_m - x_n| = \int_0^1\int_0^1 |x-y|\,\mathrm{d}g(x)\,\mathrm{d}g(y)$.
Notes: O. Strauch (1994, p. 176, Th. 1). This is also true if $x_n \in [0,1]$.

O. Strauch: *A new moment problem of distribution functions in the unit interval*, Math. Slovaca **44** (1994), no. 2, 171–211 (MR1282534 (95i:11082); Zbl. 0799.11023).

$\boxed{\textbf{2.1.5.1}}$ The sequence x_n in $[0,1)$ has an a.d.f. if and only if

$$\lim_{M,N\to\infty}\left(\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N}|x_m-x_n| - \frac{1}{2M^2}\sum_{m,n=1}^{M}|x_m-x_n| - \frac{1}{2N^2}\sum_{m,n=1}^{N}|x_m-x_n|\right)=0.$$

O. Strauch – J.T. Tóth: *Distribution functions of ratio sequences*, Publ. Math. (Debrecen) **58** (2001), 751–778 (MR1828725 (2002h:11068); Zbl. 0980.11031).

2.1.6. The sequences

$$x_n \bmod 1,\ n=1,2,\dots \quad \text{and} \quad (x_m-x_n) \bmod 1,\ m,n=1,2,\dots,$$

are simultaneously

u.d.

and for their discrepancies we have

$$D_N \le c\sqrt{D_{N^2}}(1+|\log D_{N^2}|), \quad \text{with an absolute constant } c.$$

Here D_N denotes the discrepancy of x_n mod 1, while D_{N^2} stands for the discrepancy of x_m-x_n mod 1, where the sequence x_m-x_n, $m,n=1,2,\dots,$ is ordered in such a way, that the first N^2 terms are x_m-x_n for $m,n=1,2,\dots,N$.

Notes: J.W.S. Cassels (1953), cf. [KN, p. 163, Th. 6.1].

(I) I.M. Vinogradov (1926) proved that $D_N < C\sqrt[3]{D_{N^2}}$. His proof is based on the case $k=n$ of the following identity, called Vinogradov's one by J.G. van der Corput and Ch. Pisot (1939, p. 478), who extended the original Vinogradov's result to the form: Let $k\ge 2$ be an integer such that the numbers $kx_1,\dots,kx_N$ are integers. If

$$R_N(\alpha,\beta)=A([\alpha,\beta) \bmod 1; N; x_n \bmod 1)-N(\beta-\alpha),$$
$$R^*_{N^2}(\alpha,\beta)=A([\alpha,\beta) \bmod 1; N^2; (x_m-x_n) \bmod 1)-N^2(\beta-\alpha).$$

then for every integer t we have

$$\sum_{h=0}^{k-1} R_N^2\left(\frac{h}{k},\frac{h+t}{k}\right)=\sum_{\ell=0}^{\tau-1} R^*_{N^2}\left(\frac{-\ell}{k},\frac{\ell+1}{k}\right)$$

where τ stands for the distance of t to the nearest integral multiple of k (with the convention that the empty sum vanishes). An account on Vinogradov's method can

also be found in A.O. Gel'fond and Yu.V. Linnik (1966, Ch. 7, § 2).
(II) Another proof of Vinogradov's result was given by Cassels (1950), who simultaneously specified the constant $c = 12$.
(III) Vinogradov's method and his result was further strengthened by van der Corput who proved (cf. J.F. Koksma (1936, p. 95)) that

$$D_N \leq 2^{5+\sqrt{2|\log D_{N^2}|}}\sqrt{D_{N^2}}.$$

J.G. van der Corput and C. Pisot (1939)) proved later

$$D_N < 2^{\frac{7}{2}+\sqrt{\frac{|\log D_{N^2}|}{\log 2}}}\sqrt{D_{N^2}}\,,$$

or that

$$D_N \leq 2^{\frac{7}{2}+\frac{1}{4\varepsilon}} D_{N^2}^{\frac{1}{2}-\varepsilon} \quad \text{for every } \varepsilon > 0.$$

This gives Vinogradov's result for $\varepsilon = 1/6$ and also the value $C = 2^6$.
(IV) If $k \to \infty$ then J.G. van der Corput and C. Pisot (1939, p. 478) deduced from (I) that

$$\int_0^1 R_N^2(\alpha, \alpha + t)\,\mathrm{d}\alpha = \int_0^\mu R_{N^2}^*(-\alpha, \alpha)\,\mathrm{d}\alpha,$$

where μ the distance of t to the nearest integer. (For another form of this relation cf. [KN, p. 166, Th. 6.3]).

J.W.S. Cassels: *A theorem of Vinogradoff on uniform distribution*, Proc. Cambridge Phil. Soc. **46** (1950), 642–644 (MR0045166 (13,539c); Zbl. 0038.19101).
J.W.S. Cassels: *A new inequality with application to the theory of diophantine approximation*, Math. Ann. **162** (1953), 108–118 (MR0057922 (15,293a); Zbl. 0051.28604).
A.O. Gel'fond – Yu.V. Linnik: *Elementary Methods in Analytic Number Theory*, International Series of Monographs on Pure and Applied Mathematics. 92, Pergamon Press, Oxford, New York, Toronto, 1966 (Russian original: Moscow, 1962 (MR0188134 (**32** #5575a); Zbl. 0111.04803); French translation: Gauthier – Villars, Paris 1965 (MR0188136 (**32** #5576); Zbl. 0125.29604); English translation also published by Rand McNally & Co., Chicago 1965 (MR0188135 (**32** #5575b)); Zbl. 0142.01403).
J.F. Koksma: *Diophantische Approximationen*, Ergebnisse der Mathematik und Ihrer Grenzgebiete, Vol. 4, Julius Springer, Berlin, 1936 (Zbl. 0012.39602; JFM 62.0173.01).
J.G. van der Corput – C. Pisot: *Sur la discrépance modulo un. (Première communication)*, Proc. Akad. Wet. Amsterdam **42** (1939), 476–486 (JFM 65.0170.02; Zbl. 0021.29701). (=Indag. Math. **1** (1939), 143–153).
I.M. Vinogradov: *On fractional parts of integer polynomials*, (Russian), Izv. AN SSSR **20** (1926), 585–600 (JFM 52.0182.03).

2.1.7. The sequence

$$x_n \in [0, 1), \quad n = 1, 2, \dots,$$

is u.d. if and only if the sequence

$$|x_m - x_n|, \quad m, n = 1, 2, \dots,$$

has the a.d.f.

$$g(x) = 2x - x^2.$$

Here the double sequence $|x_m - x_n|$, for $m, n = 1, 2, \dots$, is ordered to an ordinary sequence y_n in such a way that the first N^2 terms of y_n are $|x_m - x_n|$ for $m, n = 1, 2, \dots, N$.

If $D_N^{(2)}$ denotes the L^2 discrepancy of $x_1, \dots, x_N$ with respect to $g(x) = x$ and $D_{N^2}^{(2)}$ denotes the L^2 discrepancy of $|x_m - x_n|$ for $m, n = 1, 2, \dots, N$, with respect to $g(x) = 2x - x^2$, then

$$12(D_N^{(2)})^2 \le D_{N^2}^{(2)} \le 12 D_N^{(2)}.$$

NOTES: In the first inequality it is assumed that for every $1 \le m \le N$ there exists an n, $1 \le n \le N$, such that $x_n = 1 - x_m$. This criterion is also true for $x_n \in [0, 1]$.

O. STRAUCH: *On the L^2 discrepancy of distances of points from a finite sequence*, Math. Slovaca **40** (1990), 245–259 (MR1094777 (92c:11078); Zbl. 0755.11022).

2.2 Sufficient or necessary conditions for a.d.f.'s

2.2.1. van der Corput difference theorem. If the sequence of differences

$$(x_{n+h} - x_n) \bmod 1, \quad n = 1, 2, \dots,$$

is u.d. for every $h = 1, 2, \dots$, then the original sequence

$$x_n \bmod 1, \quad n = 1, 2, \dots,$$

is also

u.d.

NOTES: (I) J. G. van der Corput (1931). Several authors noticed that the assumption on h can be weakened by restricting the range of h. E. Hlawka (1984, p. 31) calls van der Corput difference theorem the **main theorem of the theory of u.d.**
(II) T. Kamae and M. Mendès France (1978) and M. Mendès France (1978) called a set H of positive integers a **van der Corput** (abbreviated vdC) one if the u.d. of $x_{n+h} - x_n \bmod 1$ for all $h \in H$ implies that also $x_n \bmod 1$ itself is u.d. For instance:
(i) $H = \{[n\alpha] \,;\, n \in \mathbb{N}\}$ is vdC for all $\alpha \ge 1$.

(ii) $H \subset \mathbb{N}$ with asymptotic density 1 is vdC.
(iii) $H = \{p-1\,;\,p \text{ prime}\}$ and $H = \{p+1\,;\,p \text{ prime}\}$ are vdC, but $H = \{p+k\,;\,p \text{ prime}\}$, for $k \neq \pm 1$ is not vdC.
(iv) Let $p(x)$ be a polynomial with integer coefficients. Then $H = \{p(n)\,;\,n \in \mathbb{N}\}$ is vdC if and only if the congruence $p(n) \equiv 0 \pmod q$ has a solution for every integers $q \geq 1$. E.g. taking $p(n) = n^k$, or $p(n) = n^2 - 1$ we get vdC sets, but for $p(n) = n^2 + 1$, or $p(n) = 2n + 1$ the resulting sets are not vdC.
(v) If $A \subset \mathbb{N}$ is infinite then the difference set $H = A - A = \{i - j > 0\,;\,i, j \in A\}$ is vdC.
(vi) If $H = \{h_1 < h_2 < \dots\}$ is lacunary (i.e. if $\frac{h_{n+1}}{h_n} \geq \alpha > 1$ for $n = 1, 2, \dots$) then H is not vdC.

T. Kamae and M. Mendès France (1978) proved that a sufficient condition for H to be a vdC set is: to every $\varepsilon > 0$ there exists a trigonometric polynomial $f(x) = \sum_{k \in H} a_k \cos(kx)$ such that $f(0) = 1$ and $f(x) > -\varepsilon$ for all x. I.Z. Ruzsa (1982) completed the theory of vdC sets and in 1984 he proved that this condition is also necessary. Kamae and Mendès France (1978) also observed that every vdC set is a **Poincaré set** (also called **recurrent set**: a subset Λ of positive integers is a Poincaré set whenever $(X, \mathcal{B}, \mu, T)$ is a dynamical system and A a measurable set of positive measure, then $\mu(T^{-m}(A) \cap A) > 0$ for some $m \in \Lambda$). J. Bourgain (1987) proved that there is a recurrence set which is not vdC. Other characterizations of recurrence sets and vdC sets can be found in Kamae and Mendès France (1978), A. Bertrand – Mathis (1986) and Ruzsa (1982), (1982/83), (1983), [DT, p. 199–200].
(III) Another general version of van der Corput's difference theorem was given by R.J. Taschner (1983).
(IV) N.M. Korobov and A.G. Postnikov (1952) proved that u.d. of differences $(x_{n+h} - x_n)$ mod 1 also implies that the subsequence x_{qn+r} mod 1, $n = 1, 2, \dots$, is u.d. for all integers $q \geq 1$ and $r \geq 0$.
(V) E. Hlawka (1960) called a property E of sequences x_n of real numbers **hereditary** if the following implication holds: If $x_{n+h} - x_n$ has the property E for every positive integer h, then x_n itself and all its subsequences of the form x_{qn+r} with integral $r \geq 0$ and $q \geq 1$ also have the property E. He found several hereditary properties different from u.d., e.g. completely u.d.
(VI) van der Corput difference theorem for well distributed sequences was proved by B. Lawton (1952).
(VII) The sequence $x_n = \log(3/2)^n$ mod 1 is u.d. but $x_{n+1} - x_n = \log 3 - \log 2$ is obviously not. A more simple example provides the sequence $n\alpha$ with irrational α.
(VIII) In 1.8.4 (II) M. Tsuji's reformulation of van der Corput difference theorem for weighted u.d. sequences is given.
(IX) **van der Corput weighted difference theorem**. B. Massé nd D. Schneider (2014) gives the following generalization: Let x_n be a sequence of real numbers, and $w_n > 0$ a sequence of weights satisfying $W_N = \sum_{n=1}^{N} w_n \to \infty$. Set

$$l_{h,k} = \limsup_{N \to \infty} \left| \frac{1}{W_N} \sum_{n=1}^{N} w_n e^{2\pi i k (x_{n+h} - x_n)} \right|.$$

If

$$\lim_{H\to\infty}\sum_{h=1}^{H} l_{h,k} = 0$$

for all k, then x_n mod 1 is u.d. with respect to weights w_n.

A. Bertrand–Mathis: *Ensembles intersectifs et récurrence de Poincaré*, Israel J. Math. **55** (1986), no. 2, 184–198 (MR0868179 (MR 87m:11071); Zbl. 0611.10032).
J. Bourgain: *Ruzsa's problem on sets of recurrence*, Israel J. Math. **59** (1987), no. 2, 150–166 (MR0920079 (89d:11012); Zbl. 0643.10045).
E. Hlawka: *Erbliche Eigenschaften in der Theorie der Gleichverteilung*, Publ. Math. Debrecen **7** (1960), 181–186 (MR0125103 (**23** #A2410); Zbl. 0109.27501).
E. Hlawka: *The Theory of Uniform Distribution*, A B Academic Publishers, Berkhamsted, 1984 (translation of the original German edition Hlawka (1979)) (MR0750652 (85f:11056); Zbl. 0563.10001).
T. Kamae – M. Mendès France: *van der Corput difference theorem*, Israel J. Math. **31** (1978), 335–342 (MR0516154 (80a:10070); Zbl. 0396.10040).
N.M. Korobov – A.G. Postnikov: *Some general theorems on the uniform distribution of fractional parts*, (Russian), Dokl. Akad. Nauk SSSR (N.S.) **84** (1952), 217–220 (MR0049246 (14,143e); Zbl. 0046.27802).
B. Lawton: *A note on well distributed sequences*, Proc. Amer. Math. Soc. **10** (1959), 891–893 (MR0109818 (**22** #703); Zbl. 0089.26902).
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M. Mendès France: *Les ensembles de van der Corput*, in: *Séminaire Delange-Pisot-Poitou, 19e année: 1977/78, Théorie des nombres, Fasc. 1, Exp. No. 12*, Secrétariat Mathématique, Paris, 1978, 5 pp. (MR0520307 (80d:10074); Zbl. 0405.10033).
I.Z. Ruzsa: *Uniform distribution, positive trigonometric polynomials and difference sets*, Seminar on Number Theory, 1981/1982, Exp. No. 18, Univ. Bordeux I, Talence 1982, 18 pp. (MR0695335 (84h:10073); Zbl. 0515.10048).
I.Z. Ruzsa: *Ensembles intersectifs*, Séminaire de Théorie des Nombres de Bordeaux 1982/1983, Univ. Bordeux I, Talence.
I.Z. Ruzsa: *Connections between the uniform distribution of a sequence and its differences*, in: *Topics in classical number theory, Vol. I, II, (Budapest 1981)*, (G.Halász ed.), Colloq. Math. Soc. János Bolyai, Vol. 34, North–Holland Publishing Co., Amsterdam, New York, 1984, pp. 1419–1443 (MR0781190 (86e:11062); Zbl. 0572.10035).
R.J. Taschner: *A general version of van der Corput's difference theorem*, Pacific J. Math. **104** no. 1, (1983), 231–239 (MR0683740 (84m:10045); Zbl. 0503.10034).
J.G. van der Corput: *Diophantische Ungleichungen I. Zur Gleichverteilung modulo Eins*, Acta Math. **56** (1931), 373–456 (MR1555330; JFM 57.0230.05; Zbl. 0001.20102).

2.2.2. Open problem. If the sequence

$$k(x_{n+h}-x_n)-h(x_{n+k}-x_n) \bmod 1, \quad n=1,2,\dots,$$

is u.d. for every $k,h=1,2,\dots$, $k>h$, then the original sequence

$$x_n \bmod 1, \quad n=1,2,\dots,$$

is also

u.d.

Notes: This problem was posed by M.H. Huxley at the Conference on Analytic and Elementary Number Theory, Vienna, July 18–20, 1996.

2.2.3. If the sequence

$$(x_{pn} - x_n) \bmod 1, \quad n = 1, 2, \dots,$$

is u.d. for all primes p, then

$$x_n \bmod 1, \quad n = 1, 2, \dots,$$

is also

u.d.

NOTES: An unpublished result attributed to G. Halász and R. Vaughan (cf. MR 84m:10045).

2.2.4. If the sequence

$$(x_{pn} - x_{qn}) \bmod 1, \quad n = 1, 2, \dots,$$

is u.d. for all primes $p \neq q$, then

$$x_n \bmod 1, \quad n = 1, 2, \dots,$$

is also

u.d.

NOTES: Attributed to P.D.T.A. Elliot or Daboussi (by M. Mendès France).

2.2.5. Let P be a set of primes such that $\sum_{p \in P} 1/p$ diverges. If the sequence

$$x_{hn} \bmod 1, \quad n = 1, 2, \dots,$$

is u.d. for every h composed only from primes taken from P then

$$x_n \bmod 1, \quad n = 1, 2, \dots,$$

is

u.d.

G. MYERSON – A.D. POLLINGTON: *Notes on uniform distribution modulo one*, J. Austral. Math. Soc. **49** (1990), 264–272 (MR1061047 (92c:11075); Zbl. 0713.11043).

2.2.6. Let D_N be the extremal discrepancy of

$$x_1, \dots, x_N \bmod 1$$

and let D_{N-j} be the extremal discrepancy of

$$x_{j+1} - x_1,\ x_{j+2} - x_2,\ \dots,\ x_N - x_{N-j} \bmod 1.$$

Then for every integer H with $1 \le H \le N$, we have

$$D_N \le cB(1 + |\log B|),$$

where

$$B^2 = \frac{1}{H}\left(1 + \frac{1}{N}\sum_{j=1}^{H-1}(N-j)D_{N-j}\right)$$

and c is an absolute constant.

NOTES: ([KN, p. 165, Th. 6.2]) J.G. van der Corput and Ch. Pisot (1939) proved that

$$D_N \le \frac{2(H-1)}{N} + 2^{\alpha}\sqrt{\omega},$$

where $\alpha = \frac{7}{2} + \sqrt{\frac{|\log \omega|}{\log 2}}$ and

$$\omega = \frac{1}{H}\left(1 + \frac{1}{N}\right) + \frac{2}{H}\sum_{j=1}^{H-1} D_{N-j} + \frac{2(H-1)}{N}.$$

J.G. VAN DER CORPUT – C. PISOT: *Sur la discrépance modulo un. III*, Nederl. Akad. Wetensch., Proc. **42** (1939), 713–722 (MR0000396 (1,66c); JFM 65.0170.02; Zbl. 0022.11605). (=Indag. Math. **1** (1939), 260–269).

2.2.7. Almost–arithmetical progressions.

NOTES: A finite sequence $x_1 < x_2 < \cdots < x_N$ in $[0, 1)$ is called an **almost–arithmetical progression**, denoted (δ, η) for $0 \le \delta < 1$, $\eta > 0$, if

- $0 \le x_1 \le \eta + \delta\eta$,
- $\eta - \delta\eta \le x_{n+1} - x_n \le \eta + \delta\eta$ for $n = 1, 2, \dots, N-1$,
- $1 - \eta - \delta\eta \le x_N < 1$.

. . . .

For an almost–arithmetical progression (δ, η) we have

$$D_N^* \le \begin{cases} \frac{1}{N} + \frac{\delta}{1+\sqrt{1-\delta^2}}, & \text{if } \delta > 0, \\ \min\left(\eta, \frac{1}{N}\right), & \text{if } \delta = 0. \end{cases}$$

NOTES: [KN, p. 118, Th. 3.1]. We have slightly modified the definition used in [KN, p. 118, Def. 3.1].

H. NIEDERREITER: *Almost–arithmetic progressions and uniform distribution*, Trans. Amer. Math. Soc. **161** (1971), 283–292 (MR0284406 (**44** #1633); Zbl. 0219.10040).

2.2.8. If x_n, $n = 1, 2, \dots$, is a monotone sequence that is u.d. mod 1, then

$$\lim_{n\to\infty} \frac{|x_n|}{\log n} = \infty.$$

NOTES: (I) H. Niederreiter (1984).
(II) This is an improvement to a result proved by F. Dress (1967/68), that if λ_n is a non–decreasing sequence of integers which satisfies $\lambda_n = o(\log n)$ then there does not exist a real number x such that $x\lambda_n$ mod 1 is u.d.
(III) A. Topuzoğlu (1981) proved this for $\lambda_n = \mathcal{O}(\log n)$.
(IV) M. Mendès France (1967/68) showed that given a $f(n)$ tending to infinity there exists a sequence of integers λ_n satisfying $\lambda_n = \mathcal{O}(f(n))$, such that the sequence $x\lambda_n$ mod 1 is u.d. for every irrational x.
(V) Actually, Niederreiter (1984) proved a more general result: If $P_N = \sum_{n=1}^{N} p_n$ where $p_n \geq 0$ are the weights (cf. 1.8.4) then the validity of the relation

$$\frac{|x_n|}{\log P_N} \to \infty$$

is necessary for the P–u.d. of the monotone sequence x_n. K. Goto and T. Kano reproved this result in (1991).

F. DRESS: *Sur l'équiréparation de certaines suites* $(x\lambda_n)$, Acta Arith. **14** (1968), 169–175 (MR0227118 (**37** #2703); Zbl. 0218.10055).
K. GOTÔ – T. KANO: *A necessary condition for monotone* (P, μ)*–u.d.* mod 1 *sequences*, Proc. Japan Acad. Ser. A Math. Sci. **67** (1991), no. 1, 17–19 (MR1103973 (92d:11075); Zbl. 0767.11032).
M. MENDÈS FRANCE: *Deux remarques concernant l'équiréparation des suites*, Acta Arith. **14** (1968), 163–167 (MR0227117 (**37** #2702); Zbl. 0177.07202).
H. NIEDERREITER: *Distribution* mod 1 *of monotone sequences*, Neder. Akad. Wetensch. Indag. Math. **46** (1984), no. 3, 315–327 (MR0763468 (86i:11041); Zbl. 0549.10038).
A. TOPUZOĞLU: *On u.d.* mod 1 *of sequences* $(a_n x)$, Nederl. Akad. Wetensch. Indag. Math. **43** (1981), no. 2, 231–236 (MR0707256 (84k:10039); Zbl. 0455.10023).

2.2.9. If x_n is a sequence that is u.d. mod 1, then

$$\limsup_{n\to\infty} n|x_{n+1} - x_n| = \infty.$$

NOTES: P.B. Kennedy (1956), cf. [KN, p. 15, Th. 2.6].

P.B. Kennedy: *A note on uniformly distributed sequences*, Quart. J. Math. Oxford Ser. 2 **7** (1956), 125–127 (MR0096922 (**20** #3404); Zbl. 0071.04401).

2.2.9.1 Let $x_n \in [0,1)$, $n = 0,1,2,\dots$ be u.d. Then we have

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=0}^{N-1} |x_{n+1} - x_n| = \infty.$$

Let $x_n, y_n \in [0,1)$, $n = 0,1,2,\dots$ be two u.d. sequences. Then we have

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |x_n - y_n| \le \frac{1}{2}.$$

In particular, we have

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |x_{n+1} - x_n| \le \frac{1}{2}.$$

Notes: Theorem 1 and Theorem 3 in F. Pillichshammer and S. Steinerberger (2009). They also found:
(a) If x_n is the van der Corput sequence and q an arbitrary base then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |x_{n+1} - x_n| = \frac{2(q-1)}{q^2}.$$

(b) Let $x_n = n\alpha \bmod 1$ with irrational α then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |x_{n+1} - x_n| = 2\{\alpha\}(1 - \{\alpha\}).$$

F. Pillichshammer – S. Steinerberger: *Average distance between consecutive points of uniformly distributed sequences*, Unif. Distrib. Theory **4** (2009), no. 1, 51–67 (MR2501478 (2009m:11116); Zbl. 1208.11088).

2.2.10. Generalized Fejér's difference theorem. Given a sequence $x(n)$ and a positive integer k, define recursively the difference operator Δ^k by $\Delta^1 x(n) = \Delta x(n) = x(n+1) - x(n)$ if $k = 1$ and $\Delta^k x(n) = \Delta(\Delta^{k-1} x(n)$ if $k > 1$. If for a $k \in \mathbb{N}$ we have that
(i) $\Delta^k x(n)$ tends monotonically to 0 as $n \to \infty$,
(ii) $\lim_{n\to\infty} n \left|\Delta^k x(n)\right| = \infty$,

then the sequence

$$x(n) \bmod 1$$

is

$$\text{u.d.}$$

NOTES: [KN, p. 29, Th. 3.4]. The generalized Fejér's difference theorem for sequences expressed in terms of differentiable functions is given in 2.6.1.

2.2.11. Fejér's difference theorem. Let x_n be a sequence such that
(i) $x_n \to \infty$, and
(ii) $\Delta x_n \downarrow 0$, where $\Delta x_n = x_{n+1} - x_n$.
Then

$$x_n \bmod 1$$

is

$$\text{u.d.}$$

if and only if
(iii) $\lim_{n\to\infty} n\Delta x_n = \infty$.
NOTES: (I) Fejér's result says that condition (iii) is sufficient.
(II) J. Cigler (1960, p. 211) proved that if x_n satisfies (i) and (ii) then either x_n mod 1 is u.d. or x_n mod 1 does not have the a.d.f. whatsoever.
(III) J.H.B. Kemperman (1973, p. 149) noted (compare this to formula (9) for $k = 1$ in Cigler (1960)) that the assumptions (i), (ii) in Cigler's result could be weakened to: x_n is strictly increasing with $x_n \to \infty$, $\Delta x_n \to 0$ such that

$$\sup_n \frac{\sum_{k=1}^{n-1} k|\Delta x_k - \Delta x_{k+1}| + n|\Delta x_n|}{x_{n+1} - x_1} < \infty.$$

(IV) J.H.B. Kemperman (1961) and (1973, Th. 3) proved that condition (iii) is necessary and sufficient for x_n mod 1 to possess the a.d.f. provided x_n satisfies (i) and (ii). Consequently, if the sequence x_n satisfies (i) and (ii) and $\liminf_{n\to\infty} |n\Delta x_n| < \infty$, then x_n mod 1 does not posses the a.d.f.
(V) Generalized Fejér's theorem is given in 2.6.1(II).
(VI) W.J. LeVeque (1953, Th. 3) proved the following variant of Fejér's theorem for u.d. modulo subdivision $\Delta = (z_n)_{n=1}^{\infty}$ (cf. p. 1 – 6): Let z_n and x_n be sequences satisfying
(i) $z_n - z_{n-1} \geq z_{n-1} - z_{n-2}$ for $n = 2, 3, \dots$ and $z_n \to \infty$,
(ii) $\Delta x_n \downarrow 0$ as $n \to \infty$ and $x_n \to \infty$,
(iii) $\lim_{k\to\infty} \frac{\#\{n\in\mathbb{N}\,;\, x_n < z_{k+1}\}}{\#\{n\in\mathbb{N}\,;\, x_n < z_k\}} = 1$.

Then x_n is u.d. mod Δ.
(VII) For the multidimensional Fejér's theorem see 3.3.2.1.

RELATED SEQUENCES: 2.2.14, 2.2.15, 2.2.16

J. CIGLER: *Asymptotische Verteilung reeller Zahlen* mod 1, Monatsh. Math. **44** (1960), 201–225 (MR0121358 (**22** #12097); Zbl. 0111.25301).
J.H.B. KEMPERMAN: *Review of an article by J. Cigler (1960)*, Math. Reviews **22** (1961)# 12097, p. 2064
J.H.B. KEMPERMAN: *Distribution modulo* 1 *of slowly changing sequences*, Nieuw Arch. Wisk. (3) **21** (1973), 138–163 (MR0387224 (**52** #8067); Zbl. 0268.10038).
W.J. LEVEQUE: *On uniform distribution modulo a subdivision*, Pacific J. Math. **3** (1953), 757–771 (MR0059323 (15,511c); Zbl. 0051.28503).

2.2.12. Let x_n be a sequence which satisfies $\lim_{n\to\infty}\Delta^k x_n = \theta$ with θ irrational. Then

$$x_n \bmod 1$$

is

$$\text{u.d.}$$

NOTES: J.G. van der Corput (1931), cf. [KN, p. 31, Exer. 3.6], for $k = 1$ [KN, p. 28, Th. 3.3] and for continuous variant cf. 2.6.5.

J.G. VAN DER CORPUT: *Diophantische Ungleichungen I. Zur Gleichverteilung modulo Eins*, Acta Math. **56** (1931), 373–456 (MR1555330; JFM 57.0230.05; Zbl. 0001.20102).

2.2.13. If

$$n(x_{n+1} - x_n - \alpha)$$

tends to a non–zero limit as $n \to \infty$ for rational α, then the sequence

$$x_n \bmod 1$$

is

$$H_\infty\text{–u.d. (cf. 1.8.5) but not u.d.}$$

P. SCHATTE: *On H_∞–summability and the uniform distribution of sequences*, Math. Nachr. **113** (1983), 237–243 (MR0725491 (85f:11057); Zbl. 0526.10043).

2.2.14. Let f be a function which is
(i) differentiable on $[0, \infty]$ and $f'(x) \downarrow 0$ as $x \to \infty$, and
(ii) unbounded for $x \to \infty$,

(iii) $xf(x) \to \infty$ as $x \to \infty$.
Let y_n be an increasing sequence of positive real numbers such that
(v) $\Delta y_n = y_{n+1} - y_n$ is non–increasing, and
(vi) $\Delta y_n / y_n \geq c/n$ for a positive constant c.
Then the sequence

$$f(y_n) \bmod 1$$

is

u.d.

NOTES: J.H.B. Kemperman (1973, p. 139–140). As an application take the sequence $y_n = cn^{\alpha}(\log n)^{\beta}$ with $0 < \alpha < 1$, or $\alpha = 1$ and $\beta \leq 0$. Note that sequences of the type $y_n = (\log n)^{\beta}$ are not covered by the result in general as the example $f(x) = (\log x)^{\gamma}$ with $\gamma > 1$ shows.

RELATED SEQUENCES: 2.2.11

J.H.B. KEMPERMAN: *Distribution modulo* 1 *of slowly changing sequences*, Nieuw Arch. Wisk. (3) **21** (1973), 138–163 (MR0387224 (**52** #8067); Zbl. 0268.10038).

2.2.15. Let x_n be a non–decreasing sequence such that
(i) $x_n \to \infty$ and $\Delta x_n \to 0$ as $n \to \infty$, where $\Delta x_n = x_{n+1} - x_n$,
(ii) there exists a constant $B \geq 1$ such that $\Delta x_n \geq B\Delta x_m$ whenever $n \geq m$,
(iii) $\liminf_{n\to\infty} n\Delta x_n < \infty$.
Then

$$x_n \bmod 1$$

cannot have the a.d.f.

NOTES: J.H.B. Kemperman (1973, Lemma 1). This result contains the necessary part of 2.2.11. The assumption $\Delta x_n \geq 0$ cannot be omitted (cf. J.H.B. Kemperman (1973, Remark on p. 143)).

RELATED SEQUENCES: 2.2.11

J.H.B. KEMPERMAN: *Distribution modulo* 1 *of slowly changing sequences*, Nieuw Arch. Wisk. (3) **21** (1973), 138–163 (MR0387224 (**52** #8067); Zbl. 0268.10038).

2.2.16. Let $\mathbf{A} = (a_{n,k})$ be a regular summation matrix with $a_{n,k} = 0$ for $k > k_n$. Let x_n be a sequence such that
(i) $x_n \neq 0$ and $\Delta x_n \to 0$, where $\Delta x_n = x_{n+1} - x_n$,
(ii) $a_{n,k}/\Delta x_k$ is monotone in k when $1 \leq k \leq k_n$ for each fixed n,
(iii) $\lim_{n\to\infty} a_{n,k_n}/\Delta x_{k_n} = 0$.

Then the sequence x_n mod 1 is

$$\mathbf{A}\text{–u.d.}$$

NOTES: J.H.B. Kemperman (1973, Th. 6). The result contains Fejér's theorem.

RELATED SEQUENCES: 2.2.11, 2.12.1

J.H.B. KEMPERMAN: *Distribution modulo* 1 *of slowly changing sequences*, Nieuw Arch. Wisk. (3) **21** (1973), 138–163 (MR0387224 (**52** #8067); Zbl. 0268.10038).

2.2.17. Let $\mathbf{A} = (a_{n,k})$ be defined through

$$a_{n,k} = \begin{cases} \dfrac{\lambda_k}{\lambda_1 + \cdots + \lambda_n}, & \text{if } 1 \le k \le n, \\ 0, & \text{if } k > n, \end{cases}$$

where we assume that

$$\lambda_n > 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda_n = +\infty.$$

If the sequence x_n is such that
(i) $\Delta x_n \to 0$ where $\Delta x_n = x_{n+1} - x_n$,
(ii) $\Delta x_n / \lambda_n$ is monotone in n,
(iii) $\lim_{n\to\infty} (\lambda_1 + \cdots + \lambda_n) \Delta x_n / \lambda_n = \infty$,
then x_n mod 1 is

$$\mathbf{A}\text{–u.d. (i.e. } \lambda_n\text{–weighted u.d.)}$$

NOTES: (I) J.H.B. Kemperman (1973, Cor. 1 to Th. 6).
(II) For an analogous result see also Tsuji (1952, p. 316) which contains the additional assumption that Δx_n is monotone.
(III) Condition (ii) can be replaced (J.H.B. Kemperman (1973, p. 148)) by the requirement that $\Delta x_k \neq 0$ and

$$\sum_{k=1}^{n-1} \left| \frac{\lambda_k}{\Delta x_k} - \frac{\lambda_{k+1}}{\Delta x_{k+1}} \right| = o(\lambda_1 + \cdots + \lambda_n).$$

(IV) Let x_n be a strictly increasing unbounded sequence and $N(t) = \sum_{x_j \le t} \lambda_j$. Then a sufficient condition that x_n mod 1 is

$$\mathbf{A}\text{–u.d.}$$

is that

$$\lim_{t\to\infty} \frac{N(t+w) - N(t)}{N(t+1) - N(t)} = w \qquad \text{if} \quad 0 < w < 1.$$

For a discussion on the possible values of the limit on the left–hand side cf. J.H.B. Kemperman (1973, pp. 157–158).
(V) In the case $\lambda_k = 1/k$ it is sufficient to require that $\Delta x_n \to 0$, $(n \log n)\Delta x_n \to \infty$, and that $n\Delta x_n$ is monotone (J.H.B. Kemperman (1973, p. 148), cf. also M. Tsuji (1952, p. 318)).
(VI) J.H.B. Kemperman (1973, Cor. 2 to Th. 6): If $\lambda_n = \Delta x_n$ and x_n is a sequence such that $x_n \to \infty$, $\Delta x_n > 0$, and $\Delta x_n \to \infty$, then x_n mod 1 is **A**–u.d.
(VII) J.H.B. Kemperman (1973, Th. 12) describes some perturbations of x_n preserving the **A**–u.d. mod 1.

RELATED SEQUENCES: 2.2.11, 2.2.16, 2.12.1, 2.2.18

J.H.B. KEMPERMAN: *Distribution modulo* 1 *of slowly changing sequences*, Nieuw Arch. Wisk. (3) **21** (1973), 138–163 (MR0387224 (**52** #8067); Zbl. 0268.10038).
M. TSUJI: *On the uniform distribution of numbers* mod 1, J. Math. Soc. Japan **4** (1952), 313–322 (MR0059322 (15,511b); Zbl. 0048.03302).

2.2.18. Let the elements λ's of matrix **A** defined in 2.2.17 also satisfy $\lambda_{n+1} = o(\lambda_1 + \cdots + \lambda_n)$ as $n \to \infty$. Let x_n be a non–decreasing sequence such that
(i) $x_n \to \infty$, $\Delta x_n \to 0$, where $\Delta x_n = x_{n+1} - x_n$,
(ii) $\Delta x_n/\lambda_n \le B\Delta x_m/\lambda_m$ whenever $n \ge m$ for some constant $B \ge 1$,
(iii) $\liminf_{n\to\infty}(\lambda_1 + \cdots + \lambda_n)\Delta x_n/\lambda_n < \infty$.
Then x_n mod 1

does not have the **A**–a.d.f.

NOTES: J.H.B. Kemperman (1973, Th. 8).

RELATED SEQUENCES: 2.12.1, 2.2.16, 2.2.17

J.H.B. KEMPERMAN: *Distribution modulo* 1 *of slowly changing sequences*, Nieuw Arch. Wisk. (3) **21** (1973), 138–163 (MR0387224 (**52** #8067); Zbl. 0268.10038).

2.2.19. Let $\lambda_n > 0$ and $\sum_{n=1}^{\infty} \lambda_n = +\infty$, and let f be a complex valued function of a real argument. Given a sequence x_n of real numbers, define

$$\mu_n(f) = (\lambda_1 + \cdots + \lambda_n)^{-1} \sum_{k=1}^{n} \lambda_k f(x_k).$$

Suppose that
(i) f is continuous and satisfies $f(x+1) = f(x)$,
(ii) $\Delta x_n \to 0$,
(iii) $\lim_{n\to\infty}(\lambda_1 + \cdots + \lambda_n)\Delta x_n/\lambda_n = u$ with u finite,

(iv) N_k is a given strictly increasing sequence of positive integers.
Then the limit

$$\lim_{k\to\infty} \mu_{N_k}(f) = \mu(f)$$

exists for every f which satisfies (i) if and only if there exists the limit

$$\lim_{k\to\infty} x_{N_k} = \xi \bmod 1.$$

Moreover, this limit $\mu(f)$ depends only on u and ξ and is given by

$$\mu(f) = \mu_\xi(f) = \int_0^\infty f(\xi - ut)e^{-t}\,\mathrm{d}t.$$

Note that the existence of the limit $\mu(f)$ for every continuous f is equivalent to the existence of an **A**–d.f. (i.e. λ_n–weighted d.f.) $g(x)$ of x_n mod 1 with respect to the given selected sequence N_k (for def. of **A**–d.f. see 1.8.3(III)). The density of $g(x)$ is

$$g'(x) = \frac{e^{\frac{\{x-\xi\}}{u}}}{u(e^{1/u}-1)}, \quad x \in [0,1].$$

NOTES: (I) J.H.B. Kemperman (1973, Th. 9). If $\lambda_n = 1$ and $u \neq 0$ then a sufficient condition for (iii) is (J.H.B. Kemperman (1973, p. 152))

$$\lim_{n\to\infty}\left(\frac{1}{\Delta x_{n+1}} - \frac{1}{\Delta x_n}\right) = \frac{1}{u}$$

(II) The case $\lambda_n = 1$, $x_n = u \log n$, is due to Pólya and Szegő (1964, Part II, Chap. 4, § 5, no. 180), cf. 2.12.1(IV).
(III) If in the hypotheses given above $u \neq 0$ and f is Riemann integrable, then the set $J[f]$ of all the accumulation points of the real sequence $\{\mu_n(f)\}$ coincides with the interval $\{\mu_\xi(f) \,|\, \xi \in I\}$, where I denotes the interval which consists of all accumulation points modulo 1 of the given sequence x_n. If $u = 0$ then $J[f] = \{f(\xi) \,|\, \xi \in I\}$.
(IV) The result also holds if s is a positive integer and x_n a sequence of points in $\mathbb{R}^s$ which satisfies (ii) and (iii) with $u \in \mathbb{R}^s$ (J.H.B. Kemperman (1973, Th. 10)).

RELATED SEQUENCES: 2.12.1, 2.2.16, 2.2.17

J.H.B. KEMPERMAN: *Distribution modulo* 1 *of slowly changing sequences*, Nieuw Arch. Wisk. (3) **21** (1973), 138–163 (MR0387224 (**52** #8067); Zbl. 0268.10038).
G. PÓLYA – G. SZEGŐ: *Aufgaben und Lehrsätze aus der Analysis*, Vol. 1 & 2, 3rd corr. ed., Grundlehren d. math. Wiss., Band 19, Springer Verlag, Berlin, Göttingen, Heidelberg, New York, 1964 (MR0170985 (**30** #1219a); MR0170986 (**30** #1219b); Zbl. 0122.29704).

2.2.20. Let f be a function defined on $[0,1]$ with a continuous second derivative. If x_n is a sequence with discrepancy D_N, then

$$\left|\sum_{n=1}^{N}\left(f\left(\left\{x_n+\frac{1}{N}\right\}\right)-f(\{x_n\})\right)-(f(1)-f(0))\right|\le$$

$$\le\left(D_N+\frac{1}{N}\right)\int_0^1 \mathrm{d}t\left|\int_0^1 \mathrm{d}\tau f''\left(\left\{t+\frac{\tau}{N}\right\}\right)\right|.$$

NOTES: (I) E. Hlawka (1980). He previously proved that if $x_n \bmod 1$, $n=1,2,\dots,$ is an u.d. sequence, and if f is a continuously differentiable function on $[0,1]$, then

$$\lim_{N\to\infty}\sum_{n=1}^{N}\left(f\left(\left\{x_n+\frac{1}{N}\right\}\right)-f(\{x_n\})\right)=f(1)-f(0)\left(=\int_0^1 f'(x)\,\mathrm{d}x\right).$$

(II) H. Rindler – J. Schoißengeier (1977) proved that the truth of the above equality for every u.d. sequence x_n in $[0,1)$ such that $x_n+\frac{1}{n}\in[0,1)$ for all $n\ge 1$ is equivalent to the Riemann integrability of f'.
(III) If f has jumps $\alpha_1,\dots,\alpha_r$ in points $\xi_1,\dots,\xi_r$ of $[0,1]$, then (I) may be replaced by

$$\lim_{n\to\infty}\sum_{n=1}^{N}\left(f\left(\left\{x_n+\frac{1}{N}\right\}\right)-f(\{x_n\})\right)=f(1)-f(0)-(\alpha_1+\dots+\alpha_r).$$

(IV) As in (I), also (II) can be proved using the mean value theorem 4.1.4.18. The general result is proved in Hlawka (1980, pp. 449–451) as a consequence of a result holding in compact connected spaces.

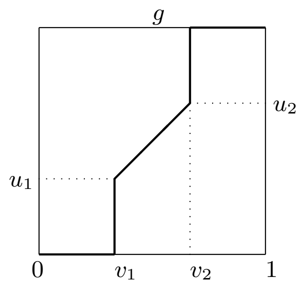
RELATED SEQUENCES: 3.1.3

E. HLAWKA: *Über einige Satze, Begriffe und Probleme in der Theorie der Gleichverteilung. II*, Österreich. Akad. Wiss. Math.–Natur. Kl. Sitzungsber. II **189** (1980), no. 8–10, 437–490 (MR0645297 (84j:10061); Zbl. 0475.10039).
H. RINDLER – J. SCHOISSENGEIER: *Gleichverteilte Folgen und differenzierbare Funktionen*, (German), Monatsh. **84** (1977), 125–131 (MR0491572 (**58** #10801); Zbl. 0371.10040).

2.2.21. Given parameters $(u_1,v_1,u_2,v_2)\in[0,1]^4$, let $h(x)$ denote the d.f. defined by

$$h(x)=\begin{cases}0, & \text{if } 0\le x\le v_1,\\ \frac{u_2-u_1}{v_2-v_1}x+u_1-v_1\frac{u_2-u_1}{v_2-v_1}, & \text{if } v_1<x\le v_2,\\ 1, & \text{if } v_2<x\le 1.\end{cases}$$

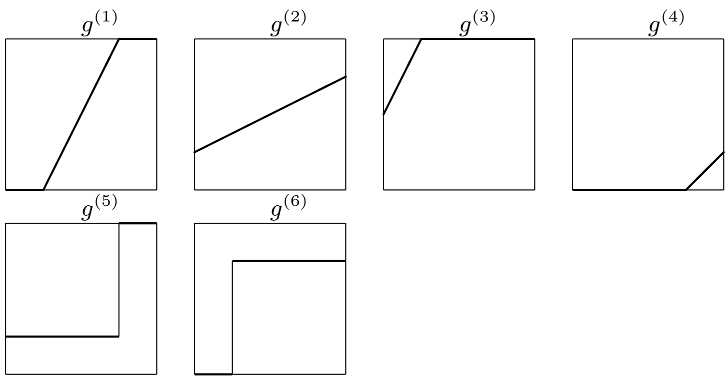
Thus always $h(0)=0$ and $h(1)=1$. Its graph is



Given $(X_1, X_2, X_3) \in [0,1]^3$ define further d.f.'s $g^{(i)}$ for $i = 1, \dots, 7$ and $i = 7^*$ by

$$
\begin{aligned}
g^{(1)} &= g(0, (1-X_1) - 3(X_1 - X_3)\,, 1\,, (1-X_1) + 3(X_1 - X_3)),\\
g^{(2)} &= g\left(X_1 - \sqrt{3(X_3 - X_1^2)}\,, 0\,, X_1 + \sqrt{3(X_3 - X_1^2)}\,, 1\right),\\
g^{(3)} &= g\left(1 - \frac{3}{2}\cdot\frac{1 + X_3 - 2X_1}{1 - X_1}\,, 0\,, 1\,, \frac{4}{3}\cdot\frac{(1-X_1)^2}{(1 + X_3 - 2X_1)}\right),\\
g^{(4)} &= g\left(0\,, 1 - \frac{4X_1^2}{3X_3}\,, \frac{3X_3}{2X_1}\,, 1\right),\\
g^{(5)} &= g\left(\frac{X_1 - X_3}{1 - X_1}\,, 0\,, \frac{X_1 - X_3}{1 - X_1}\,, \frac{(1-X_1)^2}{1 + X_3 - 2X_1}\right),\\
g^{(6)} &= g\left(\frac{X_3}{X_1}\,, 1 - \frac{X_1^2}{X_3}\,, \frac{X_3}{X_1}\,, 1\right),\\
g^{(7)} &= g\left(1 - 2X_3\,, 0\,, 1 - 2X_3\,, \frac{1}{4X_3}\right),\\
g^{(7^*)} &= g\left(2X_3\,, 1 - \frac{1}{4X_3}\,, 2X_3\,, 1\right).
\end{aligned}
$$

Their graphs are



If the areas under their graphs are 1/2 then we put $g^{(5)} = g^{(7)}$ and $g^{(6)} = g^{(7^*)}$.

Finally define the surfaces Π_i for $i = 1, \dots, 7$ in $[0,1]^3$ by

$$\Pi_1 = \Big\{ (X_1, X_2, X_3) \in [0,1]^3 \, ; \, X_2 = \frac{1}{2} - \frac{1}{2}(1 - X_1)^2 - \frac{3}{2}(X_1 - X_3)^2,$$
$$\max\Big(\frac{4}{3}X_1 - \frac{1}{3}, \frac{2}{3}X_1\Big) \le X_3 \le X_1, 0 \le X_1 \le 1 \Big\},$$
$$\Pi_2 = \Big\{ (X_1, X_2, X_3) \in [0,1]^3 \, ; \, X_2 = \frac{1}{2}X_1 + \frac{1}{2}\sqrt{\frac{1}{3}(X_3 - X_1^2)},$$
$$X_1^2 \le X_3 \le \min\Big(\frac{4}{3}X_1^2, \frac{4}{3}X_1^2 - \frac{2}{3}X_1 + \frac{1}{3}\Big), 0 \le X_1 \le 1 \Big\},$$
$$\Pi_3 = \Big\{ (X_1, X_2, X_3) \in [0,1]^3 \, ; \, X_2 = \frac{1}{2} - \frac{4}{9}\frac{(1 - X_1)^3}{(1 + X_3 - 2X_1)},$$
$$\frac{4}{3}X_1^2 - \frac{2}{3}X_1 + \frac{1}{3} \le X_3 \le \frac{4}{3}X_1 - \frac{1}{3}, \frac{1}{2} \le X_1 \le 1 \Big\},$$
$$\Pi_4 = \Big\{ (X_1, X_2, X_3) \in [0,1]^3 \, ; \, X_2 = X_1 - \frac{4}{9}\frac{X_1^3}{X_3}, \frac{4}{3}X_1^2 \le X_3 \le \frac{2}{3}X_1, 0 \le X_1 \le \frac{1}{2} \Big\},$$
$$\Pi_5 = \Big\{ (X_1, X_2, X_3) \in [0,1]^3 \, ; \, X_2 = \frac{1}{2} - \frac{1}{2}\frac{(1 - X_1)^3}{(1 + X_3 - 2X_1)},$$
$$X_1^2 \le X_3 \le X_1, 0 \le X_1 < \frac{1}{2} \Big\},$$
$$\Pi_6 = \Big\{ (X_1, X_2, X_3) \in [0,1]^3 \, ; \, X_2 = X_1 - \frac{1}{2}\frac{X_1^3}{X_3}, X_1^2 \le X_3 \le X_1 \, \frac{1}{2} < X_1 \le 1 \Big\},$$
$$\Pi_7 = \Big\{ \Big(\frac{1}{2}, \frac{1}{2} - \frac{1}{16X_3}, X_3\Big) \, ; \, \frac{1}{4} < X_3 < \frac{1}{2} \Big\}.$$

Let x_n be a sequence in $[0,1]$ for which there exist the limits in the expressions on right hand side

$$X_1 = 1 - \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} x_n,$$
$$X_2 = \frac{1}{2} - \frac{1}{2}\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} x_n^2,$$
$$X_3 = 1 - \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} x_n - \frac{1}{2}\lim_{N\to\infty} \frac{1}{N^2}\sum_{m,n=1}^{N} |x_m - x_n|.$$

If $(X_1, X_2, X_3) \in \bigcup_{1\le i\le 7} \Pi_i$, then the sequence x_n has a limit law. Moreover, if $(X_1, X_2, X_3) \in \Pi_i$ for $i = 1, \dots, 6$, then x_n has a.d.f. $g^{(i)}$, and if $(X_1, X_2, X_3) \in \Pi_7$, then x_n has a.d.f. This is either $g^{(7)}$ or $g^{(7^*)}$, depending on whether

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int_0^{x_n} g^{(7)}(t)\,\mathrm{d}t = X_1 - X_3,$$

or

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int_0^{x_n} g^{(7^*)}(t)\;\mathrm{d}t = X_1 - X_3.$$

NOTES: O. Strauch (1994) gave a complete solution to the moment problem

$$(X_1, X_2, X_3) = \left(\int_0^1 g(x)\,\mathrm{d}x, \int_0^1 xg(x)\,\mathrm{d}x, \int_0^1 g^2(x)\,\mathrm{d}x\right)$$

in d.f. $g(x)$ which implies the above conditions. An open problem is to solve the moment problem

$$(X_1, X_2, X_3, X_4) = \left(\int_0^1 g(x)\,\mathrm{d}x, \int_0^1 xg(x)\,\mathrm{d}x, \int_0^1 x^2g(x)\,\mathrm{d}x, \int_0^1 g^2(x)\,\mathrm{d}x\right).$$

E.g. for $g(x) = 2x - x^2$ it has the unique solution.

O. STRAUCH: *A new moment problem of distribution functions in the unit interval*, Math. Slovaca **44** (1994), no. 2, 171–211 (MR1282534 (95i:11082); Zbl. 0799.11023).

2.2.22. Given two different d.f.'s $g_1(x)$, and $g_2(x)$, define

$$F_{g_2}(x,y) = \int_0^x g_2(t)\,\mathrm{d}t + \int_0^y g_2(t)\,\mathrm{d}t - \max(x,y) + \int_0^1 (1-g_2(t))^2\,\mathrm{d}t,$$

$$F_{g_1,g_2}(x) = \frac{\int_0^x (g_2(t)-g_1(t))\,\mathrm{d}t - \int_0^1 (1-g_2(t))(g_2(t)-g_1(t))\,\mathrm{d}t}{\int_0^1 (g_2(t)-g_1(t))^2\,\mathrm{d}t},$$

$$F_{g_1,g_2}(x,y) = F_{g_2}(x,y) - F_{g_1,g_2}(x)F_{g_1,g_2}(y)\int_0^1 (g_2(t)-g_1(t))^2\,\mathrm{d}t.$$

Let $g_1(x) \neq g_2(x)$ be two d.f.'s. Then the set of d.f.'s $G(x_n)$ of x_n in $[0,1)$ satisfies

$$G(x_n) = \{tg_1(x) + (1-t)g_2(x)\,;\ t \in [0,1]\}$$

if and only if

(i) $\lim_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^{N} F_{g_1,g_2}(x_m,x_n) = 0$,

(ii) $\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} F_{g_1,g_2}(x_n) = 0$,

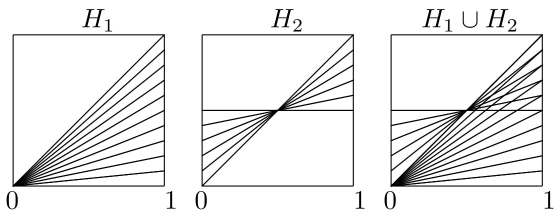
(iii) $\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} F_{g_1,g_2}(x_n) = 1$.

NOTES: O. Strauch (1997). His proof is based on Theorem 1.10.9.2. Another application is given in Strauch (1999, p. 99): Put

$$\begin{aligned}
F_1(x,y) &= 1 - \max(x,y) - \frac{3}{4}(1-x^2)(1-y^2),\\
F_2(x,y) &= \frac{x+y}{2} - \max(x,y) + \frac{1}{4} - 3(x-x^2)(y-y^2),\\
F_3(x,y) &= 1 - \max(x,y),\\
F_4(x,y) &= \frac{x+y}{2} - \max(x,y) + \frac{1}{4},
\end{aligned}$$

and

$$\begin{aligned}
H_1 &= \{tx + (1-t)c_1(x)\,;\ t \in [0,1]\},\\
H_2 &= \{tx + (1-t)h_{1/2}(x)\,;\ t \in [0,1]\}.
\end{aligned}$$



Then $G(x_n) = H_1 \cup H_2$ for a sequence x_n in $[0,1)$ if and only if
(i) $\lim_{N\to\infty} \frac{1}{N^4} \sum_{m,n,k,l=1}^N F_1(x_m,x_n)F_2(x_k,x_l) = 0$,
(ii) $\liminf_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^N F_3(x_m,x_n) = 0$,
(iii) $\liminf_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^N F_4(x_m,x_n) = 0$.
Here $c_1(x)$ is the one–jump d.f. with jump at $x = 1$ and $h_{1/2}(x)$ is the d.f. taking constant value $1/2$.

O. Strauch: *On the set of distribution functions of a sequence*, in: *Proceedings of the Conference on Analytic and Elementary Number Theory in Honor of Edmund Hlawka's 80th Birthday, Vienna, July 18–20, 1996*, (W.–G. Nowak, J. Schoißengeier eds.), Universität Wien & Universität für Bodenkultur, Vienna, 1997, pp. 214–229 (Zbl. 0886.11044).
O. Strauch: *Distribution of Sequences* (in Slovak), Mathematical Institute of the Slovak Academy of Sciences, DSc Thesis, Bratislava, Slovakia, 1999.

2.3 General operations with sequences

2.3.1. If the sequence x_n mod 1 is

u.d.

and if the sequences y_n is such that $\lim_{n\to\infty}(x_n - y_n)$ exists, then the sequence y_n mod 1 is also

u.d.

NOTES: ([KN, p. 3, Th. 1.2]) Consequently, if the sequence x_n mod 1 is u.d., then also the sequence $\alpha + x_n$ mod 1 is u.d. for every real number α ([KN, p. 3, Lemma 1.1]).

2.3.2. If x_n mod 1 is u.d. then so is the sequence

$$hx_n \bmod 1, \quad n = 1, 2, \dots,$$

for any non–zero integer h.

NOTES: This directly follows from Weyl's criterion 2.1.2. G. Myerson and A. Pollington (1990) proved that there is a sequence x_n mod 1 which is not u.d. even though hx_n mod 1 is u.d. for every integer $h \geq 2$.

G. MYERSON – A.D. POLLINGTON: *Notes on uniform distribution modulo one*, J. Austral. Math. Soc. **49** (1990), 264–272 (MR1061047 (92c:11075); Zbl. 0713.11043).

2.3.3.

• If sequences x_n and y_n satisfy

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left| \{x_n\} - \{y_n\} \right| = 0,$$

then the sets of distribution functions of x_n mod 1 and y_n mod 1 coincide.

• If every d.f. in $G(\{x_n\})$ (or in $G(\{y_n\})$) is continuous at 0 and 1, then the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |x_n - y_n| = 0 \qquad (*)$$

implies that $G(\{x_n\}) = G(\{y_n\})$.

• If every d.f. in $G(\{x_n\})$ and $G(\{y_n\})$ is continuous at 0, then the limit $(*)$ again implies that $G(\{x_n\}) = G(\{y_n\})$. The same holds in the case of continuity at 1.

NOTES: (I) O. Strauch (1999, p. 91, Chap. 6, Th. 5 and 5′). Since $0 \leq \{x_n - y_n\} = x_n - y_n - [x_n - y_n] = (x_n - [x_n - y_n]) - y_n = u_n - y_n$ and $u_n \equiv x_n \bmod 1$, the limit

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\{x_n-y_n\}=0$$

can be used instead of $(*)$.

(II) The following simple variant of the above result can be found in [KN, p. 23, Exer. 2.11]: the relation $(*)$ implies that $x_n \bmod 1$ and $y_n \bmod 1$ are simultaneously u.d.

(III) Because $x_n = n + (1/n) \bmod 1$ has a.d.f. $c_0(x)$ and $y_n = n - (1/n) \bmod 1$ has a.d.f. $c_1(x)$, the relation $(*)$ does not imply the equality $G(\{x_n\}) = G(\{y_n\})$ in general.

(IV) The above results can be modified as follows:

(i) If the all d.f.s in $G(\{x_n\})$ and $G(\{y_n\})$ are continuous at 0, then

$$\{x_n-y_n\}\to 0 \Longrightarrow G(\{x_n\})=G(\{y_n\}).$$

The same follows from the continuity at 1.

(ii) The limit $\{x_n - y_n\} \to 0$ also implies

$$\{g\in G(\{x_n\}); g \text{ is continuous at both } 0,1\} \\ =\{\tilde{g}\in G(\{y_n\}); \tilde{g} \text{ is continuous at both } 0,1\}.$$

(iii) Assume that the all d.f.s in $G(\{x_n\})$ are continuous at 0. Then

$$\{x_n-y_n\}\to 0 \Longrightarrow \{\tilde{g}\in G(\{y_n\}); \tilde{g} \text{ is continuous at } 0\}\subset G(\{x_n\}).$$

(iv) If $x_n, y_n \in [0,1)$, $n = 1,2,\dots$, then

$$|x_n-y_n|\to 0 \Longrightarrow G(x_n)=G(y_n)$$

(i.e. the continuity assumption can be omitted).

(v) If $x_n, y_n \in [0,1)$, $n = 1,2,\dots$, then

$$\frac{1}{N}\sum_{n=1}^{N}|x_n-y_n|\to 0 \Longrightarrow G(x_n)=G(y_n)$$

(i.e. the continuity assumption can be omitted).

(vi) The implication (v) follows from: If $F_N^{(1)}(x) = \frac{1}{N}\sum_{n=1}^{N} c_{[0,x)}(x_n)$ and $F_N^{(2)}(x) = \frac{1}{N}\sum_{n=1}^{N} c_{[0,x)}(y_n)$ then

$$\begin{aligned}\int_0^1 (F_N^{(1)}(x)-F_N^{(2)}(x))^2\,\mathrm{d}x &= \frac{1}{N^2}\sum_{m,n=1}^{N}|x_m-y_n| \\ &\quad -\frac{1}{2}\frac{1}{N^2}\sum_{m,n=1}^{N}|x_m-x_n| - \frac{1}{2}\frac{1}{N^2}\sum_{m,n=1}^{N}|y_m-y_n| \\ &\le \frac{1}{N}\sum_{n=1}^{N}|y_n-x_n|.\end{aligned}$$

(vii) Parent [p. 257, Ex. 5.37]: To an arbitrary sequence $x_n \in [0,1)$, $n = 1,2,\dots$, one can associate a real number α such that $\lim_{n\to\infty}(\{n!\alpha\} - x_n) = 0$. Thus by (iv) $G(\{n!\alpha\}) = G(x_n)$.

D.P. Parent: *Exercises in Number Theory*, Problem Books in Mathematics, Springer Verlag, New York, 1984 (MR0759342 (86f:11002); Zbl. 0536.10001). (French original: *Exercices de théorie des nombres*, Gauthier – Villars, Paris, 1978 (MR0485646 (**58** #5471); Zbl. 0387.10001)).
O. Strauch: *Distribution of Sequences* (in Slovak), Mathematical Institute of the Slovak Academy of Sciences, DSc Thesis, Bratislava, Slovakia, 1999.

2.3.4. Let the sequence x_n from $(0,1)$ has continuous a.d.f. $g(x)$. Then the sequence

$$y_n = \frac{1}{x_n} \bmod 1$$

has the a.d.f.

$$\widetilde{g}(x) = \sum_{n=1}^{\infty} g\left(\frac{1}{n}\right) - g\left(\frac{1}{n+x}\right)$$

Notes: (I) I.J. Schoenberg (1928), E.K. Haviland (1941), L. Kuipers (1957), a proof can be found in [KN, p. 56, Th. 7.6]. E. Hlawka (1961, 1964) considered the multi–dimensional case). G. Pólya (cf. I.J. Schoenberg (1928)) proved that for $g(x) = x$ we have

$$\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+x}\right) = \int_0^1 \frac{1-t^x}{1-t}\,\mathrm{d}t.$$

For history consult [KN, p. 66, Notes].
(II) O. Strauch (1997) gave the following generalization: Let $f : [0,1] \to [0,1]$ be a function such that, for all $x \in [0,1]$ the set $f^{-1}([0,x))$ can be expressed as a union of finitely many pairwise disjoint subintervals $I_i(x) \subseteq [0,1]$, say, with endpoints $\alpha_i(x) \le \beta_i(x)$. Then given a d.f. $g(x)$, associate with this decomposition the function

$$g_f(x) = \sum_i g\big(\beta_i(x)\big) - g\big(\alpha_i(x)\big).$$

After this preparatory definition, let $x_n \bmod 1$ be such that each its term appears only finitely many times in it, and let $g(x)$ be the d.f. of $x_n \bmod 1$ based on a sequence of indices N_k (for the def. consult 1.7). Then $g_f(x)$ is the d.f. for the same sequence of indices N_k of the sequence $f(x_n \bmod 1)$, and vice versa every d.f. of $f(x_n \bmod 1)$ has this form, i.e.

$$G(f(x_n \bmod 1)) = \{g_f\ ;\ g \in G(x_n \bmod 1)\}.$$

Related sequences: 2.22.13, 2.15.5

E.K. Haviland: *On the distribution function of the reciprocal of a function and of a function reduced modulo* 1, Amer. J. Math. **63** (1941), 408–414 (MR0003843 (2,280e); Zbl. 0025.18604).
E. Hlawka: *Cremonatransformation von Folgen modulo 1*, Monatsh. Math. **65** (1961), 227–232 (MR0130242 (**24** #A108); Zbl. 0103.27701).
E. Hlawka: *Discrepancy and uniform distribution of sequences*, Compositio Math. **16** (1964), 83–91 (MR0174544 (**30** #4745); Zbl. 0139.27903).
L. Kuipers: *Some remarks on asymptotic distribution functions*, Arch. der Math. **8** (1957), 104–108 (MR0093054 (19,1202d); Zbl. 0078.04003).
I.J. Schoenberg: *Über die asymptotische Verteilung reeller Zahlen* mod 1, Math. Z. **28** (1928), 171–199 (MR1544950; JFM 54.0212.02).
O. Strauch: *On distribution functions of* $\zeta(3/2)^n$ mod 1, Acta Arith. **81** (1997), no. 1, 25–35 (MR1454153 (98c:11075); Zbl. 0882.11044).

2.3.5. Let x_n and y_n be sequences in $[0, 1)$ such that
(i) x_n and y_n are statistically independent (cf. 1.8.8), and
(ii) x_n is u.d.

Then the sequence

$$x_n + y_n \bmod 1$$

is

$$\text{u.d.}$$

Notes: (I) This is a special case of a result proved by G. Rauzy (1976, p. 96) for compact metrisable Abelian groups. See also 2.3.6, Note (III).
(II) This Rauzy's result implies the following result proved by P. Schatte (2000, Lem. 2.3): Let x_n and y_n be two arbitrary sequences, $n = 1, 2, \dots$. Order the double sequence $x_i + y_j$, $i, j = 1, 2, \dots$, to the sequence z_n according to the scheme

$$\begin{array}{lllllll}
x_1 + y_1, & & & & & & \\
x_1 + y_2, & x_2 + y_1, & x_2 + y_2, & & & & \\
x_1 + y_3, & x_2 + y_3, & x_3 + y_1, & x_3 + y_2, & x_3 + y_3, & & \\
x_1 + y_4, & x_2 + y_4, & x_3 + y_4, & x_4 + y_1, & x_4 + y_2, & x_4 + y_3, & x_4 + y_4,
\end{array}$$

$\dots$, etc. If x_n mod 1 or y_n mod 1 is u.d., then also z_n mod 1 is u.d. (Note that the sequence $x_1, x_1, x_2, x_2, x_1, x_2, x_3, \dots$ or $y_1, y_2, y_1, y_2, y_3, y_3, y_1, \dots$ is u.d. mod 1 and are statistically independent).

G. Rauzy: *Propriétés statistiques de suites arithmétiques*, Le Mathématicien, Vol. 15, Collection SUP, Presses Universitaires de France, Paris 1976, 133 pp. (MR0409397 (**53** #13152); Zbl. 0337.10036).
P. Schatte: *On the points on the unit circle with finite b–adic expansions*, Math. Nachr. **214** (2000), 105–111 (MR1762054 (2001f:11125); Zbl. 0967.11028).

2.3.6. Given a sequence x_n, then the sequences

$$x_n \bmod 1 \quad \text{and} \quad (x_n + \log n) \bmod 1$$

are simultaneously

u.d.

NOTES: (I) G. Rauzy (1973). His proof in (1976, p. 96) starts with the statistical independence of $\log n \bmod 1$ (cf. 1.8.8) related to any u.d. sequence x_n and then he uses 2.3.5.

(II) Another proof can be found in D.P. Parent (1984, pp. 249–250, Exer. 5.11). It also works for sequences $\lambda \log n$ and $\log\log n$, but not for $\log^\tau n$, $\tau > 1$, because the sequence $\log^\tau n \bmod 1$ is u.d. (cf. 2.12.7).

(III) Rauzy (1976, p. 97) and (1973) gave the following four equivalent characterizations of sequences y_n for which the sequence $(x_n + y_n) \bmod 1$ is u.d. if and only if $x_n \bmod 1$ is u.d.:

(i) y_n is statistically independent with any u.d. sequence.

(ii) For any infinite sequence z_n of complex numbers such that $|z_n| \le 1$,

$$\lim_{N\to\infty} \frac{z_1 + \cdots + z_N}{N} = 0 \quad \text{implies} \quad \lim_{N\to\infty} \frac{e^{2\pi i y_1} z_1 + \cdots + e^{2\pi i y_N} z_N}{N} = 0.$$

(iii) To every $\varepsilon > 0$ there exists a $\theta > 1$ such that

$$\limsup_{k\to\infty} \frac{1}{\theta^k} \sum_{j=0}^{k-1} \inf_{\lambda\in\mathbb{C}} \sum_{\theta^j \le n < \theta^{j+1}} |e^{2\pi i y_n} - \lambda| \le \varepsilon.$$

(iv) For every $\varepsilon > 0$ there exists $\delta > 0$ such that for every sequence of indices n_k for which $(n_{k+1}/n_k) \to \alpha$ with $1 < \alpha < 1 + \delta$, we have

$$\limsup_{k\to\infty} \frac{1}{n_k} \sum_{h=1}^{k} \left(\inf_{y\in\mathbb{R}} \sum_{n_h \le n < n_{h+1}} ||y_n - y|| \right) \le \varepsilon$$

(see (1973)).

RELATED SEQUENCES: 2.19.7, 2.12.1, 2.12.31

D.P. PARENT: *Exercises in Number Theory*, Problem Books in Mathematics, Springer Verlag, New York, 1984 (MR0759342 (86f:11002); Zbl. 0536.10001). (French original: *Exercices de théorie des nombres*, Gauthier – Villars, Paris, 1978 (MR0485646 (**58** #5471); Zbl. 0387.10001)).

G. RAUZY: *Propriétés statistiques de suites arithmétiques*, Le Mathématicien, Vol. 15, Collection SUP, Presses Universitaires de France, Paris 1976, 133 pp. (MR0409397 (**53** #13152); Zbl. 0337.10036).

G. RAUZY: *Étude de quelques ensembles de fonctions définis par des propertiétés de moyenne*, Séminaire de Théorie des Nombres (1972–1973), 20, Lab. Théorie des Nombres, Centre Nat. Recherche Sci., Talence, 1973, 18 pp. (MR0396463 (**53** #328); Zbl. 0293.10018).

2.3.6.1 An arbitrary u.d. sequence x_n mod 1 and $\log(n \log n)$ mod 1 are statistically independent. Thus

$$x_n \bmod 1 \quad \text{and} \quad (x_n + \log(n \log n)) \bmod 1$$

are simultaneously

u.d.

Y. Ohkubo: *On sequences involving primes*, Unif. Distrib. Theory **6** (2011), no. 2, 221–238 (MR2904049; Zbl. 1313.11090)

2.3.6.2 Let p_n, $n = 1, 2, \dots$, be the increasing sequence of all primes. An arbitrary u.d. sequence x_n mod 1 and the sequence $\log p_n$ mod 1 are statistically independent. Thus

$$x_n \bmod 1 \quad \text{and} \quad (x_n + \log p_n) \bmod 1$$

are simultaneously

u.d.

Notes: Y. Ohkubo (2011). The result follows from the fact that every u.d. sequence x_n mod 1 is statistically independent with the sequence $\log(n \log n)$ mod 1 and that

$$\lim_{n\to\infty} (\log p_n - \log(n \log n)) = 0.$$

Y. Ohkubo: *On sequences involving primes*, Unif. Distrib. Theory **6** (2011), no. 2, 221–238 (MR2904049; Zbl. 1313.11090)

2.3.6.3 Let p_n, $n = 1, 2, \dots$, be the increasing sequence of all primes. An arbitrary u.d. sequence x_n mod 1 and the sequence $\frac{p_n}{n}$ mod 1 are statistically independent. Thus for every sequence x_n,

$$x_n \bmod 1 \quad \text{and} \quad \left(x_n + \frac{p_n}{n}\right) \bmod 1$$

are simultaneously

u.d.

NOTES: Y. Ohkubo (2011). The result follows from the fact that every u.d. sequence x_n mod 1 is statistically independent with the sequence $\log(n \log n)$ mod 1 and that

$$\lim_{n\to\infty} \left(\frac{p_n}{n} - \log(n \log n) \right) = -1.$$

Y. OHKUBO: *On sequences involving primes*, Unif. Distrib. Theory **6** (2011), no. 2, 221–238 (MR2904049; Zbl. 1313.11090)

2.3.7. If the sequence x_n mod 1 has continuous a.d.f. $g(x)$ then the sequence

$$g(\{x_n\})$$

is

u.d.

NOTES: I.J. Schoenberg (1928), cf. [KN, p. 68, Ex. 7.19].

I.J. SCHOENBERG: *Über die asymptotische Verteilung reeller Zahlen* mod 1, Math. Z. **28** (1928), 171–199 (MR1544950; JFM 54.0212.02).

2.3.8. Let $g(x)$ be a continuous d.f. and x_n be a van der Corput sequence (cf. 2.11.1). Then the sequence

$$y_n = \sup\{x \in [0,1] \,;\, g(x) \le x_n\}$$

has the a.d.f.

$$g(x)$$

with

$$D_N^* \le \frac{\log(N+1)}{N \log 2}.$$

NOTES: [KN, p. 137, Lemma 4.2].

2.3.9. Let the sequence x_n in $[0,1)$ have at least one irrational limit point and A_n, $n = 1, 2, \dots$, be the block of 2^n numbers

$$A_n = (\varepsilon_1 x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_n x_n \,;\, \varepsilon_i = \pm 1) \bmod 1.$$

Then the sequence of individual blocks A_n, $n = 1, 2, \dots$, is

u.d.

NOTES: This problem was proposed by A.M. Odlyzko (1987) and then solved by D.G. Cantor (1989).

D.G. Cantor: *Solution of Advanced Problems # 6542*, Amer. Math. Monthly **96** (1989), no. 1, 66–67 (MR1541447).
A.M. Odlyzko: *Solution of Advanced Problems # 6542*, Amer. Math. Monthly **96** (1989), no. 1, 66–67 (MR1541447).

2.3.9.1 Let x_n, $n = 1, 2, \dots$, be a sequence in $(0, 1]$. If every $g(x) \in G(x_n)$ is strictly increasing, then the block sequence of the 2^N-terms blocks of the form

$$\sum_{n \in X} x_n \bmod 1, \text{ with } X \text{ running over all subsets of } \{1, 2, \dots, N\}$$

is u.d., if $N \to \infty$.
Notes: O. Strauch (2009).

2.3.10. Let x_n be a u.d. sequence in $[0, 1)$ with extremal discrepancy $D_N(x_n)$, and let $g(x)$ be a d.f. with continuous derivative satisfying $0 \le g'(x) \le M$. If $A_N = (y_1^{(N)}, \dots, y_N^{(N)})$ is the block of numbers defined by

$$y_k^{(N)} = \frac{1}{N} \sum_{i=1}^{N} (1 + x_k - g(x_i)),$$

then the block sequence A_N, $N = 1, 2, \dots$, is

u.d.

and for every function f of bounded variation $V(f)$ on $[0, 1]$ one has

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(y_n^{(N)}) - \int_0^1 f(t) g'(t) \, \mathrm{d}t \right| \le (1 + M) D_N(x_n) V(f).$$

Notes: E. Hlawka and R. Mück (1972). This result was extended to the multi–dimensional case in ([a]1972), cf. 3.2.7.

E. Hlawka – R. Mück: *A transformation of equidistributed sequences*, in: *Applications of Number Theory to Numerical Analysis*, Proc. Sympos., Univ. Montréal, Montreal, Que., 1971, Academic Press, New York, 1972, pp. 371–388 (MR0447161 (**56** #5476); Zbl. 0245.10038).
[a] E. Hlawka – R. Mück: *Über eine Transformation von gleichverteilten Folgen. II*, Computing (Arch. Elektron. Rechnen) **9** (1972), 127–138 (MR0453682 (**56** #11942); Zbl. 0245.10039).

2.3.11. Assume that
(i) $f : \mathbb{R} \to \mathbb{R}$ is a continuous and almost periodic function in the sense of H. Bohr (1933) (cf. 2.4.2),
(ii) $\alpha\pi$, $\alpha \in \mathbb{R}$, cannot be written as a finite linear combination (with rational coefficients) of Fourier exponents of f,
(iii) the sequence $x_n t \bmod 1$ is u.d. for every real $t \neq 0$ (i.e. x_n is u.d. in $\mathbb{R}$).
Then the sequence

$$(\alpha x_n + f(x_n)) \bmod 1$$

is

$$\text{u.d.}$$

NOTES: (I) H. Niederreiter and J. Schoißengeier (1977). Conditions (i) and (ii) imply that $(\alpha x + g(x)) \bmod 1$ is c.u.d. This is also true if the almost periodicity of $f(x)$ in the sense of Besicovitch is assumed in (i) (cf. 2.4.4).
(II) If f in (i) is periodic with period ω, then (ii) can be replaced by the condition that $\alpha\omega$ is irrational.

H. BOHR: *Fastperiodische Funktionen*, Ergebnisse d. Math. 1, Nr. 5, Springer, Berlin, 1932 (JFM 58.0264.01; Zbl. 0278.42019 Reprint 1974 Zbl. 0278.42019). (Reprint: *Almost Periodic Functions*, New York, Chelsea Publ. Comp., 1947 (MR0020163 (8,512a))).
H. NIEDERREITER – J. SCHOISSENGEIER: *Almost periodic functions and uniform distribution* mod 1, J. Reine Angew. Math. **291** (1977), 189–203 (MR0437482 (**55** #10412); Zbl. 0338.10053).

2.3.12. Let $x_n^{(i)}$, $i = 1, 2, \dots, k$, be u.d. sequences mod 1 and let x_n be the sequence composed from the terms $x_n^{(i)}$, $i = 1, 2, \dots, k$, in such a way that the order from the original sequences remains preserved. Then

$$x_n \bmod 1$$

is

$$\text{u.d.}$$

NOTES: Given an N, let N_i denote the number of terms of $x_n^{(i)}$ in the initial segment $x_1, \dots, x_N$. The u.d. of x_n follows directly from Weyl's criterion 2.1.2, since

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} = \lim_{N\to\infty} \sum_{i=1}^{k} \frac{N_i}{N} \left(\frac{1}{N_i} \sum_{n=1}^{N_i} e^{2\pi i h x_n^{(i)}} \right) = 0.$$

2.3.13. Let y_n and z_n be two sequences in $[0,1)$ having a.d.f.'s $g_1(x)$ and $g_2(x)$, resp. Let x_n be the sequence composed from the terms of y_n and z_n in such a way that their order from the original sequences remains preserved. Given an N, let N_1 and N_2 denote the number of terms of y_n and z_n, resp., in the initial segment $x_1,\dots,x_N$. Then the set of d.f.'s $G(x_n)$ of x_n satisfies

$$G(x_n)\subset\{tg_1(x)+(1-t)g_2(x),t\in[0,1]\}\ (=H)$$

and

$$D_N^{(2)}(x_n,H)\le\left(\frac{N_1}{N}\sqrt{D_{N_1}^{(2)}(y_n,g_1)}+\frac{N_2}{N}\sqrt{D_{N_2}^{(2)}(z_n,g_2)}\right)^2$$

while

$$G(x_n)=H\quad\iff\quad\limsup_{N\to\infty}\frac{N_1}{N}=\limsup_{N\to\infty}\frac{N_2}{N}=1.$$

NOTES: O. Strauch (1997). Clearly, if y_n mod 1 and z_n mod 1 are u.d., then so is the sequence $y_1,z_1,y_2,z_2,\dots$ mod 1.

O. STRAUCH: *On the set of distribution functions of a sequence*, in: *Proceedings of the Conference on Analytic and Elementary Number Theory in Honor of Edmund Hlawka's 80th Birthday, Vienna, July 18–20, 1996*, (W.–G. Nowak, J. Schoißengeier eds.), Universität Wien & Universität für Bodenkultur, Vienna, 1997, pp. 214–229 (Zbl. 0886.11044).

2.3.14. Suppose that the sequence of blocks $A_n=(x_{n,1},\dots,x_{n,N_n})$ satisfies

(i) $\lim_{n\to\infty}\frac{A([0,x);A_n])}{N_n}=g(x)$ a.e. on $[0,1]$, and

(ii) $\lim_{n\to\infty}\frac{N_n}{N_1+\cdots+N_n}=0$.

Then the block sequence $\omega=(A_n)_{n=1}^\infty$ has the a.d.f.

$$g(x).$$

NOTES: [KN, p. 136, Lem. 41]. The case $g(x)=x$ was studied in Š. Porubský, T. Šalát and O. Strauch (1990, Prop. 1) where it is proved that:
(a) Property (i) implies that ω is almost u.d.
(b) Properties (i) and (ii) imply that ω is u.d. independently of the order in which the terms of the blocks A_n are given.
(c) If (i) but not (ii) is true then it is possible to rearrange the terms of the blocks A_n in such a way that the corresponding ω is not u.d.
(d) If (i) holds and ω is not u.d., then the terms of A_n can be rearranged in such a way that the corresponding sequence ω is u.d. Moreover, if the terms of the blocks A_n were originally ordered according to their magnitude, then there exits

such rearrangement which depends only on the number N_n of terms in A_n and not on the terms themselves.

Š. Porubský – T. Šalát – O. Strauch: *On a class of uniform distributed sequences*, Math. Slovaca **40** (1990), 143–170 (MR1094770 (92d:11076); Zbl. 0735.11034).

2.3.15. Let x_n be the sequence in $[0,1]$ with an infinite set $G(x_n)$ of d.f.'s. Let $f:[0,1]\to\mathbb{R}$ be a given continuous function of modulus $|f|\le c$. Then the sequence y_n of arithmetic means

$$y_n=\frac{1}{n}\sum_{i=1}^{n}f(x_i)$$

is dense in the interval $[m,M]$, where

$$m=\min_{g\in G}\int_0^1 f(x)\,\mathrm{d}g(x)\qquad\text{and}\qquad M=\max_{g\in G}\int_0^1 f(x)\,\mathrm{d}g(x).$$

The dispersion $d_N=\max_{x\in[m,M]}\min_{1\le n\le N}|x-y_n|$ is bounded by

$$d_N\le\max\left(y_{N_1}-m\,,M-y_{N_2}\,,\frac{c}{\min(N_1,N_2)}\right)$$

for any $N_1,N_2\le N$.

O. Strauch: *On the set of distribution functions of a sequence*, in: *Proceedings of the Conference on Analytic and Elementary Number Theory in Honor of Edmund Hlawka's 80th Birthday, Vienna, July 18–20, 1996*, (W.–G. Nowak, J. Schoißengeier eds.), Universität Wien & Universität für Bodenkultur, Vienna, 1997, pp. 214–229 (Zbl. 0886.11044).

2.3.16. Let x_n be a non–decreasing sequence of positive real numbers. Then the sequence

$$y_n=\frac{x_n}{n+x_n}$$

is

dense in the interval $[\liminf_{n\to\infty}y_n,\limsup_{n\to\infty}y_n]$.

Notes: G. Pólya and G. Szegő (1964, Part 2, Ex. 103).

G. Pólya – G. Szegő: *Aufgaben und Lehrsätze aus der Analysis*, Vol. 1 & 2, 3rd corr. ed., Grundlehren d. math. Wiss., Band 19, Springer Verlag, Berlin, Göttingen, Heidelberg, New York, 1964 (MR0170985 (**30** #1219a); MR0170986 (**30** #1219b); Zbl. 0122.29704).

2.3.17. Let x_n and y_n be two sequences of positive real numbers. If
(i) x_n and y_n are unbounded, and
(ii) $\limsup_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$,
then the double sequence

$$\frac{x_m}{y_n}, \quad m, n = 1, 2, \dots,$$

is

dense in the interval $[0, \infty)$.

NOTES: D. Andrica and S. Buzeteanu (1987, 2.1. Th.)

RELATED SEQUENCES: 2.3.22

D. Andrica – S. Buzeteanu: *Relatively dense universal sequences for the class of continuous periodical functions of period T*, Math. Rev. Anal. Numér. Théor. Approximation, Anal. Numér. Théor. Approximation **16** (1987), no. 1, 1–9 (MR0938777 (89i:11084); Zbl. 0642.26006).

2.3.18. Let x_n and y_n be two sequences of positive real numbers. If
(i) x_n and y_n are unbounded, and
(ii) $\limsup_{n\to\infty}(x_{n+1} - x_n) = 0$,
then the double sequence

$$x_m - y_n, \quad m, n = 1, 2, \dots,$$

is

dense in $(-\infty, \infty)$.

NOTES: D. Andrica and S. Buzeteanu (1987, 2.3. Coroll.)

D. Andrica – S. Buzeteanu: *Relatively dense universal sequences for the class of continuous periodical functions of period T*, Math. Rev. Anal. Numér. Théor. Approximation, Anal. Numér. Théor. Approximation **16** (1987), no. 1, 1–9 (MR0938777 (89i:11084); Zbl. 0642.26006).

2.3.19. Let $x_1, x_2, \dots, x_N$ and $y_1, y_2, \dots, y_N$ be two finite sequences in $[0, 1)$. The star discrepancy of $|x_1 - y_1|, |x_2 - y_2|, \dots, |x_N - y_N|$ with respect to the d.f. $g(x) = 2x - x^2$ and the star discrepancy of $(x_1, y_1), (x_2, y_2), \dots, (x_N, y_N)$ with respect to the d.f. $g(x, y) = xy$ satisfy

$$D_N^*(|x_n - y_n|) \le 4\sqrt{D_N^*((x_n, y_n))}.$$

NOTES: O. Strauch, M. Paštéka and G. Grekos (2003). The given constant 4 is better than $8\sqrt{2}+1$ given in [KN, p. 95, Th. 1.6.] for the isotropic discrepancy 1.11.9. They also proved that for the Kth moment, $K = 1, 2, \dots,$

$$\left| \frac{1}{N} \sum_{n=1}^{N} |x_n - y_n|^K - \frac{2}{(K+1)(K+2)} \right| \le 4\, D_N^*((x_n, y_n)).$$

O. STRAUCH – M. PAŠTÉKA – G. GREKOS: *Kloosterman's uniformly distributed sequence*, J. Number Theory **103** (2003), no. 1, 1–15 (MR2008062 (2004j:11081); Zbl. 1049.11083).

2.3.20. If the sequence $(x_1, y_1), \dots, (x_N, y_N)$ of points in $[0, 1)^2$ is invariant under the maps
(i) $(x, y) \to (y, x)$,
(ii) $(x, y) \to (1 - x, 1 - y)$,
i.e. for any $m \le N$ there exist $n_1, n_2 \le N$ such that $(x_{n_1}, y_{n_1}) = (y_m, x_m)$ and $(x_{n_2}, y_{n_2}) = (1 - x_m, 1 - y_m)$, then

$$D_N^*(|x_n - y_n|) \le 3D_N((x_n, \{y_n - x_n\})) + D_N((x_n, y_n)).$$

Here $D_N^*(|x_n - y_n|)$ denotes the star discrepancy of $|x_1 - y_1|, |x_2 - y_2|, \dots, |x_N - y_N|$ with respect to the d.f. $g(x) = 2x - x^2$ and $D_N((x_n, \{y_n - x_n\}))$ and $D_N((x_n, y_n))$ are the classical extremal discrepancies of sequences $(x_1, \{y_1 - x_1\}), \dots, (x_N, \{y_N - x_N\})$ and $(x_1, y_1), (x_2, y_2), \dots, (x_N, y_N)$, resp., with $\{y_i - x_i\}$ denoting the fractional part of $y_i - x_i$.
NOTES: O. Strauch, M. Paštéka and G. Grekos (2003). They generalized a result proved by W. Zhang (1996). The invariance under (ii) can be replaced by the condition that $x_n \ne y_n$ for $n = 1, 2, \dots, N$.

RELATED SEQUENCES: 2.20.35.

O. STRAUCH – M. PAŠTÉKA – G. GREKOS: *Kloosterman's uniformly distributed sequence*, J. Number Theory **103** (2003), no. 1, 1–15 (MR2008062 (2004j:11081); Zbl. 1049.11083).
W. ZHANG: *On the distribution of inverse modulo n*, J. Number Theory **61** (1996), no. 2, 301–310 (MR1423056 (98g:11109); Zbl. 0874.11006).

2.3.21. Let x_n and y_n be two sequences in $[0,1)$ and let $G((x_n,y_n))$ denote the set of all d.f.'s of the two–dimensional sequence (x_n,y_n). If $z_n = x_n + y_n \bmod 1$. Then the set $G(z_n)$ of all d.f.'s of z_n has the form

$$G(z_n) = \Bigg\{ g(t) = \int\limits_{0\le x+y<t} 1.\,\mathrm{d}g(x,y) + \\ + \int\limits_{1\le x+y<1+t} 1.\,\mathrm{d}g(x,y)\ ;\ g(x,y)\in G((x_n,y_n)) \Bigg\}$$

provided that the all Riemann – Stieltjes integrals exist.

NOTES: O. Strauch and O. Blažeková (2003) and for an application cf. 2.12.16.

O. STRAUCH – O. BLAŽEKOVÁ: *Distribution of the sequence p_n/n mod 1*, Math. Institute, Slovak Acad. Sci., Bratislava, Slovak Republic, 2003, 15 pp.

2.3.22. Let x_n and y_n be two sequences of positive real numbers and let $f(x)$ and $g(x)$ be two positive real functions defined on $(0,\infty)$. Assume that
(i) x_n and y_n are unbounded,
(ii) there exists a subsequence x_{k_n} such that $\limsup_{n\to\infty} \frac{x_{k_{n+1}}}{x_{k_n}} = 1$,
(iii) $f(xy) \ge f(x)f(y)$ for every $x,y>0$,
(iv) $f(x)$ is increasing and unbounded on $(0,\infty)$,
(v) $f(x)$ is continuous at $x=1$ and $f(1)=1$,
(vi) $\lim_{x\to\infty} g(x) = \infty$.
Then the double sequence

$$\frac{f(x_m)}{g(y_n)},\quad m,n = 1,2,\dots,$$

is

dense in the interval $[0,\infty)$.

RELATED SEQUENCES: 2.3.17

B. LÁSZLÓ – J.T. TÓTH: *Relatively (R)–dense universal sequences for certain classes of functions*, Real Anal. Exchange **21** (1995/96), no. 1, 335–339 (MR1377545 (97a:26013); Zbl. 0851.11016).

2.3.23. Let y_n be the normal order of x_n. Then the sequence

$$\frac{x_n}{y_n}, \quad n = 1, 2, \dots,$$

has with respect to $(-\infty, \infty)$ the a.d.f.

$$c_1(x).$$

NOTES: Note, that the sequence x_n has the normal order y_n (Hardy – Wright (1954, pp. 356–359)) if for every positive ε and almost all values n we have $(1-\varepsilon)y_n < x_n < (1+\varepsilon)y_n$. Clearly (as it is mentioned in A. Schinzel and T. Šalát (1994)) x_n has the normal order y_n if and only if x_n/y_n statistically converges to 1. Some known examples are:
(I) The normal order of $\omega(n)$ is $\log\log n$, where $\omega(n)$ denotes the number of different prime factors of n (Hardy – Wright (1954, pp. 356–359)).
(II) The normal order of $\Omega(n)$ is $\log\log n$, where $\Omega(n)$ denotes the total number of prime factors of n Hardy – Wright (1954, pp. 356–359).
(III) The normal order of $\log d(n)$ is $\log 2 \log\log n$, where $d(n)$ denotes the number of divisors of n, i.e. $d(n) = \sum_{d|n,d>0} 1$. Hardy – Wright (1954, pp. 356–359).
(IV) The normal order of $\omega(\varphi(n))$ is $(\log\log n)^2/2$, cf. Mitrinović – Sándor – Crstici (1996, p. 36).
(V) The normal order of $\omega(\sigma_k(n))$ is $d(k)(\log\log n)^2/2$, where $\sigma_k(n) = \sum_{d|n,d>0} d^k$, cf. Mitrinović – Sándor – Crstici (1996, p. 96).
(VI) $\omega(p \pm 1)$ has the normal order $\log\log p$, where p is a prime, cf. Mitrinović – Sándor – Crstici (1996, p. 171).

G.H. Hardy – E.M. Wright: *An Introduction to the Theory of Numbers*, 3nd edition ed., Clarendon Press, Oxford, 1954 (MR0067125 (16,673c); Zbl. 0058.03301).
D.S. Mitrinović – J. Sándor – J. Crstici: *Handbook of Number Theory*, Mathematics and its Applications, Vol. 351, Kluwer Academic Publishers Group, Dordrecht, Boston, London, 1996 (MR1374329 (97f:11001); Zbl. 0862.11001).
A. Schinzel – T. Šalát: *Remarks on maximum and minimum exponents in factoring*, Math. Slovaca **44** (1994), no. 5, 505–514 (MR1338424 (96f:11017a); Zbl. 0821.11004).

2.3.24. Let $\mathbf{x}_n = (x_{n,1}, \dots, x_{n,s})$ and $\mathbf{y}_n = (y_{n,1}, \dots, y_{n,s})$ be infinite sequences in $[0,1)^s$ and assume that the sequence $(\mathbf{x}_n, \mathbf{y}_n)$ is u.d. in $[0,1]^{2s}$ (i.e. $\mathbf{x}_n, \mathbf{y}_n$ are u.d. and statistically independent). Then the sequence of the inner (i.e. scalar) products

$$x_n = \mathbf{x}_n \cdot \mathbf{y}_n = \sum_{i=1}^{s} x_{n,i} y_{n,i}$$

has the a.d.f. $g_s(x)$ on the interval $[0,s]$, where

$$g_s(x) = \left|\{(\mathbf{x},\mathbf{y}) \in [0,1]^{2s}\ ;\ \mathbf{x}\cdot\mathbf{y} < x\}\right| = \\ = (-1)^s \int\limits_{\substack{x_1+\cdots+x_s<x \\ x_1\in[0,1],\dots,x_s\in[0,1]}} 1.\log x_1 \dots \log x_s \,\mathrm{d}x_1 \dots \mathrm{d}x_s,$$

and its density is

$$g_s'(x) = \begin{cases} \int_0^x g_j'(t) g_{s-j}'(x-t)\,\mathrm{d}t, & \text{if } x \in [0,j], \\ \int_0^j g_j'(t) g_{s-j}'(x-t)\,\mathrm{d}t, & \text{if } x \in [j, s-j], \\ \int_{x-s+j}^j g_j'(t) g_{s-j}'(x-t)\,\mathrm{d}t, & \text{if } x \in [s-j, s]. \end{cases}$$

For $x \in [0,1]$ we have

$$\begin{aligned} g_1(x) =& x - \log x, \\ g_2(x) =& \frac{x^2}{2}\left((\log x)^2 - 3\log x + \frac{7}{2} - \frac{1}{6}\pi^2\right), \\ g_3(x) =& \frac{x^3}{27}\Bigg(-\frac{9}{2}(\log x)^3 + \frac{99}{4}(\log x)^2 + \left(-\frac{255}{4} + \frac{9}{4}\pi^2\right)\log x \\ & + \frac{575}{8} - \frac{33}{8}\pi^2 - 9\zeta(3)\Bigg), \end{aligned}$$

while for general s (and $x \in [0,1]$) we have

$$g_s(x) = (-1)^s x^s \sum_{j=0}^{s} \binom{s}{j} (\log x)^{s-j} \widetilde{g}_j,$$

where

$$\widetilde{g}_j = \int\limits_{\substack{x_1+\cdots+x_s<1 \\ x_1\in[0,1],\dots,x_s\in[0,1]}} \log x_1 \dots \log x_j \,\mathrm{d}x_1 \dots \mathrm{d}x_s \\ = \frac{1}{(s-j)!} \cdot \int\limits_{[0,1]^j} \prod_{i=1}^{j} \big(\log x_1 + \cdots + \log x_{j-1} + \\ + \log(1-x_j)\big) x_1^{s-1} \dots x_j^{s-j} \,\mathrm{d}x_1 \dots \mathrm{d}x_j.$$

NOTES: (I) O. Strauch (2003). The formula for $g_s(x)$ with $x \in [0,1]$ was proved by L. Habsieger (Bordeaux) (personal communication). He also observed that $\widetilde{g}_j$ is a composition of integrals of the type

$$\begin{aligned}\int_0^1 (\log x)^m x^n \,\mathrm{d}x =& \frac{(-1)^m m!}{(n+1)^{m+1}},\\ \int_0^1 (\log x)^m x^n \log(1-x)\,\mathrm{d}x =&(-1)^{m+1} m! \sum_{k=1}^{\infty} \frac{1}{k(k+n+1)^{m+1}}\\ =&a_0 + a_1\zeta(2) + \cdots + a_m\zeta(m+1), \quad a_i \in \mathbb{Q}.\end{aligned}$$

The explicit formula of $g_s(x)$ for $x \in [1,s]$ is open.
(II) E. Hlawka (1982) investigated the question of the distribution of the scalar product of two vectors on an s–dimensional sphere and also the problem of the associated discrepancies.

E. HLAWKA: *Gleichverteilung auf Produkten von Sphären*, J. Reine Angew. Math. **330** (1982), 1–43 (MR0641809 (83i:10066); Zbl. 0462.10034).
O. STRAUCH: *On distribution functions of sequences generated by scalar and mixed product*, Math. Slovaca **53** (2003), no. 5, 467–478 (MR2038514 (2005d:11108); Zbl. 1061.11042).

2.3.25. Let $\mathbf{x}_n^{(i)} = (x_{n,1}^{(i)}, \dots, x_{n,s}^{(i)})$, $i = 1, \dots, s$, be infinite sequences in the s–dimensional ball $B(r)$ with the center at $(0,\dots,0)$ and radius r. Assume that these sequences are u.d. and statistically independent in $B(r)$, i.e. $(\mathbf{x}_n^{(1)}, \dots, \mathbf{x}_n^{(s)})$ is u.d. in $B(r)^s$. Then the sequence

$$x_n = \left|\det(\mathbf{x}_n^{(1)}, \dots \mathbf{x}_n^{(s)})\right|$$

has the a.d.f. $g_s(r,x)$ defined on the interval $[0, r^s]$ by

$$g_s(r,x) = \frac{\left|\{(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(s)}) \in B(r)^s\, ;\, |\det(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(s)})| < x\}\right|}{|B(r)|^s},$$

and for

$$\lambda = \frac{x}{r^s}$$

there exists $\widetilde{g}_s(\lambda)$ such that

$$g_s(r,x) = \widetilde{g}_s(\lambda) \quad \text{ if } \lambda \in [0,1].$$

Here we have

$$\widetilde{g}_1(\lambda) = \lambda,$$
$$\widetilde{g}_2(\lambda) = \frac{2}{\pi}(1+2\lambda^2)\arcsin\lambda + \frac{6}{\pi}\lambda\sqrt{1-\lambda^2} - 2\lambda^2,$$
$$\widetilde{g}_3(\lambda) = 1 + \frac{9}{4}\lambda\int_\lambda^1 \frac{\arccos x}{x}\,\mathrm{d}x - \frac{3}{4}\lambda^3\arccos\lambda - \sqrt{1-\lambda^2} + \frac{7}{4}\lambda^2\sqrt{1-\lambda^2}.$$

NOTES: (I) O. Strauch (2003). The explicit form of $\widetilde{g}_s(\lambda)$ for $s > 3$ is open.
(II) A further open question is the explicit form of the a.d.f. of the above sequence with $[0,1]^s$ instead of $B(r)$.
(III) Note that the integral in $\widetilde{g}_3(\lambda)$ cannot be expressed as a finite combination of elementary functions, cf. I.M. Ryshik and I.S. Gradstein (1951, p. 122).
(IV) The d.f.'s $\widetilde{g}_s(\lambda)$ and $g_s(x)$ from 2.3.24 form the basis of a new one–time pad cryptosystem introduced in Strauch (2002).

I.M. RYSHIK – I.S. GRADSTEIN: *Tables of Series, Products, and Integrals*, (German and English dual language edition), VEB Deutscher Verlag der Wissenschaften, Berlin, 1957 (translation from the Russian original Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951 (MR0112266 (**22** #3120))).
O. STRAUCH: *On distribution functions of sequences generated by scalar and mixed product*, Math. Slovaca **53** (2003), no. 5, 467–478 (MR2038514 (2005d:11108); Zbl. 1061.11042).
O. STRAUCH: *Some modification of one–time pad cipher*, Tatra Mt. Math. Publ. **29** (2004), 157–171 (MR2201662 (2006i:94066); Zbl. 1114.11065).

2.3.26. Let $x_n = \sum_{i=1}^n y_i$ be the sequence of the partial sums of the series $\sum_{n=1}^\infty y_n$ of real numbers y_n satisfying $\lim_{n\to\infty} y_n = 0$. Then the sequence

$$x_n$$

is

dense in the interval $[\liminf_{n\to\infty} x_n, \limsup_{n\to\infty} x_n]$.

NOTES: G. Pólya and G. Szegő (1964, Part 2, Exer. 101).

G. PÓLYA – G. SZEGŐ: *Aufgaben und Lehrsätze aus der Analysis*, Vol. 1 & 2, 3rd corr. ed., Grundlehren d. math. Wiss., Band 19, Springer Verlag, Berlin, Göttingen, Heidelberg, New York, 1964 (MR0170985 (**30** #1219a); MR0170986 (**30** #1219b); Zbl. 0122.29704).

2.3.27. Let x_n be a bounded sequence of real numbers for which a sequence of positive real numbers ε_n exists such that $\lim_{n\to\infty} \varepsilon_n = 0$ and

$$x_{n+1} > x_n - \varepsilon_n$$

for every sufficiently large n. Then the sequence

$$x_n$$

is

$$\text{dense in the interval } [\liminf_{n\to\infty} x_n, \limsup_{n\to\infty} x_n].$$

NOTES: G. Pólya and G. Szegő (1964, Part 2, Exer. 102).

G. PÓLYA – G. SZEGŐ: *Aufgaben und Lehrsätze aus der Analysis*, Vol. 1 & 2, 3rd corr. ed., Grundlehren d. math. Wiss., Band 19, Springer Verlag, Berlin, Göttingen, Heidelberg, New York, 1964 (MR0170985 (**30** #1219a); MR0170986 (**30** #1219b); Zbl. 0122.29704).

2.3.28. If the unbounded sequence x_n satisfies

$$\lim_{n\to\infty}(x_{n+1} - x_n) = 0,$$

then the sequence

$$x_n \bmod 1$$

is

$$\text{dense in the interval } [0, 1].$$

NOTES: The proof is immediate. For a generalization cf. 2.6.32.

2.3.29. Let (a_n, b_n) be points in the interval $K = [u, u+v] \times [0, 1]$ and let $\Phi(a, b)$ be a density defined on K, i.e. $\Phi(a, b) \geq 0$ and $\iint_K \Phi(a, b)\,\mathrm{d}a\,\mathrm{d}b = 1$. Define the **extremal discrepancy** $\widetilde{D}_N$ **of the sequence** (a_n, b_n) **related to** Φ by

$$\widetilde{D}_N = \sup_{J\subset K}\left|\frac{1}{N}\sum_{n=1}^{N} c_J((a_n, b_n)) - \iint_J \Phi(a, b)\,\mathrm{d}a\,\mathrm{d}b\right|,$$

where J are intervals and $c_J(x, y)$ is the characteristic function of J. If the partial derivatives $\frac{\partial\Phi}{\partial a}$ and $\frac{\partial\Phi}{\partial b}$ are bounded on K, then for every $t > (\widetilde{D}_N)^{-1/4}$ and every $M > 0$, the one–dimensional sequence

$$a_n t + b_n \bmod 1$$

has the classical extremal discrepancy

$$D_N \le C_1 \left(\frac{1}{M} + \frac{C_2}{t} + \widetilde{D}_N t M^2 \right),$$

where $C_1 > 0$ is an absolute constant and $C_2 > 0$ depends on Φ.

NOTES: E. Hlawka (1998) gave this quantitative version of a Poincaré result dealing with the planetary motions. In this connection a_n is interpreted as the angular velocity and b_n as the starting angle of the orbit of the nth planet P_n at time $t = 0$. (All angles are measured on the circle which has the unit length.) Thus, if (a_n, b_n) is distributed with density $\Phi(a, b)$, then $a_n t + b_n \bmod 1$ is u.d. as $t \to \infty$.

E. HLAWKA: *Gleichverteilung und die willkürlichen Funktionen von Poincaré*, Math. Slovaca **48** (1998), no. 5, 457–506 (MR1697611 (2000j:11120); Zbl 0956.11016).

2.3.30. Let φ_n, $n = 1, 2, \dots,$ be the sequence in $[0, 1)$ which has the limit distribution with density $\rho(\varphi)$ and the extremal discrepancy D_N^ρ with respect to $\rho(\varphi)$. Let $J_{kr} = \left[\frac{k}{s} + \frac{r}{2s}, \frac{k}{s} + \frac{r+1}{2s}\right]$, $r = 0, 1$, $k = 0, \dots, s-1$, be a two–colored decomposition of $[0, 1]$, say, using the colors 0 and 1. Define the 0–1 sequence

$$x_n = c_J(\varphi_n),$$

where $J = \bigcup_{k=1}^{s} J_{k1}$. Assuming the Lipschitz condition $|\rho(\varphi) - \rho(\varphi')| \le \alpha|\varphi - \varphi'|$ we have

$$D_N \le \frac{\alpha}{s} + s D_N^\rho,$$

where D_N is the extremal discrepancy of x_n with respect to a.d.f. $h_{1/2}(x)$.

NOTES: E. Hlawka (1998) proved this quantitative version of a Poincaré result dealing with roulettes. He identified a roulette with the couple $\rho(\varphi)$ and J_{kr}, where the density $\rho(\varphi)$ characterizes the rotation of the roulette. Thus for roulettes with various densities the resulting sequence x_n has discrete distribution close to the u.d. Hlawka also gave similar bounds for m–colored roulette and for a composition of roulettes.

E. HLAWKA: *Gleichverteilung und die willkürlichen Funktionen von Poincaré*, Math. Slovaca **48** (1998), no. 5, 457–506 (MR1697611 (2000j:11120); Zbl 0956.11016).

2.4 Subsequences

2.4.1.

NOTES: Given an infinite sequence x_n mod 1, the **spectrum** of x_n, denoted by $\mathrm{sp}(x_n)$, is defined through

$$\mathrm{sp}(x_n) = \{\alpha \in [0,1]\,;\,(x_n - n\alpha) \bmod 1 \text{ is not u.d.}\}.$$

. . . .

A necessary and sufficient condition for the sequence

$$x_{qn+r} \bmod 1$$

to be

u.d.

for all integers $q \geq 1$ and $r \geq 0$ is that $\mathrm{sp}(x_n) \cap \mathbb{Q} = \emptyset$.

NOTES: (I) M. Mendès France (1975). The definition of $\mathrm{sp}(x_n)$ can be found in his paper (1973).

(II) N.M. Korobov and A.G. Postnikov (1952) proved that the u.d. of the sequence of differences $(x_{n+h} - x_n)$ mod 1 implies the u.d. of all the subsequences x_{qn+r} mod 1, $n = 1, 2, \ldots$, with integral $q \geq 1$ and $r \geq 0$ (cf. 2.2.1).

(III) G. Myerson and A. Pollington (1990) proved that there is a sequence x_n which is u.d. mod 1 even though no subsequence of the form x_{qn+r} mod 1 with $q \geq 2$ is u.d.

(IV) Note that if the subsequence x_{qn+r} mod 1 is u.d. for a fixed positive integer q and for every $0 \leq r < q$, then x_n mod 1 is u.d.

N.M. KOROBOV – A.G. POSTNIKOV: *Some general theorems on the uniform distribution of fractional parts*, (Russian), Dokl. Akad. Nauk SSSR (N.S.) **84** (1952), 217–220 (MR0049246 (14,143e); Zbl. 0046.27802).

M. MENDÈS FRANCE: *Les suites à spectre vide et la répartition modulo 1*, J. Number Theory **5** (1973), 1–15 (MR0319909 (**47** #8450); Zbl. 0252.10033).

M. MENDÈS FRANCE: *Les ensembles de Bésineau*, in: *Séminaire Delange-Pisot-Poitou (15e année: 1973/74), Théorie des nombres, Fasc. 1, Exp. No. 7*, Secrétariat Mathématique, Paris, 1975, 6 pp. (MR0412139 (**54** #266); Zbl. 0324.10049).

G. MYERSON – A.D. POLLINGTON: *Notes on uniform distribution modulo one*, J. Austral. Math. Soc. **49** (1990), 264–272 (MR1061047 (92c:11075); Zbl. 0713.11043).

2.4.2.

NOTES: A non–decreasing unbounded sequence k_n of positive integers is called **almost periodic** if the generalized characteristic function $\chi(j) = \#\{n \in \mathbb{N}\,;\,k_n = j\}$, $j = 1, 2, \ldots$, is almost periodic in the sense of Besicovitch. Here (cf. also 2.4.4) a function $\psi : \mathbb{N} \to \mathbb{C}$ is called **almost periodic** if for every $\varepsilon > 0$ there exists a trigonometric polynomial $t(x) = \sum_{l=0}^{L} a_l e^{2\pi i \lambda_l x}$ ($L \geq 0$,$a_l \in \mathbb{C}$,$\lambda_l \in \mathbb{R}$) such that

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N-1} |\psi(n) - t(n)| < \varepsilon.$$

. . . .

If x_n has empty spectrum, i.e. $\mathrm{sp}(x_n) = \emptyset$ (in the sense of M. Mendès France cf. 2.4.1), then the sequence

$$x_{k_n} \bmod 1$$

is

$$\text{u.d.}$$

for every non–decreasing and unbounded sequence k_n of positive integer if and only if k_n is almost periodic.

NOTES: (I) Firstly noted by M. Mendès France (1973), and later also by H. Daboussi and M. Mendès France (1975) (cf. [DT, p. 103, Th. 1.113]). As examples of empty spectra they give the following instances:

- If the difference $(x_{n+k} - x_n) \bmod 1$ is u.d. for every $k = 1, 2, \dots$, then $\mathrm{sp}(x_n) = \emptyset$.
- If $q \geq 2$ is a given integer and θ is normal in the base q, then $\mathrm{sp}(\theta q^n) = \emptyset$.
- If $p(x)$ is a real polynomial of degree ≥ 2 such that $p(x) - p(0)$ has at least one irrational coefficient, then $\mathrm{sp}(p(n)) = \emptyset$.
- If f is a real entire function, other than a polynomial, and if for $|z| \to \infty$ $|f(z)| = \mathcal{O}(e^{(\log|z|)^\delta})$, where $1 < \delta < 5/4$, then again $\mathrm{sp}(p(n)) = \emptyset$.
- If f is a real function with a continuous second derivative such that
  (i) $f'(x) - f'(\varepsilon x) = \mathcal{O}(1)$ for all $\varepsilon > 0$, and
  (ii) $x^2 f''(x) \to \pm\infty$ when $x \to +\infty$,
  then $\mathrm{sp}(f(n)) = \emptyset$. For example, the conditions (i) and (ii) are fulfilled for $f(x) = (\log x)^\delta$ with $\delta > 1$, and for $f(x) = x(\log x)^\delta$ with $\delta \neq 0$.

The following examples of almost periodic sequences k_n are contained in their results (cf. [DT, p. 102–103, Lemma 1.111–2]):

- $k_n = [\beta n]$ for $\beta > 0$.
- the sequence k_n of those positive integers which are not divisible by any $q \in E$, where E is a set of positive integers such that $\sum_{q\in E} 1/q < \infty$.

(II) Y. Peres (1988) showed that the u.d. of the differences also implies the u.d. of $x_{[\alpha n]}$ for any non–zero $\alpha \in \mathbb{R}$, cf. 2.2.1(IV).

(III) H. Rindler (1973/74) and V. Losert and H. Rindler (1978) also studied strictly increasing sequences k_n of integers for which the u.d. of x_n implies the u.d. of x_{k_n}, for every sequence x_n.

H. DABOUSSI – M. MENDÈS FRANCE: *Spectrum, almost periodicity and equidistribution modulo 1*, Studia Sci. Math. Hungar. **9** (1974/1975), 173–180 (MR0374066 (**51** #10266); Zbl. 0321.10043).

V. LOSERT – H. RINDLER: *Teilfolgen gleichverteilter Folgen*, J. Reine Angew. Math. **302** (1978), 51–58 (MR0511692 (80a:10071); Zbl. 0371.10039).

M. MENDÈS FRANCE: *Les suites à spectre vide et la répartition modulo 1*, J. Number Theory **5** (1973), 1–15 (MR0319909 (**47** #8450); Zbl. 0252.10033).

Y. PERES: *Application of Banach limits to the study of sets of integers*, Israel J. Math. **62** (1988), no. 1, 17–31 (MR0947826 (90a:11088); Zbl. 0656.10050).

H. RINDLER: *Ein Problem aus der Theorie der Gleichverteilung, II*, Math. Z. **135** (1973/1974), 73–92 (MR0349614 (**50** #2107); Zbl. 0263.22009).

2.4.3. Let $h(n)$ be an increasing sequence of positive integers satisfying $h(n) \le cn$ for some constant c. If the sequence x_n in $[0,1)$ has a.d.f. $g(x)$ then the sequence

$$x_{h(n)}$$

again has the a.d.f.

$$g(x)$$

if and only if the sequence z_n defined by

$$z_n = \begin{cases} 1, & \text{if } n \in h(\mathbb{N}), \\ 0, & \text{otherwise} \end{cases}$$

is statistically independent with x_n.
NOTES: G. Rauzy (1976, p. 95, 5.1. Th.).

G. RAUZY: *Propriétés statistiques de suites arithmétiques*, Le Mathématicien, Vol. 15, Collection SUP, Presses Universitaires de France, Paris 1976, 133 pp. (MR0409397 (**53** #13152); Zbl. 0337.10036).

2.4.4.
NOTES: Following G. Rauzy (1976, p. 100) define:

(i) The strictly increasing function $h : \mathbb{N} \to \mathbb{N}$ is called **almost periodic in the sense of Besicovitch**, if for every $\varepsilon > 0$ there exists an $s \ge 1$ and complex numbers $c_1, \dots, c_s$ and real numbers $\alpha_1, \dots, \alpha_s$ mod 1 such that

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left| \chi(n) - \sum_{k=1}^{s} c_k e^{2\pi i \alpha_k n} \right| < \varepsilon,$$

where $\chi(n)$ is a characteristic function of the set $h(\mathbb{N})$ (cf. J. Marcinkiewicz (1939)).

(ii) The **Bohr spectrum (**or **Fourier – Bohr spectrum)** $\mathrm{Bsp}(\psi)$ **of the function** $\psi : \mathbb{N} \to \mathbb{C}$ is the set of all α mod 1 for which

$$\limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \psi(n) e^{-2\pi i \alpha n} \right| > 0.$$

(iii) The **Bohr spectrum** $\mathrm{Bsp}(x_n)$ **of the sequence** $x_n \in [0,1)$ is the union of all $\mathrm{Bsp}(\psi)$ with $\psi(n) = f(x_n)$, where f is continuous (note that $\mathrm{Bsp}(x_n)$ does not coincides with $\mathrm{sp}(x_n)$ defined in 2.4.1). For another definition of $\mathrm{Bsp}(x_n)$ see 3.11.

(iv) The **Bohr spectrum** Bsp(h) **of the increasing function** $h : \mathbb{N} \to \mathbb{N}$ is defined as Bsp(χ) where χ is the characteristic function of $h(\mathbb{N})$.

. . . .

Let the sequence x_n in $[0, 1)$ have the a.d.f. $g(x)$ and let $h(n)$ be almost periodic in the sense of Besicovitch. If 0 is the only common point of Bsp(h) and Bsp(x_n), then the sequence

$$x_{h(n)}$$

again has the a.d.f.

$$g(x).$$

NOTES: G. Rauzy (1976, p. 100, 6.4. Th.). He also proved (1976, p. 102, 6.5. Cor.): Let x_n mod 1 be u.d. If $x_n + n\alpha$ mod 1 is u.d. for every α, and $h(n)$ is almost periodic, such that $h(n) \le cn$ for some constant c, then

$$x_{h(n)}$$

is

$$\text{u.d.}$$

J. Marcinkiewicz: *Une remarque sur les espaces de Besicovitch*, C. R. Acad. Sci. Paris **208** (1939), 157–159 (Zbl. 0020.03104).

G. Rauzy: *Propriétés statistiques de suites arithmétiques*, Le Mathématicien, Vol. 15, Collection SUP, Presses Universitaires de France, Paris 1976, 133 pp. (MR0409397 (**53** #13152); Zbl. 0337.10036).

$\boxed{\textbf{2.4.4.1}}$ Let x_n, $n = 1, 2, \dots$, be a dense sequence in $[0, 1]$ having an a.d.f. $g(x)$. Put $d_0 = \inf_{x\in[0,1]} g'(x)$ (here inf is taken over those $x \in [0, 1]$ at which $g(x)$ is differentiable). Then for every $0 \le d \le d_0$ there exists a subsequence

$$x_{h(n)}$$

which is

$$\text{u.d.}$$

and the asymptotic density of $h(n)$ is d, i.e.

$$\lim_{n\to\infty} \frac{n}{h(n)} = d.$$

NOTES:

Y. Dupain and J. Lesca applied this result to the sequence $x_n = \theta^n$ mod 1, where θ is a Salem number (see 3.21.5). In particular, they observed, that the asymptotic density of $h(n)$ gets arbitrarily close to 1, as the degree of θ increases. In other words, x_n "approaches" the u.d. with increasing degree of θ.

Y. Dupain – J. Lesca: *Répartition des sous-suites d'une suite donnée*, Acta Arith. **23** (1973), 307–314 (MR0319884 (**47** #8425); Zbl. 0263.10021).

2.5 Transformations of sequences

Notes: Given a sequence x_n in $[0,1)$, let the sequence y_n be defined by one of the following ways:

- $y_n = x_1 + \cdots + x_n \bmod 1$,
- $y_n = \frac{x_1+\cdots+x_n}{n}$,
- $y_n = nx_n \bmod 1$,
- $\mathbf{y}_n = (x_{n+1}, \dots, x_{n+s})$,
- $\mathbf{y}_n = (x_{2n-1}, x_{2n})$,
- y_n is the sequence $F(x_m, x_n) \bmod 1$ for $m, n = 1, 2, \dots$, ordered in such way that the values $F(x_m, x_n) \bmod 1$ with $m, n = 1, 2, \dots, N$, form the first N^2 terms of y_n, $n = 1, 2, \dots$, where $F : [0,1]^2 \to \mathbb{R}$.

In every of the above cases the connection between $G(x_n)$ and $G(y_n)$ is an open problem. In what follows some results will be presented if $G(x_n)$ is a singleton and $y_n = f(x_n)$, where $f : [0,1] \to [0,1]$.

2.5.1. u.d.p. maps. The map $f : [0,1] \to [0,1]$ is called **uniform distribution preserving** (abbreviated u.d.p.) if for any u.d. sequence x_n, $n = 1, 2, \dots$, in $[0,1]$ the sequence $f(x_n)$ is also u.d.

. . . .

A Riemann integrable function $f : [0,1] \to [0,1]$ is a u.d.p. transformation if and only if one of the following conditions is satisfied:

(i) $\int_0^1 h(x)\,\mathrm{d}x = \int_0^1 h(f(x))\,\mathrm{d}x$ for every continuous $h : [0,1] \to \mathbb{R}$.

(ii) $\int_0^1 (f(x))^k\,\mathrm{d}x = \frac{1}{k+1}$ for every $k = 1, 2, \dots$.

(iii) $\int_0^1 e^{2\pi i k f(x)}\,\mathrm{d}x = 0$ for every $k = \pm 1, \pm 2, \dots$.

(iv) There exists an increasing sequence of positive integers N_k and an N_k–almost u.d. sequence x_n for which the sequence $f(x_n)$ is also N_k–almost u.d.

(v) There exists an almost u.d. sequence x_n in $[0,1)$ such that the sequence $f(x_n) - x_n$ converges to a finite limit.

(vi) There exists at least one $x \in [0,1]$ which orbit $x, f(x), f(f(x)), \dots$ is almost u.d.

(vii) f is measurable in the Jordan sense and $|f^{-1}(I)| = |I|$ for every subinterval $I \subset [0,1]$.

(viii) $\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 x\,\mathrm{d}x = \frac{1}{2}$,
$\int_0^1 (f(x))^2\,\mathrm{d}x = \int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}$,

$\int_0^1 \int_0^1 |f(x) - f(y)|\, \mathrm{d}x\, \mathrm{d}y = \int_0^1 \int_0^1 |x - y|\, \mathrm{d}x\, \mathrm{d}y = \frac{1}{3}$.

From the other properties of u.d.p. transformations let us mention:

(ix) Let f_1, f_2 be u.d.p. transformations and α a real number. Then $f_1(f_2(x))$, $1 - f_1(x)$ and $f_1(x) + \alpha \bmod 1$ are again u.d.p. transformations.

(x) Let f_n be a sequence of u.d.p. transformations uniformly converging to f. Then f is u.d.p.

(xi) Let $f : [0,1] \to [0,1]$ be piecewise differentiable. Then f is u.d.p. if and only if $\sum_{x \in f^{-1}(y)} \frac{1}{|f'(x)|} = 1$ for all but a finite number of points $y \in [0,1]$.

(xii) A piecewise linear transformation $f : [0,1] \to [0,1]$ is u.d.p. if and only if $|J_j| = |J_{j,1}| + \cdots + |J_{j,n_j}|$ for every $J_j = (y_{j-1}, y_j)$, where $0 = y_0 < y_1 < \cdots < y_m = 1$ is the sequences of ordinates of the ends of line segment components of the graph of f and $f^{-1}(J_j) = I_{j,1} \cup \cdots \cup J_{j,n_j}$.

Notes: (I) The problem to find all continuous u.d.p. is formulated in Ja.I. Rivkind (1973).

(II) The results (i)-(vii), (ix)-(xii) are proved in Š. Porubský, T. Šalát and O. Strauch (1988). The criterion (viii) is given in O. Strauch (1999, p. 116).

(III) Some parts of these results are also proved independently in W. Bosch (1988).

(IV) R.F. Tichy and R. Winkler (1991) gave a generalization for compact metric spaces.

(V) Some related results can be found in: M. Paštéka (1987), Y. Sun (1993, 1995), P. Schatte (1993), S.H. Molnár (1994) and J. Schmeling and R. Winkler (1995).

(VI) W.J. LeVeque (1953) found the following u.d.p. maps for u.d. sequences modulo subdivision $\Delta = (z_n)_{n=1}^{\infty}$ (for the def. cf. p. 1 – 6): Suppose that x_n is u.d. mod Δ and that

(i) f is a function which is differentiable except possibly at the points z_n, $n = 1, 2, \dots,$

(ii) $f(x)$ increases to ∞ as $x \to \infty$,

(iii) $\lim_{n\to\infty} \dfrac{\inf_{x\in(z_{n-1},z_n)} f'(x)}{\sup_{x\in(z_{n-1},z_n)} f'(x)} = 1.$

Then the sequence $f(x_n)$ is u.d. mod $\Delta^* = (f(z_n))_{n=1}^{\infty}$.

W. Bosch: *Functions that preserve uniform distribution*, Trans. Amer. Math. Soc. **307** (1988), no. 1, 143–152 (MR0936809 (89h:11046); Zbl. 0651.10032).

W.J. LeVeque: *On uniform distribution modulo a subdivision*, Pacific J. Math. **3** (1953), 757–771 (MR0059323 (15,511c); Zbl. 0051.28503).

S.H. Molnár: *Sequences and their transforms with identical asymptotic distribution function modulo* 1, Studia Sci. Math. Hungarica **29** (1994), no. 3–4, 315–322 (MR1304885 (95j:11071); Zbl. 0849.11053).

M. Paštéka: *On distribution functions of sequences*, Acta Math. Univ. Comenian. **50–51** (1987), 227–235 (MR0989415 (90e:11115); Zbl. 0666.10033).

Š. Porubský – T. Šalát – O. Strauch: *Transformations that preserve uniform distribution*, Acta Arith. **49** (1988), 459–479 (MR0967332 (89m:11072); Zbl. 0656.10047).

Ja.I. Rivkind: *Problems in Mathematical Analysis*, (Russian), 2nd edition ed., Izd. Vyšejšaja škola, Minsk, 1973. (For the English translation of the first edition see MR0157880 (**28** #1109) or Zbl. 0111.05203).
P. Schatte: *On transformations of distribution functions on the unit interval- a generalization of the Gauss – Kuzmin – Lévy theorem*, Z. Anal. Anwend. **12** (1993), no. 2, 273–283 (MR1245919 (95d:11098); Zbl. 0778.58042).
J. Schmeling – R. Winkler: *Typical dimension of the graph of certain functions*, Monatsh. Math. **119** (1995), 303–320 (MR1328820 (96c:28005); Zbl. 0830.28004).
O. Strauch: *Distribution of Sequences* (in Slovak), Mathematical Institute of the Slovak Academy of Sciences, DSc Thesis, Bratislava, Slovakia, 1999.
Y. Sun: *Some properties of uniform distributed sequences*, J. Number Theory **44** (1993), no. 3, 273–280 (MR1233289 (94h:11068); Zbl. 0780.11035).
Y. Sun: *Isomorphisms for convergence structures*, Adv. Math. **116** (1995), no. 2, 322–355 (MR1363767 (97c:28031); Zbl. 0867.28003).
R.F. Tichy – R. Winkler: *Uniform distribution preserving mappings*, Acta Arith. **60** (1991), no. 2, 177–189 (MR1139054 (93c:11054); Zbl. 0708.11034).

2.5.2. u.d.p. sequences of maps. Open problem. A sequence of maps $f_n(x) : [0,1] \to [0,1]$, $n = 1,2,\dots$, is called **uniform distribution preserving** (abbreviated u.d.p.) if for any u.d. sequence x_n, $n = 1,2,\dots$, the sequence $f_n(x_n)$ remains u.d. The problem is to characterize such sequences of maps.

Notes: (I) Sequences of such maps were introduced by R. Winkler (1998). He gave a complete characterization of u.d.p. sequences of maps on finite sets: Let X and Y be finite sets equipped with probability measures λ and μ, resp., such that $\lambda_i > 0$ for $i \in X$. Then the sequence $f_n : X \to Y$, $n \in \mathbb{N}$, is called **$(\boldsymbol{\lambda}, \boldsymbol{\mu})$–u.d.p. sequence of maps** if the induced sequence $f_n(x_n)$ is μ–u.d. for every λ–u.d. sequence x_n.

A sequence $f_n : X \to Y$ is (λ, μ)–u.d.p. if and only if the following conditions hold:

(i) f_n is almost constant (the definition is given below),

(ii) The set of $n \in \mathbb{N}$ for which f_n is neither a constant map nor a u.d.p. map has zero asymptotic density,

(iii) The sequence $f_n = c_n =$ const. is μ–u.d. with respect to the set C of $n \in \mathbb{N}$ for which the map f_n is a constant (i.e.

$$\lim_{N\to\infty} \frac{\#\{n \in C \cap (0,N]\,;\, c_n = j\} - \mu_j \cdot \#(C \cap (0,N])}{N} = 0$$

for all $j \in Y$).

Here, according to Winkler, the sequence f_n is **almost constant**, if for every $\varepsilon > 0$ there exists a $q > 1$ such that for every $k \in \mathbb{N}$ either f_n is a fixed function for all $n \in ([q^{k-1}], [q^k]]$, or f_n are constant functions (not necessary the same) for all $n \in ([q^{k-1}], [q^k]]$ except for a set of n's of upper asymptotic density $< \varepsilon$.

(II) A complete characterization of u.d.p. sequence $f_n(x)$ with $f_n(x) =$ constant $= y_n$, has been given by G. Rauzy (1973), cf. 2.3.5.

(III) H. Rindler [Acta Arith. **35** (1979), no. 2, 189–193; MR 84a:22010] generalized

Rauzy's result to compact metric groups.
(IV) V. Losert [Monatsh. Math. **85** (1978), no. 2, 105–113; MR **57**# 16237] found a characterization of u.d.p. $f_n(x)$ if the $f_n(x)$ are measure–preserving maps defined on a compact metric probability space, or if they are affine transformations on compact metric groups.

G. Rauzy: *Étude de quelques ensembles de fonctions définis par des propertiétés de moyenne*, Séminaire de Théorie des Nombres (1972–1973), 20, Lab. Théorie des Nombres, Centre Nat. Recherche Sci., Talence, 1973, 18 pp. (MR0396463 (**53** #328); Zbl. 0293.10018).
R. Winkler: *Distribution preserving sequences of maps and almost constants sequences*, Monatsh. Math. **126** (1998), no. 2, 161–174 (MR1639383 (99h:11088); Zbl. 0908.11035).

2.5.3. d.p. sequences of maps. A sequence of maps $f_n : [0,1] \to [0,1]$, $n = 1,2,\dots$, is called **distribution preserving** (abbreviated d.p.) if for any two sequences $x_n, x'_n \in [0,1)$, the coincidence of the sets of distribution functions $G(x_n) = G(x'_n)$ always implies $G(f_n(x_n)) = G(f_n(x'_n))$.

A sequence of maps $f_n : [0,1] \to [0,1]$ is d.p. if and only if
(i) f_n is almost constant (cf. 2.5.2),
(ii) f_n is almost equicontinuous.

Here, f_n is **almost equicontinuous** if for every $\varepsilon > 0$ there is a $\delta > 0$ such that the set of all n for which $f_n((x-\delta, x+\delta)) \not\subseteq (f_n(x)-\varepsilon, f_n(x)+\varepsilon)$ for some $x \in [0,1]$ has the upper asymptotic density $< \varepsilon$. For instance, the sequence f_n is almost constant if for every sequence of positive integers $a_0 < a_1 < a_2 < \dots$ with $\lim_{k\to\infty} a_k/a_{k-1} = 1$ there is an f_n^* such that
(i) $\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^N \sup_{x\in[0,1]} |f_n(x) - f_n^*(x)| = 0$, and
(ii) if $k = 1,2,\dots$, then either all f_n^* with $n \in (a_{k-1}, a_k]$ are constant maps, or all f_n^* with $n \in (a_{k-1}, a_k]$ coincide.

Notes: Definitions and results stem from R. Winkler (1999) and they also remain valid for compact metric spaces. Cf. also (1997).

R. Winkler: *Sets of block structure and discrepancy estimates*, J. Théor. Nombres Bordeaux **9** (1997), no. 2, 337–349 (MR1617402 (99c:11099); Zbl. 0899.11036).
R. Winkler: *Distribution preserving transformations of sequences on compact metric spaces*, Indag. Math., (N.S.) **10** (1999), no. 3, 459–471 (MR1819902 (2002c:11086); Zbl. 1027.11053).

2.5.4. f–invariant distributed sequence. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. A sequence of real numbers x_n, $n = 1,2,\dots$, is called **f–invariant distributed sequence** mod 1 if the sequences x_n mod 1 and $f(x_n)$ mod 1 have the same a.d.f. In special cases

• If x_n mod 1 and $\frac{1}{x_n}$ mod 1 have the same a.d.f. (i.e. $f(x)=\frac{1}{x}$), then x_n is said to be **reciprocal invariant distributed sequence** mod 1.

• If x_n mod 1 and $\sqrt{x_n}$ mod 1 have the same a.d.f. (i.e. $f(x)=\sqrt{x}$), then x_n is said to be **square root invariant distributed sequence** mod 1.

NOTES: (I) S.H. Molnár (1994). As an example he gives the reciprocal invariant sequence 2.24.7.
(II) Since x_n cannot be in $[0,1)$, we cannot use 2.3.4, i.e. the notion f–invariant distributed sequence is the property of the sequence x_n but not of its a.d.f.

S.H. MOLNÁR: *Sequences and their transforms with identical asymptotic distribution function modulo* 1, Studia Sci. Math. Hungarica **29** (1994), no. 3–4, 315–322 (MR1304885 (95j:11071); Zbl. 0849.11053).

2.5.5. Given the basis $q\ge 2$, fix
• the permutation $\pi:\{0,1,\dots,q-1\}\to\{0,1,\dots,q-1\}$, and
• the permutations $\pi_{\mathbf{b}}:\{0,1,\dots,q-1\}\to\{0,1,\dots,q-1\}$ for every $\mathbf{b}=(b_1,\dots,b_k)$ with $b_i\in\{0,1,\dots,q-1\}$ and every $k=1,2,\dots$.

If $x\in[0,1)$ has the q–ary representation $x=0.a_1a_2a_3\dots$ then define the map $\sigma:[0,1)\to[0,1)$, called the ***q*–ary scrambling**, by

$$\sigma(x)=0.\pi(a_1)\pi_{a_1}(a_2)\pi_{(a_1,a_2)}(a_3)\dots.$$

If x_n is a given sequence, the sequence $\sigma(x_n)$ is called the **scrambling sequence** of x_n.

NOTES: A.B. Owen (1997) discusses this mapping σ with permutations $\pi_{\mathbf{b}}$ chosen fully randomly and mutually independently and its application to the deterministic low discrepancy sequences 1.8.15, cf. J. Matoušek (1998).

J. MATOUŠEK: *On the L_2–discrepancy for anchored boxes*, J. Complexity **14** (1998), no. 4, 527–556 (MR1659004 (2000k:65246); Zbl. 0942.65021).
A.B. OWEN: *Monte–Carlo variance of scrambled net quadrature*, SIAM J. Number. Analysis **34** (1997), no. 5, 1884–1910 (MR1472202 (98h:65006); Zbl. 0890.65023).

2.6 Sequences involving continuous functions

2.6.1. Generalized Fejér's theorem. Let k be a positive integer, and let $f(x)$ be a function defined for $x\ge 1$ such that
(i) it is k times differentiable for $x\ge x_0$,
(ii) $f^{(k)}(x)$ tends monotonically to 0 as $x\to\infty$,

(iii) $\lim_{x\to\infty} x|f^{(k)}(x)| = \infty$.

Then the sequence

$$f(n) \bmod 1$$

is

$$\text{u.d.}$$

NOTES: (I) [KN, p. 29, Th. 3.5]. The case $k = 1$ is known as Fejér's theorem.

(II) Fejér's theorem is sometimes formulated under slightly different assumptions, e.g. requiring in addition to our assumptions, that $f(x)$ *has continuous derivative for sufficiently large* x (cf. [KN, p. 24, Ex. 2.22]) or under the hypotheses (cf. G. Pólya and G. Szegő (1964, Part 2, Ex. 174))

- $f(x)$ is continuously differentiable,
- $f(x)$ tends monotonically to ∞ as $x \to \infty$,
- $f'(x)$ tends monotonically to 0 as $x \to \infty$,
- $\lim_{x\to\infty} xf'(x) = \infty$.

(III) In G. Pólya and G. Szegő (1964, Part 2, Ex. 182) the following variant is proved: Let $f(x)$ be a function defined for $x \geq 1$ such that

- $f(x)$ is continuously differentiable,
- $f(x)$ tends monotonically to ∞ as $x \to \infty$,
- $f'(x)$ tends monotonically to 0 as $x \to \infty$,
- $\lim_{x\to\infty} xf'(x) = 0$,

then the sequence $f(n) \bmod 1$ is dense but not u.d.

(IV) G. Rauzy (1976, p. 43, 1.2. Coroll.) proved that if

- $f'(x)$ tends to 0 (not necessarily monotonically), and
- $\lim_{x\to\infty} xf'(x)$ exists and is finite,

then the sequence $f(n) \bmod 1$ is not u.d.

(V) J. Cigler (1960) proved that if

- $f(x)$ is twice continuously differentiable,
- $f(x)$ tends monotonically to ∞ as $x \to \infty$,
- $f'(x)$ tends monotonically to 0 as $x \to \infty$,

then the u.d. is the only $(C, 1)$ distribution which $f(n) \bmod 1$ can have.

(VI) L. Kuipers (1953) gives (III) in a slightly more general form 2.6.6.

(VII) Fejér's theorem in terms of finite differences is given in 2.2.10.

J. CIGLER: *Asymptotische Verteilung reeller Zahlen* mod 1, Monatsh. Math. **44** (1960), 201–225 (MR0121358 (**22** #12097); Zbl. 0111.25301).

L. KUIPERS: *Continuous and discrete distribution modulo* 1, Indag. Math. **15** (1953), 340–348 (MR0058690 (15,410e); Zbl. 0051.28601).(=Nederl. Akad. Wetensch., Proc. **56** (1953), 340–348).

G. PÓLYA – G. SZEGŐ: *Aufgaben und Lehrsätze aus der Analysis*, Vol. 1 & 2, 3rd corr. ed., Grundlehren d. math. Wiss., Band 19, Springer Verlag, Berlin, Göttingen, Heidelberg, New York, 1964 (MR0170985 (**30** #1219a); MR0170986 (**30** #1219b); Zbl. 0122.29704).

G. RAUZY: *Propriétés statistiques de suites arithmétiques*, Le Mathématicien, Vol. 15, Collection SUP, Presses Universitaires de France, Paris 1976, 133 pp. (MR0409397 (**53** #13152); Zbl. 0337.10036).

2.6.2. Let $f(x)$ be a monotone increasing function defined for $x \geq 1$ and continuously differentiable for $x \geq x_0$ and
(i) $\lim_{x\to\infty} f(x) = \infty$,
(ii) $\lim_{x\to\infty} xf'(x) = \infty$,
(iii) $\lim_{x\to\infty} f'(x) = 0$ monotonically.
Then the sequence

$$f(n) \bmod 1$$

has the discrepancy

$$D_N = \mathcal{O}\left(\frac{f(N)}{N} + \frac{1}{Nf'(N)}\right).$$

NOTES: This quantitative form of Fejér's theorem was proved by H. Niederreiter (1971, p. 290, Th. 4.1).

H. NIEDERREITER: *Almost–arithmetic progressions and uniform distribution*, Trans. Amer. Math. Soc. **161** (1971), 283–292 (MR0284406 (**44** #1633); Zbl. 0219.10040).

2.6.3. Let p_n be a sequence of weights with $P_N \sim g(N)$, where $\sim$ denotes the asymptotic equality and $P_N = \sum_{n=1}^{N} p_n$. Let $f(x)$ and $g(x)$ be continuously differentiable functions for $x \geq 1$ such that
(i) $f'(x)/g(x)$ decreases monotonically towards 0 with $x \to \infty$,
(ii) $g(x)f'(x)/g'(x)$ tends monotonically towards ∞ with $x \to \infty$.
Then the sequence

$$f(n), \quad n = 1, 2, \dots,$$

has weighted discrepancy

$$\sup_{0\leq\alpha<\beta\leq 1} \left| \frac{1}{P_N} \sum_{n=1}^{N} p_n c_{[\alpha,\beta)}(\{f(n)\}) - (\beta - \alpha) \right| = \mathcal{O}\left(\frac{f(N)}{g(N)} + \frac{g'(N)}{g(N)f'(N)}\right).$$

NOTES: (I) R.F. Tichy (1982). A corresponding estimate for the discrepancy can also be proved for double sequences related to the Φ–(M, N, m)–uniform distribution, where the Φ–processing is described in Tichy (1978).
(II) Y. Ohkubo (1986) proved a similar result: For $p(x) \in C^1[1, \infty)$ and $f(x) \in C^2[1, \infty]$ assume that

- $p(x)$ is positive, non–increasing such that $s_N = \sum_{n=1}^{N} p_n \to \infty$ as $N \to \infty$ where $p_n = p(n)$,

- $s(t) = \int_1^t p(x)\,\mathrm{d}x$ and $s(N)/s_N \to 1$,
- $f(x)$ is positive, strictly increasing and $f(x) \to \infty$ as $x \to \infty$,
- $f'(x) \to$ constant < 1 monotonically as $x \to \infty$,
- $f'(x)/p(x)$ is monotone for $x \geq 1$.

Then the extremal weighted discrepancy D_N with respect to weights p_n of the sequence $f(n)$ mod 1 satisfies

$$D_N = \mathcal{O}\left(\frac{1}{s(N)}\int_1^N p(x)f'(x)\,\mathrm{d}x + \frac{p(N)}{s(N)f'(N)}\right).$$

Y. Ohkubo: *Discrepancy with respect to weighted means of some sequences*, Proc. Japan Acad. **62** A (1986), no. 5, 201–204 (MR0854219 (87j:11075); Zbl. 0592.10044).
R.F. Tichy: *Gleichverteilung von Mehrfachfolgen und Ketten*, Anz. Österreich. Akad. Wiss. Math.–Natur. Kl. (1978), no. 7, 174–207 (MR0527512 (83a:10087); Zbl. 0401.10061).
R.F. Tichy: *Einige Beiträge zur Gleichverteilung modulo Eins*, Anz. Österreich. Akad. Wiss. Math.–Natur. Kl. **119** (1982), no. 1, 9–13 (MR0688688 (84e:10061); Zbl. 0495.10030).

2.6.4. Let $f(x)$ be a real function which kth difference satisfies the inequality $\Delta^k f(n) \geq r > 0$ for $n = 1, 2, \dots, N-k$, where k is an integer less than N. Then the discrepancy of the finite sequence

$$f(1), f(2), \dots, f(N) \bmod 1$$

satisfies

$$D_N < c\left(\left(\frac{\rho^2}{r}\right)^{\frac{1}{K-1}} + \left(\frac{1}{rN^k}\right)^{\frac{2}{K}} + \left(\frac{\rho}{rN}\right)^{\frac{2}{K}} \log\frac{1}{\rho}\right)\right),$$

where c is a constant, $K = 2^k$, and

$$\rho = \frac{1}{N-k}\big(\Delta^{k-1}f(N-k+1) - \Delta^{k-1}f(1)\big).$$

J.G. van der Corput – C. Pisot: *Sur la discrépance modulo un. III*, Nederl. Akad. Wetensch., Proc. **42** (1939), 713–722 (MR0000396 (1,66c); JFM 65.0170.02; Zbl. 0022.11605). (=Indag. Math. **1** (1939), 260–269).

2.6.5. Let k be a positive integer, and let $f(x)$ be a function defined for $x \geq 1$ such that

(i) it is k times differentiable for sufficiently large x, and

(ii) $\lim_{x\to\infty} f^{(k)}(x) = \theta$ with irrational θ.

Then the sequence

$$f(n) \bmod 1$$

is

$$\text{u.d.}$$

NOTES: [KN, p. 31, Exer. 3.7]. For $k = 1$ this was proved by L. Kuipers (1953) and a related result involving differences was proved by J.G. van der Corput (1931), cf. 2.2.12.

L. KUIPERS: *Continuous and discrete distribution modulo* 1, Indag. Math. **15** (1953), 340–348 (MR0058690 (15,410e); Zbl. 0051.28601).(=Nederl. Akad. Wetensch., Proc. **56** (1953), 340–348).
J.G. VAN DER CORPUT: *Diophantische Ungleichungen I. Zur Gleichverteilung modulo Eins*, Acta Math. **56** (1931), 373–456 (MR1555330; JFM 57.0230.05; Zbl. 0001.20102).

2.6.6. Let $f(x)$ be a function defined for $x \geq 0$ such that
(i) $f(x)$ is differentiable,
(ii) $|xf'(x)| \leq M$ for $0 \leq x < \infty$.

Then the sequence

$$f(n) \bmod 1$$

is

$$\text{u.d.}$$

L. KUIPERS: *Continuous and discrete distribution modulo* 1, Indag. Math. **15** (1953), 340–348 (MR0058690 (15,410e); Zbl. 0051.28601).(=Nederl. Akad. Wetensch., Proc. **56** (1953), 340–348).

2.6.7. Let α be an irrational number and $f(x)$ be a three-times differentiable function defined for $x \geq 0$ such that
(i) $f'(x) \to 0$ as $x \to \infty$,
(ii) $f''(x) \to 0$ as $x \to \infty$,
(iii) $\lim_{x\to\infty} \frac{f'''(\lambda x)}{f'''(x)} = \lambda^\rho$ for each $\lambda > 0$ and some fixed $\rho < -2$,
(iv) $f'''(x)$ is ultimately non–decreasing,
(v) $xf'(x)$ is ultimately non–increasing.

Then for the extremal discrepancy of the sequence

$$x_n = \alpha n + f(n) \bmod 1$$

we have

$$D_N \geq c\frac{(f'(N))^{1/4}}{N^{1/2}}$$

for every N with a constant $c > 0$.

Related sequences: 2.12.31, 2.19.9, 2.15.3, 2.3.6, 2.3.11, 2.10.2.

K. Goto – Y. Ohkubo: *Lower bounds for the discrepancy of some sequences*, Math. Slovaca **54** (2004), no. 5, 487–502 (MR2114620 (2005k:11153); Zbl. 1108.11054).

2.6.8. Let $f(x)$ be a function such that
(i) f is continuously differentiable for $x \ge x_0$,
(ii) $\lim_{x\to\infty} f(x) = \infty$,
(iii) $xf'(x)$ is increasing,
(iv) $0 < f'(x)x^\sigma < 1$ for $\sigma > 0$.
Then the sequence

$$x_n = f(n) \bmod 1,$$

is

$$H_\infty\text{–u.d.}$$

with discrepancy

$$D_N(H_\infty, x_n) \le c\frac{1}{\log N}.$$

Notes: R.F. Tichy (1985). For the definition of H_∞–u.d. consult 1.8.5, or P. Schatte (1983).

P. Schatte: *On H_∞–summability and the uniform distribution of sequences*, Math. Nachr. **113** (1983), 237–243 (MR0725491 (85f:11057); Zbl. 0526.10043).
R.F. Tichy: *Uniform distribution and Diophantine inequalities*, Monatsh. Math. **99** (1985), no. 2, 147–152 (MR0781691 (86f:11059); Zbl. 0538.10039).

2.6.9. Let $f(x)$ be a function defined for $x \ge 0$ such that
(i) $f(x)$ has a continuous derivative,
(ii) $\lim_{x\to\infty} f'(x)\log x \to c \ne 0$.
Then the sequence

$$f(n) \bmod 1$$

is

$$\text{u.d.}$$

Notes: L. Kuipers (1953), cf. [KN, p. 82, Th. 9.8].

L. Kuipers: *Continuous and discrete distribution modulo* 1, Indag. Math. **15** (1953), 340–348 (MR0058690 (15,410e); Zbl. 0051.28601).(=Nederl. Akad. Wetensch., Proc. **56** (1953), 340–348).

2.6.10. Let $f(x)$ be a function defined for $x \geq 0$ such that
(i) $f(x)$ is differentiable,
(ii) $\lim_{x\to\infty} f'(x) \to 0$,
(iii) $\lim_{x\to\infty} xf'(x) \to \infty$,
(iv) $\lim_{x\to\infty} \frac{f'(x)}{f'\left(x+\frac{\theta}{f(x)}\right)} \to 1$ for some fixed θ, $|\theta| \leq 1$.
Then the sequence

$$f(n) \bmod 1$$

is

$$\text{u.d.}$$

J.F. Koksma: *Asymptotische verdeling van reële getallen modulo* 1. *I, II, III*, Mathematica (Leiden) **1** (1933), 245–248, **2** (1933), 1–6, 107–114 (Zbl. 0007.33901).

2.6.11. Let $f(x)$ be a function defined for $x \geq 0$ such that
(i) $f(x)$ is differentiable,
(ii) $0 \leq f'(x) < \infty$ for $x \geq 0$,
(iii) $\lim_{x\to\infty} x^\beta f'(x) \to \alpha$, where $\alpha > 0$ and $0 < \beta < 1$.
Then the sequence

$$f(n) \bmod 1$$

is

$$\text{u.d.}$$

Notes: L. Kuipers (1953), who mentions that this follows from 2.6.10.

L. Kuipers: *Continuous and discrete distribution modulo* 1, Indag. Math. **15** (1953), 340–348 (MR0058690 (15,410e); Zbl. 0051.28601).(=Nederl. Akad. Wetensch., Proc. **56** (1953), 340–348).

2.6.12. Let $f(x)$ be a function defined for $x \geq 0$ such that
(i) $f(x)$ has the continuous derivative of a constant sign,
(ii) $f(x)$ mod 1 is c.u.d.,
(iii) $\lim_{x\to\infty} \frac{f(x)}{x} \to 0$.
Then the sequence

$$f(n) \bmod 1$$

is

u.d.

NOTES: L. Kuipers (1953), cf. [KN, p. 82, Th. 9.7] and for applications 2.13.8, 2.13.10. Here the real valued Lebesgue–measurable function $f(x)$ mod 1 defined for $0 \le x < \infty$ is called **continuously uniformly distributed** (abbreviated c.u.d.) if

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T c_I(\{f(x)\})\,\mathrm{d}x = |I|$$

for every subinterval $I \subset [0,1]$ (cf. [KN, p. 78]).

L. KUIPERS: *Continuous and discrete distribution modulo* 1, Indag. Math. **15** (1953), 340–348 (MR0058690 (15,410e); Zbl. 0051.28601).(=Nederl. Akad. Wetensch., Proc. **56** (1953), 340–348).

2.6.13. Let $w(t)$ have continuous derivatives of the first $K+2$ orders such that
(i) $w^{(K+1)}(t)$ and $w^{(K+2)}(t)$ have constant signs, and
(ii) $\lim_{t\to\infty} w^{(K)}(t)/t = 0$, and
(iii) $\lim_{t\to\infty} t\left|w^{(K+1)}(t)\right| = \infty$.
Then, for arbitrary real numbers $\alpha_0 \neq 0, \alpha_1, \dots, \alpha_K$, the sequence

$$\alpha_0 w(n) + \alpha_1 w'(n) + \cdots + \alpha_K w^{(K)}(n) \bmod 1$$

is

u.d.

NOTES: This result was proved by J. Cigler (1968) using theorems of Fejér (2.6.1) and van der Corput (Th. 2.2.1), cf. E. Hlawka (1984, pp. 36–37). Cigler calls the functions which satisfy the above conditions **tempered**. The fact that $w(n)$ mod 1 is u.d. for every tempered $w(t)$ was already known to van der Corput, cf. Hlawka (1984, p. 38).

J. CIGLER: *Some remarks on the distribution* mod 1 *of tempered sequences*, Nieuw Arch. Wisk. (3) **16** (1968), 194–196 (MR0240057 (**39** #1411); Zbl. 0167.32102).
E. HLAWKA: *The Theory of Uniform Distribution*, A B Academic Publishers, Berkhamsted, 1984 (translation of the original German edition Hlawka (1979)) (MR0750652 (85f:11056); Zbl. 0563.10001).

2.6.14. Let $f(x)$ be defined for $x \ge 1$ and twice differentiable for sufficiently large x with
(i) $f''(x)$ tending monotonically to 0 as $x \to \infty$,
(ii) $\lim_{x\to\infty} f'(x) = \pm\infty$, and

(iii) $\lim\limits_{x\to\infty} \frac{(f'(x))^2}{x^2|f''(x)|} = 0.$

Then

$$f(n) \bmod 1$$

is

$$\text{u.d.}$$

NOTES: [KN, p. 24, Exer. 2.26]

2.6.15. Let $f(x,y)$ be a real valued function with its partial derivative f_{xy} defined for $x \geq 1$, $y \geq 1$. Assume that, for $x \geq 1$, $y \geq 1$

(i) f_{xy} is continuous,

(ii) f increases in x and y,

(iii) f_x is not increasing in x and y,

(iv) $\lim\limits_{x\to\infty} f_x(x,1) = \lim\limits_{y\to\infty} f_y(1,y) = 0,$

(v) $\lim\limits_{x\to\infty} \frac{f(x,x)}{x^2} = 0,$

(vi) $\int_1^N \int_1^N f_x(x,y) f_y(x,y)\,\mathrm{d}x\,\mathrm{d}y = o(N^2),$

(vii) $\int_1^N \frac{\mathrm{d}y}{f_x(N,y)} = o(N^2)$

Let the double sequence $f(m,n)$, $m = 1,2,\dots$, $n = 1,2,\dots$, be reordered to an ordinary sequence x_n, $n = 1,2,\dots$, in such a way that for every N the initial segment x_n, $n = 1,2,\dots,N^2$, contains the terms $f(m,n)$ for $m = 1,2,\dots,N$ and $n = 1,2,\dots,N$. Then the sequence

$$x_n \bmod 1$$

is

$$\text{u.d.}$$

NOTES: [KN, p. 20, Th. 2.10].

2.6.16. Let α and β be positive real numbers and the real valued function $f(x)$ be twice differentiable for $x \geq 0$ such that

(i) $f(x)$ is increasing,

(ii) $f'(x) \to 0$ monotonically as $x \to \infty$,

(iii) $\lim_{x\to\infty} xf'(x) \to \infty$ as $x \to \infty$,

(iv) $f''(x)$ is continuous for $x > 0$.

Let the double sequence $f(\alpha m+\beta n)$, $m = 1, 2, \dots$, $n = 1, 2, \dots$, be reordered to an ordinary sequence x_n, $n = 1, 2, \dots$, in such a way that for every N the initial segment x_n, $n = 1, 2, \dots, N^2$, contains the terms $f(\alpha m + \beta n)$ for $m = 1, 2, \dots, N$ and $n = 1, 2, \dots, N$. Then the sequence

$$x_n \bmod 1$$

is

u.d.

Notes: [KN, p. 21, Ex. 2.10]. This is an applications of 2.6.15.

Related sequences: 2.12.9, 2.15.2.

2.6.17. Let $f_1(x)$, …, $f_k(x)$ be twice differentiable functions such that
(i) $\lim_{x\to\infty} f_i(x) = \infty$ for $i = 1, \dots, k$,
(ii) every derivative $f_i'(x)$ is a monotonically decreasing function and $\lim_{x\to\infty} f_i'(x) f_i^{k-1}(x) = 0$ for $i = 1, \dots, k$.
Then the multiple sequence

$$x_{\mathbf{n}} = f_1(n_1)\dots f_k(n_k) \bmod 1, \quad \mathbf{n} = (n_1, \dots, n_k) \in \mathbb{N}^k,$$

with the product ordering of $\mathbf{n} = (n_1, \dots, n_k)$ is

u.d.

with respect to the weight $f_1'(n_1)\dots f_k'(n_k)$, i.e. for $[x, y) \subset [0, 1]$

$$\lim_{\mathbf{N}\to\infty} \frac{1}{P_1(N_1)\cdots P_k(N_k)} \sum_{\mathbf{n}\le\mathbf{N}} f_1'(n_1)\dots f_k'(n_k) c_{[x,y)}(\{f_1(n_1)\dots f_k(n_k)\}) =$$

$$= y - x,$$

where $P_i(N_i) = \sum_{n\le N_i} f_i'(n)$, $\mathbf{N} = (N_1, \dots, N_k)$ and $n_1 \le N_1, \dots, n_k \le N_k$.

Notes: R.F. Tichy (1982, Satz 2.3) who generalized a result proved by J. Cigler (1960).

J. Cigler: *Asymptotische Verteilung reeller Zahlen* mod 1, Monatsh. Math. **44** (1960), 201–225 (MR0121358 (**22** #12097); Zbl. 0111.25301).
R.F. Tichy: *Einige Beiträge zur Gleichverteilung modulo Eins*, Anz. Österreich. Akad. Wiss. Math.–Natur. Kl. **119** (1982), no. 1, 9–13 (MR0688688 (84e:10061); Zbl. 0495.10030).

2.6.18. Let $f(x)$, $x \geq 1$ be a continuous increasing function with the inverse function $f^{-1}(x)$. Assume that
(i) $\lim_{n\to\infty} \Delta f^{-1}(n) = \infty$,
(ii) for every $x \in [0,1]$ there exists the limit

$$\lim_{n\to\infty} \frac{f^{-1}(n+x) - f^{-1}(n)}{\Delta f^{-1}(n)} = \underline{g}(x).$$

If

$$\liminf_{n\to\infty} \frac{f^{-1}(n)}{f^{-1}(n+x)} = \chi(x),$$

then the sequence

$$f(n) \bmod 1$$

has the lower d.f. $\underline{g}(x)$ and upper d.f. $\overline{g}(x)$ of the form

$$\overline{g}(x) = 1 - \chi(x)(1 - \underline{g}(x)).$$

Notes: (I) This was proved by J.F. Koksma (1933; 1936, Chap. 8), cf. [KN, p. 58, Th. 7.7]. In [KN, p. 59] the lower and upper d.f. of $\log_b n \bmod 1$, $b > 1$ (cf. 2.12.1) was found using this result.
(II) O. Strauch and O. Blažeková (2006) proved the following modification:

Theorem 2.6.18.1. *Let $f(x)$ be a strictly increasing function and let $f^{-1}(x)$ be its inverse. Assume further that*
(i) $\lim_{x\to\infty} f'(x) = 0$,
(ii) $\lim_{k\to\infty} \big(f^{-1}(k+1) - f^{-1}(k)\big) = \infty$,
(iii) if $w(k) \in [0,1]$ is a sequence possessing the limit, say $\lim_{k\to\infty} w(k) = w$, then $\lim_{k\to\infty} \frac{f^{-1}(k+w(k))}{f^{-1}(k)}$ also exists and its value defines the value of a new function $\psi(x) : [0,1] \to [1, \psi(1)]$ at $x = w$,
(iv) $\psi(1) > 1$.
Then

$$G(f(n) \bmod 1) = \left\{ \widetilde{g}_w(x) = \frac{\min\big(\psi(x), \psi(w)\big) - 1}{\psi(w)} + \frac{\psi(x) - 1}{\psi(w)\big(\psi(1) - 1\big)} \; ; \; w \in [0,1] \right\}.$$

The lower d.f. $\underline{g}(x)$ and the upper d.g. $\overline{g}(x)$ of $f(n) \bmod 1$ is

$$\underline{g}(x) = \frac{\psi(x) - 1}{\psi(1) - 1}, \quad \textit{and} \quad \overline{g}(x) = 1 - \frac{1}{\psi(x)}(1 - \underline{g}(x)), \quad \textit{resp.},$$

where $\underline{g}(x) = \widetilde{g}_0(x) = \widetilde{g}_1(x) \in G(f(n) \bmod 1)$ and $\overline{g}(x) = \widetilde{g}_x(x) \notin G(f(n) \bmod 1)$.

If $F_N(x)$ denotes the step d.f. of the sequence $f(n) \bmod 1$, $n = 1, 2, \dots, N$ (see 1.3) and $w(k) = \{f(N_k)\} \to w$, then $F_{N_k} \to \widetilde{g}_w(x)$ for every $x \in [0, 1]$.
The above Theorem can be applied to $f(x) = \log x$ (see 2.12.1) and to $f(x) = \log(x \log^{(i)} x)$ (see 2.12.16).

J.F. Koksma: *Asymptotische verdeling van reële getallen modulo* 1. *I, II, III*, Mathematica (Leiden) **1** (1933), 245–248, **2** (1933), 1–6, 107–114 (Zbl. 0007.33901).
J.F. Koksma: *Diophantische Approximationen*, Ergebnisse der Mathematik und Ihrer Grenzgebiete, Vol. 4, Julius Springer, Berlin, 1936 (Zbl. 0012.39602; JFM 62.0173.01).
O. Strauch – O. Blažeková: *Distribution of the sequence* p_n/n mod 1, Unif. Distrib. Theory **1** (2006), no. 1, 45–63 (MR2314266 (2008e:11092); Zbl. 1153.11038).

2.6.19. If g is a d.f. then there always exists a sequence

$$x_n \in [0, 1)$$

with the a.d.f.

$$g(x).$$

Moreover, there exist such a sequence x_n with all its terms mutually distinct.
Notes: This was first proved by R. von Mises (1933). The proof given in [KN, p. 138, Th. 4.3] goes along the following lines:
(i) If g is a d.f. then there exists a sequence g_m, $m = 1, 2, \dots$, of continuous d.f.'s which converges pointwise to g, cf. [KN, p. 138, Lemma 4.3].
(ii) Take for $y_n^{(m)}$, $n = 1, 2, \dots$, a sequence with a.d.f. g_m, $m = 1, 2, \dots$; it may be constructed for instance using 2.3.8
(iii) The constructed sequence x_n is a block sequence starting with the first term of $y_n^{(1)}$, then taking the first two terms of $y_n^{(2)}$, etc.. The proof can be finished using 2.3.14.
For a given set H of d.f.'s the necessary and sufficient conditions for the existence of a sequence $x_n \in [0, 1)$ with $G(x_n) = H$ is that H is non–empty, closed and connected, cf. 1.7.0.2.

R. von Mises: *Über Zahlenfolgen, die ein kollektiv–ähnliches Verhalten zeigen*, Math. Ann. **108** (1933), no. 1, 757–772 (MR1512874; Zbl. 0007.21801).

2.6.20. Let f be the entire function

$$f(x) = \sum_{n=0}^{\infty} \frac{v_n x^n}{n!}$$

such that

(i) there exists a constant $c > 0$ such that $0 < |v_{n+1}| \leq c|v_n|^{(n+1)/(n-1)}$ for all $n > n_0$, and
(ii) $|v_n|^{1/(n2^n)} \to 0$ as $n \to \infty$.

Then the sequence

$$f(n) \bmod 1$$

is

u.d.

G. Rauzy: *Fonctions entières et répartition modulo 1*, Bull. Soc. Math. France **100** (1972), 409–415 (MR0318089 (**47** #6638); Zbl. 0252.10035).

2.6.21. Let f be an entire function that is real on the real axis and not a polynomial such that $\log|f(z)| = \mathcal{O}(\log^\alpha|z|)$ for some $1 < \alpha < 4/3$. Then

$$f(n) \bmod 1$$

is

u.d.

Notes:
(I) R.C. Baker (1984) improved in this way previous results by G. Rauzy (1973) and G. Rhin (1975). Baker (1986) showed that no quantitative version of the u.d. can be deduced from the growth condition. In ([a]1986) he proved that given a positive function $F(x) \geq 1$ with $F(x) \to \infty$ as $x \to \infty$, an entire function $f(z) = \sum_{k=1}^{\infty} z^k/q_1 \dots q_k$ such that $\log|f(Re^{i\theta})| \leq F(R)\log R$ for $R \geq 1$ and that $D_N(f(n) \bmod 1) \geq N/F(N)$ for infinitely many N can be constructed (using positive integers $q_1, q_2, \dots$).
(II) H. Niederreiter (1978), p. 997, interpreted Rauzy's result (1973) as follows: Assume that f is an entire function that is not a polynomial, which attains real values on the real axis, and satisfies

$$\limsup_{r\to\infty} \frac{\log\log M(f;r)}{\log\log r} < \frac{5}{4},$$

where $M(f;r) = \sup_{|z|\leq r}|f(z)|$. Then

$$f(n) \bmod 1, \quad n = 1, 2, \dots$$

is

completely u.d.

Related sequences: 2.19.12, 2.4.3.

R.C. Baker: *Entire functions and uniform distribution modulo one*, Proc. London Math. Soc. (3) **49** (1984), no. 1, 87–110 (MR0743372 (86h:11055); Zbl. 0508.10023).
R.C. Baker: *Entire functions and discrepancy*, Monatsh. Math. **102** (1986), 179–182 (MR0863215 (88a:11070); Zbl. 0597.10035).
[a] R.C. Baker: *On the values of entire functions at the positive integers*, in: *Analytic and elementary number theory (Marseille, 1983)*, Publ. Math. Orsay, 86–1, Univ. Paris XI, Orsay, 1986, pp. 1–5 (MR0844580 (87m:11062); Zbl. 0582.10022).
H. Niederreiter: *Quasi–Monte Carlo methods and pseudo–random numbers*, Bull. Amer. Math. Soc. **84** (1978), no. 6, 957–1041 (MR0508447 (80d:65016); Zbl. 0404.65003).
G. Rauzy: *Fonctions entières et répartition modulo un. II*, Bull. Soc. Math. France **101** (1973), 185–192 (MR0342483 (**49** #7229); Zbl. 0269.10029).
G. Rhin: *Répartition modulo 1 de $f(p_n)$ quand f est une série entière*, in: *Actes Colloq. Marseille – Luminy 1974*, Lecture Notes in Math., Vol. 475, Springer Verlag, Berlin, 1975, pp. 176–244 (MR0392857 (**52** #13670); Zbl. 0305.10046).

2.6.22. If $\theta > 1$ is a real number and $q > \theta$ a positive integer, then the sequence

$$x_n = (q-\theta)\sum_{k=1}^{\infty}\left\{\frac{n}{q^k}\right\}\theta^k \bmod 1$$

is

u.d. if and only if θ is not a P.V. number.

Notes: M. Mendès France (1976) applied his previous result (1973) to prove this.

Related sequences: 2.9.9

M. Mendès France: *Les suites à spectre vide et la répartition modulo 1*, J. Number Theory **5** (1973), 1–15 (MR0319909 (**47** #8450); Zbl. 0252.10033).
M. Mendès France: *A characterization of Pisot numbers*, Mathematika **23** (1976), no. 1, 32–34 (MR0419373 (**54** #7394); Zbl. 0326.10032).

2.6.23. Let $f(x)$ be a function defined for $x \geq 0$. Let $h(x) = \int_0^x f(t)\,\mathrm{d}t$ and $l(x) = \int_0^x h(t)\,\mathrm{d}t$. Suppose that
(i) $f(x)$ tends monotonically to 0 as $x \to \infty$,
(ii) $h(x)$ tends to ∞ as $x \to \infty$,
Then
(1) the sequence $h(n)$ mod 1 is dense in $[0,1]$,
(2) the sequence $l(n)$ mod 1 is dense in $[0,1]$,
(3) the two–dimensional sequence $(h(n), l(n))$ mod 1 is dense in $[0,1]^2$.

Notes: F.S. Cater, R.B. Crittenden and C. Vanden Eyden (1976). Note that (1) and (2) follow from 2.6.25 and (3) from 3.3.2. They also noted the following consequences: All of the sequences

- n^σ mod 1, $0 < \sigma < 2$, $\sigma \neq 1$ (by 2.15.1 it is u.d.),

- $(\log n)^\sigma \bmod 1$, $\sigma > 0$,
- $n(\log n)^\sigma \bmod 1$, $\sigma > 0$ (by 2.12.10 it is u.d.),
- $(\arctan n)^\sigma \bmod 1$, $\sigma > 0$,
- $n(\arctan n)^\sigma \bmod 1$, $\sigma > 0$,
- $\int_1^n (t + \sin t)^\sigma \,\mathrm{d}t \bmod 1$, $0 < \sigma < 1$,
- $\int_1^n (t + \cos t)^\sigma \,\mathrm{d}t \bmod 1$, $0 < \sigma < 1$,

are dense in $[0, 1]$.

F.S. Cater – R.B. Crittenden – Ch. Vanden Eynden: *The distribution of sequences modulo one*, Acta Arith. **28** (1976), 429–432 (MR0392903 (**52** #13716); Zbl. 0319.10042).

2.6.24. Let x_n be a sequence of real numbers such that
(i) $y_n = (x_{n+1} - x_{n-1} - 2x_n) \to 0$ as $n \to \infty$, but
(ii) y_n changes signs only finitely many times.
Then the sequence

$$x_n \bmod 1$$

is either dense in $[0, 1]$ or its only limit points are

$$(s + nr) \bmod 1, n = 1, 2, \dots,$$

where s is some real and r some rational number. Indeed, one of the following three cases occurs:
(1) there is a rational r and an real s such that $(x_n - nr) \to s$,
(2) the fractional parts $\{x_n\}$ are dense in $[0, 1]$ and $(x_n - x_{n-1})$ converges,
(3) if I and J are open subintervals of $[0, 1]$ then there exists an n such that $\{x_n\} \in I$ and $\{x_{n-1}\} \in J$.

Notes: F.S. Cater, R.B. Crittenden and C. Vanden Eyden (1976). This is a discrete version of 2.6.23.

F.S. Cater – R.B. Crittenden – Ch. Vanden Eynden: *The distribution of sequences modulo one*, Acta Arith. **28** (1976), 429–432 (MR0392903 (**52** #13716); Zbl. 0319.10042).

2.6.25. Let $f(x)$ be a function defined for $x \geq 1$ and $(k + 1)$–times differentiable here such that
(i) $f^{(k)}(x) \to \infty$ as $x \to \infty$,
(ii) $f^{(k+1)}(x) > 0$,
(iii) $f^{(k+1)}(x) \to 0$ as $x \to \infty$.

Then the sequence

$$f(n) \bmod 1$$

is

$$\text{dense.}$$

RELATED SEQUENCES: 2.6.1, 2.6.1.

P. CSILLAG: *Über die Verteilung iterierter Summen von positiven Nullfolgen* mod 1, Acta Litt. Sci. Szeged **4** (1929), 151–154 (JFM 55.0129.01).

2.6.26. Let $f(x)$, $x \geq 1$, be a twice differentiable function such that
(i) $f''(x) \ll x^{-2+\varepsilon}$ for some $0 < \varepsilon < 1$, and
(ii) there are real numbers $1 = t_0 < t_1 < \cdots < t_H < \infty$ such that $f''(x)$ is of constant sign and monotone in each of the intervals $[t_{j-1}, t_j]$, $j = 1, \ldots, H$, and $[t_H, \infty)$.

Then the sequence

$$f(n) \bmod 1$$

is

$$\text{u.d.}$$

and for its discrepancy we have

$$D_N \ll \begin{cases} \dfrac{1}{N|f''(N)|^{1/2}}, & \text{if } 0 < \varepsilon < \frac{4}{5}, \\ \dfrac{\log N}{N|f''(N)|^{1/2}}, & \text{if } \varepsilon = \frac{4}{5}, \\ \dfrac{1}{N^{5(2-\varepsilon)/6}|f''(N)|^{1/2}}, & \text{if } \frac{4}{5} < \varepsilon < 1. \end{cases}$$

NOTES: This result was presented by Y. Ohkubo at the Number Theory Conference in Graz, 1998 (Austria). Its weaker version with $D_N \ll 1/N|f(N)|^{1/2}$ for $0 < \varepsilon < 1/2$ was published in Ohkubo (1999).

RELATED SEQUENCES: 2.15.3.

Y. OHKUBO: *Notes on Erdős – Turán inequality*, J. Austral. Math. Soc. A **67** (1999), no. 1, 51–57 (MR1699155 (2000d:11100); Zbl. 0940.11029).

2.6.27. Let $f(x)$ be a real valued function such that

$$\frac{\mathrm{d}^i a x^c}{\mathrm{d}x^i} \le \frac{\mathrm{d}^i f(x)}{\mathrm{d}x^i} \le \frac{\mathrm{d}^i a x^{c+\delta}}{\mathrm{d}x^i}$$

for $i = 0, 1, 2$ and for sufficiently large x with some real constants a, c, δ, where $a > 0$, $1 < c < 2$, and $0 < \delta < 1$ (small enough depending on c alone). Denote by f^{-1} the inverse function to f and let k_n be the sequence of all square–free integers and $Q(N) = \#\{n \le N\ ;\ n \text{ is square–free}\}$. Then the sequence

$$f^{-1}(k_n) \bmod 1$$

is

u.d.

and

$$Q(N)D_{Q(N)} = \mathcal{O}\Big(N^{\frac{3}{5}+\frac{c+2\delta}{5c(c+\delta)}} + N^{1-\frac{1}{2c(c+\delta)}}\Big)$$

I.E. Stux: *Distribution of squarefree integers in non–linear sequences*, Pacif. J. Math. **59** (1975), 577–584 (MR0387218 (**52** #8061); Zbl. 0297.10033).

2.6.28. Suppose that the sequence of blocks $A_n = (x_{n,1}, \dots, x_{n,q})$ is unbounded such that

$$\limsup_{n\to\infty}(x_{n,i+1} - x_{n,i}) = 0$$

for every $i = 1, \dots, q$. Given $\gamma_i > 0$, $i = 1, \dots, q$, let

$$y_n = \gamma_1 x_{n,1} + \cdots + \gamma_q x_{n,q}.$$

Then for every continuous periodical function $f : \mathbb{R} \to \mathbb{R}$ the sequence

$$f(y_n), \quad n = 1, 2, \dots,$$

is

dense in the interval $[m, M]$,

where $m = \min f(x)$ and $M = \max f(x)$ over $x \in \mathbb{R}$.

Notes: D. Andrica and S. Buzeteanu (1987, 2.5. Cor.). The authors applied this result to sequences $\sin(3n^{1/2} + n^{1/3})$ and $\cos(2n^{1/4} + 5n^{1/7})$ which are thus dense in $[-1, 1]$.

D. Andrica – S. Buzeteanu: *Relatively dense universal sequences for the class of continuous periodical functions of period T*, Math. Rev. Anal. Numér. Théor. Approximation, Anal. Numér. Théor. Approximation **16** (1987), no. 1, 1–9 (MR0938777 (89i:11084); Zbl. 0642.26006).

2.6.29. If d and e are given positive integers then there exists an (effectively computable) constant $r = r(d, e)$ such that
(i) for every polynomial P of degree d, and
(ii) any periodic function f with period T which sth derivative satisfies $f^{(s)}(0) \neq 0$ for some $s \geq r$, and
(iii) any real number α with α/T irrational,
the sequence

$$P(n)f(n^e\alpha/T) \bmod 1$$

is

dense.

Notes: D. Berend and G. Kolesnik (1990, Th. 3.2).

D. Berend – G. Kolesnik: *Distribution modulo* 1 *of some oscillating sequences*, Israel J. Math. **71** (1990), no. 2, 161–179 (MR1088812 (92d:11079) Zbl. 0726.11042).

2.6.30. Let P be a polynomial of degree $d \geq 1$, f_1, and f_2 two non–constant functions with period 1 such that

- $f_2'(x_0) = f_2''(x_0) = \cdots = f_2^{(l-1)}(x_0) = 0$, but $f_2^{(l)}(x_0) \neq 0$, for some $x_0 \in [0, 1]$ and $l \geq 2$,
- the functions f_1 and f_2 are differentiable at least $\frac{1}{2} + \frac{(7l+1)d}{l-1}$ times in some neighbourhoods of the points 0 and x_0, respectively, and
- $f_1^{(s)}(0) \neq 0$ for some $s \geq \frac{d}{l-1} + \frac{1}{2(l+1)}$.

Then for every irrational α the sequence

$$P(n)f_1(nf_2(n\alpha)) \bmod 1$$

is

dense.

Notes: D. Berend, M.D. Boshernitzan and G. Kolesnik (1995, Th. 2.3).

D. Berend – M.D. Boshernitzan – G. Kolesnik: *Distribution modulo* 1 *of some oscillating sequences. II*, Israel J. Math. **92** (1995), no. 1–3, 125–147 (MR1357748 (96j:11105); Zbl. 0867.11052).

2.6.31. Let $f(x)$ be a real valued function defined for sufficiently large x and $h(x)$ be a non–constant periodic function with period 1. Assume that
- $\lim_{x\to\infty} |f(x)| = \infty$,
- $\lim_{x,y\to\infty} \big(f(x) - f(y)\big) = 0$ as $x/y \to 1$,
- $h(x)$ satisfies the Lipschitz condition.

Then for every irrational α the sequence

$$f(n)h(n\alpha) \bmod 1$$

is

$$\text{dense in } [0,1].$$

NOTES: D. Berend, M.D. Boshernitzan and G. Kolesnik (2002, Th. 3.1). They note that the assumptions imply $f(x) = \mathcal{O}(\log x)$ and that $f(x)$ need not be continuous, e.g. $f(x) = \sum_{n\le x} \frac{1}{n}$ satisfies the conditions.

D. Berend – M.D. Boshernitzan – G. Kolesnik: *Distribution modulo* 1 *of some oscillating sequences. III*, Acta Math. Hungar. **95** (2002), no. 1–2, 1–20 (MR1906205 (2003h:11085); Zbl. 0997.11058).

2.6.32. Let x_n be an unbounded sequence of positive real numbers with $\limsup_{n\to\infty}(x_{n+1} - x_n) = 0$. Then for all continuous periodical functions $f : \mathbb{R} \to \mathbb{R}$ the sequence

$$f(x_n), \quad n = 1, 2, \dots,$$

is

$$\text{dense in the interval } [\min_{x\in\mathbb{R}} f(x), \max_{x\in\mathbb{R}} f(x)].$$

NOTES: M. Somos (1976) and D. Andrica and S. Buzeteanu (1987, 2.4. Th.).

D. Andrica – S. Buzeteanu: *Relatively dense universal sequences for the class of continuous periodical functions of period T*, Math. Rev. Anal. Numér. Théor. Approximation, Anal. Numér. Théor. Approximation **16** (1987), no. 1, 1–9 (MR0938777 (89i:11084); Zbl. 0642.26006).
M. Somos: *Solution of the problem E2506*, Amer. Math. Monthly **83** (1976), no. 1, 60 (MR1537958).

2.6.33. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous periodical function with an irrational period T, then the sequence

$$f(n), \quad n = 1, 2, \dots,$$

is

$$\text{dense in the interval } [\min_{x\in\mathbb{R}} f(x), \max_{x\in\mathbb{R}} f(x)].$$

D. ANDRICA: *A supra unor siruri care an multimile punclelor limită intervale*, Gaz. Mat. (Bucharest) **84** (1979), no. 11, 404–405.

2.6.34. Let s_n be an increasing sequence of positive real numbers which is multiplicatively closed and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous periodical function with period T. Then, for every real α for which α/T is irrational, the sequence

$$f(\alpha s_n), \quad n = 1, 2, \dots,$$

is

dense in the interval $[\min_{x\in\mathbb{R}} f(x), \max_{x\in\mathbb{R}} f(x)]$.

NOTES: D. Andrica and S. Buzeteanu (1987, 4.10. Th.). This is a generalization of 2.8.3.

D. ANDRICA – S. BUZETEANU: *Relatively dense universal sequences for the class of continuous periodical functions of period T*, Math. Rev. Anal. Numér. Théor. Approximation, Anal. Numér. Théor. Approximation **16** (1987), no. 1, 1–9 (MR0938777 (89i:11084); Zbl. 0642.26006).

2.6.35. Let U denote the union of the all Hardy fields. If for $f \in U$ we have $|f(x)| < x^n$ for all x large enough and some $n \geq 1$ then the sequence

$$f(n) \bmod 1$$

is

u.d.

if and only if for every polynomial $p(x) \in \mathbb{Q}[x]$ the limit

$$L(p) = \lim_{x\to\infty} \frac{f(x) - p(x)}{\log x}$$

is infinite, i.e. if $L(p) = \pm\infty$.

The sequence

$$f(n) \bmod 1$$

is

dense in $[0, 1]$

if and only if for every polynomial $p(x) \in \mathbb{Q}[x]$ the limit

$$L(p) = \lim_{x\to\infty} \big(f(x) - p(x)\big)$$

is infinite, i.e. if $L(p) = \pm\infty$.

NOTES: M.D. Boshernitzan (1994, Th. 1.3, 1.4). He gave the following definition: Denote by B the set of the so–called germs at $+\infty$, that is the real valued functions defined for all sufficiently large real variable x. A subfield of the ring B closed under differentiation is called a **Hardy field**. Examples of Hardy fields:

- $\mathbb{R}(x)$, the field of real rational functions.
- L, the field of Hardy's logarithmico–exponential functions (introduced by G. Hardy (1912, 1924)) which consists of all functions defined for all sufficiently large x and which can be expressed using ordinary arithmetical symbols in terms of finite combinations of the functional symbols log, exp operating on x and on real constants.

The union U of all Hardy fields has the following properties (cf. also Boshernitzan (1987)):

- U is closed under differentiation and integration.
- If $f \in U$ is a non–zero function then one of the relations $f(x) > 0$ or $f(x) < 0$ holds for all sufficiently large x.
- The non–constant functions in U must be strictly monotone for large x.
- If $f \in U$ then the limit $\lim_{x\to\infty} f(x)$, finite or infinite, always exists.
- Non–linear functions in U must ultimately be either convex or concave.
- If $f \in U$, then $|f(x)|^\alpha x^\beta \log^\gamma x \in U$, for any α, β, γ.

Note that the above limits $L(p)$ exist in all cases and that this theorem includes as a very special case the classical result of H. Weyl saying that $f(n) \bmod 1$ is u.d. if f is a polynomial with at least one irrational coefficient. For another Boshernitzan's example see 2.12.17. He also formulates an open problem on the asymptotic behavior of $\Gamma(\log x) \bmod 1$.

RELATED SEQUENCES: 2.12.17

M.D. BOSHERNITZAN: *Second order differential equations over Hardy fields*, J. London Math. Soc. (2) **35** (1987), no. 1, 109–120 (MR0871769 (88f:26001); Zbl. 0616.26002).
M.D. BOSHERNITZAN: *Uniform distribution and Hardy fields*, J. Anal. Math. **62** (1994), 225–240 (MR1269206 (95e:11085); Zbl. 0804.11046).
G.H. HARDY: *Properties of logarithmico–exponential functions*, Proc. London Math. Soc. **10** (1911), 54–90 (MR1576038; JFM 42.0437.02).
G.H. HARDY: *Orders of Infinity*, 2nd ed., Cambridge Tracts in Math. and Phys., Vol. 12, Cambridge, 1924 (JFM 50.0153.04).

2.6.36.

NOTES: If U is the union of the all Hardy fields (cf. 2.6.35) let

$$U^+ = \{f \in U\ ;\ \lim_{x\to\infty} f(x) = \infty\}.$$

The following implication is true: If $f \in U^+$, then $\log f \in U$.

. . . .

Assume that

- $f(x) \in U^+$ with $f(x) = \mathcal{O}(\log x)$,

- $h(x)$ is a non–constant periodic function with period 1,
- $h(x)$ satisfies the Lipschitz condition (i.e. $|h(x) - h(y)| \leq c|x - y|$ for every x, y, where $c > 0$ is an appropriate constant).

Then for every irrational α the sequence

$$f(n)h(n\alpha) \bmod 1$$

is

$$\text{dense in } [0, 1].$$

Related sequences: 2.6.30

D. Berend – M.D. Boshernitzan – G. Kolesnik: *Distribution modulo* 1 *of some oscillating sequences. III*, Acta Math. Hungar. **95** (2002), no. 1–2, 1–20 (MR1906205 (2003h:11085); Zbl. 0997.11058).

2.6.37. Assume that
- $f(x) \in U^+$ (cf. 2.6.36),
- $\lim_{x\to\infty} \frac{\log f(x)}{\log x} < \infty$, and $\lim_{x\to\infty} \frac{f(x)}{\log x} = \infty$,
- m is a positive integer,
- $h(x)$ is a periodic function with period 1 which is k times continuously differentiable for sufficiently large k (depending on m),
- $h^{(i)}(x)$ has for every $i \leq k$ only finitely many zeros in $[0, 1]$,
- $|h^{(i)}(x)| + |h^{(i+1)}(x)| + \cdots + |h^{(i+m)}(x)| \geq c > 0$ for all x, every $i \leq k - m$ and some absolute constant $c > 0$.

Then the sequence

$$f(n)h(n\alpha) \bmod 1$$

is

$$\text{u.d. for every non–Liouville number } \alpha.$$

Notes: D. Berend, M.D. Boshernitzan and G. Kolesnik (2002, Th. 2.2). They noted (2002, Cor. 2.1) that if the assumption $\lim_{x\to\infty} \frac{f(x)}{\log x} = \infty$ is omitted, then the sequence $f(n)h(n\alpha) \bmod 1$ is dense in $[0, 1]$ for every irrational α.

D. Berend – M.D. Boshernitzan – G. Kolesnik: *Distribution modulo* 1 *of some oscillating sequences. III*, Acta Math. Hungar. **95** (2002), no. 1–2, 1–20 (MR1906205 (2003h:11085); Zbl. 0997.11058).

2.7 Sequences of iterations

2.7.1. Let $f : [0,1) \to [0,1)$ be a one–to–one and piecewise linear map defined by

$$f(x) = \begin{cases} b + \frac{1-b}{a}x, & \text{if } 0 \le x < a, \\ \frac{b}{1-a}(x-a), & \text{if } a \le x < 1, \end{cases}$$

where $a, b \in [0,1)$ and $a + b < 1$. If $(1-b)/a$ and $b/(1-a)$ are multiplicatively independent over $\mathbb{Q}$ (i.e. $\log((1-b)/a)$ and $\log(b/(1-a))$ are linearly independent over $\mathbb{Q}$), then the sequence of iterations (i.e. the orbit of x)

$$f(x), f(f(x)), f(f(f(x))), \dots, f^{(n)}(x), \dots,$$

is

$$\text{dense in } [0,1] \text{ for any } x \in [0,1).$$

Moreover,
(i) $f(x)$ belongs to $\mathbb{Q}$ for $x \in [0,1)$ if and only if $x \in \mathbb{Q}$,
(ii) the rotation number of f is $c = \log((1-b)/a)/\big(\log((1-b)/a) - \log(b/(1-a))\big)$,
(iii) f is an automorphism of the unit circle.

Notes: M.D. Boshernitzan (1993) gives an example with $a = 2/5$ and $b = 1/5$, i.e.

$$f(x) = \begin{cases} \frac{1}{5} + 2x, & \text{if } 0 \le x < \frac{2}{5}, \\ \frac{1}{3}(x - \frac{2}{5}), & \text{if } \frac{2}{5} \le x < 1. \end{cases}$$

Related sequences: 2.19.12

M.D. Boshernitzan: *Dense orbits of rationals*, Proc. Amer. Math. Soc. **117** (1993), no. 4, 1201–1203 (MR1134622 (93e:58099); Zbl. 0772.54031).

2.7.2. Let k be a positive integer, $\beta \neq 0$ a real number, $q(x)$ a polynomial, and $f(x)$ a function defined on $[1,\infty)$ such that
- the degree of $q(x)$ does not exceed $k+1$,
- $h(x)$ is positive, decreasing and differentiable, and
- $\lim_{x\to\infty} h(x) = 0$, $\int_1^\infty h(x)\,\mathrm{d}x = \infty$, $\int_1^\infty h^2(x)\,\mathrm{d}x < \infty$.

If

$$H_0(x) = \int_0^x h(t)\,\mathrm{d}t, \dots, H_k(x) = \int_1^x H_{k-1}(t)\,\mathrm{d}t,$$

then the weighted discrepancy D_N (cf. 1.10.6) with respect to the weights $p_n = h(n)$ of the sequence

$$x_n = (\beta H_k(n) + q(n)) \bmod 1$$

satisfies

$$D_N \le c(\beta,k)\frac{1}{H_0(N)^{\alpha_k}}, \quad \text{where } \alpha_k = \frac{3!}{2^{k+1}(k+3)!}.$$

NOTES: E. Hlawka (1983). An improvement in the case $h(x) = 1/x$ was given by Y. Ohkubo (1995).

E. HLAWKA: *Gleichverteilung und das Konvergenzverhalten von Potenzreihen am Rande des Konvergenzkreises*, Manuscripta Math. **44** (1983), no. 1–3, 231–263 (MR0709853 (85c:11060); Zbl. 0516.10030).
Y. OHKUBO: *The weighted discrepancies of some slowly increasing sequences*, Math. Nachr. **174** (1995), 239–251 (MR1349048 (96h:11074); Zbl. 0830.11028).

2.7.3. Given the base $q \ge 2$, let $x = \sum_{j=0}^{\infty} a_j q^{-j-1}$ and $y = \sum_{j=0}^{\infty} b_j q^{-j-1}$ be the q–adic digit expansion of $x, y \in [0,1]$ (for the sake of uniqueness we assume an infinite number of non–zero digits in expansions if x, y are non–zero). Define

$$x \oplus y = \sum_{j=0}^{\infty} c_j q^{-j-1}$$

by

$$c_j = a_j + b_j + \varepsilon_{j-1} - q\varepsilon_j,$$

where

$$\varepsilon_j = \begin{cases} -1, & \text{if } j = -1, \\ 1, & \text{if } a_j + b_j + \varepsilon_{j-1} \ge q, \\ 0, & \text{otherwise.} \end{cases}$$

If $T_y(x) = x \oplus y$ then for every $y \in [1/q, 1)$ and every $x \in [0,1]$, the sequence of iterates

$$T_y^{(n)}(x) = T_y(T_y^{(n-1)}(x)), \quad n = 0, 1, 2, \dots,$$

is

u.d.

with discrepancy

$$D_N^* \le \frac{1 + (q-1)[\log_q(Nq)]}{N},$$

i.e. it is a low discrepancy sequence.

NOTES: B. Lapeyre and G. Pagès (1989). They note that $T_{1/q}^{(n)}(1/q)$, $n = 0, 1, 2, \dots$, (with $T_y^{(0)}(x) = 0$) is the classical van der Corput sequence 2.11.3. For the multidimensional case cf. 3.3.3.

B. LAPEYRE – G. PAGÈS: *Familles de suites à discrépance faible obtenues par itération de transformations de* $[0, 1]$, C. R. Acad. Sci. Paris, Série I **308** (1989), no. 17, 507–509 (MR0998641 (90b:11076); Zbl. 0676.10038).

2.7.4. If $a_{n+1} = \sin(a_n)$ with any starting point $a_1 \in (0, \pi)$ then the sequence

$$n^\sigma a_n \bmod 1, \quad \frac{1}{2} < \sigma < \frac{3}{2}$$

is

$$\text{u.d.}$$

NOTES: This follows from 2.3.3, 2.14.7 and from the expression

$$a_n = \frac{\sqrt{3}}{\sqrt{n}} - \frac{3\sqrt{3}}{10} \cdot \frac{\log n}{n\sqrt{n}} + \frac{9\sqrt{3}}{50} \cdot \frac{\log n}{n^2\sqrt{n}} + o\left(\frac{\log n}{n^{5/2}}\right)$$

given by E. Ionascu and P. Stănică (2004). It seems that the coefficients in the expression do not depend on the starting point a_1.

E. IONASCU – P. STĂNICĂ: *Effective asymptotic for some nonlinear recurrences and almost doubly–exponential sequences*, Acta Math. Univ. Comenian. **73** (2004), no. 1, 75–87 (MR2076045 (2005f:11018); Zbl. 1109.11013).

$\boxed{\textbf{2.7.4.1}}$ Let

$$x_{n+1} = x_n - x_n^2, \quad n = 1, 2, \dots$$

with the initial term $x_1 \in (0, 1)$. Then the sequences

$$n^2 x_n \bmod 1, \qquad \frac{1}{x_n} \bmod 1,$$

$n = 1, 2, \dots$, have the same d.f.s as the sequence $\log n \bmod 1$.

NOTES: This follows from the fact that $G(\log n + \alpha \bmod 1) = G(\log n \bmod 1)$ and from the expansions

$$x_n = \frac{1}{n} - \frac{\log n}{n^2} - \frac{v}{n^2} + \frac{(\log n)^2}{n^3} + (2v - 1)\frac{\log n}{n^3} + o\left(\frac{\log n}{n^3}\right),$$

$$\begin{aligned}\frac{1}{x_n} =& n + \log n + v + \frac{\log n}{n} + \frac{v-(1/2)}{n} - \frac{1}{2}\frac{(\log n)^2}{n^2}\\ &+ (3/2 - v)\frac{\log n}{n^2} + \left(\frac{3}{2}v - \frac{1}{2}v^2 - \frac{5}{6}\right)\frac{1}{n^2} + \frac{1}{3}\frac{(\log n)^3}{n^3}\\ &+ (-2+v)\frac{(\log n)^2}{n^3}\left(\frac{19}{6} - 4v + v^2\right)\frac{\log n}{n^3} + o\left(\frac{\log n}{n^3}\right)\end{aligned}$$

found by E. Ionascu and P. Stănică (2004).

E. Ionascu – P. Stănică: *Effective asymptotic for some nonlinear recurrences and almost doubly–exponential sequences*, Acta Math. Univ. Comenian. **73** (2004), no. 1, 75–87 (MR2076045 (2005f:11018); Zbl. 1109.11013).

2.8 Sequences of the form $a(n)\theta$

Notes: If $a(n)$ is an increasing sequence of positive integers, then the set of all x for which $a(n)x$ mod 1 is not u.d. has zero Lebesgue measure (H. Weyl (1916)) and if $a(n)$ is a polynomial with integral coefficient, then this set is enumerable. If $a(n+1) - a(n) <$constant, then this set has Hausdorff dimension zero (P. Erdős and S.J. Taylor (1957)).
For every bounded Lebesgue measurable $f(x)$ on $[0,1]$ the limit

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(\{a(n)x\}) = \int_0^1 f(t)\,\mathrm{d}t$$

holds for almost all x (with respect to Lebesgue measure) for the sequence $a(n)$ of all integers generated multiplicatively by a finite set $b_1,\dots,b_k$ of pairwise coprime integers > 1 ordered by magnitude and each $a(n)$ is taken only once (cf. J.M. Marstrand (1970), R. Nair (1990)). Note that for every of the following sequences $a(n)$ there exists a bounded Lebesgue measurable function $f(x)$ defined on $[0,1]$ such that the limit $N^{-1}\sum_{n=1}^{N} f(\{a(n)x\}) \to \int_0^1 f(t)\,\mathrm{d}t$ fails to hold almost everywhere:
(i) $a(n) = n$,
(ii) $a(m)$ and $a(n)$ are coprime for $m \neq n$,
(iii) $A([0,x);a(n)) \sim cx^\alpha$, for some positive real c and α,
(iv) $a(n) = 2^{2^n}$.
The cases (i)–(iii) are from Marstrand (1970) and (iv) from Nair (2003). In the first case (i) we can take for $f(x)$ the indicator function of a measurable set $E \subset [0,1]$ with a positive measure. This disproves Khintchine's conjecture (1923).

P. Erdős – S.J. Taylor: *On the set of points of convergence of a lacunary trigonometric series and the equidistribution properties of related sequences*, Proc. London Math. Soc. (3) **7** (1957), 598–615 (MR0092032 (19,1050b); Zbl. 0111.26801).

A. Khintchine (A.J. Chinčin): *Ein Satz über Kettenbrüche, mit arithmetischen Anwendungen*, Math. Z. **18** (1923), 289–306 (MR1544632; JFM 49.0159.03).
J.M. Marstrand: *On Khinchin's conjecture about strong uniform distribution*, Proc. London Math. Soc. (3) **21** (1970), 540–556 (MR0291091 (**45** #185); Zbl. 0208.31402).
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R. Nair: *On a problem of R.C. Baker*, Acta Arith. **109** (2003), no. 4, 343–348 (MR2009048 (2004g:11062); Zbl. 1042.11049).
H. Weyl: *Über die Gleichverteilung von Zahlen mod. Eins*, Math. Ann. **77** (1916), 313–352 (JFM 46.0278.06).

2.8.1. $n\theta$ sequences

(I) If θ is an irrational number then the sequence

$$x_n = n\theta \bmod 1, \quad n = 1, 2, \dots,$$

is

u.d.

Notes: We shall write $D_N(\theta)$ and $D_N^*(\theta)$ instead of D_N and D_N^*, resp., for discrepancies of $x_n = n\theta \bmod 1$.

(II) Let $\theta = [a_0; a_1, a_2, \dots]$ be an irrational number with bounded partial quotients, say $a_i \le K$ for $i = 1, 2, \dots$. Then

$$ND_N(\theta) \le 3 + \left(\frac{1}{\log((1+\sqrt{5})/2)} + \frac{K}{\log(K+1)} \right) \log N$$

and

$$ND_N(\theta) < C(K) \log(N+1)$$

for all $N \ge 1$ where

$$C(K) = \begin{cases} 2/\log 2, & \text{if } K = 1, 2, 3, \text{ and} \\ \frac{K+1}{\log(K+1)}, & \text{if } K \ge 4. \end{cases}$$

(III)

$$c_1 \sum_{i=1}^{m} a_i \le ND_N^*(\theta) \le c_2 \sum_{i=1}^{m} a_i$$

for all irrationals $\theta = [0; a_1, \dots]$ and $q_m \le N < q_{m+1}$.

(IV) Let $\theta = [0; a_1, \dots]$ be an irrational number with convergents p_n/q_n. Every positive integer N can be written in the form (the so–called **Ostrowski expansion**) $N = \sum_{i=0}^m b_i q_i$, where the integer m is uniquely determined while the digits b_i, $0 \le i \le m$, satisfy $b_m > 0$, $b_0 < a_1$, $0 \le b_i \le a_{i+1}$, and if $b_i = a_{i+1}$, then $b_{i-1} = 0$ for $0 < i \le m$. Then

$$ND_N^*(\theta) = \max\left\{\sum_{\substack{j=0\\2|j}}^m b_j\left(1-\frac{b_j}{a_{j+1}}\right), \sum_{\substack{j=0\\2\nmid j}}^m b_j\left(1-\frac{b_j}{a_{j+1}}\right)\right\} + \mathcal{O}(m),$$

where the $\mathcal{O}$–constant is absolute. This implies

$$ND_N^*(\theta) = \mathcal{O}(\log N) \Longleftrightarrow \limsup_{m\to\infty} \frac{1}{m}\sum_{j=1}^m a_j < \infty.$$

(IV') If $\theta = [0; a_1, \dots]$ is an irrational number then for $i, j \ge 0$ and $m \ge 0$ define

- $s_{ij} = q_{\min(i,j)}(q_{\max(i,j)}\theta - p_{\max(i,j)})$,
- $\varepsilon_i = \frac{1}{2}(1-(-1)^{a_{i+1}}) \prod_{\substack{0\le j\le i\\ j\equiv i \,(\mathrm{mod}\ 2)}} (-1)^{a_{j+1}}$,
- $N_m = \frac{1}{2}\sum_{i=0}^m (a_{i+1} + (-1)^m \varepsilon_i) q_i$.

Then

$$4 \max_{1\le N<q_{m+1}} ND_N(\theta) = \sum_{i=0}^m a_{i+1} - \sum_{0\le i\le m} \sum_{\substack{0\le j\le m\\ j\equiv i\,(\mathrm{mod}\ 2)}} \varepsilon_i\varepsilon_j |s_{ij}| + \mathcal{O}(1)$$

with an absolute implicit constant.

(IV") **Central limit theorem.** Let α be any quadratic irrational and $I = [0, x)$ any interval with rational endpoint $0 < x < 1$. There are effectively computable constants $C_1 = C_1(\alpha, x)$ and $C_2 = C_2(\alpha, x)$ such that

$$\frac{1}{N}\#\left\{n \le N : A \le \frac{(A([0,x); n : \{k\alpha\}) - nx) - C_1\log N}{C_2\sqrt{\log N}} \le B\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\int_A^B e^{-u^2/2}\,\mathrm{d}u + O\big((\log N)^{-1/10}\log\log N\big).$$

Notes: In the following the Koksma classification of irrational numbers will be used.

(i) The irrational θ is said to be of **type** $< \psi$ if $\psi(q)q\|q\theta\| \geq 1$ holds for all positive integers q, where $\|x\| = \min\big(\{x\}, 1-\{x\}\big)$. If ψ is a constant function, then θ is said to be of **constant type**.

(ii) The irrational number θ is said to be of **finite type** γ, where γ is a real number, if γ is the supremum of all σ for which $\liminf_{q\to\infty} q^{\sigma}\|q\theta\| = 0$. In all cases $\gamma \geq 1$. If the supremum of such σ is infinity, then θ is said to be of **infinite type**.

(V) If θ is of finite type γ then for every $\varepsilon > 0$

$$D_N(\theta) = \mathcal{O}(N^{(-1/\gamma)+\varepsilon}), \quad \text{and} \quad D_N(\theta) = \Omega(N^{(-1/\gamma)-\varepsilon}).$$

(VI) Thus, if θ is an algebraic irrational then $\gamma = 1$ and

$$D_N(\theta) = \mathcal{O}(N^{-1+\varepsilon}).$$

NOTES: (I) The u.d. of $n\theta$ mod 1 was independently established by Bohl (1909), W. Sierpiński (1910,[a]1910), and H. Weyl in (1909–1910), cf. (1916). M. Lerch (1904) proved that $\sum_{n=1}^{N}(\{n\theta\} - 1/2) = \mathcal{O}(\log N)$. The sequence $n\theta$ mod 1 is also called the **Weyl sequence**.[1] The subject was taken up again by E. Hecke, A. Ostrowski, G.H. Hardy and J.E. Littlewood, and H. Behnke. A detailed account of the history can be found in the book [KN, pp. 21–23, 157–158, Notes] or in E. Hlawka and Ch. Binder (1986).

(II) H. Niederreiter (1978), [KN, p. 125, Th. 3.4], Niederreiter (1992, p. 27, Cor. 3.5).

(III) H. Behnke (1924).

(IV) J. Schoißengeier (1984, §9, Cor. 3 and 5). As applications he shows (1984, § 9, Ex. and Coroll. 4)

$$10^6 D^*_{10^6}(\pi) = 41.064561094, \qquad \limsup_{N\to\infty} \left(\frac{\log\log N}{\log N}\right)^2 N D^*_N(e) = \frac{1}{8}.$$

If $\theta = \frac{1+\sqrt{5}}{2}$ the lim sup was computed by Y. Dupain (1979)

$$\limsup_{N\to\infty} \frac{N D^*_N(\theta)}{\log N} = \frac{3}{20} \cdot \frac{1}{\log\theta}.$$

The Ostrowski expansion $\sum_{i=0}^{m} b_i q_i$ of N with respect to basis $\theta = [a_0; a_1, a_2, \dots]$ was often used, e.g. A. Ostrowski (1922), R. Descombes (1956), V.T. Sós (1958).

(IV') C. Baxa and J. Schoißengeier (1994).

(IV") J. Beck [p. 12](2014).

[1] The sequence $n\theta$ mod 1 is called golden for $\theta = (1+\sqrt{5})/2$, and for $\theta = (3-\sqrt{5})/2$ it is called silver (cf. Steinhaus (1956)). Often under the silver number the number $1+\sqrt{2}$ is understood. Both sequences provide the best possible u.d. mod 1.

(V) [KN, p. 123–124, Th. 3.2-3.]. The known Koksma classification of irrational numbers can be found e.g. in [KN, p. 121, Def. 3.2–3.].
(VI) [KN, p. 124, Examp. 3.1]. D.P. Parent (1984, pp. 253–254, Exer. 5.26) proved only $D_N^*(\theta) = \mathcal{O}(N^{-1/2})$.
(VII) The famous Steinhaus conjecture or three–gaps theorem says[2]: Let θ be a real number and N a positive integer. If the points $0, \{\theta\}, \dots, \{N\theta\}, 1$ are arranged in ascending order then the distances between the consecutive points can have at most three distinct lengths, and if there are three, one equals the sum of the other two. This conjecture was probably first proved by N.B. Slater (1950), and later by K. Florek (1951), H. Steinhaus (1956), V.T. Sós ([a]1958), J. Surányi (1958) and S. Świerczkowski (1959).[3] Other proofs were given by P. Szüsz and P. Erdős, cf. [KN, p. 22, Notes] for additional information. J.H. Halton (1965, Th. 2, Cor. 3) proved the following quantitative result: Let p_n/q_n be the nth convergent of the continued fraction of $\theta = [a_0; a_1, a_2, \dots]$ and $r_n = |q_n\theta - p_n|$. The Steinhaus three–gaps are: r_{n+1}, $r_n - ir_{n+1}$ and $r_n - (i-1)r_{n+1}$, where the integers n, i (and j) are uniquely determined by conditions $N = q_n + iq_{n+1} + j$, $1 \le i \le a_{n+2}$, $1 \le j \le q_{n+1}$. In N.B. Slater (1967) a summary of these results can be found. The finite–gaps theorems are not valid for $n^2\theta$, cf. 2.14.1.

• N.B. Slater (1967) calls the *Steinhaus three gaps problem* as *the step problem* and under *the gap problem* he understands the following problem: Let I be an interval in $(0,1)$, $A = \{n \in \mathbb{N}; \{n\alpha\} \in I\} = \{a_1 < a_2 < \dots\}$ and $\Delta = \{a_{n+1} - a_n; n = 1, 2, \dots\}$. The problem is to find Δ. If $|I| \le 1/2$, Slater proved that $\Delta = \{a, b, a+b\}$, for $\{a, b\}$ see 4.1.3.
(VII') J.F. Geelen, R.J. Simpson (1993) prove the following two-dimensional Steinhaus theorem: If α, β are real numbers and $M \le N$ positive integers then the point $\{m\alpha + n\beta\}$, $m = 0, 1, \dots, M-1$, $n = 0, 1, \dots, N-1$, partition the unit interval into MN subintervals having at most $M+3$ distinct widths. The bound $M+3$ can be attained, if $M > 1$.
(VIII) If θ is an irrational number and the points $0, \{\theta\}, \dots, \{N\theta\}, 1$ are arranged in the ascending order $0 < \{n_1\theta\} < \{n_2\theta\} < \dots < \{n_N\theta\} < 1$.[4] Then the Steinhaus three–gaps can also be determined as follows, cf. Świerczkowski (1959): $\{n_1\theta\}$, $1 - \{n_N\theta\}$, and $\{n_1\theta\} + 1 - \{n_N\theta\}$ the last one only if $N < n_1 + n_N - 1$. Thus

[2] The first conjecture in this direction goes back to J. Oderfeld and C. Rajski in connection with their empirical investigation of the sequence for $\theta = (\sqrt{5}-1)/2$, cf. Steinhaus (1956).
[3] Świerczkowski also proved the Oderfeld conjecture that if $\theta = (\sqrt{5}-1)/2$ and F_m is the greatest Fibonacci number which does not exceed N then θ^m, θ^{m-1}, and θ^{m-2} are the possible values of the three gaps for the sequence $n\theta \bmod 1$, $n = 1, 2, \dots$.
[4] J. Beck [p. 14](2014), **Lemma on Restricted Permutations:** Let α be an arbitrary irrational, and let π be a permutation of the set of integers $1, 2, \dots, N$ such that

$$\{\pi(1)\alpha\} < \{\pi(2)\alpha\} < \dots < \{\pi(N)\alpha\} < 1.$$

Then the whole permutation π can be reconstructed from the knowledge of $\pi(1)$ and $\pi(N)$ (the point is that we do not need to know α).

for dispersion $d_N(\theta)$ of the sequence $x_n = n\theta \bmod 1$, $n = 1, 2, \dots, N$, we get (see 1.10.11)

$$\begin{aligned} d_N(\theta) &= \max\Big(\frac{1}{2}\max_{1\le i\le N}\big(\{n_i\theta\} - \{n_{i-1}\theta\}\big), \{n_1\theta\}, 1 - \{n_N\theta\}\Big) \\ &= \max\big(\{n_1\theta\}, 1 - \{n_N\theta\}\big). \end{aligned}$$

Let

$$\widetilde{d}_N(\theta) = \max\Big(\max_{1\le i\le N}(\{n_i\theta\} - \{n_{i-1}\theta\}), \{n_1\theta\}, 1 - \{n_N\theta\}\Big).$$

R.L. Graham and J.H. van Lint (1968) proved that

$$\sup_\theta \liminf_{N\to\infty} N\widetilde{d}_N(\theta) = \frac{1+\sqrt{2}}{2}, \qquad \inf_\theta \limsup_{N\to\infty} N\widetilde{d}_N(\theta) = 1 + \frac{2\sqrt{5}}{5}$$

and that these limits are attained for $\theta = 1 + \sqrt{2}$ and $\theta = (1 + \sqrt{5})/2$, resp. They also proved that $\limsup_{N\to\infty} N\widetilde{d}_N(\theta)$ is finite if and only if the partial quotients of the simple continued fraction of θ are bounded.
H. Niederreiter (1984) proved that $d_N(\theta) = \mathcal{O}(1/N)$ if θ has bounded partial quotients. If

$$D(\theta) = \limsup_{N\to\infty} Nd_N(\theta)$$

then he proved that
$D(\theta) \ge 3 - \sqrt{3}$,
$D(\theta) = (5 + 3\sqrt{5})/10$ for $\theta = (1 + \sqrt{5})/2$,
$D(\theta) = (1 + \sqrt{2})/2$ for $\theta = 1 + \sqrt{2}$.
and G. Ji and H. Lu (1996) found explicit values
$D(\theta) = (2d + (d + b)\sqrt{d})/4d$ for $\theta = (b + \sqrt{d})/2$ and $d \equiv b \bmod 4, d > 0$.
Niederreiter (1984) notes an analogy between the Markov spectrum and $D(\theta)$ as θ runs through the all irrationals with bounded partial quotients. Define the **Markov constant** $M(\theta)$ as $(M(\theta))^{-1} = \liminf_{n\to\infty} n\|n\theta\|$. He posed the question whether $M(\theta_1) < M(\theta_2)$ implies $D(\theta_1) < D(\theta_2)$ which was disproved by V. Drobot (1986) by producing a counterexample of two quadratic irrationals. A. Tripathi (1993) gave some new families of counterexamples, e.g. for $(3 + \sqrt{21})/2) = [\overline{3, 1}]$ and $(6 + \sqrt{48})/3 = [\overline{4, 3}]$ we have
$M((3 + \sqrt{21})/2) = \sqrt{21} = 4.5\cdots < M((6 + \sqrt{48})/3) = \sqrt{192}/3 = 4.6\dots$,
$D((3+\sqrt{21})/2) = (5.5/\sqrt{21})+1/2 = 1.7\cdots > D((6+\sqrt{48})/3) = (16/\sqrt{192})+1/2 = 1.6\dots$.
J. Schoißengeier (1993) found the exact values of
$D(\theta) = \limsup_{N\to\infty} Nd_N(\theta)$,
$\liminf_{N\to\infty} Nd_N(\theta)$,
$\limsup_{N\to\infty} N\widetilde{d}_N(\theta)$,
$\liminf_{N\to\infty} N\widetilde{d}_N(\theta)$,

for a wide class of irrational $\theta's$ and he proved that θ has bounded partial quotients if and only if $1 < \liminf_{N\to\infty} N\widetilde{d}_N(\theta)$.
H. Jager and J. de Jonge (1994) introduced the quantity

$$\limsup_{N\to\infty} N(\{n_1\theta\} + 1 - \{n_N\theta\})$$

for irrational θ and they found its smallest value $(2/\sqrt{5}) + 1$ for $\theta = (\sqrt{5}+1)/2$ and its smallest accumulation point $(\sqrt{5}+4)/3$.
(IX) The investigation of the sum $C_N(\theta) = -N/2 + \sum_{n=1}^N \{n\theta\}$ has a long history. The first result seems to have been provided by M. Lerch (1904) who showed $C_N(\theta) = \mathcal{O}(\log N)$ for irrational θ with continued fraction expansion $[a_0; a_1, \dots]$ having bounded partial quotients, thereby answering problems proposed by J. Franel (1898, 1899) [L'Intermédiaire Math. **5** (1898), 77; **6** (1899), 149]. The subject was taken up again by W. Sierpiński, E. Hecke, A. Ostrowski, G.H. Hardy and J.E. Littlewood, and H. Behnke. It was shown by Ostrowski (1922) that $C_N(\theta)$ is unbounded for every irrational θ. Obviously $|C_N(\theta)| \le ND_N^*(\theta)$. J. Schoißengeier (1986) proved that if $\frac{1}{t}\sum_{1\le j\le t} a_j \le A$ for all t, then

$$|C_N(\theta)| < \frac{A}{2\log\tau}\log N + A\cdot\left(\frac{\log\sqrt{5}}{2\log\tau} - \frac{1}{2}\right)$$

for $N \ge 2$, where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio. T.C. Brown and P.J.–S. Shiue (1995) proved that if $\frac{1}{t}\sum_{1\le j\le t} a_j \le A$ for infinitely many t, then there exists a positive constant d_A such that each of $C_N(\theta) > d_A \log N$ and $C_N(\theta) < -d_A\log N$ holds for an infinitely many N. Note that $d_A \ge 1/\big(7\cdot 64(A+1)^2\log(A+1)\big)$. They also give an example: If $\theta = [0; a_1, a_2 \dots]$ with $a_{2n+1} = 1$, $a_{2n} = n^2$ for $n \ge 0$ then there exists a constant C such that $C_N(\theta) > C$ for all $N \ge 1$ (obviously $C < 0$). V.T. Sós (1957) showed that there exists a real θ with $C_N(\theta)$ bounded below (or above) and noted that there is a θ with $C_N(\theta) > -\varepsilon$ for all N, where ε is an arbitrarily small positive number. However it is impossible for $C_N(\theta)$ to be positive for every N.
(i) J. Beck [p. 79](2014): Let $M_N(\theta) = \frac{1}{N}\sum_{n=1}^N C_n(\theta)$. Then for any irrational $\theta > 0$ and any integer $N \ge 1$ we have

$$M_N(\theta) = \frac{-a_1 + a_2 - a_3 + \cdots + (-1)^k a_k}{12} + O\big(\max_{1\le j\le k} a_j\big),$$

where $q_k \le N < q_{k+1}$, $[a_0; a_1, \dots, a_{k-1}] = p_k/q_k$ and the O-constant is ≤ 10.

Let us change the value $C_n(\theta)$ to $S_\alpha(n) = \sum_{k=1}^n \Big(\{k\alpha\} - \frac{1}{2}\Big)$.
(ii) J. Beck [p. 26](2014) **Ostrowski's large fluctuation result:** Suppose that $\alpha = [a_0; a_1, a_2, \dots]$ with $a_i \le A$ for all i. Then there are positive constants $0 < c_1 < 1$ and $c_2 > 0$ such that, for every sufficiently large N, the interval $c_1 N < n < N$ contains integers n_1, n_2 with the properties $S_\alpha(n_1) > c_2 \log N$, $S_\alpha(n_2) < -c_2\log N$.
(iii) J. Beck [p. 20](2014) **Central limit theorem:** There are effectively computable

constants $C_3 = C_3(\alpha)$ and $C_4 = C_4(\alpha)$ such that

$$\frac{1}{N}\#\left\{n \le N : A \le \frac{S_\alpha(n) - C_3 \log N}{C_4\sqrt{\log N}} \le B\right\}$$
$$= \frac{1}{\sqrt{2\pi}}\int_A^B e^{-u^2/2}\,\mathrm{d}u + O\big((\log N)^{-1/10}\log\log N\big).$$

(IX')(i) P. Borwein (1978) solved problem of H.D. Ruderman (1977) that the series $\sum_{n=1}^{\infty}(-1)^{[n\sqrt{2}]}/n$ converges.
(ii) P. Bundschuh (1977) proved that the series $\sum_{n=1}^{\infty}(-1)^{[n\alpha]}/n$ converges for numbers α with bounded by partial quotients b_i of $\alpha/2 = [b_0; b_1, b_2, \dots]$.
(iii) J. Schoißengeier (2007) proved that the series
$\sum_{n=1}^{\infty}(-1)^{[n\alpha]}/n$ and
$\sum_{k=0,2\nmid q_k}^{\infty}(-1)^k(\log b_{k+1})/q_k$
converges simultaneously, where $\frac{p_k}{q_k}$ are convergents of $\alpha/2 = [b_0; b_1, b_2, \dots]$.
(iv) A.E. Brouwer and J. van de Lune (1976) proved that $S_\alpha(n) = \sum_{j=1}^{n}(-1)^{[j\alpha]} \ge 0$ for all n if and only if the partial quotients a_{2i} of $\alpha = [a_0; a_1, a_2, \dots]$ are even for all $i \ge 0$.
(v) J. Arias de Reyna and J. van de Lune (2008) defined the sequence $t_0 = 0, t_1, t_2, \dots$ of those n for which $S_\alpha(n) = \sum_{j=1}^{n}(-1)^{[j\alpha]}$ assumes a value for the first time, i.e., is larger/smaller than ever before. They proved that $S_\alpha(n)$ is not bounded, so that the corresponding sequence t_k is actually an infinite sequence. They also proved that for every $j \ge 1$ there is an index k such that $t_j - t_{j-1} = Q_k$, where P_k/Q_k is a convergent of $\alpha = [a_0; a_1, a_2, \dots]$. They also give a fast algorithm for the computation of $S_\alpha(n)$ in case of an irrational α and for very large n in terms of $\alpha/2 = [b_0; b_1, b_2, \dots]$, e.g., $S_{\sqrt{2}}(10^{1000}) = -10$, $S_{\sqrt{2}}(10^{10000}) = 166$, $S_\pi(10^{10000}) = 11726$.
(vi) J. Arias de Reyna and J. van de Lune (2008) proposed the following **Problem:** Determine whether the t_k is recurrent sequence and whether the sequence $\mathrm{sign}(S(t_k))$ is purely periodic.
(X) For the Abel discrepancy of the sequence $n\theta$, $n = 0, 1, 2, \dots,$ we have (cf. H. Niederreiter (1975, Th. 6.9)): Let θ be an irrational of type $< \psi$. Then the Abel discrepancy satisfies

$$D_r(n\theta) \le C\left(\frac{1}{m+1} + (1-r)\left(\log^2 m + \psi(m) + \sum_{h=1}^{m}\frac{\psi(h)}{h}\right)\right)$$

for all $0 < r < 1$ and for all positive integers m, where the constant C only depends on θ. Let θ be an irrational of finite type γ. Then the Abel discrepancy satisfies

$$D_r(n\theta) = \mathcal{O}((1-r)^{(1/\gamma)-\varepsilon}), \qquad D_r(n\theta) = \Omega((1-r)^{(1/\gamma)+\varepsilon})$$

for every $\varepsilon > 0$.
(XI) The sequence $x_n = n\theta \bmod 1$ satisfies the the recurrence relation $x_{n+1} \equiv$

$x_n + \theta \bmod 1$, $x_0 \equiv 0 \bmod 1$ and it is a prominent candidate for pseudorandom number generators.
(XII) For $\theta = \frac{\sqrt{m^2+4}-m}{2}$, $m = 1, 2, \dots$, L. Ramshaw (1981) proved that

$$\limsup_{N\to\infty} \frac{ND_N(\theta)}{\log N} = \begin{cases} \dfrac{m}{-4\log\theta}, & \text{for even } m, \\ \dfrac{(m^2+3)m}{-4(m^2+4)\log\theta}, & \text{for odd } m. \end{cases}$$

(XIII) W.J. LeVeque (1953) proved that the sequence $n\theta$ is u.d. mod Δ (for the def. see p. 1 – 6) for every $\theta > 0$ provided the subdivision $\Delta = (z_n)_{n=0}^{\infty}$ (i.e. $z_0 = 0$, z_n increases to infinity) satisfies:
(i) $z_0 = 0$ and $z_n - z_{n-1}$ increases to infinity,
(ii) $\lim_{n\to\infty} \frac{z_n}{z_{n-1}} = 1$.
Note that the assumption (ii) is necessary.
P. Kiss (1985) proved that for every irrational θ the sequence $n\theta \bmod \Delta$ is not u.d. but only almost u.d. for every subdivision $\Delta = (z_n)_{n=0}^{\infty}$, where z_n is a linear recurring sequence of the order r defined by the recurrence relation $z_n = a_1 z_{n-1} + a_2 z_{n-2} + \cdots + a_r z_{n-r}$ for $n \ge r$, where
- $a_1, \dots, a_r$ are integers, $a_r \neq 0$,
- the initial integer terms $z_1, \dots, z_{r-1}$ are not all zero, but $z_0 = 0$,
- if α_i are roots of the characteristic polynomial $p(x) = x^r - a_1 x^{r-1} - \cdots - a_r$, then $|\alpha_1| > |\alpha_i|$, $i = 2, 3, \dots, r$, and
- z_n strictly increases.

(XIV) An integral sequence a_n, $a_n \in \mathbb{Z}$, is called **well distributed** if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i a_{n+k} t} = 0$$

holds uniformly for $k \ge 0$ and for every non–integral real number t.
If θ is an irrational number then a necessary and sufficient condition for the sequence $b_n = \#\{1 \le j \le n\, ;\, \{n\theta\} \in I = (\alpha, \beta)\}$ to be well distributed for all intervals $I = (\alpha, \beta)$, where $\beta - \alpha$ is not an integer multiple of θ reduced mod 1, is that the continued fraction expansion of θ has bounded partial quotients (cf. W.A. Veech (1971)). See also [DT, p. 131].
(XV) (i) Let $D(N, I) = \big|\#\{n \le N; \{n\alpha\} \in I\} - N|I|\big|$ be the local discrepancy function of $n\alpha \bmod 1$ with α be irrational. If $|I| = \{h\alpha\}$ for some integer $h = h(I)$, then E. Hecke (1921) and A. Ostrowski (1921, 1927, 1930) proved that $D(N, I)$ is bounded, while Ostrowski (1930) gave the bound $D(N, I) < |h(I)|$. H. Kesten (1966/1967) proved that the condition $|I| = \{h(I)\alpha\}$ is also necessary for the boundedness of $D(N, I)$.
(ii) For a simpler proof of (i) see H. Furstenberg, H.B. Keynes and L. Shapiro (1973), K. Petersen (1973), K. Petersen and L. Shapiro (1973). For further references consult [KN, p. 128, Notes] and [DT, p. 131, Notes].
(iii) A.V. Shutov (2006) improved (i) using the “generalized Fibonacci expansion” $T = \sum_{i=-1}^{M} \varepsilon_i L_i(\alpha)$ which was originally proposed by V.G. Zhuravlev for the golden

section $\alpha = \frac{1+\sqrt{5}}{2}$. Here, given $\alpha = [a_0, a_1, a_2, \dots]$, define $\omega_i(\alpha)$ as the ith term of the binary sequence $0^{a_1-1}1^{a_2}0^{a_3}\dots$. Then define $E_0(\alpha) = G_0(\alpha) = 1$ and by induction $E_{i+1}(\alpha) = E_i(\alpha)$ and $G_{i+1}(\alpha) = G_i(\alpha) + E_i(\alpha)$ if $\omega_i(\alpha) = 0$, or $G_{i+1}(\alpha) = G_i(\alpha)$ and $E_{i+1}(\alpha) = G_i(\alpha) + E_i(\alpha)$ if $\omega_i(\alpha) = 1$. Then put $L_i(\alpha) = E_i(\alpha)$ if $\omega_i(\alpha) = 0$ and $L_i(\alpha) = G_i(\alpha)$ if $\omega_i(\alpha) = 1$. Now, given a positive integer T find M such that $E_M + G_M \le T < E_{M+1} + G_{M+1}$ and put $\varepsilon_M = 1$ and $T_M = T$. Then compute $T_{M-1}, \dots, T_{-1}$ and $\varepsilon_{M-1}, \dots, \varepsilon_{-1}$ in such a way that if $T_i \ge E_i + G_i$, then $T_{i-1} = T_i - L_i(\alpha)$ and $\varepsilon_i = 1$, and if $T_i < E_i + G_i$ then $T_{i-1} = T_i$ and $\varepsilon_i = 0$. Shutov (2006) proved (Theorem 7.8):

(iii1) If $|I| = \{h(I)\alpha\}$ and $|h(I)| = \sum_{i=-1}^{M} \varepsilon_i L_i(\alpha)$, then $D(N, I) < \sum_{k=0}^{M} \varepsilon_k(k+1)$. This implies (Shutov (2006), Theorem 9.1): Let $\alpha_0 = \min(\{\alpha\}, 1-\{\alpha\})$ have partial quotients bounded by K. Then for every interval I with bounded $D(N, I)$ as $N \to \infty$ we have $D(N, I) < (K+2)(1 + (K-1)\psi(|h(I)|))$, where $\psi(x) = \log_{\frac{\sqrt{5}+1}{2}}(\sqrt{5}x + \sqrt{5} + \frac{1}{2})$.

(iv) A.V. Shutov (2007) proved for intervals of the form $|I| = \{h\alpha\}$ that:

(iv1) $D(N, I) = O(\log h)$ for all α's with bounded partial quotients;

(iv2) $D(N, I) = O(\log^2 h)$ for all $\alpha = [a_0; a_1, a_2, \dots]$ with bounded means $\frac{1}{n}\sum_{i=1}^{n} a_i$;

(iv3) $D(N, I) = O(\log^2 h(\log\log h)^{1+\varepsilon})$ for any $\varepsilon > 0$ and almost all α.

(v) I. Oren (1981) proved that (i) also holds for a finite union of intervals.

(vi) Some generalizations can be found in G. Rauzy (1983–1984) and S. Ferenczi (1992).

(vii) **Strong form of Hecke's lemma.** Let $I \subset (0, 1)$ be an arbitrary half-open interval of length $|I| = \{q_k\alpha\}$ for some integer $k \ge 0$, where q_k is the k-th convergent denominator of α. Then for any integer $N \ge 1$ the local discrepancy satisfies

$$\left|\#\{n \le N; \{n\alpha\} \in I\} - N|I|\right| < 2$$

(see J. Beck (2014), p. 14).

(XVI) (i) G.H. Hardy and J.E. Littlewood (1946) were the first who studied the relation

(2) $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(\{n\alpha\}) = \int_0^1 f(x)dx$

for irrational α's and certain non-Riemann-integrable functions $f(x)$.

(ii) V.A. Oskolkov (1990) proved that (2) holds if and only if $\lim_{m\to\infty} \frac{1}{q_m} f(\{q_m\alpha\}) = 0$, where p_m/q_m denotes the mth convergent of α, provided that

- $f(x)$ is defined on $[0, 1]$,
- $f(0+) = f(1-0) = \infty$,
- the improper integral $\int_0^1 f(x)dx$ exists,
- $f(x)$ is non-increasing on $(0, h)$ and non-decreasing on $(1-h, 0)$, $h \in (0, 1/2)$.

(iii) Ch. Baxa and J. Schoißengeier (2002) extended Oskolkov's result in such a way that two-elements set $\{0, 1\}$ of singularities they replaced by a finite set of rational points and in this case (2) holds if and only if (3) $\lim_{N\to\infty} \frac{f(\{N\alpha\})}{N} = 0$.

(iv) Baxa (2005) removed the assumption of the rationality of singularities. He defined a new class of functions $f(x)$, integrable in $[0,1]$, and monotone in some neighbourhood of its each singularity and he proved that these functions (2) $\Leftrightarrow$ (3) if α has bounded partial quotients, or if $\liminf_{m\to\infty}||\beta q_m|| > 0$ at each singularity β of f. Moreover, he found that if $\limsup_{m\to\infty}||\beta q_m|| > 0$ at all singularities β of f then for every k, $k = 1, 2, \dots$, there exist infinite sets M_k of positive integers such that $\lim_{N\in M_k, N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(\{n\alpha\}) = \int_0^1 f(x)dx$.

Related sequences: 2.15

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2.8.1.1 Let a_n, $n = 1, 2, \dots$, be sequences of positive integers such that $a_1 = 1$, $a_2 = 2$, $a_2 = 2$, $a_3 = 2$ and for every $n = 1, 2, \dots$

$$a_{n+3} = \begin{cases} 1 + \left[5^{(na_1a_2\dots a_{n+2})^{1/3}}\right], & \text{if } n = 3k, 501 \mid k, \\ a_n + [2\log^4 a_n], & \text{if } n = 3k, 501 \nmid k, \\ 4 + 3^{3^{a_1a_2\cdots a_{n+1}}}, & \text{if } n = 3k+1, 503 \mid k, \\ a_n + [\log a_n \log^3 \log^2 a_n], & \text{if } n = 3k+1, 503 \nmid k, \\ 5 + 4^{4^{n^{a_1a_2\cdots a_n}}}, & \text{if } n = 3k+2, 505 \mid k, \\ a_n + [\log a_n \log\log^2 a_n \log^3 \log\log a_n], & \text{if } n = 3k+2, 505 \nmid k. \end{cases}$$

Then

$$\sum_{n=1}^{\infty} \frac{1}{a_n}$$

is an irrational number.

J. Hančl – P. Rucki – J. Šustek: *A generalization of Sándor's theorem using iterated logarithms*, Kumamoto J. Math. **19** (2006), 25–36 (MR2211630 (2007d:11080); Zbl. 1220.11087).

2.8.1.2 For every sequence c_n, $n = 1, 2, \dots$, of positive integers the number

$$\sum_{n=1}^{\infty} \frac{1}{2^{2^n} c_n}$$

is irrational.

P. Erdős: *Some problems and results on the irrationality of the sum of infinite series*, J. Math. Sci. **10** (1975), 1–7 (MR0539489 (80k:10029); Zbl. 0372.10023).

2.8.1.3 Let a_n, $n = 1, 2, \dots$, be defined by

$$a_n = \begin{cases} 2^{2^n}, & \text{if } n \text{ is prime,} \\ 3^{2^n}, & \text{if } n \text{ is composite.} \end{cases}$$

Then for every sequence c_n, $n = 1, 2, \dots$, of positive integers the number

$$\sum_{n=1}^{\infty} \frac{1}{a_n c_n},$$

is irrational.

J. Hančl – J. Štěpnička – J. Šustek: *Linearly unrelated sequences and problem of Erdős*, Ramanujan J. **17** (2008), no. 3, 331–342 (MR2456837 (2009i:11089); Zbl. 1242.11049).

2.8.1.4 The sequence

$$\{(n+K)\alpha\}, \{(n+2K)\alpha\}, \dots, \{(n+NK)\alpha\},$$

has discrepancy

$$D_N(\{(n+kK)\alpha\}) = D_N(\{(kK)\alpha\}) \le K D_N(\{k\alpha\}),$$

where $D_N(\{k\alpha\})$ is the extremal discrepancy of the sequence

$$\{1\alpha\}, \{2\alpha\}, \dots, \{N\alpha\}.$$

CH. MAUDUIT – A. SÁRKÖZY: *On finite pseudorandom binary sequences. II. The Champernowne, Rudin – Shapiro, and Thue – Morse sequences, a further construction*, J. Number Theory **73** (1998), no. 2, 256–276 (MR1657960 (99m:11084); Zbl. 0916.11047).

2.8.2. Let α be an irrational number having bounded partial quotients in its continued fraction and let M be an upper bound for these partial quotients. Then the infinite symmetrized sequence

$$\alpha 1,\ -\alpha 1,\ \alpha 2,\ -\alpha 2,\ \alpha 3,\ -\alpha 3,\ \dots \bmod 1$$

is

u.d.

with L^2 discrepancy satisfying

$$D_N^{(2)} \le \frac{1}{N^2}\left(1+\left(4M+\frac{9}{2}\right)\left(\frac{2\log\left(\frac{N}{M+1}\right)}{3\log 2}\right)+\frac{24}{\pi^2}-\frac{4}{3}\right)$$

for $N \ge 8(M+1)$.

NOTES: P.D. Proinov (1983, 1985). He gives the first construction of a sequence from $[0,1)$ for which the L^2 discrepancy has the least possible order. Note that K.F. Roth (1954) (cf. H. Niederreiter (1973)) proved that there exists a constant c, $\left(c \ge \frac{1}{2^{16}\log 2}\right)$ such that

$$D_N^{(2)} \ge c\frac{\log N}{N^2}$$

for every infinite sequence x_n and for all sufficiently large N. Proinov (1983) proved that $D_N^{(2)} \le 100\frac{\log N}{N^2}$ for $\alpha = \frac{\sqrt{5}-1}{2}$ and later (1986) he improved this to $D_N^{(2)} \le 9.1521\frac{\log N}{N^2}$.

RELATED SEQUENCES: 2.11.6

H. NIEDERREITER: *Application of diophantine approximations to numerical integration*, in: *Diophantine Approximation and Its Applications (Washington, D.C., 1972)*, (C.F. Osgood ed.), Academic Press, New York, 1973, pp. 129–199 (MR0357357 (**50** #9825); Zbl. 0268.65014).
P.D. PROINOV: *Estimation of L^2 discrepancy of a class of infinite sequences*, C. R. Acad. Bulgare Sci. **36** (1983), no. 1, 37–40 (MR0707760 (86a:11030); Zbl. 0514.10039).
P.D. PROINOV: *On the L^2 discrepancy of some infinite sequences*, Serdica **11** (1985), no. 1, 3–12 (MR0807713 (87a:11071); Zbl. 0584.10033).
P.D. PROINOV: *On irregularities of distribution*, C. R. Acad. Bulgare Sci. **39** (1986), no. 9, 31–34 (MR0875938 (88f:11068); Zbl. 0616.10042).
K.F. ROTH: *On irregularities of distribution*, Mathematika **1** (1954), 73–79 (MR0066435 (16,575c); Zbl. 0057.28604).

2.8.3. Let s_n be an increasing sequence of positive integers which is multiplicatively closed (i.e. its terms form a multiplicative semigroup in $\mathbb{R}$) which satisfy

$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = 1.$$

Then, for every irrational α the sequence

$$s_n\alpha \bmod 1, \quad n = 1, 2, \dots,$$

is

$$\text{dense in } [0, 1].$$

NOTES:
(I) H. Furstenberg (1967). The sequence s_n is called a non–lacunary multiplicative semigroug of integers. It is not formed by the powers of a single integer. M.D. Boshernitzan (1994) gave a short elementary proof for this and D. Berend (1983) proved a multi–dimensional analogue.
(II) D. Berend's (1986) extension:

Let K be a real algebraic number field and S a subsemigroup of the multiplicative group of K such that

(i) $S \subset (-\infty, -1) \cup (1, \infty)$,

(ii) there exist multiplicatively independent elements $\lambda, \mu \in S$ (i.e. there exist no integers m and n, not both vanishing, with $\lambda^m = \mu^n$),

(iii) $\mathbb{Q}(S) = K$.

Then for every $\alpha \notin K$ the set $S\alpha \bmod 1$ is dense in $[0, 1]$. If, moreover

(iv) $S \not\subset PS(K)$,

then $S\alpha \bmod 1$ is dense in $[0, 1]$ for every $\alpha \neq 0$.

Here $PS(K)$ denotes the semigroup of the all Pisot or Salem numbers of degree m over $\mathbb{Q}$, where $m = [K : \mathbb{Q}]$.

Furthermore, if $S\alpha \bmod 1$ is dense in $[0, 1]$ for every $\alpha \notin K$ or for all $\alpha \neq 0$, then S has a subsemigroup generated by two elements having the properties (i)-(iii).

D. BEREND: *Dense* (mod 1) *semigroups of algebraic numbers*, J. Number Theory **26** (1987), no. 3, 246–256 (MR901238 (88e:11102); Zbl. 0623.10038).
D. BEREND: *Multi–invariant sets on tori*, Trans. Amer. Math. Soc. **280** (1983), no. 2, 125–147 (MR0716835 (85b:11064); Zbl. 0532.10028).
M.D. BOSHERNITZAN: *Elementary proof of Furstenberg's Diophantine result*, Proc. Amer. Math. Soc. **122** (1994), no. 1, 67–70 (MR1195714 (94k:11085); Zbl. 0815.11036).
H. FURSTENBERG: *Disjointness in ergodic theory, minimal sets and a problem in diophantine approximation*, Math. Systems Theory **1** (1967), no. 1, 1–49 (MR0213508 (**35** #4369); Zbl. 0146.28502).

2.8.4. Let P be a subset of primes containing at least two distinct primes and let s_n be the increasing sequences of the all positive integers divisible only by primes from P. Then

$$\alpha(s_n)^k \bmod 1$$

is

$$\text{dense in } [0,1] \text{ for every irrational } \alpha \text{ and every } k = 1, 2, \dots.$$

NOTES: This follows from 2.6.34, cf. D. Andrica and S. Buzeteanu (1987, 4.9. Cor.).

D. ANDRICA – S. BUZETEANU: *Relatively dense universal sequences for the class of continuous periodical functions of period T*, Math. Rev. Anal. Numér. Théor. Approximation, Anal. Numér. Théor. Approximation **16** (1987), no. 1, 1–9 (MR0938777 (89i:11084); Zbl. 0642.26006).

2.8.5. Open problem. For a given sequence q_n of positive integers, find conditions on reals θ such that the sequence

$$q_n\theta \bmod 1$$

is

$$\text{u.d.}$$

NOTES: (I) If the strictly increasing sequence q_n of integers has the property that x_{q_n} is u.d. provided x_n is u.d, then $q_n\theta$ mod 1 is u.d. for every irrational θ. Examples of such sequences can be found in 2.4, and especially in H. Rindler (1973/74) and V. Losert and H. Rindler (1978).

(II) Ch. Mauduit (1984) gave a sufficient condition for an integer sequence q_n (which are recognizable by a finite automaton) that for all irrational θ the sequence $q_n\theta$ mod 1 is u.d.

(III) D. Berend (1990) proved conditions (necessary and sufficient, cf. 2.8.6) for q_n for which $q_n\theta$ mod 1 is dense or u.d., where q_n are elements of an additive semigroup generated by a strictly increasing sequence of integers.

(IV) I.Z. Ruzsa (1983) proved Niederreiter's conjecture that there exists an increasing sequence q_n of positive integers which is u.d. modulo each positive integer m, but nevertheless q_nx mod 1 is not u.d. for any x.

(V) M. Mendès France (1967/68) showed that for any $f(n)$ which tends to infinity there exists an integer sequence q_n satisfying $q_n = \mathcal{O}(f(n))$ such that the sequence q_nx mod 1 is u.d. for any irrational x. F. Dress (1967/68) proved that the sequence q_n cannot be non–decreasing if $q_n = \mathcal{O}(\log n)$, cf. 2.2.8.

(VI) Constructions of such θ are given in 2.8.8.

(VII) For a given real sequence q_n consider the so–called **normal set associate to** q_n

$$B(q_n) = \{\theta \in \mathbb{R}\,;\, q_n\theta \bmod 1 \text{ is u.d. }\}.$$

J. Lesca and M. Mendès France (1970) proved: Let a mapping $h : \mathbb{N} \to \mathbb{N}$ be fixed. If for all increasing sequences q_n of positive integers $B(h(q_n)) = B(q_n)$, then $h(n) = n + \text{const.}$ for all large n.
(VIII) O. Strauch (1992) proved that for the mean value of the L^2 discrepancy $D_N^{(2)}(q_n\theta)$ we have

$$\int_0^1 D_N^{(2)}(q_n\theta)\,\mathrm{d}\theta = \frac{1}{N^2}\left(\frac{1}{12}\sum_{m,n=1}^{N}\frac{(q_m,q_n)^2}{q_mq_n} + \frac{1}{12}\sum_{\substack{m,n=1\\ q_m=q_n}}^{N} 1\right).$$

For the multi–dimensional sequence $q_n\boldsymbol{\theta}$, cf. 3.4.3
(IX) Y. Bugeaud (2009): Let θ be an irrational number, X be a finite non-empty set in $[0,1]$, and let λ_n, $n = 1,2,\dots$, be an arbitrary sequence of real numbers such that $\lambda_n \ge 1$ and $\lambda_n \to \infty$. Then there exists a sequence q_n, $n = 1,2,\dots$, of positive integers such that $q_n \le n\lambda_n$ and the set of limit points of $q_n\theta$ mod 1 is equal to X.

D. Berend: *IP–sets on the circle*, Canad. J. Math. **42** (1990), no. 4, 575–589 (MR 92c:11076; Zbl. 0721.11025).
Y. Bugeaud: *On sequences* $(a_n\xi)_{n\ge1}$ *converging modulo 1*, Proc. Amer. Math. Soc. **137** (2009), no. 8, 2609–2612 (MR2497472 (2010c:11089); Zbl. 1266.11084)).
F. Dress: *Sur l'équiréparation de certaines suites* $(x\lambda_n)$, Acta Arith. **14** (1968), 169–175 (MR0227118 (**37** #2703); Zbl. 0218.10055).
J. Lesca – M. Mendès France: *Ensembles normaux*, Acta Arith. **17** (1970), 273–282 (MR0272724 (**42** #7605); Zbl. 0208.05703).
V. Losert – H. Rindler: *Teilfolgen gleichverteilter Folgen*, J. Reine Angew. Math. **302** (1978), 51–58 (MR0511692 (80a:10071); Zbl. 0371.10039).
Ch. Mauduit: *Automates finis et équirépartion modulo* 1, C. R. Acad. Sci. Paris. Sér. I Math. **299** (1984), no. 5, 121–123 (MR0756306 (85i:11066); Zbl. 0565.10030).
M. Mendès France: *Deux remarques concernant l'équiréparation des suites*, Acta Arith. **14** (1968), 163–167 (MR0227117 (**37** #2702); Zbl. 0177.07202).
H. Rindler: *Ein Problem aus der Theorie der Gleichverteilung, II*, Math. Z. **135** (1973/1974), 73–92 (MR0349614 (**50** #2107); Zbl. 0263.22009).
I.Z. Ruzsa: *On the uniform and almost uniform distribution of* (a_nx) mod 1, Séminaire de Théorie des Nombres de Bordeaux 1982/1983, Exp. No. 20, Univ. Bordeux I, Talence, 1983, 21 pp. (MR0750320 (86c:11051); Zbl. 0529.10046).
O. Strauch: *An improvement of an inequality of Koksma*, Indag. Mathem., N.S. **3** (1992), 113–118 (MR1157523 (93b:11098); Zbl. 0755.11023).

$\boxed{\textbf{2.8.5.1}}$ **Glasner sets.** Following D. Berend and Y. Peres (1993) a strictly increasing sequence of positive integers k_n, $n = 1,2,\dots$, is called a **Glasner set** if for every infinite set $A \subset [0,1)$ and every $\varepsilon > 0$ there exists a k_n such that the dilation $k_nA \bmod 1 = \{k_nx \bmod 1 : x \in A\}$ is ε–dense in $[0,1]$, i.e. k_nA mod 1 intersects every subinterval of $[0,1]$ of length ϵ. The following sequences k_n, $n = 1,2,\dots$, are Glasner sets:
(i) $k_n = n$ (D. Berend and Y. Peres (1993)).

(ii) $k_n = P(n)$, where $P(x)$ is a non-constant polynomial with integer coefficients (D. Berend and Y. Peres (1993)).
(iii) $k_n = P(p_n)$, where p_n is the increasing sequence of all primes and polynomial $P(x)$ is as in (ii) (N. Alon and Y. Peres (1992)).

A strictly increasing sequence of positive integers k_n, $n = 1, 2, \dots$, is said to have **quantitative Glasner property** if for every $\varepsilon > 0$ there exists an integer $s(\varepsilon)$ such that for any finite set $A \subset [0, 1)$ of cardinality at least $s(\varepsilon)$ there exists a k_n such that the dilation $k_n A$ mod 1 is ε–dense in $[0, 1)$. The following sequences k_n, $n = 1, 2, \dots$, share this property:

(iv) $k_n = n$ with $s(\varepsilon) = [\varepsilon^{-2-\gamma}]$, where $\gamma > 0$ is arbitrary and $\varepsilon \le \varepsilon_0(\gamma)$.
(v) $k_n = P(n)$, where $P(x)$ is a non-constant polynomial with integer coefficients.
(vi) $k_n = P(p_n)$ as in (iii) with $s(\varepsilon) = [\varepsilon^{-2d-\delta}]$, where $d = \deg P(x)$, $\delta > 0$ arbitrary and $\varepsilon < \varepsilon_0(P(x), \delta)$.
(vii) k_n, $n = 1, 2, \dots$, satisfying
($*$) uniformly distributed for each positive integer m (i.e. the relative density of $k_m \equiv i \pmod{m}$ is $1/m$ for each $i = 0, 1, \dots, m-1$), and
($**$) the sequence $k_n\alpha$ mod 1 is u.d. in $[0, 1]$ for each irrational α.
Here $s(\varepsilon) = [\varepsilon^{-2-3(\log\log(1/\varepsilon))^{-1}}] + 1$, for every $\varepsilon < \varepsilon_0$, where ε_0 depends on the sequence k_n, $n = 1, 2, \dots$.
(viii) $k_n = [f(n)]$, where $f(x)$, $x \in \mathbb{R}$, is a non-polynomial entire real function such that $|f(z)| = O(e^{(\log|z|)^\alpha})$ with $\alpha < 4/3$ and with $s(\varepsilon)$ is as in (vii).
(ix) $k_n = [f(p_n)]$ with p_n denoting the increasing sequence of all primes and f as in (viii) and with $s(\varepsilon)$ as in (vii).
(x) $k_n = [n^\alpha]$ with $\alpha = 1$ or any non-integral $\alpha > 1$ and with $s(\varepsilon)$ as in (vii).

N. Alon – Y. Peres: *Uniform dilations*, Geom. Funct. Anal. **2** (1992), no. 1, 1–28 (MR1143662 (93a:11061); Zbl. 0756.11020).
D. Berend – Y. Peres: *Asymptotically dense dilations of sets on the circle*, J. Lond. Math. Soc., II. Ser. **47** (1993), no. 1, 1–17 (MR1200973 (94b:11068); Zbl. 0788.11028).
S. Glasner: *Almost periodic sets and measures on the torus*, Israel J. Math **32** (1979), no. 2–3, 161–172 (MR0531259 (80f:54038); Zbl. 0406.54023).
H.H. Kamarul – R. Nair: *On certain Glasner sets*, Proc. R. Soc. Edinb., Sect. A, Math. **133** (2003), no. 4, 849–853 (MR2006205 (2005g:11014); Zbl. 1051.11042).
R. Nair: *On asymptotic distribution on the a–adic integers*, Proc. Indian Acad. Sci., Math. Sci. **107** (1997), no. 4, 363–376 (MR1484371 (98k:11110); Zbl. 0908.11036).
R. Nair – S.L. Velani: *Glasner sets and polynomials in primes*, Proc. Amer. Math. Soc. **126** (1998), no. 10, 2835–2840 (MR1452815 (99a:11095); Zbl. 0913.11031).

2.8.6. Let b_n, $n = 1, 2, \dots$, be a sequence of integers and let q_n, $n = 1, 2, \dots$, be the sequence of integers representable as the finite sum

$$b_{n_1} + b_{n_2} + \dots + b_{n_k}, \quad \text{where } n_1 < n_2 < \dots < n_k \text{ and } k = 1, 2, \dots,$$

and ordered lexicographically (the sequence may assume some values more than once). Then it is true:

(I) If b_n is strictly increasing such that $\liminf_{n\to\infty}(b_{n+1}-b_n)<+\infty$, then the sequence $q_n\theta \bmod 1$ is u.d for every irrational θ.

(II) If $\sum_{n=1}^{\infty}\frac{b_n}{b_{n+1}}<+\infty$ then there exists an irrational θ such that $q_n\theta \bmod 1$ is not dense in $[0,1]$.

(III) If $\sum_{n=1}^{\infty}\left(\frac{b_n}{b_{n+1}}\right)^2<+\infty$ then there exists an irrational θ such that $q_n\theta \bmod 1$ is not u.d.

(IV) If $b_n=n!$ then there exists an irrational θ such that $q_n\theta \bmod 1$ is not dense in $[0,1]$.

(V) If $b_n=n!-1$ then the sequence $q_n\theta \bmod 1$ is dense in $[0,1]$ for every irrational θ, but there exists an irrational θ such that $q_n\theta \bmod 1$ is not u.d.

(VI) If $b_n=[u^n]$, where $u>1$ is rational, then the sequence $q_n\theta \bmod 1$ is u.d. for any irrational θ.

(VII) If $b_n=F_n$, $n=2,3,\dots$, the Fibonacci sequence, then for any irrational θ the sequence $q_n\theta \bmod 1$ is dense in $[0,1]$. On the other hand if we omit F_2, i.e. take $b_n=F_n$ for $n=3,4,\dots$, then there exists an irrational θ such that $q_n\theta \bmod 1$ is not dense.

D. Berend: *IP–sets on the circle*, Canad. J. Math. **42** (1990), no. 4, 575–589 (MR 92c:11076; Zbl. 0721.11025).

2.8.7. Let q_n be an increasing sequence of positive integers generated (multiplicatively) by a finite sequence $Q_1,Q_2,\dots,Q_k$ of pairwise coprime integers ≥ 2, i.e. any q_n has the form $q_n=Q_1^{\alpha_1}\dots Q_k^{\alpha_k}$ for some non–negative integers $\alpha_1,\dots,\alpha_k$. If θ is a given real number then the sequence

$$x_n=q_n\theta \bmod 1$$

is

u.d.

if there exist two positive constant c and σ such that for every subinterval $I\subset[0,1]$, $|I|>0$, we have

$$\limsup_{N\to\infty}\frac{A(I;N;x_n)}{N}\leq c|I|(1-\log|I|)^{\sigma}.$$

Notes: D.A. Moskvin (1970) extended thus a previous result of A.G. Postnikov (1952), cf. 2.18.19. Moskvin (1970) also proved that any subsequence $x_n=q_n\theta \bmod 1$

of the form $x_{n^*} = q_{n^*}\theta \bmod 1$, where q_{n^*}'s are generated by integers of the form $Q_1^{m_1}, \dots, Q_k^{m_k}$ where $m_1, \dots, m_k$ are fixed positive integers, is again u.d.

D.A. Moskvin: *The distribution of fractional parts of a sequence that is more general than the exponential function*, Izv. Vyš. Učebn. Zaved. Matematika **12(103)** (1970), 72–77 (MR0289425 (**44** #6616; Zbl. 0216.31902)).
A.G. Postnikov: *On distribution of the fractional parts of the exponential function*, Dokl. Akad. Nauk. SSSR (N.S.) (Russian), **86** (1952), 473–476 (MR0050637 (14,359d); Zbl. 0047.05202).

$\boxed{\textbf{2.8.7.1}}$ Let q_n, $n = 1, 2, \dots$, be an increasing sequence of positive integers having positive upper asymptotic density. Then for every irrational number α the sequence

$$q_n\alpha \bmod 1, n = 1, 2, \dots,$$

has infinitely many limit points.

A. Dubickas: *On the limit points of* $(a_n\xi)_{n=1}^{\infty}$ mod 1 *for slowly increasing integer sequence* $(a_n)_{n=1}^{\infty}$, Proc. Amer. Math. Soc. **137** (2009), no. 2, 449–456 (MR2448563 (2009h:11123)).

$\boxed{\textbf{2.8.7.2}}$ Let q_n, $n = 1, 2, \dots$, be an increasing sequence of positive integers of the form $p^k + q^m$, where $p < q$ are two fixed primes and k, m run over all non-negative integers. Then the question whether the sequence

$$q_n\alpha \bmod 1, n = 2, 3, \dots$$

is everywhere dense in $[0, 1]$ is open.

Notes: Proposed by D. Meiri (1998) and discussed in A. Dubickas (2009).

A. Dubickas: *On the limit points of* $(a_n\xi)_{n=1}^{\infty}$ mod 1 *for slowly increasing integer sequence* $(a_n)_{n=1}^{\infty}$, (), Proc. Amer. Math. Soc. **137** (2009), no. 2, 449–456 (MR2448563 (2009h:11123)).
D. Meiri: *Entropy and uniform distribution of orbits in* $\mathbb{T}^n$, Israel J. Math. **105** (1998), 155–183 (MR1639747 (99f:58129)).

2.8.8. Let f_n be a sequence of positive integers and $q_n(c, k) = f_n + cn^k$, where $c \neq 0$ and k are positive integers. Let (c, k) be such a pair that there is a sequence of positive integers a_n with $q_n(c, k)a_n = q_{n+1}(c, k) - q_{n-1}(c, k)$. Define $\alpha = [b_1, \dots, b_m, a_1, a_2, \dots]$, where $\frac{q_2(c,k)}{q_1(c,k)} = [b_m, \dots, b_1]$ is the continued fraction expansion of $\frac{q_2(c,k)}{q_1(c,k)}$. Then the sequence

$$\alpha f_n$$

is

$$\text{u.d.}$$

P. Gerl: *Konstruktion gleichverteilter Punktfolgen*, Monatsh. Math. **69** (1965), 306–317 (MR0184922 (**32** #2393); Zbl. 0144.28801).

2.8.9. Let $f(x) = \alpha_k x^k + \alpha_{k-1}x^{k-1} + \cdots + \alpha_0$ be a polynomial with real coefficients and $k > 1$. Then the sequence

$$x_n = tf(n) \bmod 1, \quad n = 1, 2, \dots,$$

is

$$\text{u.d.}$$

for every real $t \neq 0$ (i.e. $f(n)$ is u.d. in $\mathbb{R}$, cf. 1.5) if and only if at least two of the coefficients $\alpha_k, \alpha_{k-1}, \dots, \alpha_1$ are linearly independent over $\mathbb{Q}$.
Notes: [KN, p. 283, Ex. 5.4].

2.8.10. If t_n is a sequence of positive real numbers satisfying

$$\frac{t_{n+1}}{t_n} \geq 5^{1/3}, \quad \text{for } n = 1, 2, \dots,$$

then there exist positive numbers θ and β such that

$$\{t_n\theta\} \in [\beta, 1 - \beta], \quad n = 1, 2, \dots.$$

Notes: E. Strzelecki (1975). The result is related to a problem proposed by P. Erdős and S.J. Taylor (1957).

P. Erdős – S.J. Taylor: *On the set of points of convergence of a lacunary trigonometric series and the equidistribution properties of related sequences*, Proc. London Math. Soc. (3) **7** (1957), 598–615 (MR0092032 (19,1050b); Zbl. 0111.26801).
E. Strzelecki: *On sequences* $\{\xi t_n \pmod 1\}$, Canad. Math. Bull **18** (1975), no. 5, 727–738 (MR0406949 (**53** #10734); Zbl. 0326.10033).

2.8.11. Assume that $f(x) \in C^2[1, \infty]$ satisfies
- $f(x)$ is positive, strictly increasing and $f(x) \to \infty$ as $x \to \infty$,
- $f'(x) \to$ constant < 1 monotonically as $x \to \infty$,

If α is an irrational number of finite type γ (cf. 2.8.1(V)) then the sequence

$$f(n)\alpha \bmod 1$$

has extremal weighted discrepancy D_N with respect to weights $f'(n)$ satisfying

$$D_N = \mathcal{O}\left(\frac{1}{f(N)}\int_1^N (f'(x))^2\,\mathrm{d}x\right)^{\frac{1}{\gamma}-\varepsilon}$$

for every $\varepsilon > 0$. If α is an irrational of constant type, then

$$D_N = \mathcal{O}\left(\frac{(\log F(N))^2}{F(N)}\right),$$

where $f(N)/F(N) = \int_1^N (f'(x))^2\,\mathrm{d}x$.

Y. Ohkubo: *Discrepancy with respect to weighted means of some sequences*, Proc. Japan Acad. **62** A (1986), no. 5, 201–204 (MR0854219 (87j:11075); Zbl. 0592.10044).

2.8.12. Open problem. Characterize the distribution of the sequence

$$x_n = \begin{cases} \{n\alpha\}\alpha, & \text{if } \{n\alpha\} < 1-\alpha, \\ (1-\{n\alpha\})(1-\alpha), & \text{if } \{n\alpha\} \geq 1-\alpha, \end{cases}$$

for $0 < \alpha < 1$.

Notes:

(I) A.F. Timan (1987) proved that the series $\sum_{n=1}^{\infty} \frac{x_n}{n^r}$ converges for all $\alpha \in (0,1)$ if and only if $r > 1$.

(II) S. Steinerberger's **Solution:** For irrational $0 < \alpha < 1$ we have $x_n = f(\{n\alpha\})$, where

$$f(x) = \begin{cases} x\alpha, & \text{if } x \in [0, 1-\alpha], \\ (1-x)(1-\alpha), & \text{if } x \in [1-\alpha, 1]. \end{cases}$$

Then a.d.f. $g(x)$ of x_n is

$$g(x) = |f^{-1}([0,x))| = \begin{cases} 1, & \text{if } x \in [\alpha(1-\alpha), 1], \\ \frac{x}{\alpha(1-\alpha)}, & \text{others}. \end{cases}$$

A.F. Timan: *Distribution of fractional parts and approximation of functions with singularities by Bernstein polynomials.*, J. Approx. Theory **50** (1987), no. 2, 167–174 (MR0888298 (88m:11054); Zbl. 0632.41012).
S. Steinerberger: *Personal communication.*

2.8.13. Let $s > 0, a \geq 0, b \geq 0$ be integers and θ an irrational number. Then the double sequence

$$m + \theta n, \quad m, n = 1, 2, \dots, \quad m \equiv a \pmod{s}, \quad n \equiv b \pmod{s},$$

is

dense in $\mathbb{R}$.

NOTES: D. Andrica and S. Buzeteanu (1987, 3.2. Lemma) and S. Hartman (1949).

D. ANDRICA – S. BUZETEANU: *Relatively dense universal sequences for the class of continuous periodical functions of period T*, Math. Rev. Anal. Numér. Théor. Approximation, Anal. Numér. Théor. Approximation **16** (1987), no. 1, 1–9 (MR0938777 (89i:11084); Zbl. 0642.26006).
S. HARTMAN: *Sur une condition supplémentaire dans les approximations diophantiques*, Colloq. Math. **2** (1949), no. 1, 48–51 (MR0041174 (12,807a); Zbl. 0038.18802).

2.8.14. Let α, β be two real numbers such that $1, \alpha, \beta$ are linearly independent over $\mathbb{Q}$, and $I \subset [0, 1]$ an interval with $|I| > 0$. If $n_1 < n_2 < \dots$ denotes the sequence of integers n such that $\{n\alpha\} \in I$, then the sequence

$$\beta n_k \bmod 1, \quad k = 1, 2, \dots,$$

is

u.d.

NOTES: D.P. Parent (1984, p. 254, Ex. 5.27).

RELATED SEQUENCES: 2.16.2

D.P. PARENT: *Exercises in Number Theory*, Problem Books in Mathematics, Springer Verlag, New York, 1984 (MR0759342 (86f:11002); Zbl. 0536.10001). (French original: *Exercices de théorie des nombres*, Gauthier – Villars, Paris, 1978 (MR0485646 (**58** #5471); Zbl. 0387.10001)).

2.8.15. Let θ be an irrational number. Then for every $k = 2, 3, \dots$ the sequence

$$x_n = \frac{\{\theta n\}^k + (-1)^{k+1}[\theta n]^k}{\theta k!} \bmod 1$$

is

u.d.

J. BREZIN: *Applications of nilmanifold theory to diophantine approximations*, Proc. Amer. Math. Soc. **33** (1972), no. 2, 543–547 (MR0311587 (**47** #149); Zbl. 0249.22007).

2.8.16. Let $q_n \geq 2$ be a given sequence of positive integers with $\lim_{n\to\infty} q_n = \infty$ and θ be a real number. Then the sequence

$$x_n = \theta q_1 \dots q_n \bmod 1$$

is

u.d.

if and only if θ can be expressed in the form

$$\theta = a_0 + \sum_{n=1}^{\infty} \frac{[\vartheta_n q_n]}{q_1 \cdots q_n},$$

where ϑ_n is a sequence u.d. in $[0,1)$. Moreover, x_n is also u.d. if $\vartheta_n = \vartheta'_n + k_n$, where

(i) k_n is a sequence of positive integers,
(ii) ϑ'_n is u.d. in $[0,1)$,
(iii) $\lim_{n\to\infty} \frac{k_n}{q_n} = 0$.

Notes: (I) This was proved by N.M. Korobov (1950, Th. 3).
(II) T. Šalát (1968) relaxed the condition $\lim_{n\to\infty} q_n = \infty$ to $\sum_{n=1}^{N} 1/q_n = o(N)$ as $N \to \infty$.
(III) Note that, for a given $\vartheta \in [0,1)$, the series $\theta = a_0 + \sum_{n=1}^{\infty} \frac{[\vartheta_n q_n]}{q_1 \cdots q_n}$ coincides $\big($if $a_0 = [\theta]\big)$ with the Cantor expansion $\theta = a_0 + \sum_{n=1}^{\infty} \frac{[\theta_n q_n]}{q_1 \dots q_n}$, where $\theta_1 = \{\theta\}$, $\theta_{n+1} = \{\theta_n q_n\}$, $[\theta_n q_n] = [\vartheta_n q_n]$. Note also that θ_n and ϑ_n may be distinct.
(IV) Korobov (1950, Th. 4) also proved that if $q_n = q \geq 2$ and

$$\theta = a_0 + \sum_{n=1}^{\infty} \frac{[\vartheta_n q]}{q^n},$$

and ϑ_n is a completely u.d. sequence in $[0,1)$, then the sequence

$$\theta q^n \bmod 1$$

is

u.d.

(cf. 2.18.15). He mentioned that for $q_n = q$, the u.d. mod 1 of $\theta q_1 \dots q_n = \theta q^n$ implies the expression $\theta = a_0 + \sum_{n=1}^{\infty} \frac{[\vartheta_n q]}{q^n}$, where ϑ_n is u.d. in $[0,1]$. For the reverse implication we need that the q_n's are unbounded.

Related sequences: 2.8.17, 2.8.18, 2.18.15.

N.M. Korobov: *Concerning some questions of uniform distribution*, (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. **14** (1950), 215–238 (MR0037876 (12,321a); Zbl. 0036.31104).
T. Šalát: *Zu einigen Fragen der Gleichverteilung* (mod 1), Czechoslovak Math. J. **18(93)** (1968), 476–488 (MR0229586 (**37** #5160); Zbl. 0162.34701).

2.8.17. Let $0 < \lambda < 1$ and

$$\theta = \sum_{n=1}^{\infty} \frac{[n^{1+\lambda}]}{n!}.$$

Then the sequence

$$\theta n! \bmod 1$$

is

$$\text{u.d.}$$

Notes:
(I) N.M. Korobov (1950) proved this as an application of 2.8.16 with $q_n = n+1$ and $\vartheta_n = \vartheta'_n + k_n = \{n^\lambda\} + [n^\lambda] = n^\lambda$.
(II) Parent [p. 257, Ex. 5.37]: To an arbitrary sequence $x_n \in [0,1)$, $n = 1, 2, \dots$, one can associate a real number α such that $\lim_{n\to\infty}(\{n!\alpha\} - x_n) = 0$. Thus by 2.3.3 $G(\{n!\alpha\}) = G(x_n)$.
(III) A. Aleksenko proved: Suppose that the sequence n_k, $k = 1, 2, \dots$, of positive numbers satisfies $\frac{n_{k+1}}{kn_k} \geq \rho > 0$, where ρ is a constant. Then there exists a real α for which the sequence $\{\alpha n_k\}$, $k = 1, 2, \dots$ has discrepancy $D_N = O\left(\frac{\log N}{N}\right)$. From there she deduced that there exists a real α such that the discrepancy of $\{\alpha n!\}$, $n = 1, 2, \dots, N$, is $D_N = O\left(\frac{\log N}{N}\right)$.

A. Aleksenko: *On the sequence $\alpha n!$*, Unif. Distrib. Theory **9** (2014), no. 2, 1–6 (MR3430807; Zbl. 1340.11061).
N.M. Korobov: *Concerning some questions of uniform distribution*, (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. **14** (1950), 215–238 (MR0037876 (12,321a); Zbl. 0036.31104).
D.P. Parent: *Exercises in Number Theory*, Problem Books in Mathematics, Springer Verlag, New York, 1984 (MR0759342 (86f:11002); Zbl. 0536.10001). (French original: *Exercices de théorie des nombres*, Gauthier – Villars, Paris, 1978 (MR0485646 (**58** #5471); Zbl. 0387.10001)).

2.8.18. Let

$$\theta = \sum_{n=1}^{\infty} \frac{[4^n \log^\lambda n]}{2^{n^2}}.$$

If $\lambda > 1$ then the sequence

$$\theta 2^{n^2} \bmod 1$$

is

u.d.

but if $0 < \lambda \leq 1$ it is

dense but not u.d.

NOTES: N.M. Korobov (1950) via an application of 2.8.16 with $q_n = 2^{2n-1}$ and $\vartheta_n = \vartheta'_n + k_n = \{\log^\lambda n\} + [\log^\lambda n] = \log^\lambda n$.

N.M. KOROBOV: *Concerning some questions of uniform distribution*, (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. **14** (1950), 215–238 (MR0037876 (12,321a); Zbl. 0036.31104).

2.8.19. If α, β are real numbers and P, Q positive integers, then the elements of the sequence

$$i\alpha + j\beta \bmod 1, \qquad i = 0, 1, \dots, P-1, \quad j = 0, 1, \dots, Q-1,$$

partition the unit interval $[0, 1]$ into PQ subintervals which have at most $P + 3$ distinct lengths.

NOTES: (I) J.F. Geelen and R.J. Simpson (1993). This generalizes the Steinhaus three–gaps theorem, cf. 2.8.1.

(II) If one of the numbers α and β is irrational, then the double infinite sequence

$$x_{i,j} = i\alpha + j\beta \bmod 1, \qquad i = 0, 1, 2, \dots, \quad j = 0, 1, 2, \dots,$$

is (cf. [KN, p. 18, Ex. 2.9])

u.d.

J.F. GEELEN – R.J. SIMPSON: *A two–dimensional Steinhaus theorem*, Australas. J. Combin **8** (1993), 169–197 (MR1240154 (94k:11083); Zbl. 0804.11020).

2.9 Sequences involving sum–of–digits functions

See also: 3.5, 3.11.2

NOTES: Let $q \geq 2$ be a positive integer and let $n = \sum_{j=0}^{k(n)} a_j(n) q^j$ be the q–adic digit expansion of n with integral digits $0 \leq a_j(n) < q$. The q–ary **sum–of–digits function $s_q(n)$ in base q** is defined by

$$s_q(n) = \sum_{j=0}^{k(n)} a_j(n).$$

For the multi–dimensional sequences involving sum–of–digits function consult 3.5.

2.9.1. If $s_q(n)$ denotes sum–of–digits function in base q then the sequence

$$s_q(n)\theta \bmod 1, \quad n = 0, 1, 2, \dots,$$

is

u.d.

if and only if θ is irrational. If $\theta = [b_0; b_1, b_2, \dots]$ has bounded partial quotients b_i, then

$$D_N \le c(q, \theta) \frac{\log\log\log N}{\sqrt{\log N}},$$

for all N. It θ is irrational then there exists a constant $c(q) > 0$ such that

$$D_N > c(q) \frac{1}{\sqrt{\log N}}$$

for all N.

NOTES: (I) M. Mendès France (1967, Th. III.5.1) was the first who turned the attention to problems of this type, cf. [KN, p. 76, Notes].

(II) Discrepancy bounds were proved by M. Drmota and G. Larcher (2001). More precisely, they proved that there are positive constants $c'(q)$ and $d(q)$ such that

$$D_N \le c'(q) \frac{1}{\sqrt{\log N}} \sum_{1 \le i < i_0 + 1} b_i,$$

where i_0 is defined by $q_{i_0} \le d(q)\sqrt{\log N. \log\log N} < q_{i_0+1}$ and q_i are the denominators of the convergents $p_i/q_i = [b_0; b_1, b_2, \dots, b_i]$.

They also posed the following **open problem:** are there irrational numbers θ such that for the discrepancy D_N of $s_q(n)\theta \bmod 1$ we have

$$D_N \le c \frac{1}{\sqrt{\log N}}$$

for every N?

(III) Previously, R.F. Tichy and G. Turnwald (1987) found the discrepancy bound

$$D_N \le c(q, \theta) \frac{\sqrt{\log\log N}}{\sqrt{\log N}},$$

for θ with bounded partial quotients, cf. 2.9.3.

G. Larcher (1993) proved the best possible general lower bound of the form $D_N \ge c(\log N)^{-v/2}$ with some fixed v depending on θ.

(IV) Given a sequence $x_n \in [0, 1)$, a subset $X \subset [0, 1]$ is called a **bounded remainder set** if there exists a $t \in [0, 1]$ such that $|A(X; N; x_n) - tN|$ is bounded as a function of N. P. Liardet (1987) proved that the only intervals I mod 1 which are bounded remainder sets for the sequence $s_q(n)\theta$ with irrational θ are the trivial ones $|I| = 0$ and $|I| = 1$.

Related sequences: 2.19.10

M. Drmota – G. Larcher: *The sum–of–digits–function and uniform distribution modulo 1*, J. Number Theory **89** (2001), 65–96 (MR1838704 (2002e:11094); Zbl. 0990.11053).
G. Larcher: *Zur Diskrepanz verallgemeinerter Ziffernsummenfolgen*, Österreich. Akad. Wiss. Math.–Natur. Kl. Sitzungsber. II **22** (1993), no. 1–10, 179–185 (MR1268811 (95d:11096); Zbl. 0790.11057).
P. Liardet: *Regularities of distribution*, Compositio Math. **61** (1987), 267–293 (MR0883484 (88h:11052); Zbl. 0619.10053).
M. Mendès France: *Nombres normaux. Applications aux fonctions pseudo–aléatoires*, J. Analyse Math. **20** (1967), 1–56 (MR0220683 (**36** #3735); Zbl. 0161.05002).
R.F. Tichy – G. Turnwald: *On the discrepancy of some special sequences*, J. Number Theory **26** (1987), no. 1, 68–78 (MR0883534 (88g:11048); Zbl. 0628.10052).

2.9.2. If θ is irrational and α real, then the sequence

$$s_q([n\alpha])\theta \bmod 1$$

is

u.d.

M. Mendès France: *Les suites à spectre vide et la répartition modulo 1*, J. Number Theory **5** (1973), 1–15 (MR0319909 (**47** #8450); Zbl. 0252.10033).

2.9.3.

Notes: If η is a real number, we say that the real number θ is of **finite approximation type** η if for every $\varepsilon > 0$ we have $\|h\theta\| \geq c(\theta,\varepsilon)h^{-\eta-\varepsilon}$ for all positive integers h, where $\|x\|$ denotes the distance of x to the nearest integer (see 2.8.1(ii)).

. . . .

If $s_q^{(d)}(n)$ denotes the sum of dth powers of the digits of positive integer n in its q–adic digit expansion, then for the discrepancy of the sequence

$$s_q^{(d)}(n)\theta \bmod 1$$

we have

- $D_N \leq c(q,\theta,\varepsilon)(\log N)^{-\frac{1}{2\eta}+\varepsilon}$ for every $\varepsilon > 0$ and every $N = 1, 2, \dots,$ if θ is of finite approximation type η.
- $D_N \geq (\log N)^{-\frac{1}{2\eta}+\varepsilon}$ for every $\varepsilon > 0$ and infinitely many N, if θ is not of finite approximation type η' for any $\eta' < \eta$.

- $D_N \geq c'(q,d,\theta)(\log N)^{-\frac{1}{2}}$ for every irrational θ and infinitely many N.

R.F. Tichy – G. Turnwald: *On the discrepancy of some special sequences*, J. Number Theory **26** (1987), no. 1, 68–78 (MR0883534 (88g:11048); Zbl. 0628.10052).

2.9.4. Assume that

- $f : \mathbb{N} \to \mathbb{Z}$ is a function such that $f(0) = 0$,
- b_n, $n = 0, 1, \dots$, is an arbitrary sequence of integers,
- $F : \mathbb{N} \to \mathbb{Z}$ satisfies $F(0) = 0$ and $F(n) = \sum_{j=0}^{k(n)} f(a_j)b_j$ if $n = \sum_{j=0}^{k(n)} a_j q^j$ is the q–adic digit expansion of n.

Then the sequence

$$F(n)\theta \bmod 1$$

is

u.d.

if and only if

- $f(a) \neq 0$ for some $1 \leq a \leq q-1$, and
- $\sum_{n=1}^{\infty} \|b_n h\theta\|^2 = \infty$ for all $h \in \mathbb{N}$, where $\|\cdot\|$ is the distance to the nearest integer.

G. Larcher: *On the distribution of sequences connected with digit–representation*, Manuscripta Math. **61** (1988), no. 1, 33–44 (MR0939138 (89f:11104); Zbl. 0647.10034).

2.9.5. Let $q_1, \dots, q_m$ be coprime positive integers ≥ 2. If $s_{q_i}(n)$ denotes the sum of the q_i–digits of n, then the sequence

$$\sum_{j=1}^{m} \alpha_j (s_{q_j}(n))^2 \bmod 1$$

is

u.d. provided at least one of the α_j's is irrational.

Related sequences: 2.9.1

J. Coquet: *Sur les fonctions q–multiplicatives pseudo–aléatoires*, C.R. Acad. Sci. Paris, Ser. A–B **282** (1976), no. 4, Ai, A175–A178 (MR0401691 (**53** #5518); Zbl. 0316.10032).

2.9.6. If $s_q(n)$ is the sum of the q–adic digits when n is expressed in the q–adic digit expansion in base q then the sequence

$$x_n = \alpha_1 s_q(n) + \alpha_2 s_q\big([n\sqrt{2}\,]\big) + \alpha_3 s_q\big([n\sqrt{3}\,]\big) \bmod 1$$

is

u.d.

if and only if at least one of the real numbers $\alpha_1, \alpha_2, \alpha_3$ is irrational.

J. Coquet: *Sur certain suites pseudo–alétoires. III*, Monatsh. Math. **90** (1980), no. 1, 27–35 (MR0593829 (92d:10072); Zbl 0432.10030).

2.9.7. Let $q \geq 2$, $n \geq 0$ be integers, and let $s_q(n)$ be the sum of q–adic digits in the q–adic digit expansion of n in the base q. If α_1, α_2 are real numbers and h_1, h_2 distinct positive integers not divisible by q, then the sequence

$$\alpha_1 s_q(h_1 n) + \alpha_2 s_q(h_2 n) \bmod 1$$

is

u.d.

if and only if at least one of the numbers α_1, α_2 is irrational.

Notes: J. Coquet (1983). He also proved that if h_1, h_2 are distinct odd positive integers and b_1, b_2 are integers ≥ 2, then for any integers a_1, a_2 the set $\{n \in \mathbb{N};\, s_2(h_1 n) \equiv a_1 \pmod{b_1}$ and $s_2(h_2 n) \equiv a_2 \pmod{b_2}\}$ has asymptotic density $(b_1 b_2)^{-1}$.

J. Coquet: *Sur la représentation des multiples d'un entier dans une base*, in: *Hubert Delange colloquium (Orsay, 1982)*, Publ. Math. Orsay, Vol.83-4, Univ. Paris XI, Orsay, 1983, pp. 20–37 (MR0728398 (85m:11045); Zbl 0521.10045).

2.9.8.

Let one of α_1, α_2 be irrational, and $\omega(n)$ denote the number of distinct prime divisor of n. If $q \geq 2$ and $n \geq 0$ are integers, and $s_q(n)$ stands for the sum of q–adic digits in the q–adic digit expansion of n in the base q, then the sequence

$$\alpha_1 s_q(n) + \alpha_2 \omega(n) \bmod 1$$

is

u.d.

J. Coquet: *Sur la représentation des multiples d'un entier dans une base*, in: *Hubert Delange colloquium (Orsay, 1982)*, Publ. Math. Orsay, Vol.83-4, Univ. Paris XI, Orsay, 1983, pp. 20–37 (MR0728398 (85m:11045); Zbl 0521.10045).

2.9.9. Let $q \geq 2$ be a positive integer, $\theta > 1$ real, and let $n = \sum_{j=0}^{k(n)} a_j(n)q^j$ be the q–adic digit expansion of n. Then the sequence

$$x_n = \sum_{j=0}^{k(n)} a_j(n)\theta^j \bmod 1$$

is

u.d. if and only if θ is not a P.V. number.

NOTES: M. Mendès France (1967/68). J. Coquet and M. Mendès France (1977) gave the following generalization: Let $\theta_1 > 1, \dots, \theta_m > 1$ be real numbers, $q_1 \geq 2, \dots, q_m \geq 2$ be distinct primes and $n = \sum_{j=0}^{\infty} a_{j,k}(n)q_k^j$ be the q_k–adic expansion of n for $k = 1, \dots, m$. Then the sequence

$$x_n = \sum_{k=1}^{m}\sum_{j=0}^{\infty} a_{j,k}(n)\theta_k^j \bmod 1$$

is

u.d if and only if at least one $\theta_1, \dots, \theta_m$ is not a P.V. number.

RELATED SEQUENCES: 2.6.22

J. COQUET – M. MENDÈS FRANCE: *Suites à spectre vide et suites pseudo–aléatoires*, Acta Arith. **32** (1977), no. 1, 99–106 (MR0435019 (**55** #7981); Zbl. 0303.10047).
M. MENDÈS FRANCE: *Deux remarques concernant l'équiréparation des suites*, Acta Arith. **14** (1968), 163–167 (MR0227117 (**37** #2702); Zbl. 0177.07202).

2.9.10. Let $Q = (q_n)_{n=0}^{\infty}$ be a sequence of positive integers subject to the conditions:

(i) $q_0 = 1$, $q_{n+1} > q_n$ for all $n = 0, 1, \dots,$

(ii) there exist $\alpha > 1$ and $a \in \mathbb{N}$ such that for all n the inequality $q_{n+i+1} \geq \alpha q_{n+i}$ holds for some and $i \in \{0, 1, \dots, a-1\}$.

Every non–negative integer n can be uniquely represented in the form $n = \sum_{j=0}^{k(n)} a_j(n)q_j$, where the digits $a_j(n)$ are non–negative integers, if we impose the additional condition that $\sum_{j=0}^{i-1} a_j(n)q_j < q_i$ for all $i = 1, \dots, k(n)$. Let $s_Q(n) = \sum_{j=0}^{k(n)} a_j(n)$ denote the sum–of–digits function at n in this expansion. Then for all irrational θ the sequence

$$s_Q(n)\theta \bmod 1$$

is

u.d.

NOTES: (I) Firstly proved by J. Coquet (1982) and later also by Ch. Radoux (1990). Assuming additionally that

(iii) q_{n+1}/q_n tends to a value > 1 as $n \to \infty$,

(iv) Q is a linear recurring sequence, i.e. $q_{n+m} = \sum_{j=0}^{m-1} a_j q_{n+j}$ for $n = 1, 2, \dots$ with m fixed and integral a_j,

Coquet (1983) proved that the sequence $s_Q(n)\theta$ mod 1 has empty Bohr spectrum (see 2.4.4) for every irrational θ.

(II) The sequence Q satisfying (i) and (ii) is called the **scale** and Radoux (1990) gives two examples: the sequence $[\alpha^n]$, $n = 0, 1, \dots$, and the Fibonacci one.

J. COQUET: *Représentations lacunaires des entiers naturels*, Arch. Math. (Basel) **38** (1982), no. 2, 184–188 (MR0650350 (83h:10092); Zbl 0473.10033).

J. COQUET: *Représentations lacunaires des entiers naturels. II*, Arch. Math. (Basel) **41** (1983), no. 3, 238–242 (MR0721055 (86i:11040); Zbl. 521:10043).

CH. RADOUX: *Suites à croissance presque géométrique et répartition modulo 1*, Bull. Soc. Math. Belg. Sér. A, Ser. A **42** (1990), no. 3, 659–671 (MR1316216 (96a:11072); Zbl. 0733.11024).

2.9.11. Let $G = (G_k)_{k=0}^{\infty}$ be a linear recurring sequence, say,

$$G_{k+d} = a_1 G_{k+d-1} + a_2 G_{k+d-2} + \cdots + a_d G_k$$

with integral coefficients and integral initial values. If $d = 1$ then we assume that $G_0 = 1$ and $a_1 > 1$. If $d \geq 2$ then $a_1 \geq a_2 \geq \cdots \geq a_d > 0$, $G_0 = 1$, and $G_k \geq a_1 G_{k-1} + \cdots + a_k G_0 + 1$ for $k = 1, 2, \dots, d-1$. Given $n \in \mathbb{N}$, the greedy algorithm yields the digits $0 \leq \varepsilon_k(n) < G_{k+1}/G_k$ such that $n = \varepsilon_0(n)G_0 + \cdots + \varepsilon_{k(n)}(n)G_{k(n)}$ (the so–called G**–expansion** of n, or the **generalized Zeckendorf expansion**). If $s_G(n)$ denotes the sum of the digits in the G–expansion of n and if θ is irrational then the sequence

$$x_n = s_G(n)\theta \bmod 1$$

is

u.d.

and if θ is of the approximation type η and $a_1 \geq 2$, then

$$D_N = \mathcal{O}\big(\log N^{-(1/2\eta)+\varepsilon}\big)$$

for all $N = 1, 2, \dots$ and $\varepsilon > 0$. If θ is irrational then the sequence x_n has empty spectrum in the sense of Mendès France (cf. 2.4.1 for the def. of the spectrum; the emptiness means that $s_G(n)\theta + n\alpha \bmod 1$ is u.d. for every α) and the sequence

$$s_G(n)\theta + n\theta_1 \bmod 1$$

is

w.d. (see 1.5) if at least one of the numbers θ and θ_1 is irrational.

NOTES: P.J. Grabner and R.F. Tichy (1990). The estimate of the discrepancy D_N can be found in P.J. Grabner (1990). Emptiness of the spectrum of x_n and the w.d. of $s_G(n)\theta+n\theta_1 \bmod 1$ was proved by P.J. Grabner, P. Liardet and R.F. Tichy (1995).

RELATED SEQUENCES: 2.9.1

P.J. GRABNER: *Ziffernentwicklungen bezüglich linearer Rekursionen*, Österreich. Akad. Wiss. Math.–Natur. Kl. Sitzungsber. II **199** (1990), no. 1–3, 1–21 (MR1101092 (92h:11066); Zbl. 0721.11026).
P.J. GRABNER – P. LIARDET – R.F. TICHY: *Odometres and systems of numeration*, Acta Arith. **70** (1995), no. 2, 103–123 (MR1322556 (96b:11108); Zbl. 0822.11008).
P.J. GRABNER – R.F. TICHY: *Contributions to digit expansions with respect to linear recurrences*, J. Number Theory **36** (1990), no. 2, 160–169 (MR1072462 (92f:11111); Zbl. 0711.11004).

2.9.12. Let $n = \sum_{k=0}^{\infty} \varepsilon_k(n)F_k$ be the Zeckendorf representation of the positive integer n, where F_k is the kth Fibonacci number, $\varepsilon_k(n) = 0$ or 1, and $\varepsilon_k(n)\varepsilon_{k+1}(n) = 0$. Denote $s_F^{(1)}(n) = \sum_{k=1}^{\infty} \varepsilon_k(n)$ and define $s_F^{(j+1)}(n) = s^{(1)}\big(s_F^{(j)}(n)\big)$ for $j = 1, 2, \dots$. If $q(x) - q(0)$ is a real polynomial with at least one irrational coefficient, then the sequence

$$q\big(s_F^{(j)}(n)\big) \bmod 1, \quad n = 1, 2, \dots,$$

is

u.d. for every $j = 1, 2, \dots$.

J. COQUET: *Sur certaines suites uniformément équiréparties modulo un. II*, Bull. Soc. Roy. Sci. Liège **48** (1979), no. 11–12, 426–431 (MR0581914 (81j:10053a); Zbl. 0437.10025).
J. COQUET: *Sur certaines suites uniformément équiréparties modulo* 1, Acta Arith. **36** (1980), no. 2, 157–162 (MR0581914 (81j:10053a); Zbl. 0357.10026).
E. ZECKENDORF: *Représentation des nombres naturels par une somme des nombres de Fibonacci ou de nombres de Lucas*, Bull. Soc. Roy. Sci. Liège **41** (1972), 179–182 (MR0308032 (**46** #7147); Zbl. 0252.10011).

2.9.13. Let $[a_0; a_1, a_2, \dots]$ be the continued fraction expansion of an irrational number α, and let $q_0 = 1, q_1, q_2, \dots$ be the denominators of the convergents. Let $n = \sum_{k=0}^{\infty} \varepsilon_k(n)q_k$ be the so–called Ostrowski expansion (cf. 2.8.1) of n, i.e. $\varepsilon_k(n)$ are integers which satisfy $0 \le \varepsilon_k(n) \le a_{k+1}$ for

$k \geq 1$, $0 \leq \varepsilon_0(n) < a_1$, and $\varepsilon_{k-1}(n) = 0$ whenever $\varepsilon_k(n) = a_{k+1}$. Define the sum–of–digits function by $\sigma_\alpha(n) = \sum_{k=0}^{\infty} \varepsilon_k(n)$.

(I) The sequence

$$x_n = \sigma_\alpha(n)\theta \bmod 1$$

is

u.d. if and only if θ is irrational.

(II) If $s_q(n)$ denotes the sum of q–adic digits in the q–adic digit expansion in base $q \geq 2$ of the positive integer n then the sequence

$$y_n = s_q(n)\gamma + \sigma_\alpha(n)\theta \bmod 1$$

is

u.d. if and only if one of the numbers γ or θ is irrational.

(III) Let $\Phi : \mathbb{N} \to \mathbb{R}$ be a function with $\Phi(0) = 0$. Then the sequence

$$z_n = \sum_{k=0}^{\infty} \Phi(\varepsilon_k(n)) \bmod 1$$

is

u.d. if $\Phi(1)$ is irrational and $\Phi(n) \bmod 1$ is u.d.

Consequences:

(i) If the sequence a_n is unbounded and $\Phi(n) \bmod 1$ is u.d., then z_n is u.d.

(ii) If the sequence a_n is bounded and $\Phi(1)$ is irrational, then z_n is u.d.

(iii) If $a_n \geq 3$ for infinitely many n, $\Phi(1)$ is rational and $\Phi(2)$ is irrational then the sequence

$$u_n = \sum_{k=0}^{\infty} \Phi(\varepsilon_k(n)) + \theta \sum_{k=0}^{\infty} \varepsilon_k(n) \bmod 1$$

is

u.d. for every real θ.

Notes: (I) J. Coquet (1982) proved that x_n is w.d. for irrational θ.
(II) The u.d. of y_n was proved by J. Coquet and P. Toffin (1981) and by J. Coquet, G. Rhin and P. Toffin (1981). Coquet (1982) proved that if at least one of the numbers γ or θ is irrational the sequence y_n is w.d.
(III) Kawai (1984).

Related sequences: 3.5.2.

J. Coquet: *Répartition de la somme des chiffres associéte à une fraction continue*, Bull. Soc. Roy. Sci. Liège **51** (1982), no. 3–4, 161–165 (MR0685812 (84e:10060); Zbl. 0497.10040).
J. Coquet: *Représentation des entiers naturels et suites uniformément équiréparties* , Ann. Inst. Fourier (Grenoble) **32** (1982), no. 1, xi, 1–5 (MR0658939 (83h:10071); Zbl 0463.10039).
J. Coquet – G. Rhin – P. Toffin: *Représentation des entiers naturels et indépendence statistique. II*, Ann. Inst. Fourier (Grenoble) **31** (1981), no. 1, ix, 1–15 (MR0613026 (83e:10071b); Zbl. 0437.10026).
J. Coquet – P. Toffin: *Représentation des entiers naturels et indépendence statistique*, Bull. Sci. Math **105** (1981), no. 3, 289–298 (MR0629711 (83e:10071a); Zbl. 0463.10040).
H. Kawai: *α–additive functions and uniform distribution modulo one*, Proc. Japan Acad. Ser. A Math. Sci. **60** (1984), no. 8, 299–301 (MR0774578 (86d:11056); Zbl. 0556.10037).

2.9.14.

Notes: A number system with the base q of an order $\mathbf{O}$ of an algebraic number field is called **canonical** if every element $z \in \mathbf{O}$ has the unique representation of the form $z = \sum_{j=0}^{k(z)} a_j q^j$ with $a_j \in \{0, 1, 2, \dots, |N(q)| - 1\}$. The corresponding **sum–of–digits function** is defined by

$$s_q(z) = \sum_{j=0}^{k(z)} a_j.$$

The only bases of canonical number systems of the ring of Gaussian integers $\mathbb{Z}[i]$ are the Gaussian integers $q = -b \pm i$ with positive integers b.

. . . .

Arrange the Gaussian integer $\mathbb{Z}[i]$ into a sequence z_n, $n = 1, 2, \dots$, according to their norm $|\cdot|$ and let q be a canonical base in $\mathbb{Z}[i]$. For irrational θ the sequence

$$s_q(z_n)\theta \bmod 1$$

is

almost u.d.

(cf. 1.5) with respect to the sequence of indices $[\pi N]$, $N = 1, 2, \dots$, $\pi = 3.14\dots$, i.e.

$$\lim_{N\to\infty} \frac{\#\{z \in \mathbb{Z}[i]\,;\ |z| < \sqrt{N}, \{s_q(z)\theta\} \in I\}}{\pi N} = |I|$$

for all intervals $I \subset [0, 1]$.

Notes: P.J. Grabner and P. Liardet (1999). For the characterization of canonical bases cf. I. Kátai and J. Szabó (1975).

Related sequences: 3.5.3

P.J. Grabner – P. Liardet: *Harmonic properties of the sum–of–digits function for complex base*, Acta Arith. **91** (1999), no. 4, 329–349 (MR1736016 (2001f:11126); Zbl. 0949.11004).
I. Kátai – J. Szabó: *Canonical number systems for complex integers*, Acta Sci. Math. (Szeged) **37** (1975), no. 3–4, 255–260 (MR0389759 (**52** #10590); Zbl. 0309.12001).

2.10 Sequences involving q–additive functions

Notes: Let positive integer $q > 1$ be fixed and $n = \sum_{k=0}^{\infty} a_k q^k$ be the q–adic digit expansion of n. A function $f : \mathbb{Z}_0^+ \to \mathbb{Z}_0^+$ is called q–**additive** if $f(n) = \sum_{k=0}^{\infty} f(a_k q^k)$ and **strongly** q–**additive** or **completely** q–**additive** if $f(n) = \sum_{k=0}^{\infty} f(a_k)$ for all $n = 0, 1, 2, \dots,$ while $f(0) = 0$ in both cases. Every sum–of–digits function $s_q(n)$ (see 2.9) is strongly q–additive. Circle sequences involving q–additive functions are discussed in 3.11.2.
The notion of the q–additive function was introduced by A.O. Gelfond (1968).

A.O. Gelfond: *Sur les nombres qui ont des propriétés additives et multiplicatives données*, Acta Arith. **13** (1968), 259–265 (MR0220693 (**36** #3745); Zbl. 0155.09003).

2.10.1. Suppose that $f(n)$ is strongly q–additive and that $f(b) \neq 0$ for some $1 \leq b \leq q-1$. Then the sequence

$$\alpha f(n) \bmod 1$$

is

u.d. if and only if α is irrational.

If α is an irrational of finite approximation type η then for every $\varepsilon > 0$ there exists a constant $c = c(q, \alpha, \varepsilon, f)$ such that

$$D_N \leq \frac{c}{(\log N)^{\frac{1}{2\eta} - \varepsilon}} \qquad \text{for all } N \geq 1.$$

Notes: [DT, pp. 91–92, Th. 1.99, 1.100]. M. Drmota and G. Larcher (2001, p. 68) noted that for all N and all irrational α we have $D_N > c(q)\frac{1}{\sqrt{\log N}}$ with a constant $c(q) > 0$.

M. Drmota – G. Larcher: *The sum–of–digits–function and uniform distribution modulo 1*, J. Number Theory **89** (2001), 65–96 (MR1838704 (2002e:11094); Zbl. 0990.11053).

2.10.2. Suppose that $f(n)$ is strongly q–additive (for the definition cf. 2.10.1) and that $f(b) \neq 0$ for some $1 \leq b \leq q-1$. Then the sequence

$$\alpha f(n) + \beta n \bmod 1$$

is

$$\text{u.d. for every } \alpha \notin \mathbb{Q} \text{ and for every } \beta \in \mathbb{R}.$$

NOTES: Proved by M. Mendès France (1973), see [DT, p. 101, Th. 1.108] and for more general functions f see 2.6.7.

M. MENDÈS FRANCE: *Les suites à spectre vide et la répartition modulo 1*, J. Number Theory **5** (1973), 1–15 (MR0319909 (**47** #8450); Zbl. 0252.10033).

2.10.3. Let m_n denote the sequence of all squarefree positive integers and let $f(n)$ be a strongly q–additive function (for the def. see 2.10.1) such that $f(b) \neq 0$ for some $1 \leq b \leq q-1$. Then the sequence

$$\alpha f(m_n) \bmod 1$$

is

$$\text{u.d. if and only if } \alpha \notin \mathbb{Q}.$$

NOTES: Proved by M. Mendès France (1973), see [DT, p. 104, Coroll. 1.115]. Note that $f(n)$ has empty spectrum, cf. 2.10.2, 2.4.2 and 2.4.1.
RELATED SEQUENCES: 2.16.3.

M. MENDÈS FRANCE: *Les suites à spectre vide et la répartition modulo 1*, J. Number Theory **5** (1973), 1–15 (MR0319909 (**47** #8450); Zbl. 0252.10033).

2.10.4. If f is a real q–additive function then the sequence

$$f(n), \quad n = 0, 1, 2, \dots,$$

has the a.d.f.

$$g(x) \text{ defined on } (-\infty, \infty),$$

if and only if the following two series converge
(i) $\sum_{k=0}^{\infty} \left(\sum_{a=1}^{q-1} f(aq^k) \right)$,
(ii) $\sum_{k=0}^{\infty} \left(\sum_{a=1}^{q-1} (f(aq^k))^2 \right)$.
The characteristic function $h(t) = \int_{-\infty}^{\infty} e^{itx} \, \mathrm{d}g(x)$ of $g(x)$ is given by

$$h(t) = \prod_{k=0}^{\infty} \frac{1}{q} \left(1 + \sum_{a=1}^{q-1} e^{itf(aq^k)} \right).$$

NOTES: H. Delange (1972). J. Coquet (1975) proved that if f_1 is q_1–additive and f_2 is q_2–additive, and if the a.d.f.'s of f_1 and f_2 exist, then also the a.d.f. of $f_1 + f_2$ exists. Moreover, if $\gcd(q_1, q_2) = 1$, then the characteristic function of $f_1 + f_2$ is equal to the product of the characteristic functions of f_1 and f_2.

J. COQUET: *Sur les fonctions q–multiplicatives presque–périodiques*, C. R. Acad. Sci. Paris Sér. A–B **281** (1975), no. 2–3, Ai, A63–A65 (MR0384736 (**52** #5609); Zbl. 0311.10050).
H. DELANGE: *Sur les fonctions q–additives ou q–multiplicatives*, Acta Arith. **21** (1972), 285–298 (MR0309891 (**46** #8995); Zbl. 0219.10062).

2.10.5.

NOTES: Let Q_n, $n = 0, 1, 2, \dots$, be an increasing sequence of positive integers with $Q_0 = 1$. We can expand every positive integer n with respect to this sequence, i.e. $n = \sum_{k=0}^{\infty} a_k Q_k$ and this expansion is finite and unique, if for every K we have $\sum_{k=0}^{K-1} a_k Q_k < Q_K$. A function $f(n)$ is called Q–**additive** if $f(n) = \sum_{k=0}^{\infty} f(a_k Q_k)$ for all $n = 0, 1, 2, \dots$ (cf. 2.10.1).

. . . .

Suppose that $n = \sum_{k=0}^{K} a_k Q_k$ is the Q–adic digit expansion of n where Q_n, $n = 0, 1, \dots$, is a linear recurring sequence such that

- $Q_{n+d} = q_0 Q_{n+d-1} + \cdots + q_{d-1} Q_n$ with integral coefficients $q_0 \ge q_1 \ge \cdots \ge q_{d-1}$,
- the initial values are $Q_0 = 1$ and $Q_k = q_0 Q_{k-1} + \cdots + q_{k-1} Q_0 + 1$ for $0 < k < d$,
- the dominating root α of the characteristic equation $x^d - q_0 x^{d-1} - \cdots - q_{d-1} = 0$ is a P.V. number.

Furthermore, assume that

- $f(n)$ is Q–additive,
- $\sum_{n=0}^{\infty} \left| \sum_{i=0}^{q_s - 1} f(q_0 Q_{n+d-1}) + \cdots + q_{s-1} Q_{n+d-s} + i Q_{n+d-s-1} \right|$ converges for every $s = 0, \dots, d-1$,
- $\sum_{n=0}^{\infty} \sum_{i=0}^{q_0} f(i Q_n)^2$ converges.

Then the sequence

$$f(n), \quad n = 0, 1, 2, \dots,$$

has the a.d.f.

$$g(x)$$

defined on $(-\infty, \infty)$ and its characteristic function $h(t) = \int_{-\infty}^{\infty} e^{itx} \, \mathrm{d}g(x)$ can be computed as the limit

$$h(t) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{itf(n)}.$$

G. Barat – P.J. Grabner: *Distribution properties of G–additive functions*, J. Number Theory **60** (1996), no. 1, 103–123 (MR1405729 (97k:11112); Zbl. 0862.11048).

2.10.6. Suppose that $n = \sum_{k=0}^{K} a_k Q_k$ is the Q–adic digit expansion of n where Q_n, $n = 0, 1, \dots$, is a linear recurring sequence such that

- $Q_{n+d} = q_0 Q_{n+d-1} + \dots + q_{d-1} Q_n$ with integral coefficients $q_0 \geq q_1 \geq \dots \geq q_{d-1}$,
- the initial values are $Q_0 = 1$ and $Q_k = q_0 Q_{k-1} + \dots + q_{k-1} Q_0 + 1$ for $0 < k < d$,
- the dominating root α of the characteristic equation $x^d - q_0 x^{d-1} - \dots - q_{d-1} = 0$ is a P.V. number.

Furthermore, assume that $|\beta| < 1$ is a real number, and $f(n)$ an arithmetical function which satisfy

- $f(S(n)) = \beta f(n)$ for every $n = 0, 1, 2, \dots$, where
- $S(\sum_{k=0}^{K} a_k Q_k) = \sum_{k=0}^{K} a_k Q_{k+1}$ is the shift adjoint operator.

Then the sequence

$$f(n), \quad n = 0, 1, 2, \dots,$$

has the a.d.f.

$$g(x)$$

defined on $(-\infty, \infty)$ and which has the following properties:

- if $|\alpha\beta| < 1$, then $g(x)$ has zero derivative everywhere and the set where the derivative does not vanish has Hausdorff dimension at most $\log \alpha / (-\log |\beta|)$,
- if $f(n)$ is not identically 0, then $g(x)$ is continuous,
- if β is positive then $g(x)$ is the unique solution of the functional equation

$$g(x) = \sum_{l=0}^{\infty} \frac{1}{\alpha^{l+1}} \sum_{m \in P_l} g\left(\frac{x - f(m)}{\beta^{l+1}} \right)$$

  and if β is negative then $g(x)$ is the unique solution of the equation

$$g(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{\alpha^{l+1}} \sum_{m \in P_l} g\left(\frac{x - f(m)}{\beta^{l+1}} \right) + \sum_{l=0}^{\infty} \frac{p_{2l}}{\alpha^{2l+1}},$$

  where $P_l = \{m \in \mathbb{N}\,;\, m = \sum_{k=0}^{l} a_k Q_k, a_k \geq q_{d-1}, 0 \leq a_l < q_{d-1}\}$ and $p_l = \# P_l$.

G. Barat – P.J. Grabner: *Distribution properties of G–additive functions*, J. Number Theory **60** (1996), no. 1, 103–123 (MR1405729 (97k:11112); Zbl. 0862.11048).

2.11 van der Corput sequences

2.11.1. van der Corput sequence. Let $n = \sum_{j=0}^{s} a_j 2^j$ be the dyadic expansion of n for $n = 0, 1, 2, \dots$. Then the sequence

$$x_n = \gamma_2(n) = \sum_{j=0}^{s} a_j 2^{-j-1} \quad (\text{i.e. } n = a_s a_{s-1} \dots a_0 \to x_n = 0.a_0 a_1 \dots a_s)$$

is

u.d.

and

$$ND_N \le \frac{\log(N+1)}{\log 2} \quad \text{[J.G. van der Corput (1936)]}$$

$$ND_N^* = ND_N \le \frac{\log N}{3\log 2} + 1 \quad \text{[R. Béjian an H. Faure (1977)]}$$

$$\limsup_{N\to\infty} ND_N - \frac{\log N}{3\log 2} = \frac{4}{9} + \frac{\log 3}{3\log 2} \quad \text{[R. Béjian an H. Faure (1977)]}$$

$$DI_N < 4\frac{\sqrt{\log N}}{N} \quad (N \ge 2) \quad \text{[P.D. Proinov and V.S. Grozdanov (1988)]}$$

$$N^2 D_N^{(2)} \le \left(\frac{\log N}{6\log 2}\right)^2 + \left(\frac{11}{3} + \frac{2\log 3}{\log 2}\right)\frac{\log N}{36\log 2} + \frac{1}{3} \quad (N \ge 1)$$

[H. Faure (1990)]

NOTES: (I) J.G. van der Corput (1936). His sequence is also a $(0,1)$–sequence in base $q = 2$ (for def. cf. 1.8.18) and van der Corput's construction actually provides the basis of almost all constructions of s–dimensional (t, s)–sequences (cf. 3.19.2).
(II) I.M. Sobol' (1957) defined the van der Corput sequence independently. He gave (1966, 1967) the following generalization: For two dyadic rationals $\gamma = \frac{c}{2^l}$ and $\delta = \frac{d}{2^m}$ in $[0, 1)$ with dyadic expansions $\gamma = 0.c_1c_2\dots$ and $\delta = 0.d_1d_2\dots$ define $\gamma \oplus \delta = 0.e_1e_2\dots$, by $e_i = c_i + d_i \pmod 2$ $\big($e.g. $7/8 \oplus 11/16 = 0.111 \oplus 0.1011 = 0.0101 = 5/16\big)$. For a given sequence y_n, $n = 0, 1, 2, \dots$, of dyadic rationals in $[0, 1)$ define

$$x_n = a_0 y_0 \oplus a_1 y_1 \oplus \cdots \oplus a_s y_s,$$

where $n = \sum_{j=0}^{s} a_j 2^j$. Sobol' calls x_n the *DR*–**sequence** and y_n as the **directed sequence** of x_n. Let $y_n = 0.y_{n,1}y_{n,2}\dots$ be the dyadic expansion of y_n and define the **directed matrix** by $Y = (y_{n,j})$ for $n = 0, 1, 2, \dots$, $j = 1, 2, \dots$. The DR–sequence x_n, $n = 0, 1, 2, \dots$, is a $(0, 1)$–sequence in base $q = 2$ and thus it is

u.d.

if and only if

$$\det(Y_m) \equiv 1 \pmod 2 \quad \text{for } m = 1, 2, \dots,$$

where $Y_m = (y_{n,j})$, $n = 0, 1, 2, \dots, m-1$, $j = 1, 2, \dots, m$, cf. (1969, p. 123, Th. 7′).
(III) S. Haber (1996) proved that

$$\limsup_{N\to\infty} \frac{N^2 D_N^{(2)}}{\log^2 N} = \frac{1}{(6\log 2)^2}.$$

(IV) H. Faure (1990) improved this to

$$\limsup_{N\to\infty} \left(N^2 D_N^{(2)} - \left(\frac{\log N}{6\log 2}\right)^2 - \left(\frac{11}{3} + \frac{2\log 3}{\log 2}\right)\frac{\log N}{36\log 2} \right) = \\ = \frac{7}{81} + \frac{11\log 3}{108\log 2} + \left(\frac{\log 3}{6\log 2}\right)^2.$$

(V) Béjian and Faure (1977) gave the explicit formula for the extremal discrepancy

$$D_N = \sum_{j=1}^{\infty} \|N/2^j\|,$$

(VI) Similarly, Faure (1990) proved the formula for the L^2 discrepancy

$$4N^2 D_N^{(2)} = \left(\sum_{j=1}^{\infty} \|N/2^j\|\right)^2 + \sum_{j=1}^{\infty} \|N/2^j\|^2$$

which holds for all $N \geq 1$. Here $\|x\| = \min(\{x\}, 1 - \{x\})$ and discrepancies are over N points $x_0, \dots, x_{N-1}$.
(VII) For more information cf. [KN, p. 129], [DT, p. 368] and H. Niederreiter (1992, p. 25).
(VIII) J. Beck [p. 29](2014): van der Corput sequence $x_0, x_1, x_2, \dots$ has the following three properties

Property A: The set $\{x_i : 0 \leq i < 2^k\}$ of the first 2^k elements is the equidistant set $\{j2^{-k} : 0 \leq j < 2^k\}$ possibly in different order.

Property B: Let $I \subset (0, 1)$ be an arbitrary half-open subinterval of length 2^{-k} for some integer $k \geq 1$, and let n be an arbitrary integer divisible by 2^k. Then the number of elements of the set $\{x_i : 0 \leq i < 2^k\}$ that fall into interval I is exactly $n2^{-k}$.

Property C: If $2^k \leq n < 2^{k+1}$ then the consecutive points of the set $\{x_i : 0 \leq i < n\}$ have at most two distances: 2^{-k} and 2^{-k-1}.

(IX) J. Beck [p. 30](2014) Central limit theorem for the van der Corput sequence:

Put $S(n) = \sum_{k=0}^{n-1}\left(x_k - \frac{1}{2}\right)$. Then for any integer $m \geq 2$ and any real numbers $-\infty < A < B < \infty$ we have

$$\frac{1}{2^m}\#\left\{0 \leq n < 2^m : A \leq \frac{S(n)+m/8}{\sqrt{m/4}} \leq B\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\int_A^B e^{-u^2/2}\,\mathrm{d}u + O(m^{-1/10}\log m).$$

J. Beck: *Probabilistic Diophantine approximation (Randomness in lattice point counting)*, Springer Monographs in Mathematics, Springer, Cham, 2014 (MR3308897; Zbl. 1304.11003).
R. Béjian – H. Faure: *Discrépance de la suite de van der Corput*, C. R. Acad. Sci. Paris Sér. A–B **285** (1977), A313–A316 (MR0444600; ((56 #2950))).
H. Faure: *Discrépance quadratique de la suite van der Corput et de sa symétrique*, Acta Arith. **55** (1990), 333–350 (MR1069187 (91g:11085); Zbl. 0705.11039).
S. Haber: *On a sequence of points of interest for numerical quadrature*, J. Res. Nat. Bur. Standards, Sec. B **70** (1966), 127–136 (MR0203938 (**34** #3785); Zbl. 0158.16002).
H. Niederreiter: *Random Number Generation and Quasi–Monte Carlo Methods*, CBMS–NSF Regional Conference Series in Applied Mathematics, 63, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992, vi+241 pp. (MR1172997 (93h:65008); Zbl. 0761.65002).
P.D. Proinov – V.S. Grozdanov: *On the diaphony of the van der Corput – Halton sequence*, J. Number Theory **30** (1988), no. 1, 94–104 (MR0960236 (89k:11065); Zbl. 0654.10050).
I.M. Sobol': *Multidimensional integrals and the Monte–Carlo method*, (Russian), Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), no. 4, 706–709 (MR0092205 (19,1079b); Zbl. 0091.14601).
I.M. Sobol': *Distribution of points in a cube and integration nets*, (Russian), Uspechi Mat. Nauk **21** (1966), no. 5(131), 271–272 (MR0198678 (**33** #6833)).
I.M. Sobol': *Distribution of points in a cube and approximate evaluation of integrals*, (Russian), Ž. Vyčisl. Mat. i Mat. Fiz. **7** (1967), 784–802 (MR0219238 (**36** #2321)).
I.M. Sobol': *Multidimensional Quadrature Formulas and Haar Functions*, (Russian), Library of Applied Analysis and Computational Mathematics, Izd. "Nauka", Moscow, 1969 (MR0422968 (**54** #10952); Zbl. 0195.16903).
J.G. van der Corput: *Verteilungsfunktionen III – VIII*, Proc. Akad. Amsterdam **39** (1936), 10–19, 19–26, 149–153, 339–344, 489–494, 579–590 (JFM 61.0204.01, 61.0204.02, 62.0206.06, 62.0207.01, 62.0207.02, 62.0207.03; Zbl. 0013.16001, 0013.20306, 0014.01106, 0014.01107, 0014.20803).

2.11.2. van der Corput sequence in the base q. Let $q \geq 2$ be an integer and

$$n = a_{k(n)}(n)q^{k(n)} + \cdots + a_1(n)q + a_0(n),\ a_j(n) \in \{0,1,\dots,q-1\},\ a_{k(n)} > 0,$$

be the q–adic digit expansion of integer n in the base q. Then the van der Corput sequence $\gamma_q(n)$, $n = 0,1,2,\dots$, in the base q defined by

$$\gamma_q(n) = \frac{a_0(n)}{q} + \frac{a_1(n)}{q^2} + \cdots + \frac{a_{k(n)}(n)}{q^{k(n)+1}}$$

is

u.d.

and if $N > q$ then

$$D_N^* < \frac{1}{N}\left(\frac{q\log(qN)}{\log q}\right),$$

and

$$\limsup_{N\to\infty}\frac{ND_N^*}{\log N} = \limsup_{N\to\infty}\frac{ND_N}{\log N} = \begin{cases}\dfrac{q^2}{4(q+1)\log q}, & \text{for even } q,\\ \dfrac{q-1}{4\log q}, & \text{for odd } q.\end{cases}$$

For its diaphony we have

$$DI_N < C(q)\frac{\sqrt{\log((q-1)N+1)}}{N},$$

where

$$C(q) = \pi\sqrt{\frac{r^2-1}{3\log q}}$$

NOTES: (I) This sequence was introduced by J.H. Halton (1960) for an arbitrary $q \geq 2$.
(II) The estimation for the discrepancy D_N^* holds for $\gamma_q(n)$ with $n = 1, 2, \dots, N$, cf. L.–K. Hua and Y. Wang (1981, p. 72, Lem. 4.3), but not for $n = 0, 1, \dots, N-1$.
(III) The asymptotic result for discrepancy was established by H. Faure (1981).
(IV) The estimation for the diaphony was proved by P.D. Proinov and V.S. Gorazdov (1988).
(V) The $\gamma_q(n)$ is called the **radical inverse function** of the natural q–adic digit expansion of n. It can be defined recursively (cf. I.M. Soboľ (1961)): In $\gamma_q(n) = 0.a_0a_1\dots a_m00\dots$ we find $a_k < q-1$ with minimal k and then $\gamma_q(n+1) = 0.0\dots0(a_k+1)a_{k+1}a_{k+2}\dots a_m\dots$, or in other words $\gamma_q(n+1) = \gamma_q(n) + q^{-k} + q^{-(k-1)} - 1$.
(VI) Another recursive expression is: $\gamma_q(0) = 0$, $\gamma_q(q^k) = q^{-k-1}$ for $k = 0, 1, 2, \dots$ and $\gamma_q(q^k + j) = \gamma_q(q^k) + \gamma_q(j)$ for $j = 1, 2, \dots, q^{k+1} - q^k - 1$. For a related programming scheme, cf. I.M. Soboľ (1969, p. 176).
(VII) The distribution of $(\gamma_q(n), \dots, \gamma_q(n+s))$ in $[0,1]^s$ is an open problem. However, for $s = 1$ we have, see O. Blažeková (2007),

$$D_N(\gamma_q(n), \gamma_q(n+1)) = \frac{1}{4} + O(D_N(\gamma_q(n))),$$

$$D_N^*((\gamma_q(n), \gamma_q(n+1))) = \max\left(\frac{1}{q}\left(1-\frac{1}{q}\right), \frac{1}{4}\left(1-\frac{1}{q}\right)^2\right) + O(D_N(\gamma_q(n))).$$

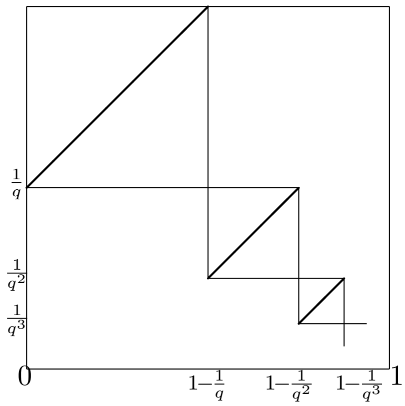
Consequently, van der Corput sequence is not pseudo-random.

RELATED SEQUENCES: 3.18.1.1, 3.18.1.2.

(VIII) The local discrepancy function $D(N, I) = \left|A(I; N; \gamma_q(n)) - N|I|\right|$ as a function of an interval $I \subset [0, 1]$ is bounded for $N \to \infty$ if and only if $|I|$ has finite q-ary expansion. This was proved by:

- W.M. Schmidt (1974) and L. Shapiro (1978) for $q = 2$;
- P. Hellekalek (1980) for $q \geq 2$;
- H. Faure (1983), who extended this result for generalized van der Corput sequences and in (2005) for digital $(0, 1)$-sequences.

(IX) The graph of von Neumann-Kakutani transformation $T : [0, 1] \to [0, 1]$ is given by



where the line segments are

$$Y = X - 1 + \frac{1}{q^i} + \frac{1}{q^{i+1}}, \quad X \in \left[1 - \frac{1}{q^i}, 1 - \frac{1}{q^{i+1}}\right]$$

for $i = 0, 1, \dots$. We have

(i) The sequence of iterates (also called generalized van der Corput sequence)

$$x, T(x), T(T(x)), T(T(T(x))), \dots$$

is u.d. for every $x \in [0, 1)$. Moreover, it is a low discrepancy sequence, see P. Grabner, P. Hellekalek and P. Liardet (2011) and G. Pagés (1992).

(ii) The iterates

$$0, T(0), T(T(0)), T(T(T(0))), \dots$$

form van der Corput sequence $\gamma_q(n)$, $n = 0, 1, 2, \dots$, and thus every point $(\gamma_q(n), \gamma_q(n + 1))$, $n = 0, 1, 2, \dots$, lies on the graph of T.

(X) P. Grabner, P. Hellekalek and P. Liardet (2012): The van der Corput

sequence $\gamma_q(n)$ is not only u.d. but also well-distributed (for definition cf. 1.5).
(XI) The sth iteration of von Neumann-Kakutani transformation T has the form (cf. V. Baláž, J. Fialová, M. Hoffer, M.R. Iacó and O. Strauch (2015)) :

$$T^s(x) = \begin{cases} x + \frac{s}{q} & \text{if } x \in \left[0, 1 - \frac{s}{q}\right], \\ x + \frac{s-1}{q} - 1 + \frac{1}{q^i} + \frac{1}{q^{i+1}} & \text{if } x \in \left[1 - \frac{s-1}{q} - \frac{1}{q^i}, 1 - \frac{s-1}{q} - \frac{1}{q^{i+1}}\right] \\ & \quad \cup \left[1 - \frac{s-2}{q} - \frac{1}{q^i}, 1 - \frac{s-2}{q} - \frac{1}{q^{i+1}}\right] \\ & \quad \ldots\ldots \\ & \quad \cup \left[1 - \frac{s-l+1}{q} - \frac{1}{q^i}, 1 - \frac{s-l-1}{q} - \frac{1}{q^{i+1}}\right] \\ & \quad \ldots\ldots \\ & \quad \cup \left[1 - \frac{1}{q^i}, 1 - \frac{1}{q^{i+1}}\right], \text{ where} \\ & \qquad i = 1, 2, \ldots \end{cases}$$

Related sequences: 2.11.1.

V. Baláž– J. Fialová – M. Hoffer – M.R. Iacó – O. Strauch: *The asymptotic distribution function of the 4-dimensional shifted van der Corput sequence*, Tatra Mt. Math. Publ. **64** (2015), 75–92 (MR3458785; Zbl 06545459).
O. Blažeková: *Pseudo-randomnes of van der Corput's sequences* , Math. Slovaca **59** (2009), no. 3, 291–298 (MR2505811 (2010c:11095); Zbl. 1209.11075)
H. Faure: *Discrépances de suites associées à un système de numération (en dimension un)*, Bull. Soc. Math. France **109** (1981), 143–182 (MR0623787 (82i:10069); Zbl. 0488.10052).
H. Faure: *Étude des restes pour les suites de van der Corput généralisées*, J. Number Theory **16** (1983), no. 3, 376–394 (MR0707610 (84g:10082); Zbl. 0513.10047).
H. Faure: *Discrepancy and diaphony of digital* $(0,1)$*-sequences in prime base*, Acta Arith. **117** (2005), no. 2, 125–148 (MR2139596 (2005m:11141); Zbl. 1080.11054).
P. Grabner – P. Hellekalek – P. Liardet: *The dynamical point of view of low-discrepancy sequences*, Unif. Distrib. Theory **7** (2012), no. 1, 11–70 (MR2943160; Zbl. 1313.11093)
P. Hellekalek: *On regularities of the distribution of special sequences*, Monatsh. Math. **90** (1980), no. 4, 291–295 (MR0596894 (82a:10063); Zbl. 0435.10032).
P. Hellekalek: *Regularities in the distribution of special sequences*, J. Number Theory **18** (1984), no. 1, 41–55 (MR0734436 (85e:11052); Zbl. 0531.10055).
J.H. Halton: *On the efficiency of certain quasi–random sequences of points in evaluating multi–dimensional integrals*, Numer. Math. **2** (1960), 84–90 (MR0121961 (**22** #12688); Zbl. 0090.34505).
L.–K. Hua – Y. Wang: *Applications of Number Theory to Numerical Analysis*, Springer Verlag & Science Press, Berlin, Heidelberg, New York, Beijing, 1981 (MR0617192 (83g:10034); Zbl. 0465.10045). (Chinese edition: Science Press, Beijing, 1978 (MR0617192 (83g:10034); Zbl. 0451.10001)).
G. Pagés: *Van der Corput sequences, Kakutani transforms and one-dimensional numerical integration*, J. Comput. Appl. Math. **44** (1992), 21–39.(MR1199252 (94c:11066); Zbl. 0765.41033).
P.D. Proinov – V.S. Grozdanov: *On the diaphony of the van der Corput – Halton sequence*, J. Number Theory **30** (1988), no. 1, 94–104 (MR0960236 (89k:11065); Zbl. 0654.10050).
W.M. Schmidt: *Irregularities of distribution VIII*, Trans. Amer. Math. Soc. **198** (1974), 1–22.(MR0360504 (**50** #12952); Zbl. 0278.10036)
L. Shapiro: *Regularities of distribution*, in: *Studies in probability and ergodic theory*, Math. Suppl. Stud., 2, Academic Press, New York, London, 1978, pp. 135–154 (MR0517257 (80m:10039); Zbl. 0446.10045).

I.M. Sobolʹ: *Evaluation of multiple integrals*, (Russian), Dokl. Akad. Nauk SSSR **139** (1961), no. 4, 821–823 (English translation: Sov. Math., Dokl. **2** (1961), 1022-1025 (MR0140186 (**25** #3608); Zbl 0112.08001)).
I.M. Sobolʹ: *Multidimensional Quadrature Formulas and Haar Functions*, (Russian), Library of Applied Analysis and Computational Mathematics, Izd. "Nauka", Moscow, 1969 (MR0422968 (**54** #10952); Zbl. 0195.16903).

2.11.2.1 **Subsequences of van der Corput sequence.**

(i) Let F_n denote the nth Fibonacci number. Then the sequence $\gamma_q(F_n)$, $n = 0, 1, \dots$, in base q is u.d. if and only if $q = 5^k$ for some $k \in \mathbb{N}$.

(ii) The sequence $\gamma_q([\log F_n])$, $n = 0, 1, \dots$, is u.d. in any base q.

(iii) Let α be irrational or $\alpha = 1/d$ for some nonzero integer d. Then, the sequence $\gamma_q([n\alpha])$, $n = 0, 1, \dots$, is u.d. in any base q.

(iv) Let $s_{\tilde{q}}(n)$ denote the $\tilde{q}$-ary sum-of-digits function, cf. 2.9. Then, the sequence $\gamma_q(s_{\tilde{q}}(n))$, $n = 0, 1, \dots$, is u.d. in any base q.

(v) The subsequence $\gamma_q(p_n)$, $n = 1, 2, \dots$, with primes p_n in base $q \geq 2$ is not u.d.

Notes:
(I) The items (i)–(iv) were proved by R. Hofer, P. Kritzer, G. Larcher, and F. Pillichshammer (2009).
(II) The u.d. of $\gamma_5(F_n)$ was also proved P. Hellekalek and H. Niederreiter (2011) using another method.

R. Hofer – P. Kritzer – G. Larcher – F. Pillichshammer: *Distribution properties of generalized van der Corput-Halton sequences and their subsequences*, Int. J. Number Theory **5** (2009), 719–746 (MR2532267 (2010d:11082); Zbl. 1188.11038).
P. Hellekalek – H. Niederreiter: *Constructions of uniformly distributed sequences using the b-adic method*, Unif. Distrib. Theory **6** (2011), no. 1 185–200.(MR2817766; Zbl. 1333.11071)

2.11.3. Generalized van der Corput sequences in the base q. Let $q \geq 2$, n be integers and $n = \sum_{j=0}^{\infty} a_j(n) q^j$, $a_j \in \{0, 1, \dots, q-1\}$, the q–adic digit expansion of n. If π is a permutation on $\{0, 1, 2, \dots, q-1\}$ then a **generalized van der Corput sequence in the base** q is defined by

$$x_n = \sum_{j=0}^{\infty} \pi(a_j(n)) q^{-j-1}$$

for $n = 0, 1, 2, \dots$. It is

u.d.

The sequence can also be defined recursively

$$x_0 = \frac{\pi(0)}{q-1}, \quad \text{and} \quad x_{qn+r} = \frac{1}{q}(x_n + \pi(r))$$

for $n = 0, 1, 2, \dots$ and $0 \le r \le q-1$.

NOTES: H. Niederreiter (1992, pp. 25–26). Currently, the best choice of parameters was found by H. Faure (1992), who used $q = 36$ and a specific permutation π of $0, 1, 2, \dots, 35$ given below, and showed that the resulting generalized van der Corput sequence satisfies

$$\limsup_{N\to\infty} \frac{ND_N}{\log N} = \frac{23}{35 \log 6} = 0.366\dots.$$

Faure's permutation π is (here the kth number stands for $\pi(k)$):

$$(0, 25, 17, 7, 31, 11, 20, 3, 27, 13, 34, 22, 5, 15, 29, 9, 23, 1, 18,$$
$$32, 8, 28, 14, 4, 21, 33, 12, 26, 2, 19, 10, 30, 6, 16, 24, 35).$$

At present, this sequence yields the smallest value of the upper limit on the left–hand side for any known sequence of elements of $[0, 1)$. For the star discrepancy, the current record is a generalized van der Corput sequence in the base $q = 12$ constructed by H. Faure (1981), which satisfies

$$\limsup_{N\to\infty} \frac{ND_N^*}{\log N} = \frac{1919}{3454 \log 12} = 0.223\dots.$$

RELATED SEQUENCES: 2.11.1, 2.11.2.

H. FAURE: *Discrépances de suites associées à un système de numération (en dimension un)*, Bull. Soc. Math. France **109** (1981), 143–182 (MR0623787 (82i:10069); Zbl. 0488.10052).
H. FAURE: *Good permutations for extreme discrepancy*, J. Number Theory **42** (1992), no. 1, 47–56 (MR1176419 (93j:11049); Zbl. 0768.11026).
H. NIEDERREITER: *Random Number Generation and Quasi–Monte Carlo Methods*, CBMS–NSF Regional Conference Series in Applied Mathematics, 63, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992, vi+241 pp. (MR1172997 (93h:65008); Zbl. 0761.65002).

2.11.4. Generalized van der Corput sequences for Cantor expansion. Let r_n be an integral sequence with $r_n \ge 2$. Let $n = \sum_{j=0}^{\infty} a_j R_j$, be the corresponding expansion of n (cf. e.g. 2.9.10) where $a_j \in \{0, 1, \dots, r_{j+1} - 1\}$ and $R_j = r_0 r_1 \dots r_j$. If π_n is a permutation of the set $\{0, 1, \dots, r_n - 1\}$, $n = 1, 2, \dots$, then

$$x_n = \sum_{j=0}^{\infty} \frac{\pi_{j+1}(a_j)}{R_{j+1}}$$

is

u.d.

and if π_n, $n = 1, 2, \dots$, is the identity permutation then

$$D_N = \mathcal{O}\left(\frac{\log N}{N}\right)$$

if and only if

$$\sum_{j=0}^{n} r_j = \mathcal{O}(n).$$

NOTES: Generalized van der Corput sequences were introduced by H. Faure (1981, 1983) and the above estimate was proved by E.Y. Atanassov (1989).

RELATED SEQUENCES: 2.11.1, 2.11.2, 2.11.3

E.Y. ATANASSOV: *Note on the discrepancy of the van der Corput generalized sequences*, C. R. Acad. Bulgare Sci. **42** (1989), no. 3 41–44 (MR1000628 (90h:11069); Zbl. 0677.10038).
H. FAURE: *Discrépance de suites associées à un système de numération (en dimension s)*, Acta Arith. **41** (1982), 337–351 (MR0677547 (84m:10050); Zbl. 0442.10035).
H. FAURE: *Étude des restes pour les suites de van der Corput généralisées*, J. Number Theory **16** (1983), no. 3, 376–394 (MR0707610 (84g:10082); Zbl. 0513.10047).

2.11.5. Zaremba sequence. Let $\gamma_2(n)$, $n = 0, 1, 2, \dots$, be the van der Corput sequence 2.11.1. Then the **Zaremba sequence** x_n defined by

$$\gamma_2(0), 1 - \gamma_2(1), \gamma_2(2), 1 - \gamma_2(3), \gamma_2(4), 1 - \gamma_2(5), \dots$$

is

u.d.

J.H. HALTON – S.K. ZAREMBA: *The extreme and L^2 discrepancies of some plane set*, Monatsh. Math. **73** (1969), 316–328 (MR0252329 (**40** #5550); Zbl. 0183.31401).

2.11.6. Let $\gamma_q(n)$, $n = 0, 1, 2, \dots$, be the van der Corput sequence in the base $q \geq 2$ (cf. 2.11.2). Then the symmetrized sequence x_n defined by

$$\gamma_q(0),\ 1 - \gamma_q(0),\ \gamma_q(1),\ 1 - \gamma_q(1),\ \gamma_q(2),\ 1 - \gamma_q(2),\ \dots$$

has the L^2 discrepancy with the least possible order, namely

$$D_N^{(2)} = \mathcal{O}\left(\frac{\log N}{N^2}\right),$$

where the implied constant depends only on q.

NOTES: The first construction of a sequence in $[0,1)$ for which $D_N^{(2)}$ has this least possible order of magnitude was given by P.D. Proinov (1983), cf. Notes in 2.8.2. The symmetrized sequence is defined in P.D. Proinov and V.S. Grozdanov (1987). If $q = 2$ then H. Faure (1990) proved the expression $N^2 D_N^{(2)} = \sum_{j=1}^{\infty} \|N/2^j\|^2(1 - 2\|2^j x_{N+1}\|)$ for $x_0, \dots, x_{N-1}$, from which he deduced the inequalities

$$0.089 < \limsup_{N\to\infty} \frac{N^2 D_N^{(2)}}{\log N} < 0.103.$$

H. FAURE: *Discrépance quadratique de la suite van der Corput et de sa symétrique*, Acta Arith. **55** (1990), 333–350 (MR1069187 (91g:11085); Zbl. 0705.11039).
P.D. PROINOV: *Estimation of L^2 discrepancy of a class of infinite sequences*, C. R. Acad. Bulgare Sci. **36** (1983), no. 1, 37–40 (MR0707760 (86a:11030); Zbl. 0514.10039).
P.D. PROINOV – V.S. GROZDANOV: *Symmetrization of the van der Corput – Halton sequence*, A. R. Acad. Bulgare Sci. **40** (1987), no. 8, 5–8 (MR0915437 (89c:11121); Zbl. 0621.10035).

2.11.7. ***Q*–adic van der Corput sequence** for a special Q–adic digit expansion. Let a and d be positive integers and $Q_n = Q_{n+d} = a(Q_{n+d-1} + \cdots + Q_n)$ be recurring sequence such that

- $Q_0 = 1$ and $Q_k = a(Q_{k-1} + \cdots + Q_0) + 1$ for $0 < k < d$,
- α is the dominating root of the characteristic equation $x^d - a(x^{d-1} + \cdots + 1) = 0$.

If $n = \sum_{k=0^K} a_k Q_k$ is the Q–adic digit expansion of n then the ***Q*–adic van der Corput sequence** defined by

$$f(n) = f\left(\sum_{k=0}^{K} a_k Q_k\right) = \sum_{k=0}^{K} \frac{a_k}{\alpha^{k+1}}$$

is

u.d. in $[0,1]$

having the star discrepancy

$$D_N^* = \mathcal{O}\left(\frac{\log N}{N}\right).$$

NOTES: (I) G. Barat and P.J. Grabner (1996, Prop. 13) and for proof they used the theory of Q–additive functions, cf. 2.10.

(II) They also give another example (1996, Prop. 14): Let a be a positive integer and Q_n be defined by the recurrence such that:

- $Q_{n+2} = (a+1)Q_{n+1} + aQ_n$,
- $Q_0 = 1$ and $Q_2 = a+2$,
- α is the dominating root of the characteristic equation $x^2 - (a+1)x - a = 0$.

Let $n = \sum_{k=0}^{K} a_k Q_k$ be the Q–adic digit expansion of n. Define the function f by
- $f\left(\sum_{k=0}^{K} a_k Q_k\right) = \sum_{k=0}^{K} f(a_k)/\alpha^k$,
- $f(x) = x/a$ for $0 \le x \le a$, and
- $f(a+1) = a/(\alpha - 1)$.

Then the sequence

$$f(n), \quad n = 0, 1, \dots,$$

is

$$\text{u.d. in } [0,1].$$

(III) Barat and Grabner noticed that for the recurrence relation $Q_{n+3} = 3\,Q_{n+2} + Q_{n+1} + Q_n$, with initial values $Q_0 = 1$, $Q_1 = 4$ and $Q_2 = 14$, the sequence $f(n)$ defined by

$$f\left(\sum_{k=0}^{K} a_k Q_k\right) = \sum_{k=0}^{K} \frac{a_k}{\alpha^{k+1}}$$

is

$$\text{not u.d. } \bmod 1.$$

RELATED SEQUENCES: 2.11.2

G. BARAT – P.J. GRABNER: *Distribution properties of G–additive functions*, J. Number Theory **60** (1996), no. 1, 103–123 (MR1405729 (97k:11112); Zbl. 0862.11048).

2.11.7.1 **β-adic van der Corput sequence.** Let β be an arbitrary positive number greater than 1. Then every $x \in [0,1)$ has an expansion $x = \sum_{k=1}^{\infty} a_k(x)/\beta^k$ (abbreviated $x = 0.a_1a_2\dots$), where the digits $a_k(x)$ may take on the values $0, 1, \dots, [\beta]$ and can be computed by the following algorithm

$$x = \frac{[\beta x] + \{\beta x\}}{\beta} = \frac{[\beta x]}{\beta} + \frac{\beta\{\beta x\}}{\beta^2} = \frac{[\beta x]}{\beta} + \frac{[\beta\{\beta x\}]}{\beta^2} + \frac{\beta\{\beta\{\beta x\}\}}{\beta^3}, \text{ etc.}$$

The sequence of all finite β-expansion (ordered by the magnitude, see below)

$$x_n = 0.a_1a_2\dots a_k, \quad n = 0, 1, 2, \dots$$

is

$$\text{u.d.}$$

and if β is a P.V. number with irreducible β-polynomial (see Notes II) then for the extremal discrepancy we have

$$D_N = \mathcal{O}\left(\frac{\log N}{N}\right).$$

Here x_n are ordered in such a way that if $n < n'$ and $x_n = 0.a_1a_2\dots a_k$, $x_{n'} = 0.a'_1a'_2\dots a'_{k'}$ then either $k < k'$ or $k = k'$ and there exists some j such that $a_j < a'_j$ and $a_i = a'_i$ for $i > j$.

Notes:

(I) The notion of β-expansion of real numbers was introduced by A. Rényi (1957) and further developed by W. Parry (1960).

(II) If the β-expansion of 1 is finite $1 = 0.a_1a_2\dots a_k$ or eventually periodic $1 = 0.a_1a_2\dots a_{k-i}a_{k-i+1}\dots a_ka_{k-i+1}\dots a_k\dots$ then β is called a **Parry number**. In this case 1 is the dominant root of the so-called β-polynomial defined by $x^k - a_1x^{k-1} - \dots - a_k$, or by $(x^k - a_1x^{k-1} - \dots - a_k) - (x^{k-i} - a_1x^{k-i-1} - \dots - a_{k-i})$ with minimal k, respectively. A. Bertrand (1977) and K. Schmidt (1980) proved that all P.V. numbers are Parry numbers (cf. W. Steiner (2006)).

(III) β-adic van der Corput sequence was introduced independently by G. Barat and P.J. Grabner (1996) and S. Ninomiya (1998 [a],[b]). who proved that this sequence is a low discrepancy sequence.

(IV) For the local discrepancy function $D(N,I) = \left|A(I;N;x_n) - N|I|\right|$ W. Steiner (2006) proved:

(i) If β is a Parry number and $D(N,I)$ is bounded as $N \to \infty$, then the length $|I|$ of interval I belongs to $\mathbb{Q}(\beta)$, the field generated by β over $\mathbb{Q}$.

(ii) If β is a P.V. number with an irreducible β-polynomial, then $D(N,[0,y))$ is bounded as $N \to \infty$ if and only if the β-expansion of y is finite or eventually periodic with the same minimal period as that of the expansion of 1.

(V) W. Steiner (2009) defined abstract van der Corput sequences using abstract numeration systems and he explicitly computed their discrepancy.

G. Barat – P.J. Grabner: *Distribution properties of G–additive functions*, J. Number Theory **60** (1996), no. 1, 103–123 (MR1405729 (97k:11112); Zbl. 0862.11048).

A. Bertrand: *Développements en base de Pisot et répartition modulo 1*, C. R. Acad. Sci., Paris, Sér. A **285** (1977), 419–421 (MR0444600 (**56** #2950); Zbl. 0362.10040).

[a] S. Ninomiya: *Constructing a new class of low-discrepancy sequences by using the β-adic transformation*, Math. Comput. Simulation **47** (1998), no. 2–5, 403–418 (MR1641375 (99i:65009)).

[b] S. Ninomiya: *On the discrepancy of the β-adic van der Corput sequence*, J. Math. Sci. Univ. Tokyo **5** (1998), no. 2, 345–366 (MR1633866 (99h:11087); Zbl. 0971.11043).

W. Parry: *On the β-expansions of real numbers*, Acta Math. Acad. Sci. Hungar. **11** (1960), 401–416 (MR0142719 (**26** #288); Zbl. 0099.28103)).

A. Rényi: *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar. **8** (1957), 477–493 (MR0097374 (**20** #3843); Zbl. 0079.08901).

K. Schmidt: *On periodic expansion of Pisot numbers and Salem numbers*, Bull. London Math. Soc. **12** (1980), no. 4, 269–278 (MR0576976 (82c:12003); Zbl. 0494.10040).

W. Steiner: *Regularities of the distribution of β-adic van der Corput sequences*, Monatsh. Math. **149** (2006), 67–81 (MR2260660 (2007g:11085); Zbl. 1111.11039).

W. Steiner: *Regularities of the distribution of abstract van der Corputt sequences*, Unif. Distrib. Theory **4** (2009), no. 2, 81–100 (MR2591843 (2011c:11123); Zbl. 1249.11076).

2.11.7.2 **Kakutani sequence of partition:** Let $x_{n,1} < x_{n,2} < \cdots < x_{n,k(n)}$ be a partition of $[0,1]$ in the n-th step. Let the partition in the $(n+1)$st step is obtained by subdividing of every interval $[x_{n,i}, x_{n,i+1}]$ of maximal length into two parts in proportion $\alpha/(1-\alpha)$. Then the sequence of blocks

$$X_n = (x_{n,1}, x_{n,2}, \dots, x_{n,k(n)}), n = 1, 2, \dots$$

is

u.d.

for any $\alpha \in (0,1)$. (For the definition of block sequences see Part 1.8.23.

S. Kakutani: *A problem of equidistribution on the unit interval* $[0,1]$, in: *Measure Theory Oberwolfach 1975 (Proceedings of the Conference Held at Oberwolfach 15–20 June, 1975*, (A. Doldan and B. Eckmann eds.), Lecture Notes in Mathematics, 541, Springer Verlag, Berlin, Heidelberg, New York, 1976, pp. 369–375 (MR0457678 (**56** #15882); Zbl. 0363.60023).

2.11.7.3 ***LS*-sequences of partitions:** Kakutani sequence of partition can be generalized in a natural way in several ways:

ρ-refinements: Let ρ denote a non-trivial finite partition of $[0,1)$. Then the ρ-refinement of a partition π, denoted by $\rho\pi$, is given by subdividing all intervals of maximal length homotetically to ρ.

ρ_{LS}-refinements is the ρ-refinement of the trivial partition $\pi = \{[0,1)\}$ where ρ consists of $L+S$ intervals such that the first $L > 0$ have length α and the remaining ones $S > 0$ have length α^2, where L, S are positive integers and $0 < \alpha < 1$. The sequence of successive of ρ_{LS}-refinements of the trivial partition π is called LS **sequence of partitions**.

Notes:
(I) Necessarily, $L\alpha + S\alpha^2 = 1$ holds. For every n the partition $\rho^n\pi$ consists only of intervals having either length α^n or α^{n+1}.
(II) This sequence of partitions has been introduced by I. Carbone (2012).
(III) If $S \geq 1$ then by I. Carbone (2012) a LS sequence is a low-discrepancy sequences if and only if $L > S - 1$.
(IV) If $L = S = 1$ then $\alpha = \frac{\sqrt{5}-1}{2}$ and we obtain the so-called **Kakutani-Fibonacci** sequence and the discrepancy is of the order $\frac{1}{k(n)}$.

CH. AISTLEITNER – M. HOFER – V. ZIEGLER: *On the uniform distribution modulo* 1 *of multidimensional LS-sequence*, Ann. Mat. Pura Appl. (4) **193** (2014), no. 5, 1329–1344 (MR3262635; Zbl 1323.11049).
M. DRMOTA – M. INFUSINO: *On the discrepancy of some generalized Kakutani's sequences of partitions*, Unif. Distrib. Theory **7** (2012), no. 1, 75–104 (MR2943162; Zbl 1313.11084).
I. CARBONE: *Discrepancy of LS-sequences of partitions and points*, Ann. Mat. Pura Appl. (4) **191** (2012), no. 4, 819–844 (MR2993975; Zbl 1277.11080).

2.12 Sequences involving logarithmic function

See also: 2.13.5, 2.13.7, 2.16.8, 2.19.7, 2.19.8, 2.19.9, 2.24.4, 2.24.5, 2.24.6

2.12.1. The sequence

$$x_n = \log n \bmod 1$$

has the set of d.f.'s

$$G(x_n) = \left\{ g_u(x) = \frac{e^{\min(x,u)} - 1}{e^u} + \frac{1}{e^u}\frac{e^x - 1}{e - 1}\ ;\ u \in [0,1] \right\},$$

where $\{\log N_k\} \to u$ implies $F_{N_k}(x) \to g_u(x)$.
The lower and upper d.f. of $\log n \bmod 1$ are

$$\underline{g}(x) = \frac{e^x - 1}{e - 1}, \qquad \overline{g}(x) = \frac{1 - e^{-x}}{1 - e^{-1}},$$

and $\underline{g} \in G(x_n)$ but $\overline{g} \notin G(x_n)$.

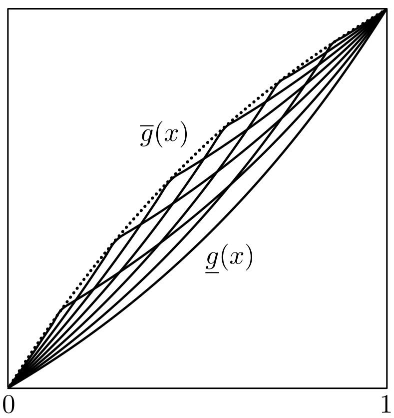


Figure 1: Distribution functions of $\log n \mod 1$

If the elements of the matrix $\mathbf{A} = (a_{N,n})_{N,n\geq 1}$ are

$$a_{N,n} = \begin{cases} \dfrac{1/n}{\sum_{i=1}^{N} 1/i}, & \text{if } n \leq N, \\ 0, & \text{if } n > N, \end{cases}$$

then the sequence $\log n$ mod 1 is

$\mathbf{A}$–u.d. (i.e. logarithmically u.d.)

NOTES: (I) The set of d.f.'s was found by A. Wintner (1935, relation (7)).
(II) The lower and upper d.f.'s can be found using 2.6.18. Similarly, for $\log_b n$ mod 1, $b > 1$, we have (cf. [KN, p. 59])

$$\underline{g}(x) = \frac{b^x - 1}{b - 1}, \qquad \overline{g}(x) = \frac{1 - b^{-x}}{1 - b^{-1}}.$$

(III) The sequence $c \log n$ with any real constant c is also not u.d. (cf. [KN, p. 24, Exer. 2.13]). A proof can be found in D.P. Parent (1984, pp. 281–282, Solution 5.18) which gives

$$\lim_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} e^{ic\log n} \right| = \frac{1}{|1 + ic|}.$$

(IV) In G. Pólya and G. Szegő (1964, Part 2, Ex. 179) it is proved that the derivative (density) $g'(x)$ of any $g(x) \in G(c \log n \bmod 1)$, $c > 0$, has the form

$$g'(x) = \begin{cases} \dfrac{\log q}{q - 1}\, q^{x-\alpha+1}, & \text{if } 0 \leq x < \alpha, \\ \dfrac{\log q}{q - 1}\, q^{x-\alpha}, & \text{if } \alpha < x \leq 1, \end{cases}$$

where $q = e^{1/c}$ and $\alpha \in (0, 1)$. If $\alpha = 0$ or $\alpha = 1$ then

$$g'(x) = \frac{\log q}{q - 1}\, q^x$$

and $c \log n$ mod 1 is (λ, λ')–distributed with $\lambda = \frac{\log q}{q-1}$ and $\lambda' = q\frac{\log q}{q-1}$, cf. J. Chauvineau (1967/68).
The connection between $g(x)$ and α is: if $\lim_{k\to\infty}\{c \log N_k\} = \alpha$ then we have $\lim_{k\to\infty} F_{N_k}(x) = g(x)$.
(IV') O. Strauch and O. Blažeková (2006): The result (IV) can be rewriten in the form. Given any base $b > 1$, the sequence $\log_b n$ mod 1, $n = 1, 2, \dots$, has the following set of d.f.'s

$$G(x_n) = \left\{ g_u(x) = \frac{b^{\min(x,u)} - 1}{b^u} + \frac{1}{b^u}\frac{b^x - 1}{b - 1}; u \in [0, 1] \right\}.$$

The lower and upper d.f. of $\log_b n$ mod 1 are given by

$$\underline{g}(x) = \frac{b^x - 1}{b - 1}, \qquad \overline{g}(x) = \frac{1 - b^{-x}}{1 - b^{-1}},$$

where $\underline{g} \in G(x_n)$ but $\overline{g} \notin G(x_n)$.
Moreover $\{\log_b N_k\} \to u$ implies $F_{N_k}(x) \to g_u(x)$. Note that in G. Pólya and G. Szegő (1964) this implication does not appear.
(V) The u.d. of $\log n$ mod 1 under the above mentioned $\mathbf{A}$ (the so–called u.d. of the logarithmically weighted means) was proved by M. Tsuji (1952).
(VI) B.D. Kotlyar (1981) also proved that $\log_b n$ is not u.d.
(VII) R. Giuliano Antonini (1989, 1991) proved the u.d. of $\log_{10} n$ mod 1 with respect to positive weights p_n, $P_N = \sum_{n=1}^N p_n \to \infty$, for which there exits a function H on $\mathbb{R}^+$ such that:

- $H(n) = e^{P_n}$, and either
- $H(y) = y^\alpha L(y)$ for some $\alpha > 0$ and for a slowly oscillating function $L(y)$ (i.e. $\lim_{y\to\infty} L(xy)/L(y) = 1$ for every $x > 0$), or, if $\alpha = 0$
- $H(y) = L(y)$, where $L(y)$ is a slowly oscillating function such that

$$\lim_{y\to\infty} \frac{L(x_1 y) - L(x_2 y)}{L(x_3 y) - L(x_4 y)} \frac{\log(x_1/x_2)}{\log(x_3/x_4)} = 1$$

for each positive reals x_1, x_2, x_3, x_4 such that $x_1 \neq x_2$ and $x_3 \neq x_4$.
The p_n–weighted u.d. of $\log_{10} n$ mod 1 can be interpreted in such a way that the sequence $n = 1, 2, \dots$ obeys the p_n–weighted Benford's law, i.e. if $A(a)$ is the set of all $n \in \mathbb{N}$ having the first decimal digit equal to a, then $\frac{1}{P_N}\sum_{n=1}^N p_n c_{A(a)}(n) \to \log_{10}\left(1 + \frac{1}{a}\right)$. Here $c_{A(a)}(x)$ is the characteristic function of $A(a)$ and p_n–weighted u.d. sequences are def. in 1.8.4.
(VIII) J. van de Lune (1969) considered the distribution of $\frac{\log n}{P(n)}$, where $P(n)$ is the largest prime factor of n (see [DT, p. 153, Notes]).
(IX) If $\{\log N_k\} \to u$, then the Weyl limit relation (see p. 1 – 9) implies

$$\frac{1}{N_k}\sum_{n=1}^{N_k} e^{2\pi i j \log n} \to \int_0^1 e^{2\pi i j x}\, \mathrm{d}g_u(x),$$

for $j = 0, \pm 1, \pm 2, \dots$. Since the all d.f.'s of $x_n = \log n$ mod 1 are continuous, then 1.8.8(IV) implies that the asymptotic density of any sequence $N_1 < N_2 < \dots$ of positive integers for which $\{\log N_k\} \to u$ is zero. For example, we can take $N_k = [e^{K_k + u_k}]$, where $K_k \in \mathbb{N}$, $K_k \to \infty$ and $u_k \to u$.
(X) A.I. Pavlov (1981) proved for the lower and appear asymptotic density of those n which r initial digits in base b are $K = k_1 k_2 \cdots k_r$, that

$$\liminf_{N\to\infty} \frac{\#\{n \le N; n \text{ has the first } r \text{ digits } = K\}}{N} = \frac{1}{K(b-1)},$$
$$\limsup_{N\to\infty} \frac{\#\{n \le N; n \text{ has the first } r \text{ digits } = K\}}{N} = \frac{b}{(K+1)(b-1)}.$$

V. Baláž, K. Nagasaka and O. Strauch (2010) using properties of distribution functions of the sequence $\log_b n \bmod 1$ proved the following: If $x_1 = \log_b(k_1.k_2k_3\cdots k_r)$ and $x_2 = \log_b(k_1.k_2k_3\cdots(k_r+1))$ then for a sequence N_i such that

$$\lim_{i\to\infty} \log_b N_i \bmod 1 = u$$

we have

$$\lim_{i\to\infty} \frac{\#\{n \le N_i; n \text{ has the first } r \text{ digits } = K\}}{N_i} = g_u(x_2) - g_u(x_1),$$

and consequently

$$\liminf_{N\to\infty} \frac{\#\{n \le N; n \text{ has the first } r \text{ digits } = K\}}{N} = \min_{u\in[0,1]} (g_u(x_2) - g_u(x_1)),$$
$$\limsup_{N\to\infty} \frac{\#\{n \le N; n \text{ has the first } r \text{ digits } = K\}}{N} = \max_{u\in[0,1]} (g_u(x_2) - g_u(x_1)),$$

where the minimum is attained at $u = x_1$ and the maximum at $u = x_2$, in which case we get Pavlov results.

Related sequences: 2.3.6, 2.19.7, 2.12.31, 2.6.18, 2.2.16

V. Baláž – K. Nagasaka – O. Strauch, *Benford's law and distribution functions of sequences in* $(0,1)$, Math. Notes, **88** (2010), no. 3-4, 449–463, (translated from Mat. Zametki **88** (2010), no. 4, 485–501) (MR2882211; Zbl. 1242.11055).

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M. Tsuji: *On the uniform distribution of numbers* mod 1, J. Math. Soc. Japan **4** (1952), 313–322 (MR0059322 (15,511b); Zbl. 0048.03302).

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A. Wintner: *On the cyclical distribution of the logarithms of the prime numbers*, Quart. J. Math. Oxford (1) **6** (1935), 65–68 (Zbl. 0011.14904).

2.12.1.1 Sequences which satisfy Benford's law

This is a continuation of 2.12.1.

• P. Diaconis (1977): A sequence x_n, $n = 1, 2, \dots$, of positive real numbers satisfies **Benford's law** (abbreviated B.L.) [5] in base b, if for every $s = 1, 2, \dots$, and every s-digits integer $D = d_1 d_2 \cdots d_s$ we have the density

$$\lim_{N\to\infty} \frac{\#\{n \le N;\ \text{leading block of } s \text{ digits (beginning with} \ne 0) \text{ of } x_n = D\}}{N}$$
$$= \log_b\left(\frac{D+1}{b^{s-1}}\right) - \log_b\left(\frac{D}{b^{s-1}}\right).$$

Immediately

Theorem 2.12.0.1. *A sequence x_n, $n = 1, 2, \dots$, satisfies B.L. in base b if and only if the sequence $\log_b x_n \bmod 1$ is u.d. in $[0, 1)$.*

NOTES:
(I) Historical comments. B.L. or *the first digit problem* appeared in the following original definitions:

Newcomb (1881): The law of probability of the occurrence of numbers is such that all mantissæ of their logarithms are equally probable.

Benford (1938): The frequency of first digits follows closely the logarithmic relation $F_a = \log_{10}\left(\frac{a+1}{a}\right)$, where F_a is the frequency of the digit $a \in \{1, 2, \dots, 9\}$ in the first place of used numbers.

Thus an infinite sequence $x_n \ge 1$ of real numbers satisfies **Benford's law**, if the frequency (the asymptotic density) of occurrences of a given first digit $a \in \{1, 2, \dots, 9\}$ (0 as a possible first digit is not admitted), when x_n is expressed in the decimal form, is given by $\log_{10}\left(1 + \frac{1}{a}\right)$ for every $a = 1, 2, \dots, 9$. Since a is the first digit of x_n if and only if $\log_{10} x_n \bmod 1 \in [\log_{10} a, \log_{10}(a+1))$, Benford's law for sequence x_n follows from the u.d. of $\log_{10} x_n \bmod 1$. F. Benford (1938) compared the empirical frequency of occurrences of a with $\log_{10}((a+1)/a)$ in twenty different domains such as the areas of 335 rivers; the size of 3259 U.S. populations; the street address of first 342 persons listed in American Men of Sciences, etc. which led him to the conclusion that "the logarithmic law applies particularly to those outlaw numbers that are without known relationships ..." Actually F. Benford rediscovered S. Newcomb's observation from (1881) and Benford's law is a special case of Zipf's law.
(II) Examples. The sequence of Fibonacci numbers F_n, factorials $n!$, and n^n, and n^{n^2} satisfy B.L., but the sequence n, and the sequence of all primes p_n do not (consult (IV) and (V) below),
(III) General scheme of solution of the First Digit Problem: Let $g(x)$ be a d.f. of

[5] precisely generalized or strong B.L.

$\log_b x_n \bmod 1$ and $\lim_{i\to\infty} F_{N_i}(x) = g(x)$. Then for $D = d_1 d_2 \cdots d_s$

$$\lim_{N_i\to\infty} \frac{\#\{n \le N_i;\ \text{first } s \text{ digits (starting a non-zero digit) of } x_n = D\}}{N_i}$$
$$= g\Big(\log_b\Big(\frac{D+1}{b^{s-1}}\Big)\Big) - g\Big(\log_b\Big(\frac{D}{b^{s-1}}\Big)\Big).$$

(IV) Natural numbers. By (III), if
$f(n) = \log_b n^r$, $n = 1, 2, \dots$, then $f^{-1}(x) = b^{x/r}$ and
$\lim_{k\to\infty} \dfrac{f^{-1}(k+w)}{f^{-1}(k)} = \dfrac{b^{(k+w)/r}}{b^{k/r}} = b^{w/r} = \psi(w)$, then
$G(\log_b n^r \bmod 1) = \left\{ g_w(x) = \dfrac{1}{b^{w/r}} \cdot \dfrac{b^{x/r}-1}{b^{1/r}-1} + \dfrac{\min(b^{x/r}, b^{w/r}) - 1}{b^{w/r}}; w \in [0,1] \right\}$.
If $\lim_{i\to\infty}\{f(N_i)\} = \lim_{i\to\infty}\{\log_b(N_i^r)\} = w$, then we have

$$\lim_{i\to\infty} \frac{\#\{n \le N_i;\ \text{first } s \text{ digits of } n^r \text{ are } d_1 d_2 \dots d_s\}}{N_i}$$
$$= g_w\big(\log_b d_1.d_2 \dots (d_s+1)\big) - g_w\big(\log_b d_1.d_2 \dots d_s\big).$$

(V) Primes. Applying (III) to the sequence
$f(p_n) = \log_b p_n^r$, $n = 1, 2, \dots$, where p_n is the nth prime and $r > 0$, we have
$G(\log_b p_n^r \bmod 1) = \left\{ g_w(x) = \dfrac{1}{b^{w/r}} \cdot \dfrac{b^{x/r}-1}{b^{1/r}-1} + \dfrac{\min(b^{x/r}, b^{w/r}) - 1}{b^{w/r}}; w \in [0,1] \right\}$.
If $\{f(p_{N_i})\} = \{\log_b(p_{N_i}^r)\} \to w$ then

$$\lim_{i\to\infty} \frac{\#\{n \le N_i;\ \text{first } s \text{ digits of } p_n^r = d_1 d_2 \dots d_s\}}{N_i}$$
$$= g_w\big(\log_b d_1.d_2 \dots (d_s+1)\big) - g_w\big(\log_b d_1.d_2 \dots d_s\big).$$

(VI) Summary. From (IV) and (V) there follows that both sequences
$\log_b n^r \bmod 1$, $n = 1, 2, \dots$,
$\log_b p_n^r \bmod 1$, $n = 1, 2, \dots$,
have the same set of distribution functions, namely
$\left\{ g_w(x) = \dfrac{1}{b^{w/r}} \cdot \dfrac{b^{x/r}-1}{b^{1/r}-1} + \dfrac{\min(b^{x/r}, b^{w/r}) - 1}{b^{w/r}}; w \in [0,1] \right\}$.
Since $\lim_{r\to\infty} \dfrac{b^{x/r}-1}{b^{1/r}-1} = x$, we get $\lim_{r\to\infty} g_w(x) = x$ for every $w \in [0,1]$.
Thus, with $r \to \infty$ the sequences n^r and p_n^r tend to B.L.
This is a qualitative proof of results given in S. Eliahou, B. Massé and D. Schneider (2013).
(VII) Directly from the u.d. theory there follows

(i) If a sequence $x_n > 0$, $n = 1, 2, \dots$, satisfies B.L. in a base b, then $\limsup_{n\to\infty} n\left|\log \frac{x_{n+1}}{x_n}\right| = \infty$.

(ii) Let $x_n > 0$, $n = 1, 2, \dots$. If the ratio sequence $\frac{x_{n+k}}{x_n}$, $n = 1, 2, \dots$, satisfies B.L. in a base b for every $k = 1, 2, \dots$, then the original sequence x_n, $n = 1, 2, \dots$, also satisfies B.L. in this base b.

(iii) The positive sequences x_n and $\frac{1}{x_n}$, $n = 1, 2, \dots$, satisfy B.L. in a base b simultaneously.

(iv) The positive sequences x_n and nx_n, $n = 1, 2, \dots$, satisfy B.L. in a base b simultaneously.

(v) If a sequence $0 < x_1 \le x_2 \le \dots$ satisfies B.L. in an integer base $b > 1$ then $\lim_{n\to\infty} \frac{\log x_n}{\log n} = \infty$.

(vi) Given a sequence $x_n > 0$, $n = 1, 2, \dots$ such that
$\lim_{n\to\infty} x_n = \infty$ monotonically and $\lim_{n\to\infty} \log \frac{x_{n+1}}{x_n} = 0$ monotonically,
then x_n satisfies B.L. in every base b if and only if $\lim_{n\to\infty} n \log \frac{x_{n+1}}{x_n} = \infty$.

(vii) If a positive sequence x_n satisfies $\lim_{n\to\infty} \log_b \frac{x_{n+1}}{x_n} = \theta$ with θ irrational, then x_n satisfies B.L. in base b.

(viii) If a sequence x_n satisfies B.L., then the asymptotic density of n's for which x_n has in the rth place the given digit a is

$$\sum_{k_1=1}^{b-1} \sum_{k_2=0}^{b-1} \cdots \sum_{k_{r-1}=0}^{b-1} \big(\log_b(k_1.k_2k_3 \dots k_{r-1}(a+1)) - \log_b(k_1.k_2k_3 \dots k_{r-1}a) \big).$$

(VIII) V. Baláž, K. Nagasaka and O. Strauch (2010): Assume that every d.f. $g(x) \in G(x_n)$ is continuous at $x = 0$. Then the sequence x_n satisfies B.L. in the base b if and only if for every $g(x) \in G(x_n)$ we have

$$x = \sum_{i=0}^{\infty} \left(g\left(\frac{1}{b^i}\right) - g\left(\frac{1}{b^{i+x}}\right) \right) \text{ for } x \in [0, 1].$$

(i) Examples of solutions to (VIII):

$$g(x) = \begin{cases} x & \text{if } x \in \left[0, \frac{1}{b}\right], \\ 1 + \frac{\log x}{\log b} + (1-x)\frac{1}{b-1} & \text{if } x \in \left[\frac{1}{b}, 1\right]. \end{cases}$$

$$g^*(x) = \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{b^2}\right], \\ 2 + \frac{\log x}{\log b} & \text{if } x \in \left[\frac{1}{b^2}, \frac{1}{b}\right], \\ 1 & \text{if } x \in \left[\frac{1}{b}, 1\right] \end{cases}$$

$$g^{**}(x)=\begin{cases}0 & \text{if } x\in\left[0,\frac{1}{b^3}\right],\\ 3+\frac{\log x}{\log b} & \text{if } x\in\left[\frac{1}{b^3},\frac{1}{b^2}\right],\\ 1 & \text{if } x\in\left[\frac{1}{b^2},1\right]\end{cases}$$

RELATED SEQUENCES: 2.19.7.2, 2.19.7.1, 2.12.26.

V. BALÁŽ – K. NAGASAKA – O. STRAUCH, *Benford's law and distribution functions of sequences in* $(0,1)$, Math. Notes, **88** (2010), no. 3-4, 449–463, (translated from Mat. Zametki **88** (2010), no. 4, 485–501) (MR2882211; Zbl. 1242.11055).
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2.12.2. The sequence

$$x_n=\log^{(k)} n \bmod 1,$$

where $\log^{(k)} n=\log\log\dots\log n$ with $k>1$ has the set of d.f.'s given by

$$G(x_n)=\{c_\alpha(x)\,;\,\alpha\in[0,1]\}\cup\{h_\alpha(x)\,;\,\alpha\in[0,1]\},$$

where $c_\alpha(x)$ denotes the one–jump d.f. and $h_\alpha(x)$ the constant one.

NOTES:

(I) O. Strauch (1995) settled the general case. B.D. Kotlyar (1981) proved that $\log_2\log_2 n \bmod 1$ is not u.d., and G. Myerson proved that $\log^{(k)} n \bmod 1$ is maldistributed.

(II) For the sake of simplicity, take $k=2$, i.e. $x_n=\{\log\log n\}$, and let $N_1<N_2<\dots$ be a sequence of indices. If $\{\log\log N_k\}\to v>0$, then

$$F_{N_k}(x)=\frac{A([0,x);N_k;x_n)}{N_k}\to c_v(x)$$

and the Weyl limit relation (see p. 1 – 9) implies

$$\frac{1}{N_k}\sum_{n=1}^{N_k}e^{2\pi ij\log\log n}\to\int_0^1 e^{2\pi ijx}\,\mathrm{d}c_v(x)=e^{2\pi ijv},$$

for $j=0,\pm1,\pm2,\dots$. Since for every $v\in[0,1]$ the set $G(\log\log n \bmod 1)$ contains a d.f. which is discontinuous at v (e.g. $c_v(x)$), then by 1.8.8(IV) there exists a sequence N_k such that $\{\log\log N_k\}\to v$ and which positive upper asymptotic density. E.g. the sequence $N_k=[e^{e^{J_k+v_k}}]$ with $J_k\in\mathbb{N}$, $J_k\to\infty$ and $v_k\to v$ can be used. If $\{\log\log N_k\}\to 0$ we take a subsequence N'_k of N_k for which

$$\frac{e^{e^{J'_k}}}{e^{e^{J'_k+v'_k}}}\to t\in[0,1],$$

where $J'_k = [\log\log N'_k]$, and $v'_k = \{\log\log N'_k\}$. In this case $F_{N'_k} \to h_{1-t}(x)$.

B.D. Kotlyar: *A method for calculating the number of lattice points*, (Russian), Ukrain. Math. Zh. **33** no. 5, (1981), 678–681, 718 (MR0633747 (83b:10044); Zbl 0479.10024).
G. Myerson: *A sampler of recent developments in the distribution of sequences*, in: *Number theory with an emphasis on the Markoff spectrum (Provo, UT 1991)*, (A.D. Pollington and W. Moran eds.), Lecture Notes in Pure and App.Math., Vol. 147, Marcel Dekker, New York, Basel, Hong Kong, 1993, pp. 163–190 (MR1219333 (94a:11112); Zbl. 0789.11043).
O. Strauch: *Uniformly maldistributed sequence in a strict sense*, Monatsh. Math. **120** (1995), 153–164 (MR1348367 (96g:11095); Zbl. 0835.11029).

2.12.3. If $x_0 = 1$ and

$$x_n = \log_2(2n-1) \bmod 1,$$

for $n = 1, 2, \dots$, then

$$\liminf_{N\to\infty} Nd_N^* = \frac{1}{\log 4}, \qquad \text{and} \qquad \limsup_{N\to\infty} Nd_N = \frac{1}{\log 4},$$

where d_N^* and dispersion d_N are defined in 1.10.11.
Notes: The number $\frac{1}{\log 4}$ is the upper bound for lim inf and also the lower bound for lim sup for all one–to–one sequences x_n, $n = 0, 1, 2, \dots$, for which $x_0 = 1$ and $x_1 = 0$. The upper bound has been found by many authors: N.G. de Bruijn and P. Erdős (1949), A. Ostrowski (1957, [a]1957), A. Schönhage (1957) and G.H. Toulmin (1957). For details cf. 1.10.11(II).
Similarly, if $x_1 = 1$, and $x_n = \log_2(2n-3) \bmod 1$ for $n = 2, 3, \dots$, then we know the exact value

$$d_N = \frac{\log N - \log(N-1)}{\log 4}$$

if $N \ge 2$ (I. Ruzsa, see H. Niederreiter (1984, p. 1172)).

N.G. de Bruijn – P. Erdős: *Sequences of points on a circle*, Nederl. Akad. Wetensch., Proc. **52** (1949), 14–17 (MR0033331 (11,423i); Zbl. 0031.34803). (=Indag. Math. **11** (1949), 46–49).
H. Niederreiter: *On a measure of denseness for sequences*, in: *Topics in classical number theory, Vol. I, II (Budapest 1981)*, (G.Halász ed.), Colloq. Math. Soc. János Bolyai, Vol. 34, North–Holland Publishing Co., Amsterdam, New York, 1984, pp. 1163–1208 (MR0781180 (86h:11058); Zbl. 0547.10045).
A. Ostrowski: *Zum Schubfächerprinzip in einem linearen Intervall*, Jber. Deutsch. Math. Verein. **60** (1957), Abt. 1, 33–39 (MR0089232 (19,638a); Zbl. 0077.26703).
[a] A. Ostrowski: *Eine Verschärfung des Schubfächerprinzips in einem linearen Intervall*, Arch. Math. **8** (1957), 1–10 (MR0089233 (19,638b); Zbl. 0079.07302).A. Schönhage: *Zum Schubfächerprinzip im linearen Intervall*, Arch. Math. **8** (1957), 327–329 (MR0093511 (**20** #35); Zbl. 0079.07303).
G.H. Toulmin: *Subdivision of an interval by a sequence of points*, Arch. Math **8** (1957), 158–161 (MR0093513 (**20** #37); Zbl. 0086.03801).

2.12.4. The sequence

$$x_n = \left\{1 + (-1)^{\left[\sqrt{\left[\sqrt{\log_2 n}\right]}\right]}\left\{\sqrt{\left[\sqrt{\log_2 n}\right]}\right\}\right\}$$

has the set of d.f.'s

$$G(x_n) = \{c_\alpha(x)\ ;\ \alpha \in [0,1]\}.$$

O. Strauch: *Uniformly maldistributed sequence in a strict sense*, Monatsh. Math. **120** (1995), 153–164 (MR1348367 (96g:11095); Zbl. 0835.11029).

2.12.5. The sequence

$$n \log^{(k)} n \bmod 1$$

is

$$\text{u.d. for every } k = 1, 2, \dots.$$

Notes: Cf. [KN, p. 24, Exer. 2.25]. By [KN, p. 132, Exer. 3.21] the sequence $n \log\log en \bmod 1$ has discrepancy $D_N = \mathcal{O}(N^{-1/5}(\log N)^{1/5}(\log\log N)^{2/5})$.

2.12.6. The sequence

$$n^2 \log\log n \bmod 1$$

is

$$\text{u.d.}$$

Notes: Cf. [KN, p. 31, Exer. 3.13].

2.12.7. The sequence

$$\alpha \log^\tau n \bmod 1, \alpha > 0, \tau > 1$$

is

$$\text{u.d.}$$

with

$$D_N = \mathcal{O}(\log^{1-\tau} N).$$

Notes: Cf. [KN, p. 130, Exer. 3.3].

H. Niederreiter: *Almost–arithmetic progressions and uniform distribution*, Trans. Amer. Math. Soc. **161** (1971), 283–292 (MR0284406 (**44** #1633); Zbl. 0219.10040).

2.12.8. The sequence

$$\alpha \log^\tau n \bmod 1, \alpha > 0, 0 < \tau < 1$$

is

dense but not u.d.

Notes: G. Pólya and G. Szegő (1964, Part 2, Ex. 183).

Related sequences: 2.12.7

G. Pólya – G. Szegő: *Aufgaben und Lehrsätze aus der Analysis*, Vol. 1 & 2, 3rd corr. ed., Grundlehren d. math. Wiss., Band 19, Springer Verlag, Berlin, Göttingen, Heidelberg, New York, 1964 (MR0170985 (**30** #1219a); MR0170986 (**30** #1219b); Zbl. 0122.29704).

2.12.9. Let α and β be positive reals and $\tau > 1$. Let the double sequence $\log^\tau(\alpha m + \beta n)$, $m = 1, 2, \dots$, $n = 1, 2, \dots$, be rearranged to an ordinary sequence x_n, $n = 1, 2, \dots$, in such a way, that for every N, the initial segment x_n, $n = 1, 2, \dots, N^2$, contains $\log^\tau(\alpha m + \beta n)$ for $m, n = 1, 2, \dots, N$. Then the sequence

$$x_n \bmod 1$$

is

u.d.

Notes: [KN, p. 25, Exer. 2.30]. The result follows directly from 2.6.16.

Related sequences: 2.12.7

2.12.10. The sequence

$$n^\sigma g(\log n) \bmod 1, \quad \sigma > 0,$$

where $g(x)$ is a non–constant linear combination of arbitrary powers of x, is

u.d.

Notes: (cf. [KN, p. 31, Exer. 3.15])

2.12.11. The sequence

$$n^2 \log n \bmod 1$$

is

u.d.

Related sequences: The sequence of type 2.12.10 (cf. [KN, p. 31, Exer. 3.12]).

2.12.12. The sequence

$$\alpha n^\sigma \log^\tau n \bmod 1, \quad \alpha \neq 0,\ \sigma > 0,\ \sigma \notin \mathbb{N},\ \tau \in \mathbb{R}$$

is

u.d.

NOTES: (I) This is a special case of the sequence 2.6.1 (cf. [KN, p. 31, Exer. 3.10]). The sequence is of the type 2.12.10.
(II) Y. Ohkubo (1986) proved that the sequence $\alpha n^\sigma \log^\tau n \bmod 1$ has logarithmic discrepancy (cf. 1.10.7) $D_N = \mathcal{O}(1/\log N)$ if $\alpha > 0$, $0 \leq \sigma < 1$ and τ are such that $\lim_{n\to\infty} n^\sigma \log^\tau n = \infty$.

Y. OHKUBO: *Discrepancy with respect to weighted means of some sequences*, Proc. Japan Acad. **62** A (1986), no. 5, 201–204 (MR0854219 (87j:11075); Zbl. 0592.10044).

2.12.13. The sequence

$$\alpha n^k \log^\tau n \bmod 1, \quad k \in \mathbb{N},\ \alpha \neq 0,\ \tau < 0 \text{ or } \tau > 1$$

is

u.d.

RELATED SEQUENCES: This a special case of the sequence 2.6.1 (cf. [KN, p. 31, Exer. 3.11]). The sequence is of the type 2.12.10.

2.12.14. The sequence

$$\alpha n \log^\tau n \bmod 1, \quad \alpha \neq 0,\ 0 < \tau \leq 1$$

is

u.d.

RELATED SEQUENCES: The sequence is of the type 2.12.10. What concerns the discrepancy, the sequence $n \log n \bmod 1$ has discrepancy $D_N = \mathcal{O}(N^{-1/5}(\log N)^{2/5})$, cf. [KN, p. 132, Exer. 3.20].

2.12.15. The sequence

$$\alpha n^2 \log^\tau n \bmod 1, \quad \alpha \neq 0,\ 0 < \tau \leq 1$$

is

u.d.

RELATED SEQUENCES: The sequence is of the type 2.12.10.

2.12.16. The sequence

$$x_n = \log(n \log n) \bmod 1$$

is

everywhere dense in $[0,1]$, but it is not u.d.

More precisely, the sequence x_n has the same set of d.f.'s as the sequence $\log n \bmod 1$ from 2.12.1, i.e.

$$G(\{\log(n \log n) \bmod 1\}) = \\ = \left\{ g_u(x) = \frac{e^{\min(x,u)} - 1}{e^u} + \frac{1}{e^u}\frac{e^x - 1}{e - 1}\ ;\ u \in [0,1] \right\}.$$

NOTES: (I) The non–uniformity is a consequence of Niederreiter's theorem 2.2.8 and by 2.6.18 the lower and upper d.f. of x_n are

$$\underline{g}(x) = \frac{e^x - 1}{e - 1}, \qquad \overline{g}(x) = \frac{1 - e^{-x}}{1 - e^{-1}}.$$

That the set of d.f.'s of x_n coincides with that of $\log n \bmod 1$ (cf. 2.12.1) was proved by O. Strauch and O. Blažeková (2003) using Theorem 2.6.18.1 from 2.6.18 Note (II).
(II) Strauch and Blažeková (2003) present two methods for finding $G(\{\log(n \log n)\})$. The first one leads to the results mentioned in the previous Note (I). The second one applies Th. 2.3.21 to d.f.'s of $(\log n, \log\log n) \bmod 1$, cf. 3.13.5. This method also gives the following functional equation

$$g_w(x) = \begin{cases} g_u(1 + x - v) - g_u(1 - v), & \text{if } 0 \le x \le v, \\ g_u(x - v) + 1 - g_u(1 - v), & \text{if } v < x \le 1, \end{cases}$$

where $w = (u + v) \bmod 1$.
(III) Theorem 2.6.18.1 in 2.6.18 Note (II) also gives

$$G(\log(n \log^{(i)} n) \bmod 1) = G(\log n \bmod 1)$$

for $i = 1, 2, \dots$, where $\log^{(i)} n$ is the ith iterated logarithm $\log\log \dots \log n$.
(IV) Strauch and Blažeková (2003) also showed that x_n has the same distribution as $p_n/n \bmod 1$ from 2.19.19.

O. STRAUCH – O. BLAŽEKOVÁ: *Distribution of the sequence* p_n/n mod 1, Math. Institute, Slovak Acad. Sci., Bratislava, Slovak Republic, 2003, 15 pp.

2.12.17. Let $\alpha \neq 0$, β, γ, and δ be real numbers. Then the sequence

$$x_n = \alpha n^\beta (\log^\gamma n) \log^\delta (\log n) \bmod 1$$

is

u.d.

if and only if (at least) one of the following conditions holds:

1. β is a positive and non–integeral,
2. β is a positive integer and either α is irrational, or $\gamma \neq 0$, or $\delta \neq 0$,
3. $\beta = 0$ but $\gamma > 1$,
4. $\beta = 0$, $\gamma = 1$ and $\delta > 0$.

The sequence x_n is

dense but not u.d. in the interval $[0,1]$

if and only if one of the following conditions holds:

1. $\beta = 0$, $0 < \gamma < 1$,
2. $\beta = 0$, $\gamma = 1$ and $\delta \leq 0$,
3. $\beta = 0$, $\gamma = 0$ and $\delta > 0$.

Related sequences: 2.6.35.

M.D. Boshernitzan: *Uniform distribution and Hardy fields*, J. Anal. Math. **62** (1994), 225–240 (MR1269206 (95e:11085); Zbl. 0804.11046).

2.12.18. If $\gamma > 0$ and α is irrational, then the sequence

$$(\log^{1+\gamma} n) \cos(2\pi n\alpha) \bmod 1$$

is

u.d.

and if $\alpha \in \bigcup_{0<u<1} \{x \in \mathbb{R}; \liminf_{q\to\infty} q^{1/u} \|xq\| > 0\}$ (i.e. α is non–Liouville), then

$$D_N \ll (\log N)^{-\gamma/2}.$$

D. Berend – M.D. Boshernitzan – G. Kolesnik: *Distribution modulo* 1 *of some oscillating sequences. III*, Acta Math. Hungar. **95** (2002), no. 1–2, 1–20 (MR1906205 (2003h:11085); Zbl. 0997.11058).

2.12.19. If α is irrational, then the sequence

$$n(\log n)\cos(2\pi n\alpha) \bmod 1$$

is

u.d.

and if $\liminf_{q\to\infty} q^{1/u}\|\alpha q\| > 0$ for some $0 < u < 1$, then

$$D_N \ll (\log N)^{-\beta},$$

with $\beta = u.8^{-1-1/(2u)}/(3+2u)$.

D. Berend – M.D. Boshernitzan – G. Kolesnik: *Distribution modulo* 1 *of some oscillating sequences. III*, Acta Math. Hungar. **95** (2002), no. 1–2, 1–20 (MR1906205 (2003h:11085); Zbl. 0997.11058).

2.12.20. If α is irrational, then the sequence

$$n^\beta(\log^\gamma n)\cos(2\pi n\alpha) \bmod 1$$

is

dense in $[0,1]$

provided that either $\beta > 0$ or $\beta = 0, \gamma > 0$.

D. Berend – M.D. Boshernitzan – G. Kolesnik: *Distribution modulo* 1 *of some oscillating sequences. III*, Acta Math. Hungar. **95** (2002), no. 1–2, 1–20 (MR1906205 (2003h:11085); Zbl. 0997.11058).

2.12.21. The sequence

$$\log F_n \bmod 1, \quad F_n \text{ is the } n\text{th Fibonacci number},$$

is

u.d.

Notes: (I) J.L. Brown, Jr. and R.L. Duncan (1972). The same holds for Lucas numbers.
(II) Fibonacci numbers F_n can be defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \quad F_1 = F_2 = 1,$$

or directly by Binet's formula

$$F_n = \frac{\tau^n - (-\tau)^n}{\tau + \tau^{-1}}, \text{ where } \tau = \frac{1+\sqrt{5}}{2} \text{ is the } \textbf{golden mean}$$

and consequently, as the closest integer to $\tau^n/\sqrt{5}$, or more compactly

$$F_n = i^{n-1}\frac{\sin(nz_0)}{\sin z_0}, \quad \text{where } z_0 = \frac{\pi}{2} + i\log\tau,$$

cf. M.R. Schroeder (1997).

J.L. Brown, Jr. – R.L. Ducan: *Modulo one uniform distribution of certain Fibonacci–related sequences*, Fibonacci Quart. **10** (1972), no. 3, 277–280, 294 (MR0304291 (**46** #3426); Zbl. 0237.10033).
M.R. Schroeder: *Number Theory in Science and Communication. With Applications in Cryptography, Physics, Digital Information, Computing and Self–similarity*, 3rd ed., Springer Verlag, Berlin, 1997 (MR1457262 (99c:11165); Zbl. 0997.11501).

2.12.22. If $b > 1$ is a positive integer, then the sequence

$$\log_b F_n \bmod 1, \quad F_n \text{ is the } n\text{th Fibonacci number},$$

is

$$\text{u.d. in } [0,1].$$

Notes:
(I) Consequently, Fibonacci numbers satisfy strong Benford's law in any base b (see 2.12.26).
(II) Rediscovered by L.C. Washington (1981).
(III) L. Kuipers (1982) proved the density of $\log_b F_n \bmod 1$ in $[0,1]$. Also see [KN, p. 31, Exer. 3.4].
(IV) R.L. Duncan (1967) proved earlier that $\log_{10} F_n \bmod 1$ is u.d.
Related sequences: 2.24.5.

R.L. Duncan: *An application of uniform distribution to the Fibonacci numbers*, Fibonacci Quart. **5** (1967), 137–140 (MR0240058 (**39** #1412); Zbl. 0212.39501).
L. Kuipers: *A property of the Fibonacci sequence* $(F_m), m = 0, 1, \dots$, Fibonacci Quart. **20** (1982), no. 2, 112–113 (MR0673290 (83k:10012); Zbl. 0481.10036).
L.C. Washington: *Benford's law for Fibonacci and Lucas numbers*, Fibonacci Q. **19** (1981), 175–177 (MR0614056 (82f:10009); Zbl. 0455.10004).

$\boxed{\textbf{2.12.22.1}}$ Let x_n be a sequence generated by the linear recurrence relation

$$x_{n+k} = a_{k-1}x_{n+k-1} + \cdots + a_1x_{n+1} + a_0x_n, \quad n = 1, 2, \dots,$$

where $a_0, a_1, \dots, a_{k-1}$ are non-negative rational numbers with $a_0 \neq 0$, k is a fixed integer, and $x_1, x_2, \dots, x_k$ are initial values. Let its characteristic polynomial

$$x^k - a_{k-1}x^{k-1} - \cdots - a_1x - a_0$$

have k distinct roots $\beta_1, \beta_2, \dots, \beta_k$ satisfying $0 < |\beta_1| < \cdots < |\beta_k|$ and such that none of them has magnitude equal to 1, then

$$\log x_n \bmod 1, n = 1, 2, \dots$$

is

u.d. in $[0, 1]$.

Furthermore, if in the general solution $x_n = \sum_{j=1}^{k} \alpha_j \beta_j^n$ of the recurrence j_0 is the largest value of j for which $\alpha_j \neq 0$ and if $\log_b \beta_{j_0}$ is irrational, then also

$$\log_b x_n \bmod 1, n = 1, 2, \dots$$

is

u.d. in $[0, 1]$.

NOTES:
(I) J.L. Brown, Jr. and R.L. Ducan (1970).
(II) I.e. x_n satisfies strong Benford law in the base b, see 2.12.26.
(III) This implies that Fibonacci and Lucas numbers obey strong Benford law, cf. 2.12.21, 2.12.22, a fact which was often rediscovered, e.g. L.C. Washington (1981), etc.

J.L. BROWN, JR. – R.L. DUCAN: *Modulo one uniform distribution of the sequence of logarithms of certain recursive sequences*, Fibonacci Quart. **8** (1970), 482–486 (MR0360444 (**50** #12894); Zbl. 0214.06802).
L.C. WASHINGTON: *Benford's law for Fibonacci and Lucas numbers*, Fibonacci Q. **19** (1981), 175–177 (MR0614056 (82f:10009); Zbl. 0455.10004).

2.12.23. The sequence

$$x_n = e^{c \log^\tau n} \bmod 1, \quad c > 0,\ 1 < \tau < \frac{3}{2},$$

is

u.d.

with discrepancy

$$D_N^* = \mathcal{O}\left(e^{-c_1 (\log N)^{3-2\tau}}\right),$$

where both $c_1 > 0$ and the $\mathcal{O}$–constant depend only on τ.

NOTES: This was proved by A.A. Karacuba (1971) (cf. also Karacuba (1975, p. 72 or 1983, p. 103) and [KN, p. 30, Notes]).

A.A. Karacuba (A.A. Karatsuba): *Estimates for trigonometric sums by the method of I.M. Vinogradov and their applications*, (Russian), Trudy Mat. Inst. Steklov. **112** (1971), 241–255, 388 (English translation: Proc. Steklov Inst. Math. **112** Amer. Math. Soc., Providence, R.I., (1973), pp. 251–265 (MR0330068 (**48** #8407); Zbl. 0259.10040)).
A.A. Karacuba (A.A. Karatsuba): *Principles of Analytic Number Theory*, (Russian), Izdat. Nauka, Moscow, 1975 (MR0439767 (**55** #12653); Zbl. 0428.10019). (2nd edition 1983).

2.12.24. Let α be an arbitrary real algebraic number of degree ≥ 2 and c satisfies $0 < c < 1$. If $c' > 0$ and $I \subset [0,1]$ is an subinterval of the length

$$|I| \geq e^{-c' \log^{1-c} N}$$

then for the sequence

$$x_n = \alpha e^{[\log^c n] \log n} \bmod 1$$

we have

$$A(I; N; x_n) \geq N e^{-c_1 (\log^{1-c} N + \log^c N \log\log N)}$$

for $N \geq N_1$, where N_1 and $c_1 > 0$ are constants which depend on c and c'.
Notes: A.A. Karacuba (2001). He noted that the lower bound remains valid if α is an irrational number with bounded partial quotients.

A.A. Karacuba (A.A. Karatsuba): *On the fractional parts of rapidly increasing functions*, (Russian), Izv. Ross. Akad. Nauk Ser. Mat. **65** (2001), no. 4, 89–110 (English translation: Izv. Math. **65** (2001), no. 4, 727–748 (MR1857712 (2002i:11066); Zbl. 1028.11045)).

2.12.25. The sequence

$$\log n! \bmod 1$$

is

u.d.

and for any $\varepsilon > 0$ we have

$$D_N \leq c.N^{-1/2+\varepsilon}$$

with a constant $c = c(\varepsilon)$.
Notes: The u.d. was proved by P. Diaconis (1977, Th. 3) and for discrepancy cf. K. Goto and T. Kano (1985, Th. 3).

P. Diaconis: *The distribution of leading digits and uniform distribution* mod 1, Anals of Prob. **5** (1977), 72–81 (MR0422186 (**54** #10178); Zbl. 0364.10025).

K. Goto – T. Kano: *Uniform distribution of some special sequences*, Proc. Japan Acad. Ser. A Math. Sci. **61** (1985), no. 3, 83–86 (MR0796473 (87a:11069); Zbl. 0573.10023).

2.12.26. The sequence

$$\log_{10} n! \bmod 1$$

is

$$\text{u.d.}$$

Notes: (I) P. Diaconis (1977, Th. 3). He noted that the u.d. of $\log_{10} n! \bmod 1$ implies that the sequence $n!$ obeys the Benford's law. See also S. Kunoff (1987).
(II) **The first digit problem:** The infinite sequence $x_n \geq 1$ of real numbers obeys the **Benford's law**, if the frequency (the asymptotic density) of the occurrence of a given first digit a, when x_n is expressed in the decimal form

$$x_n = a_{k(n)}(n)a_{k(n)-1}(n)\dots a_0(n).a_{-1}(n)a_{-2}(n)\dots$$

is given by

$$\lim_{N\to\infty} \frac{\#\{n \leq N\,;\, a_{k(n)}(n) = a\}}{N} = \log_{10}\left(1 + \frac{1}{a}\right)$$

for every $a = 1, 2, \dots, 9$ (0 as a possible first digit is not admitted). One writes

$$x_n = 10^{k(n)} \cdot a_{k(n)}(n).a_{k(n)-1}(n)a_{k(n)-2}(n)\dots,$$

where $a_{k(n)}(n).a_{k(n)-1}(n)a_{k(n)-2}(n)\dots$ is the **mantissa of $\boldsymbol{x_n}$**. Since

$$\log_{10} x_n \equiv \log_{10}\big(a_{k(n)}(n).a_{k(n)-1}(n)a_{k(n)-2}(n)\dots\big) \bmod 1$$

and

$$a_{k(n)}(n) = a \Longleftrightarrow \{\log_{10} x_n\} \in [\log_{10} a, \log_{10}(a+1),$$

the Benford's law for x_n follows from the u.d. of $\log_{10} x_n \bmod 1$. The definition can be extended to any sequence $x_n \neq 0$ requiring that the frequency with which the non–zero digit a appears as the first digit is $\log_{10}\big(1 + \frac{1}{a}\big)$. The u.d. of $\log_{10}|x_n| \bmod 1$ again implies this law.
(III) It was S. Newcomb (1881) who firstly noted "*That the ten digits not occur with equal frequency must be evident to anyone making use of logarithm tables*".
(IV) F. Benford (1938) compared the empirical frequency of a with $\log_{10}((a+1)/a)$ in twenty different tables having lengths running from 91 entries (atomic weights) to 5000 entries in a mathematical handbook which led him to the conclusion that "the logarithmic law applies particularly to those outlaw numbers that are without known relationships ..."
(V) J. Cigler suggested (cf. R.A. Raimi (1976) and P. Diaconis (1976)) to call x_n a **strong Benford sequence** if $\log_{10} x_n \bmod 1$ is u.d. and a **weak Benford**

sequence if $\log_{10} x_n \bmod 1$ is logarithmically weighted u.d. (cf. 1.8.4) The sequence x_n is a strong Benford sequence if and only if

$$\lim_{N\to\infty} \frac{\#\{n \le N \,;\, a_{k(n)}(n)a_{k(n)-1}(n)\dots a_{k(n)-l}(n) = a_l a_{l-1}\dots a_0\}}{N} =$$
$$= \log_{10}(a_l.a_{l-1}\dots a_0 + 0.0\dots 01) - \log_{10}(a_l.a_{l-1}\dots a_0) =$$
$$= \log_{10}\left(1 + \frac{1}{a_l a_{l-1}\dots a_0}\right)$$

for every initial string of digits $a_l a_{l-1}\dots a_0 = a_l 10^l a_{l-1} 10^{l-1} + \dots + a_0$.
(VI) É. Janvresse and T. de la Rue (2003-04) proved that d.f. $g(t) = \log_{10} t$ is the unique d.f. defined on $[1,10]$ satisfying

$$g(t) = \int_1^t \left(1 - \frac{t}{x}\right) \mathrm{d}g(x) + \frac{10}{9}\int_1^{10} \frac{\mathrm{d}g(x)}{x}$$

for every $t \in [1,10]$.
(VII) A similar Benford's law can be defined with respect to base e.
(VIII) The bibliography given in Raimi (1976) is almost complete until 1976. For another comprehensive survey on Benford's law consult P. Schatte (1988), and some results on the subject can be found in K. Nagasaka, S. Kanemitsu, J.–S. Shiue (1990).
(IX) For the weighted Benford's law consult 2.12.1(VII).
(X) Strong or generalized Benford's law of the sequence x_n for the base b is equivalent to the u.d. of $\log_b x_n \bmod 1$.

Related sequences: Benford sequences are: 2.12.22, 2.12.27, 2.12.28, 2.24.4, 2.24.3, 2.24.4, 2.24.5, 2.19.8

F. Benford: *The law of anomalous numbers*, Proc. Amer. Phil. Soc. **78** (1938), 551–572 (Zbl. 0018.26502; JFM 64.0555.03).
P. Diaconis: *The distribution of leading digits and uniform distribution* mod 1, Anals of Prob. **5** (1977), 72–81 (MR0422186 (**54** #10178); Zbl. 0364.10025).
É. – T. de la Rue: *From uniform distribution to Benford's law*, Laboratoire de Mathématiques Raphaël Salem, Université de Rouen, 2003-04, 10 pp. (Publication de l'umr 6085). (MR2122815 (2006b:60161); Zbl. 1065.60095).
S. Kunoff: *$N!$ has the first digit property*, Fibonacci Quart. **25** (1987), no. 4, 365–367 (MR0911988 (88m:11059); Zbl. 0627.10007).
K. Nagasaka – S. Kanemitsu – J.–S. Shiue: *Benford's law: the logarithmic law of first digit*, in: *Number theory, Vol. I (Budapest, 1987)*, Colloq. Math. Soc. János Bolyai, Vol. 51, North–Holland Publishing Co., Amsterdam, 1990, pp. 361–391 (MR1058225 (92b:11048); Zbl. 0702.11045).
S. Newcomb: *Note on the frequency of use of the different digits in natural numbers*, Amer. J. Math. **4** (1881), 39–41 (MR1505286 ; JFM 13.0161.01).
R.A. Raimi: *The first digit problem*, Amer. Math. Monthly **83** (1976), no. 7, 521–538 (MR0410850 (**53** #14593); Zbl. 0349.60014).
P. Schatte: *On mantissa distribution in computing and Benford's law*, J. Inform. Process. Cybernet. **24** (1988), no. 9, 443–455 (MR0984516 (90g:60016); Zbl. 0662.65040).

2.12.27. Let $\theta = [0; a_1, a_2, \dots]$ denote the continued fraction expansion of $\theta \in (0,1)$, and let $p_n(\theta)$ and $q_n(\theta)$ denote the numerator and denominator of the nth convergent, resp. If θ is a quadratic irrational number, then the sequences

$$\log p_n(\theta) \bmod 1 \quad \text{and} \quad \log q_n(\theta) \bmod 1$$

are

$$\text{u.d.}$$

NOTES: (I) S. Kanemitsu, K. Nagasaka, G. Rauzy and J.–S. Shiue (1988) have stated the result without proof in the terms of Benford's law for q_n (cf. 2.12.26). H. Jager and P. Liardet (1988) gave the first proof of this fact. Actually, they proved that every subsequence of the form $\log p_{a+bn}(\theta) \bmod 1$ and $\log q_{a+bn}(\theta) \bmod 1$ is u.d. and if $\log_{10}\theta$ is irrational, then also $\log_{10} p_n(\theta) \bmod 1$ and $\log_{10} q_n(\theta) \bmod 1$ are u.d.
(II) P. Schatte (1990) extended the result to the nth denominator of the **regular Hurwitzian continued fractions**, i.e. for continued fraction expansions of the form

$$\begin{aligned}\theta = [0; b_1, \dots, b_h, \overline{f_1(x), \dots, f_k(x)}]_{x=0}^{\infty} &= \\ &= [0; b_1 \dots, b_h, f_1(0), \dots, f_k(0), f_1(1), \dots, f_k(1), \dots]\end{aligned}$$

where the elements $b_1, \dots, b_h$ are positive integers, and $f_1(x), \dots, f_k(x)$ are polynomials with rational coefficients assuming positive integral values at $x = 0, 1, 2, \dots$. For instance, the continued fractions of $e^{j/q}$ is Hurwitzian for $j = 1, 2$ and arbitrary $q \in \mathbb{N}$.

RELATED SEQUENCES: 2.24.4

H. JAGER – P. LIARDET: *Distributions arithmétiques des dénominateures de convergents de fractions continues*, Nederl. Akad. Wetensch. Indag. Math. **50** (1988), no. 2, 181–197 (MR0952514 (89i:11085); Zbl. 0655.10045).
S. KANEMITSU – K. NAGASAKA – G. RAUZY – J.–S. SHIUE: *On Benford's law: the first digit problem*, in: *Probability theory and mathematical statistics (Kyoto, 1986)*, Lecture Notes in Math., 1299, Springer Verlag, Berlin, New York, 1988, pp. 158–169 (MR0935987 (89d:11059); Zbl. 0642.10007).
P. SCHATTE: *On Benford's law for continued fractions*, Math. Nachr. **148** (1990), 137–144 (MR1127337 (92m:11077); Zbl. 0728.11036).

2.12.28. The sequence of blocks X_n, $n = 1, 2, \dots$, with blocks

$$X_n = \left(\log\binom{n}{0}, \log\binom{n}{1}, \dots, \log\binom{n}{n}\right) \bmod 1$$

is

$$\text{u.d.}$$

and thus the block sequence $\omega = (X_n)_{n=1}^{\infty}$ is u.d.

Notes: It was P.B. Sarkar (1973) who firstly conjectured that binomial coefficients $\binom{n}{k}$, $k = 0, 1, 2, \dots, n$, satisfy the Benford law (see 2.12.26 (II)). He computed the initial digits of these blocks for $n = 1, 2, \dots, 500$. This conjecture was firstly proved by P. Diaconis (1977) in the form that the block sequence X_n mod 1, $n = 1, 2, \dots,$ is as u.d. He proved that $\left|\sum_{k=0}^{n} e^{2\pi i h \log\binom{n}{k}}\right| = \mathcal{O}(n^{\frac{1}{2}} \log n)$. See also O. Strauch (1999, p. 169).

P. Diaconis: *The distribution of leading digits and uniform distribution* mod 1, Anals of Prob. **5** (1977), 72–81 (MR0422186 (**54** #10178); Zbl. 0364.10025).
P.B. Sarkar: *An observation on the significant digits of binomial coefficients and fatorials*, Sankhyã **B35** (1973), 363–364
O. Strauch: *Distribution of Sequences* (in Slovak), Mathematical Institute of the Slovak Academy of Sciences, DSc Thesis, Bratislava, Slovakia, 1999.

2.12.29. The 0–1 sequence

$$x_n = \frac{1 + (-1)^{[\log \log n]}}{2}, \qquad n > 1,$$

has the set of d.f.'s

$$G(x_n) = \{h_\alpha(x)\ ;\ \alpha \in [0, 1]\}.$$

Note that here $x_n \in [0, 1]$, the corresponding $G(x_n)$ is defined on p. 1 – 11 .

O. Strauch: *On the set of distribution functions of a sequence*, in: *Proceedings of the Conference on Analytic and Elementary Number Theory in Honor of Edmund Hlawka's 80th Birthday, Vienna, July 18–20, 1996*, (W.–G. Nowak, J. Schoißengeier eds.), Universität Wien & Universität für Bodenkultur, Vienna, 1997, pp. 214–229 (Zbl. 0886.11044).

2.12.30. The sequence

$$x_n = \frac{1}{n} \sum_{i=2}^{n} \frac{1 + (-1)^{[\log \log i]}}{2}$$

is

dense in $[0, 1]$

and for dispersion d_N we have

$$d_N \le \frac{1}{N^{\frac{1}{e^2} - \frac{1}{e^3}}}.$$

Related sequences: This is a special case of the sequence 2.3.15, since 2.12.29 satisfies the conditions of 2.3.15.

O. Strauch: *On the set of distribution functions of a sequence*, in: *Proceedings of the Conference on Analytic and Elementary Number Theory in Honor of Edmund Hlawka's 80th Birthday, Vienna, July 18–20, 1996*, (W.–G. Nowak, J. Schoißengeier eds.), Universität Wien & Universität für Bodenkultur, Vienna, 1997, pp. 214–229 (Zbl. 0886.11044).

2.12.31. If α is irrational and $\beta \neq 0$ is real, then the sequence

$$x_n = \alpha n + \beta \log n \bmod 1$$

is

u.d.

and if α is an irrational of a finite type $\eta \geq 1$ then

$$D_N \ll N^{-\frac{1}{\eta+1/2}+\varepsilon}$$

for every $\varepsilon > 0$. If irrational α is of a constant type (cf. 2.8.1(V)), then

$$D_N \leq C(\beta) N^{-\frac{2}{3}} \log N.$$

If α, $\beta \neq 0$ are real, and $\mathbf{A}$ is the matrix defined in 2.12.1 (the so–called logarithmically weighted means) then this sequence is

A–u.d.

and for its logarithmic discrepancy we have

$$L_N \leq c(\beta)(\log N)^{-1}.$$

Notes: (0) The u.d. of x_n follows from a result proved by G. Rauzy (1973, 1976), cf. 2.3.6, saying that sequences y_n and $y_n + \beta \log n$ are simultaneously u.d. mod 1 for every sequence y_n.

(I) E. Hlawka (1983) proved (cf. [DT, pp. 252–253]) that x_n is logarithmically u.d. thus extending a result by M. Tsuji 2.12.1 for $\log n$ mod 1.

(II) R.F. Tichy (1983) extended the results of Hlawka finding a bound for the logarithmic discrepancy. R.F. Tichy and G. Turnwald (1986) proved that the logarithmic discrepancy is of the order $\mathcal{O}(\log\log^2 N/\log N)$ with the $\mathcal{O}$-constant depending only on β. They conjectured that the $\log\log^2 N$–term is superfluous.

(III) The conjecture mentioned in (II) was proved by R.C. Baker and G. Harman (1990). They used the following theorem (cf. [DT, p. 253, Th. 2.41]) which is applicable to more general classes of sequences including the case $\alpha n + \beta n^{1-\delta}, 0 < \delta < 1$.

Theorem 2.12.31.1. *Assume that f is a real valued twice differentiable function defined on $[1, \infty)$, and that there exist positive constants c, K, δ and a positive integer H such that*

- *$x(f'(x)-\lambda)$ is of class H (see below) for every real λ,*
- *f' is bounded on bounded intervals,*
- *if $x \geq 1$ then either $cx^{-2} \leq f''(x) \leq Kx^{-1-\delta}$ or $cx^{-2} \leq -f''(x) \leq Kx^{-1-\delta}$.*

Then the sequence $f(n) \bmod 1$ has the logarithmic discrepancy

$$L_N \leq c(c,K,\delta,H)(\log N)^{-1}.$$

Here f is said **to be of class** H if there are $1 = t_0 < t_1 < \cdots < t_H$ such that f is monotone in each of the intervals $[t_0,t_1],\dots,[t_{H-1},t_H]$, and $[t_H,\infty)$.
(IV) The above upper bound for the classical extremal discrepancy D_N of x_n was given by Y. Ohkubo (1999) by applying a version of 2.6.26 similar to (III). Ohkubo (1995) extended the Baker – Harman theorem for generalized p_n–weighted discrepancy and also for functions f which are $(i+2)$ times continuously differentiable.
(V) In 3.13.6 a multi–dimensional analogue can be found.
(VI) If α is an irrational with bounded partial quotients and

$$x_n = \alpha n + \beta(\log n)^\gamma$$

with $\gamma \geq 1$, and $\beta > 0$ then the following lower bound of its extremal discrepancy

$$D_N \geq c\frac{(\log N)^{(\gamma-1)/4}}{N^{3/4}}$$

holding for every N and with a constant $c > 0$ was proved by K. Goto and Y. Ohkubo (2004), cf. 2.6.7.

Related sequences: 2.19.9, 2.3.6, 2.3.11, 2.15.3, 2.6.7, 2.10.2.

R.C. Baker – G. Harman: *Sequences with bounded logarithmic discrepancy*, Math. Proc. Cambridge Philos. Soc. **107** (1990), no. 2, 213–225 (MR1027775 (91d:11091); Zbl. 0705.11040).
K. Goto – Y. Ohkubo: *Lower bounds for the discrepancy of some sequences*, Math. Slovaca **54** (2004), no. 5, 487–502 (MR2114620 (2005k:11153); Zbl. 1108.11054).
E. Hlawka: *Gleichverteilung und das Konvergenzverhalten von Potenzreihen am Rande des Konvergenzkreises*, Manuscripta Math. **44** (1983), no. 1–3, 231–263 (MR0709853 (85c:11060); Zbl. 0516.10030).
Y. Ohkubo: *The weighted discrepancies of some slowly increasing sequences*, Math. Nachr. **174** (1995), 239–251 (MR1349048 (96h:11074); Zbl. 0830.11028).
Y. Ohkubo: *Notes on Erdős – Turán inequality*, J. Austral. Math. Soc. A **67** (1999), no. 1, 51–57 (MR1699155 (2000d:11100); Zbl. 0940.11029).
G. Rauzy: *Étude de quelques ensembles de fonctions définis par des propertiétés de moyenne*, Séminaire de Théorie des Nombres (1972–1973), 20, Lab. Théorie des Nombres, Centre Nat. Recherche Sci., Talence, 1973, 18 pp. (MR0396463 (**53** #328); Zbl. 0293.10018).
G. Rauzy: *Propriétés statistiques de suites arithmétiques*, Le Mathématicien, Vol. 15, Collection SUP, Presses Universitaires de France, Paris 1976, 133 pp. (MR0409397 (**53** #13152); Zbl. 0337.10036).
R.F. Tichy: *Diskrepanz bezüglich gewichteter Mittel und Konvergenzverhalten von Potenzreihen*, Manuscripta Math. **44** (1983), no. 1–3, 265–277 (MR0709854 (85c:11066); Zbl. 0507.10040).
R.F. Tichy – G. Turnwald: *Logarithmic uniform distribution of* $(\alpha n+\beta \log n)$, Tsukuba J. Math. **10** (1986), no. 2, 351–366 (MR0868660 (88f:11069); Zbl. 0619.10031).

2.12.32. If f is a continuous periodical function with period T, then the sequence

$$f(\log n)$$

is

dense in the interval $[m, M]$,

where $m = \min f(x)$ and $M = \max f(x)$ both with x running over $x \in \mathbb{R}$.

NOTES: D. Andrica and S. Buzeteanu (1987, 2.6. Applications). They apply the result to sequences $\sin(\log n)$ and $\cos(\log n)$ to show that they are dense in $[-1, 1]$.

RELATED SEQUENCES: 2.6.32.

D. ANDRICA – S. BUZETEANU: *Relatively dense universal sequences for the class of continuous periodical functions of period T*, Math. Rev. Anal. Numér. Théor. Approximation, Anal. Numér. Théor. Approximation **16** (1987), no. 1, 1–9 (MR0938777 (89i:11084); Zbl. 0642.26006).

2.12.33. Let s_n be an increasing sequence of positive numbers which is multiplicatively closed and which satisfies

$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = 1.$$

Then for every continuous periodical function with period T, the sequence

$$f(\log s_n)$$

is

dense in the interval $[m, M]$,

where $m = \min f(x)$ and $M = \max f(x)$ both with x running over $x \in \mathbb{R}$.

NOTES: D. Andrica and S. Buzeteanu (1987, 4.7. Th.). Compare with 2.8.3.

RELATED SEQUENCES: 2.6.32, 2.6.34.

D. ANDRICA – S. BUZETEANU: *Relatively dense universal sequences for the class of continuous periodical functions of period T*, Math. Rev. Anal. Numér. Théor. Approximation, Anal. Numér. Théor. Approximation **16** (1987), no. 1, 1–9 (MR0938777 (89i:11084); Zbl. 0642.26006).

2.12.34. Let $q \geq 2$ be an integer. A sequence s_n, $n = 1, 2, \ldots$, of positive integers is called **extendable in the base** q if for every finite block of q–adic digits, there exists an s_n whose initial digits in q–adic digit expansion coincide with the given block. The sequence s_n is extendable in base q if and only if

$$\log_q s_n \bmod 1$$

is

$$\text{dense in } [0,1].$$

NOTES: [KN, p. 24, Exer. 2.14]. Examples:
(i) If k is a positive integer then the sequence $s_n = n^k$, $n = 1, 2, \dots$, is extendable in any base $q \geq 2$ [KN, p. 24, Exer. 2.15].
(ii) Let $q \geq 2$ and k be positive integers such that k is not a rational power of q. Then the sequence $s_n = k^n$, $n = 1, 2, \dots$, is extendable in the base q [KN, p. 24, Exer. 2.16],
(iii) The sequence $s_n = n^n$, $n = 1, 2, \dots$, is extendable in any base q [KN, p. 24, Exer. 2.17].
(iv) The sequence $s_n = F_n$, $n = 1, 2, \dots$, of Fibonacci numbers is extendable in any base q [KN, p. 31, Exer. 3.4], cf. 2.12.22.

2.13 Sequences involving trigonometric functions

See also: 2.7.4, 2.12.18, 2.12.19, 2.12.20, 2.12.32, 2.14.9

2.13.1. The sequence

$$\sin n \bmod 1$$

has the a.d.f.

$$g(x) = \frac{1}{\pi}\arcsin x + \frac{1}{2} - \frac{1}{\pi}\arcsin(1-x).$$

NOTES: The a.d.f $g(x)$ can be found transforming the u.d. sequence $n/2\pi \bmod 1$ using function $\sin 2\pi x$.

2.13.2. The sequence

$$n\theta + \sin 2\pi\sqrt{n} \bmod 1, \quad \theta \text{ irrational},$$

is

$$\text{u.d.}$$

NOTES: (cf. [KN, p. 31, Exer. 3.2])

2.13.3. The sequence

$$n^2\theta + \sin 2\pi\sqrt{n} \bmod 1, \quad \theta \text{ irrational},$$

is

u.d.

NOTES: (cf. [KN, p. 31, Exer. 3.8])

2.13.4. If α/π is irrational, then the sequence

$$x_n = n\cos(n\cos n\alpha) \bmod 1$$

is

dense

and for any non–trivial interval $I \subset [0,1]$

$$|\{n \le N\,;\ \{n\cos(n\cos n\alpha)\} \in I\}| \gg N^{2/3}.$$

If $\alpha = \frac{p}{q}\pi$, $(p,q) = 1$, then the sequence x_n is

u.d.

if q is odd. If q is even then the sequence x_n has the a.d.f.

$$g(x) = (1 - \tfrac{1}{q})x + \tfrac{1}{q}c_0(x).$$

NOTES: D. Berend, M.D. Boshernitzan and G. Kolesnik (1995, Prop. 2.3).

D. BEREND – M.D. BOSHERNITZAN – G. KOLESNIK: *Distribution modulo* 1 *of some oscillating sequences. II*, Israel J. Math. **92** (1995), no. 1–3, 125–147 (MR1357748 (96j:11105); Zbl. 0867.11052).

2.13.5. If α is real, then the sequence

$$x_n = (\log n)\cos(n\alpha) \bmod 1$$

is

dense in $[0,1]$.

NOTES:
(I) Proposition 2.4 in D. Berend, M.D. Boshernitzan and G. Kolesnik (1995). The authors also claim that it can shown that there are uncountably many α's for which this sequence is not u.d. They also note that $\log n$ can be replaced by a function from a more general class of functions having regular growth at infinity (e.g. belonging

to a Hardy field, cf. 2.6.35) but approaching infinity slower, for instance $\sqrt{\log n}$ or $\log\log n$, but that their proof fails for $(\log n)^{1+\varepsilon}$.
(II) Ch. Aistleitner, M. Hofer and M. Madritsch (2013): Let α be such that the discrepancy D_N of the sequence

$$\frac{\alpha}{2\pi}n \bmod 1, \quad n = 1, 2, \dots, N,$$

is of asymptotic order $D_N = o\big(\frac{1}{\log N}\big)$. Then the sequence x_n is u.d. in $[0, 1]$.
(III) Ch. Aistleitner, M. Hofer and M. Madritsch (2013): Let $\frac{\alpha}{2\pi} = \frac{p}{q}$, where p, q are co-prime integers, and let $N_1 < N_2 < \dots$ be a fixed integer sequence such that

$$\lim_{k\to\infty} \{\cos(\alpha i)\log N_k\} = \beta_i \quad \text{for } i = 1, \dots, q. \tag{1}$$

Then there exists a d.f. of x_n of the form $g(x) = \lim_{k\to\infty} F_{N_k}(x)$ with $F_N(x) = \frac{\{n\le N; x_n\in[0,x)\}}{N}$ given by

$$g(x) = \frac{1}{q}\sum_{i=1}^{q} h_{q,\beta_i,c_i}(x), \tag{2}$$

where

$$h_{q,\beta_i,c_i}(x) = \begin{cases} f_{\beta_i,c_i}(x + 1 - \nu_i) - f_{\beta_i,c_i}(1 - \nu_i), & \text{if } 0 \le x \le \nu_i \text{ and } c_i > 0,\\ f_{\beta_i,c_i}(x - \nu_i) + 1 - f_{\beta_i,c_i}(1 - \nu_i), & \text{if } \nu_i \le x \le 1 \text{ and } c_i > 0,\\ f_{\beta_i,c_i}(x + \nu_i) - f_{\beta_i,c_i}(\nu_i), & \text{if } 0 \le x \le 1 - \nu_i \text{ and } c_i < 0,\\ f_{\beta_i,c_i}(x - (1 - \nu_i)) + 1 - f_{\beta_i,c_i}(\nu_i), & \text{if } 1 - \nu_i \le x \le 1 \text{ and } c_i < 0,\\ \mathbf{1}_{\{(0,1]\}}(x), & \text{if } c_i = 0, \end{cases}$$

with

$$f_{\beta,c}(x) = \begin{cases} g_{\beta,c}(x), & \text{if } c > 0,\\ 1 - g_{\beta,|c|}(1 - x), & \text{if } c < 0,\\ \mathbf{1}_{\{(0,1]\}}(x), & \text{if } c = 0, \end{cases}$$

and

$$g_{\beta,c}(x) = \frac{e^{\frac{\min(x,\beta)}{c}} - 1}{e^{\frac{\beta}{c}}} + \frac{1}{e^{\frac{\beta}{c}}}\frac{e^{\frac{x}{c}} - 1}{e^{\frac{1}{c}} - 1},$$

and $\nu_i = \{|c_i|\log(q)\}$, $c_i = \cos(\alpha i)$. Moreover, the set $G(x_n)$ is the set of all d.f.'s of the form (2) for those $(\beta_1, \dots, \beta_q)$ for which a subsequence $(N_k)_{k\ge1}$ satisfying (1) exists.
The authors also note that for an arbitrary q, it is a difficult problem to determine all possible vectors $(\beta_1, \dots, \beta_q)$ for which there exists a sequence $N_1 < N_2 < \dots$ such that (1) holds, due to the fact that there can exist non-trivial linear relations between the values $\cos(\alpha i)$, $i = 1, \dots, q$ (cf. K. Gristmair (1997)).

Ch. Aistleitner – M. Hofer – M. Madritsch: *On the distribution functions of two oscillating sequences*, Unif. Distrib. Theory **8** (2013), no. 2, 157–169 (MR3155465; Zbl. 1313.11087).
D. Berend – M.D. Boshernitzan – G. Kolesnik: *Distribution modulo 1 of some oscillating sequences. II*, Israel J. Math. **92** (1995), no. 1–3, 125–147 (MR1357748 (96j:11105); Zbl. 0867.11052).
K. Girstmair: *Some linear relations between values of trigonometric functions at $k\pi/n$*, Acta Arith. **81** (1997), no. 4, 387–398 (MR1472818 (98h:11133); Zbl. 0960.11048).

2.13.6. The sequence

$$(\cos n)^n$$

is

$$\text{dense in } [-1,1].$$

Notes:
(I) The original problem posed by M. Bencze and F. Popovici (1996) was solved by J. Bukor (1997) and as a consequence of a Diophantine approximation lemma by F. Luca (1999).
(II) S. Hartman (1949) proved that if $\frac{\alpha}{\pi}$ is irrational, then

$$\liminf_{n\to\infty}(\cos\alpha n)^n = \liminf_{n\to\infty}(\sin\alpha n)^n = -1.$$

(III) Ch. Aistleitner, M. Hofer and M. Madritsch (2013): Let

$$x_n = \cos(\alpha n)^n \bmod 1, \quad n = 1,2,\dots.$$

If $\frac{\alpha}{2\pi}\notin\mathbb{Q}$ put $a = 3/4$, and if $\frac{\alpha}{2\pi} = \frac{p}{q}\in\mathbb{Q}$ with p,q co-prime integers, let

$$a = \begin{cases} \frac{q+1}{2q} + \frac{q-1}{4q}, & \text{if } 4 \mid (q-1),\\ \frac{q-1}{2q} + \frac{q+1}{4q}, & \text{if } 4 \nmid (q-1) \end{cases}$$

for q odd and let

$$a = \begin{cases} \frac{1}{2} + \frac{q-2}{4q}, & \text{if } 4\nmid q \text{ and } 8 \mid (q-2),\\ \frac{1}{2} + \frac{q+2}{4q}, & \text{if } 4\nmid q \text{ and } 8 \nmid (q-2),\\ \frac{q+2}{2q} + \frac{1}{4}, & \text{if } 4\mid q \text{ and } 8\nmid q,\\ \frac{q+2}{2q} + \frac{q-4}{4q}, & \text{if } 8 \mid q \end{cases}$$

for q even. Then the a.d.f. of x_n is given by

$$g_a(x) = \begin{cases} 0, & \text{if } x = 0,\\ a, & \text{if } 0 < x < 1,\\ 1, & \text{if } x = 1. \end{cases}$$

Ch. Aistleitner – M. Hofer – M. Madritsch: *On the distribution functions of two oscillating sequences*, Unif. Distrib. Theory **8** (2013), no. 2, 157–169 (MR3155465; Zbl. 1313.11087).
M. Bencze – F. Popovici: *OQ. 45*, Octogon Math. Mag.(Brasov) **4** (1996), 77
J. Bukor: *On a certain density problem*, Octogon Mathematical Magazine (Brasov) **5** (1997), no. 2, 73–75.
Quoted in: 2.13.6
S. Hartman: *Sur une condition supplémentaire dans les approximations diophantiques*, Colloq. Math. **2** (1949), no. 1, 48–51 (MR0041174 (12,807a); Zbl. 0038.18802).
F. Luca: $\{(\cos(n))^n\}_{n\geq 1}$ *is dense in* $[-1,1]$, Bull. Math. Soc. Sci. Math. Roum., Nouv. Sér. **42(90)** (1999), no. 4, 369–376 (MR1879621 (2002k:11118); Zbl. 1053.11529).

2.13.6.1 The sequence

$$P(n)\cos n\alpha \bmod 1, \quad n=1,2,\dots,$$

is

completely u.d.

for any non-constant polynomial $P(x)$ and any α such that $\cos\alpha$ is transcendental.

Notes:
(I) D. Berend and G. Kolesnik (2011).
(II) If $\cos\alpha$ is not transcendental Berend and Kolesnik (2011) proved: Let α be such that $e^{i\alpha}$ is either a transcendental number or an algebraic number of degree d which is not a root of unity. Then the sequence

$$\big(P(n)\cos n\alpha, P(n+1)\cos(n+1)\alpha,\dots,P(n+d-1)\cos(n+d-1)\alpha\big) \bmod 1, \qquad n=1,2,\dots, \tag{1}$$

is

u.d.

for any non-constant polynomial $P(x)$.
(III) **Open problem** (Berend and Kolesik (2011)): Let $P(x)=x$, $\alpha=\arccos 3/5$, i.e. $e^{i\alpha}=(3+4i)/5$. If $x_n=P(n)\cos n\alpha = n\frac{(3+4i)^n-(3-4i)^n}{2.5^n}$ then (1) implies that the sequence (x_n,x_{n+1}) mod 1 is u.d., but the authors showed that $(x_n,x_{n+1},x_{n+2},x_{n+3},x_{n+4})$ mod 1 is not u.d. They ask whether the sequences (x_n,x_{n+1},x_{n+2}) mod 1 and $(x_n,x_{n+1},x_{n+2},x_{n+3})$ mod 1) are u.d.

D. Berend – G. Kolesnik: *Complete uniform distribution of some oscillating sequences*, J. Ramanujan Math. Soc. **26** (2011), no. 2, 127–144 (MR2815328 (2012e:11134); Zbl. 1256.11041).

2.13.7. The sequence

$$x_n=\cos(n+\log n) \bmod 1, \quad n=1,2,\dots$$

is

$$\text{not u.d.}$$

NOTES:
(I) L. Kuipers (1953).
(II) S. Steinerberger (2012) proved that the sequence x_n has the same a.d.f $g(x)$ as the sequence $\cos n \bmod 1$, as follows: The sequence
$\cos(n+\log n) = \cos 2\pi\left(\frac{n}{2\pi} + \frac{1}{2\pi}\log n\right) = \cos 2\pi z_n$, where $z_n = (\frac{n}{2\pi} + \frac{1}{2\pi}\log n) \bmod 1$ is u.d., since both $\frac{n}{2\pi}$ and $\frac{n}{2\pi} + \frac{1}{2\pi}\log n$ are u.d. mod 1 simultaneously, see 2.3.6. Put $f(x) = \cos 2\pi x \bmod 1$. Then a.d.f. $g(x)$ of x_n is

$$g(x) = |f^{-1}([0,x))| = \frac{1}{2} - \frac{1}{\pi}\arccos x + 1 - \frac{1}{\pi}\arccos(x-1).$$

L. KUIPERS: *Continuous and discrete distribution modulo* 1, Indag. Math. **15** (1953), 340–348 (MR0058690 (15,410e); Zbl. 0051.28601).(=Nederl. Akad. Wetensch., Proc. **56** (1953), 340–348).
S. STEINERBERGER: *Solution to the Problem 1.10(iii)*, in: *Unsolved Problems of the journal Uniform Distribution Theory as of June 10, 2012*, (O. Strauch ed.), p. 22 (`http://www.boku.ac.at/MATH/udt/unsolvedproblems.pdf`).

2.13.8. The sequence

$$(\sqrt{n} + \sin n) \bmod 1$$

is

$$\text{u.d.}$$

L. KUIPERS: *Continuous and discrete distribution modulo* 1, Indag. Math. **15** (1953), 340–348 (MR0058690 (15,410e); Zbl. 0051.28601).(=Nederl. Akad. Wetensch., Proc. **56** (1953), 340–348).

2.13.9. Open problem. Characterize the values B, r, x and α for which the sequence

$$Br^n \cos(nx - \alpha) \bmod 1$$

is

$$\text{u.d.}$$

NOTES: (I) W.J. LeVeque (1953). He investigated more general form

$$f_n(x) = u_n f(v_n x - \alpha),$$

where u_n and v_n are sequences of real numbers and f is a periodic function of period ω and $0 \le x \le \omega$. He proved only metric results (e.g. the complex sequence $z^n \bmod 1$ is u.d. for almost all complex z with $|z| > 1$).
(II) B. Reznick (1999) studied the sequence

$$x_n = \left|r^n \sin(\pi(n\theta - \beta))\right|$$

and proved:

(i) The sequence x_n increases and decreases infinitely often (indeed each event occurs with a positive density) unless the following conditions are met: $\theta = \frac{k}{m}$, $\gcd(k, m) = 1$, $m\beta \notin \mathbb{Z}$, and r is sufficiently large or sufficiently small. In this case x_n is monotone increasing or monotone decreasing.

(ii) The sequence

$$\frac{x_{n+1}}{x_n} = \left| r\left(\cos(\pi\theta) - \sin(\pi\theta)\cos(\pi(n\theta - \beta))\right)\right|$$

has the a.d.f.

$$g(x) = \frac{1}{\pi}\cot^{-1}\left(\frac{r^2 - x^2}{2r(\sin(\pi\theta))x}\right)$$

defined on $(-\infty, \infty)$.

W.J. LeVeque: *The distribution modulo 1 of trigonomertic sequences*, Duke Math. J. **20** (1953), 367–374 (MR0057925 (15,293d); Zbl. 0051.28504).
B. Reznick: *On the monotonicity of* $(|Im(z^n)|)$, J. Number Theory **78** (1999), no. 1, 144–148 (MR1706901 (2001a:11134); Zbl. 0935.11027).

2.13.10. The sequence

$$x_n = \int_1^n \left(\int_0^x \frac{\sin y}{y}\,\mathrm{d}y\right)\frac{\mathrm{d}x}{\sqrt{x}} \bmod 1$$

is

u.d.

Notes: L. Kuipers (1953) applied 2.6.12.

L. Kuipers: *Continuous and discrete distribution modulo* 1, Indag. Math. **15** (1953), 340–348 (MR0058690 (15,410e); Zbl. 0051.28601).(=Nederl. Akad. Wetensch., Proc. **56** (1953), 340–348).

2.13.11. The sequence

$$x_n = \left(\sqrt{n} + \sin\frac{1}{n}\right) \bmod 1$$

is

u.d.

Notes: L. Kuipers (1953) applied 2.6.11, but the result follows from the fact that $\sin\frac{1}{n} \to 0$ and that $\sqrt{n} \bmod 1$ is u.d.

L. Kuipers: *Continuous and discrete distribution modulo* 1, Indag. Math. **15** (1953), 340–348 (MR0058690 (15,410e); Zbl. 0051.28601).(=Nederl. Akad. Wetensch., Proc. **56** (1953), 340–348).

2.13.12. The **square–root spiral** is a piecewise linear planar path W with vertices $P_0, P_1, \dots$, which polar coordinates (x, φ) are of the form $P_n = (x_n, \varphi_n)$, where $P_0 = (0,0)$, and

$$x_n = \sqrt{n}, \qquad \varphi_n = \sum_{j=1}^{n-1} \arctan(1/\sqrt{j})$$

for $n = 1, 2, \dots$. Let

- $f(n) = an^2 + bn + c_n$ be integer valued (i.e. $f(n) \in \mathbb{N}$ for $n = 1, 2, \dots$) with c_n bounded, $a > 0$, and $\sqrt{a}$ not a rational multiple of π,
- $S_0, S_1, \dots$ be the successive points of the intersection of W with a fixed ray $\varphi = \alpha$, $0 \le \alpha < 2\pi$, emanating from P_0,
- $P_{h(n)}P_{h(n)+1}$ be the segment of W containing the point S_n, and
- $|PQ|$ be the length of the segment PQ.

Then for the corresponding sequences we have

(i) φ_n is u.d. mod 2π with discrepancy $D_N = \mathcal{O}(1/\sqrt{N})$,

(ii) $\varphi_{1+k} - \varphi_1, \varphi_{2+k} - \varphi_2, \dots$ is u.d. mod 2π with discrepancy $D_N = \mathcal{O}(\sqrt{N+k}/N)$,

(iii) $\varphi_{f(n)}$, $n = 1, 2, \dots$, is u.d. mod 2π,

(iv) $(\varphi_{np_1}, \dots, \varphi_{np_s})$ for indices $n = 1, 2, \dots$, and distinct primes $p_1, \dots, p_s$ is u.d. mod 2π,

(v) $|P_0S_n|$ is u.d. mod 1,

(vi) $|P_{h(n)}S_n|$ is u.d. mod 1,

(vii) $h(n) = \pi^2 n^2 + (\alpha + d)\pi n + d_n$ with d_n bounded and u.d. mod 1, while the constant d does not depend on α.

Notes: The u.d. mod 2π of φ_n was proved by W. Ness (1966) and the discrepancy bound in (i) was found by E. Hlawka (1980). He also proved (ii) and (iv). The results (iii), (v), (vi) and (vii) were proved by E. Teuffel (1981) who also proved in Teuffel (1958) that the equation $\varphi_{n+k} - \varphi_n = j\pi$ cannot be solved in positive integers n, k, j.

E. Hlawka: *Gleichverteilung und Quadratwurzelschnecke*, Monatsh. Math. **89** (1980), no. 1, 19–44 (MR0566292 (81h:10069); Zbl. 0474.68092).
W. Ness: *Ein elementargeometrisches Beispiel für Gleichverteilung*, Praxis Math. **8** (1966), 241–243.(Zbl. 0289.50009)
E. Teuffel: *Ein Eigenschaft der Quadratwurzelschnecke*, Math. – Phys. Semesterber. **6** (1958), 148–152 (MR0096160 (**20** #2655); Zbl. 0089.00803).

E. Teuffel: *Einige asymptotische Eigenschaften der Quadratwurzelschnecke*, Math. Semesterber. **28** (1981), no. 1, 39–51 (MR0611459 (82j:10085); Zbl. 0464.10025).

2.14 Sequences involving polynomials

2.14.1. Let $p(x)$ be a polynomial with real coefficients. Then the sequence

$$p(n) \bmod 1, \quad n = 1, 2, \dots,$$

is

u.d.

if and only if the polynomial $p(x) - p(0)$ has at least one irrational coefficient.

Notes: (I) This fundamental result was proved by H. Weyl (1914), (1916), [KN, p. 27, Th. 3.2]. The weaker case $n^k\theta \bmod 1$ was studied earlier by G.H. Hardy and J.E. Littlewood (1914).

(IIA) A complicated bound of discrepancy was found by I.M. Vinogradov (1926) of which a more shapely form can be found in [1947, Chapt. VIII]: Let $p(x) = \alpha_{k+1}x^{k+1} + \cdots + \alpha_1 x$ be a polynomial with real coefficients $\alpha_{k+1}, \dots, \alpha_1$, $k \geq 11$, and let for some index s, $1 \leq s \leq k+1$, we have

$$\alpha_s = \frac{a}{q} + \frac{\theta}{q^2}; \quad (a, q) = 1, |\theta| < 1.$$

Then $D_N = \mathcal{O}(N^{-\rho})$, where

$$\rho = \frac{\tau}{3k^2 \log \frac{12k(k+1)}{\tau}}$$

and τ is defined for given constants c_1 and c_2 (e.g. $c_1 = c_2 = 1$) by relations

$$\begin{cases} q = c_1 N^{\tau}, & \text{if } 1 < q \leq c_1 N; \\ \tau = 1, & \text{if } c_1 N \leq q \leq c_2 N^{s-1}; \\ q = c_2 N^{s-\tau}, & \text{if } c_2 N^{s-1} \leq q < c_2 N^s, \end{cases}$$

and $\tau \geq \tau_0$ for some fixed sufficiently small positive τ_0.

(IIB) Concerning the discrepancy J.G. van der Corput and Ch. Pisot (1939) proved: Let $p(x) = \alpha\frac{x^k}{k!} + \alpha_1 x^{k-1} + \cdots + \alpha_k$ be a polynomial of degree $k \geq 1$ with real coefficients α_i and let $\left|\alpha - \frac{a}{q}\right| \leq \frac{\tau}{q^2}$, where $\tau \leq 1$ and $\frac{a}{q}$ is an irreducible fraction with $q > 0$. Then the extremal discrepancy D_N of $p(1), p(2), \dots, p(N) \bmod 1$ $(N \geq 3)$ satisfies

$$D_N \leq c(\log N)^{\omega} \left(\left(\tau + \frac{q}{N}\right) \left(\frac{1}{q} + \frac{1}{N^{k-1}}\right) \right)^{\frac{1-\varepsilon}{2^{k-1}}}$$

for any $\varepsilon > 0$, where c is a constant and ω depends only on k and ε.
(IIC) Yu.V. Linnik (1943) proved: If $p(x) = a_0x^k + \cdots + a_k$ is a polynomial with integral coefficients and $\alpha = \frac{a}{q} + \frac{\theta}{q^2}$, $(a, q) = 1$, $|\theta| < 1$, $N \le q < N^{k-1}$, then

$$\left|\sum_{n=1}^{N} e^{2\pi\alpha p(n)}\right| \le c.N^{1-\frac{1}{22400k^2 \log k}}.$$

(IID) M. Weber noticed that the bounds (IIA), (IIB) and (IIC) does not give a good discrepancy bound for the sequence $n^2\alpha \bmod 1$, $n = 1, 2, \dots, N$. However, using the step by step method described in [KN, pp. 122–125] for computation of the extremal discrepancy $D_N(n\alpha)$ and a quantitative version of the van der Corput difference theorem [KN, p. 165, Th. 6.2] it is possible to prove (O. Strauch): If α is an algebraic irrational then $D_N(n^2\alpha) = \mathcal{O}(N^{-(1/6)+\varepsilon})$ for every $\varepsilon > 0$.
(III) Let θ be an irrational number and $k > 1$ a positive integer. The sequence $n^k\theta \bmod 1$, $n = 1, 2, \dots, N$, induces a partition of $[0, 1]$ into intervals $I_0, I_1, \dots, T_j$ and let $T_k(N)$ denote that number of distinct lengths that these intervals can assume. In contrast to the Steinhaus three–gaps theorem for $n\theta \bmod 1$ (cf. 2.8.1) V. Drobot (1987) showed that for $k > 1$, $T_k(N) \to \infty$ as $N \to \infty$, more precisely that

$$T_k(N) \ge Ne^{-(1+\varepsilon)\log 2^k \frac{\log N}{\log\log N}}$$

for $N \ge N(\varepsilon)$.
(IV) The well distribution of this sequence was proved by B. Lawton (1959, Th. 2) and M. Mendès France (1967, p. 14).
(V) If $p(x) = \alpha_kx^k + \alpha_{k-1}x^{k-1} + \cdots + \alpha_0$ is a polynomial with real coefficients α_i, then the sequence $p(n)$, $n = 1, 2, \dots$, is u.d. in $\mathbb{R}$ (for def. cf. p. 1 − 6) if and only if the system $\alpha_k, \alpha_{k-1}, \dots, \alpha_1$ is of rank at least two over the rationals (cf. [KN, p. 284]).

Related sequences: For the u.d. of $p(p_n) \bmod 1$ where n is replaced by the nth prime p_n consult 2.19.4, 3.8.3.

V. Drobot: *Gaps in the sequence $n^2\theta$* (mod 1), Internat. J. Math. Sci. **10** (1987), no. 1, 131–134 (MR0875971 (88e:11068); Zbl. 0622.10026).
G.H. Hardy – J.E. Littlewood: *Some problems of Diophantine approximation I: The fractional part $n^k\theta$*, Acta Math. **37** (1914), 155–191 (MR1555098; JFM 45.0305.03).
B. Lawton: *A note on well distributed sequences*, Proc. Amer. Math. Soc. **10** (1959), 891–893 (MR0109818 (**22** #703); Zbl. 0089.26902).
Yu.V. Linnik: *On Weyl's sums*, Rec. Math. [Mat. Sbornik] N.S. **12(54)** (1943), 28–39 (MR0009776 (5,200a); Zbl. 0063.03578).
M. Mendès France: *Nombres normaux. Applications aux fonctions pseudo–aléatoires*, J. Analyse Math. **20** (1967), 1–56 (MR0220683 (**36** #3735); Zbl. 0161.05002).
J.G. van der Corput – C. Pisot: *Sur la discrépance modulo un. (Deuxème communication)*, Nederl. Akad. Wetensch., Proc. **42** (1939), 554–565 (MR0000395 (1,66b); JFM 65.0170.02; Zbl. 0022.11604). (=Indag. Math. **1** (1939), 184–195).
I.M. Vinogradov: *On fractional parts of integer polynomials*, (Russian), Izv. AN SSSR **20** (1926), 585–600 (JFM 52.0182.03).

I.M. Vinogradov: *The Method of Trigonometrical Sums in the Theory of Numbers*, (Russian), Trav. Inst. Math. Stekloff, Vol. 23, (1947) (MR0029417 (10,599a); Zbl. 0041.37002) *Translated, revised and annotated by K.F. Roth and A. Davenport*, Interscience Publishers, London, New York, 1954 (MR0062183 (15,941b); Zbl. 0055.27504).
H. Weyl: *Über ein Problem aus dem Gebiet der diophantischen Approximationen*, Nachr. Ges. Wiss. Göttingen, Math.–phys.Kl. (1914), 234–244 (JFM 45.0325.01).
H. Weyl: *Über die Gleichverteilung von Zahlen mod. Eins*, Math. Ann. **77** (1916), 313–352 (JFM 46.0278.06).

2.14.2. Let $p(z) = a_0 + \cdots + a_N z^N$ be a polynomial of degree N with complex coefficients which satisfies $|p(z)| \le M_N$ on $|z| = 1$. If $z_1, \dots, z_N$ are its roots then the finite sequence

$$x_n = \frac{\arg z_n}{2\pi}, \quad n = 1, \dots, N,$$

has discrepancy

$$D_N(x_n) \le \frac{16}{\sqrt{N}} \sqrt{\log \frac{M_N}{\sqrt{|a_0 a_N|}}}$$

and more precisely

$$D_N(x_n) \le 13 \max\left(1, \log \frac{2N}{\log C_N}\right) \frac{\log C_N}{N},$$

with $C_N = \max(M_N, B_N, N)$, where B_N is such that $\max_{1\le j\le N} |p'(z_j)| \ge 1/B_N$, and $M_N, B_N > 1$.

Notes: (I) The first bound is due to P. Erdős and P. Turán (1948, 1950). They pointed out that a similar result cannot hold in terms $M_N(\theta)$ (where $M_N(\theta)$ denotes the upper bound of $|p(z)|$ on $|z| = \theta$, where θ is fixed and such that $0 < \theta < 1$), and that it does hold if it is further postulated that all the roots of $p(z)$ are outside $|z| = 1$. If $M_N(\theta) = \sqrt{a_0 a_N}\, e^{\frac{N}{g(N,\theta)}}$, and $N \ge g(N, \theta) \ge 2$, then

$$D_N(x_n) \le c \frac{\log(4\theta^{-1})}{\log g(N, \theta)},$$

where c is a numerical constant. Erdős and Turán (1950) showed that u.d. of x_n implies two known theorems: E. Schmidt's Theorem on the maximum number of real roots and Szegő's one on the u.d. of the roots of partial sums of a power series whose radius of convergence is 1.

(II) The Erdős – Turán result allows the following reformulation: Let $p(z)$ be a monic polynomial of degree N all of whose zeros z_n lie in $[-1, 1]$ and let $\max_{z\in[-1,1]} |p(z)| \le$

$A_N/2^N$. Then the extremal discrepancy $D_N(z_n)$ of $z_1, \dots, z_N$ with respect to the d.f. (called the **arcsine** or **equilibrium measure** on $[-1,1]$)

$$g(x) = \frac{1}{\pi}\int_{-1}^{x} \frac{\mathrm{d}t}{\sqrt{1-t^2}},$$

i.e.

$$D_N(z_n) = \sup_{[x,y)\subset[-1,1]} \left| \frac{A([x,y);N;z_n)}{N} - \big(g(y)-g(x)\big) \right|$$

satisfies

$$D_N(z_n) \le \frac{8}{\log 3}\sqrt{\frac{\log A_N}{N}}.$$

H.–P. Blatt (1992) improved this to

$$D_N(z_n) \le c\log C_N \frac{\log N}{N}$$

with $c > 0$ a constant, $C_N = \max(A_N, B'_N, N)$ where $|p'(z_n)| \ge 1/(2^N B'_N)$ for $n = 1, 2, \dots, N$. Definite improvement was given by V. Totik (1993)

$$D_N(z_n) \le c\frac{\log C_N}{N}\log\left(\frac{N}{\log C_N}\right).$$

H.–P. Blatt and H.N. Mhaskar (1993) extended this to monic polynomials with zeros on a smooth Jordan arc and V.V. Andrievskii, H.–P. Blatt and H.N. Mhaskar (2001) studied distribution of zeros of a class of orthogonal polynomials, which includes the so–called Pollaczek polynomials.
(III) The second estimate for $D_N(x_n)$ if x_n is the sequence given above was found by F. Amoroso and M. Mignotte (1996). They also proved an upper estimate for $\max_{|z|=1}|p(z)|$ in terms of $D_N(x_n)$:

$$\log\left(\max_{|z|=1}|p(z)|\right) \le ND_N(x_n)\left(3 + \log\frac{1}{D_N(x_n)}\right)$$

provided the polynomial $p(z)$ with complex coefficients is such that $p(0) = 1$ and all its zeros are on the unit circle.
(IV) Let $a = b + c$, where a, b, c are coprime positive integers. A. Borisov (1998) defined the abc**-polynomials** by

$$f_{abc}(x) = \frac{bx^a - ax^b + c}{(x-1)^2}.$$

An application of the above mentioned Erdős – Turán theorem to $f_{abc}(x)$ yields that

$$D_N(x_n) \le 12\sqrt{\frac{\log(N+1)}{N}},$$

where $N = a - 2 = \deg f_{abc}(x)$.
(V) P. Borwein, T. Erdélyi and G. Kós (1999) proved: There is an absolute constant $c > 0$ such that every polynomial $p(z) = \sum_{j=0}^{N} a_j z^j$, with $|a_j| \le 1$, $|a_0| = 1$, and $a_j \in \mathbb{C}$, has (i) at most $cN|I|$ zeros on a subarc I of the length $|I|$ of the unit circle if $|I| \ge 1/\sqrt{N}$, while (ii) it has at most $c\sqrt{N}$ zeros if $|I| \le 1/\sqrt{N}$. Here the length is normalized so that the unit circle has length 1. The bounds are essentially sharp.

F. Amoroso – M. Mignotte: *On the distribution of the roots of polynomials*, Ann. Inst. Fourier (Grenoble) **46** (1996), no. 5, 1275–1291 (MR1427125 (98h:11101); Zbl. 0867.26009).
V.V. Andrievskii – H.–P. Blatt – H.N. Mhaskar: *A local discrepancy theorem*, Indag. Mathem., N.S. **12** (2001), no. 1, 23–39 (MR1908137 (2003g:11084); Zbl. 1013.42017).
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A. Borisov: *On some polynomials allegedly related to the abc conjecture*, Acta Arith. **84** (1998), no. 2, 109–128 (MR1614326 (99f:11140); Zbl 0903.11025).
P. Borwein – T. Erdélyi – G. Kós: *Littlewood–type problems on* $[0,1]$, Proc. London Math. Soc., III. Ser. **79** (1999), no. 1, 22–46 (MR1687555 (2000c:11111); Zbl. 1039.11046).
P. Erdős – P. Turán: *On a problem in the theory of uniform distribution I, II*, Nederl. Akad. Wetensch., Proc. **51** (1948), 1146–1154, 1262–1269 (MR0027895 (10,372c); Zbl. 0031.25402; MR0027896 (10,372d); Zbl. 0032.01601).(=Indag. Math. **10** (1948), 370–378, 406–413).
P. Erdős – P. Turán: *On the distribution of roots of polynomials*, Ann. of Math. (2) **51** (1950), 105–119 (MR0033372 (**11**,431b); Zbl. 0036.01501).
V. Totik: *Distribution of simple zeros of polynomials*, Acta Math. **170** (1993), no. 1, 1–28 (MR1208561 (95i:41011); Zbl. 0888.41003).

2.14.3. Let $F(z) = a\prod_{k=1}^{n}(z - r_k e^{i\phi_k})$ $(0 \le \phi_k < 2\pi)$ be a separable polynomial with integer coefficients of degree n. If $|\log a| \le \delta n$, and $|r_k - 1| \le \varepsilon$, for $k = 1, 2, \dots, n$, then the finite sequence of the arguments of roots

$$\phi_1, \phi_2, \dots, \phi_n$$

in the interval $[0, 2\pi)$ has the discrepancy

$$D_n \le c\sigma,$$

where $c > 0$ is an absolute constant, and

$$\sigma = \max\left(\frac{\log(n+1)}{\sqrt{n}}, \sqrt{\delta \log\frac{1}{\delta}}, \sqrt{\varepsilon \log\frac{1}{\varepsilon}}\right)$$

Ju.F. Belotserkovskij (Bilu): *Uniform distribution of algebraic numbers near the unit circle*, Vestsi Akad. Navuk BSSR Ser. Fiz.–Mat. Navuk **1988** (1988), no. 1, 49–52, 124 (MR0937893 (89f:11110); Zbl. 0646.10040).

2.14.4. Let

$$A_n = (x_{n,1}, x_{n,2}, \dots, x_{n,n}), \quad -1 < x_{n,i} < 1,$$

be the sequence of the all roots of the nth Legendre polynomial $P_n(x)$. Then the sequence of single blocks A_n has the a.d.f.

$$g(x) = 1 - \tfrac{1}{\pi} \arccos x$$

with respect to $[-1, 1]$.
NOTES: That is,

$$g(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} c_{[-1,x)}(x_{n,i}) = 1 - \frac{1}{\pi} \arccos x$$

for $x \in [-1, 1]$, cf. G. Pólya and G. Szegő (1964, Part II, Ex. 194).

G. PÓLYA – G. SZEGŐ: *Aufgaben und Lehrsätze aus der Analysis*, Vol. 1 & 2, 3rd corr. ed., Grundlehren d. math. Wiss., Band 19, Springer Verlag, Berlin, Göttingen, Heidelberg, New York, 1964 (MR0170985 (**30** #1219a); MR0170986 (**30** #1219b); Zbl. 0122.29704).

2.14.5. Let $\theta \in (\sqrt{2}, 2)$ be such that θ^2 is not a root of a polynomial with coefficients from $\{-1, 0, 1\}$. Then the block sequence $A(\theta) = (A_n)_{n=1}^{\infty}$ with the nth block

$$A_n = \left(\sum_{i=0}^{n} a_i \theta^i \; ; \; a_i \in \{-1, 1\} \right)$$

is

dense in $\mathbb{R}$.

NOTES: Y. Peres and B. Solomyak (2000). They also proved that the set of all such $\theta \in (\sqrt{2}, 2)$ for which $A(\theta)$ is dense is of the full measure and that it is a residual subset of the interval $(\sqrt{2}, 2)$. Note that the sequence cannot be dense if $\theta \notin (\sqrt{2}, 2)$.

Y. PERES – B. SOLOMYAK: *Approximation by polynomials with coefficients* ± 1, J. Number Theory **84** (2000), no. 2, 185–198 (MR1795789 (2002g:11107); Zbl. 1081.11509).

2.14.6. Let k and l be positive integers such that $k \geq l$, $l \geq 80^2$, and p be a given prime number and $q(x) = a_1 x + \cdots + a_l x^l$ be a polynomial with integral coefficients $a_1, \dots, a_l$ and $(a_l, p) = 1$. Let $N = \left[p^{\alpha k}\right] + 1$ for an α which satisfies $\sqrt{\log l / l} \leq \alpha \leq 1$. Then the discrepancy of the finite sequence

$$\frac{q(n)}{p^k} \bmod 1, \quad n = 1, \dots, N,$$

can be estimated by

$$D_N^* = \mathcal{O}\left(N^{1-\frac{1}{cl}}\right),$$

where c is an absolute constant and the $\mathcal{O}$–constant depends only on k and l.

A.A. Karacuba (A.A. Karatsuba): *Distribution of fractional parts of polynomials of a special type* (Russian), Vestnik Moskov. Univ., Ser. I Mat., Mech. (1962), no. 3, 34–39 (MR0138613 (**25** #2056); Zbl 0132.03304).

2.14.7. Let $P(x) = \gamma_1 x^{\alpha_1} + \cdots + \gamma_q x^{\alpha_q}$ be a generalized polynomial with γ_i, α_i, $i = 1, 2, \dots, q$, real. Then the sequence

$$P(n) \bmod 1, \quad n = 1, 2, \dots,$$

is

u.d.

if and only if one of the α's is not integral or if one of the γ's is irrational.

Notes: D.P. Parent (1984, pp. 285–286, Solution 5.23).

Related sequences: 2.6.28, 2.14.8, 2.14.9.

D.P. Parent: *Exercises in Number Theory*, Problem Books in Mathematics, Springer Verlag, New York, 1984 (MR0759342 (86f:11002); Zbl. 0536.10001). (French original: *Exercices de théorie des nombres*, Gauthier – Villars, Paris, 1978 (MR0485646 (**58** #5471); Zbl. 0387.10001)).

2.14.8. Let $P(x) = \gamma_1 x^{\alpha_1} + \cdots + \gamma_q x^{\alpha_q}$ be a generalized polynomial with real $\gamma_i > 0$ and $\alpha_i \in (0, 1)$ for $i = 1, 2, \dots, q$. If f is a continuous periodical function with period T, then the sequence

$$f(P(n))$$

is

dense in the interval $[m, M]$,

where $m = \min f(x)$ and $M = \max f(x)$ both with x running over $x \in \mathbb{R}$.

Notes: D. Andrica and S. Buzeteanu (1987, 2.6. Applications). They also mention that the problem of density of $\sin(P(n))$ in $[-1, 1]$ for $P(x) = \gamma_1 x^{\alpha_1} + \cdots + \gamma_q x^{\alpha_q} + \gamma_0$ with $\gamma_q \neq 0$, $\alpha_i > 1$ and at least one α_i irrational, is open, see 2.14.9.

Related sequences: 2.6.28.

D. Andrica – S. Buzeteanu: *Relatively dense universal sequences for the class of continuous periodical functions of period T*, Math. Rev. Anal. Numér. Théor. Approximation, Anal. Numér. Théor. Approximation **16** (1987), no. 1, 1–9 (MR0938777 (89i:11084); Zbl. 0642.26006).

2.14.9. Let $P(x) = \gamma_1 x^{\alpha_1} + \cdots + \gamma_q x^{\alpha_q} + \gamma_0$ be a generalized polynomial with real coefficients $\gamma_1, \dots, \gamma_q, \gamma_0$, $\gamma_q \neq 0$, and non–zero rational exponents $\alpha_1, \dots, \alpha_q$ such that at least one of the numbers $\gamma_1/\pi, \dots, \gamma_q/\pi$ is irrational. Then the sequences

$$\sin(P(n)) \quad \text{and} \quad \cos(P(n))$$

are

$$\text{dense in } [-1, 1]$$

and

$$\tan(P(n))$$

is

$$\text{dense in } \mathbb{R}.$$

Notes: D. Andrica and S. Buzeteanu (1987, 3.12. Applications).

Related sequences: 2.14.8.

D. Andrica – S. Buzeteanu: *Relatively dense universal sequences for the class of continuous periodical functions of period T*, Math. Rev. Anal. Numér. Théor. Approximation, Anal. Numér. Théor. Approximation **16** (1987), no. 1, 1–9 (MR0938777 (89i:11084); Zbl. 0642.26006).

2.15 Power sequences

2.15.1. The sequence

$$\alpha n^\sigma \bmod 1,\ \alpha \neq 0,\ \sigma > 0, \quad \text{where } \sigma \text{ is not an integer,}$$

is

$$\text{u.d.}$$

If moreover $\alpha > 0$ and $0 < \sigma < 1$ then

$$D_N = \mathcal{O}(N^{\tau-1}), \qquad \text{where } \tau = \max(\sigma, 1-\sigma).$$

Notes: (I) The u.d. of αn^σ mod 1 was first shown by P. Csillag (1930). This result follows from Theorem 2.6.1 (cf. [KN, p. 31, Exer. 3.9]). The sequence n^σ is u.d. in $\mathbb{R}$, cf. the def. on p. 1 – 6 .
(II) For the estimation of D_N cf. [KN, p. 130, Exer. 3.1] and this result goes back to H. Niederreiter (1971). J. Schoißengeier (1981) showed that a sharper (and explicit) result can be proved for the discrepancy D_N if $\alpha > 0$ and $1/2 < \sigma < 1$. Namely that

$D_N = \mathcal{O}(N^{\sigma-1-\varepsilon})$ for some $\varepsilon > 0$. He also asks what is the best possible $\varepsilon > 0$.
(III) u.d. of αn^σ mod 1 was also proved in G. Pólya and G. Szegő (1964, p. 72, No. 175).
(IV) If x_n is a real sequences then the **autocorrelation function** (compare with 3.11) has the form

$$\psi(k) = \lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{1}{2} - \{x_n\}\right)\left(\frac{1}{2} - \{x_{n+k}\}\right).$$

D.L. Jagerman (1963) proved that if $x_n = n^\sigma$ mod 1 and $0 < \sigma < 1$, then $\psi(k) = 1/12$ for $k = 1, 2, \dots$. If $x_n = n^2\theta$ mod 1 with irrational θ then he also proved that $\psi(k) = 0$ for all $k = 1, 2, \dots$. The autocorrelation function of $x_n = n\theta$ mod 1 does not vanish identically.

Related sequences: 2.15.3, 2.19.2.

P. Csillag: *Über die gleichmässige Verteilung nichtganzer positiver Potenzen* mod 1, Acta Litt. Sci. Szeged **5** (1930), 13–18 (JFM 56.0898.04).
D.L. Jagerman: *The autocorrelation function of a sequence uniformly distributed modulo* 1, Ann. Math. Statist. **34** (1963), 1243–1252 (MR0160309 (**28** #3523); Zbl. 0119.34503).
H. Niederreiter: *Almost–arithmetic progressions and uniform distribution*, Trans. Amer. Math. Soc. **161** (1971), 283–292 (MR0284406 (**44** #1633); Zbl. 0219.10040).
G. Pólya – G. Szegő: *Aufgaben und Lehrsätze aus der Analysis*, Vol. 1 & 2, 3rd corr. ed., Grundlehren d. math. Wiss., Band 19, Springer Verlag, Berlin, Göttingen, Heidelberg, New York, 1964 (MR0170985 (**30** #1219a); MR0170986 (**30** #1219b); Zbl. 0122.29704).
J. Schoissengeier: *On the discrepancy of sequences* (αn^σ), Acta Math. Acad. Sci. Hungar. **38** (1981), 29–43 (MR0634563 (83i:10067); Zbl. 0484.10032).

2.15.2. Let α and β be positive real numbers and $0 < \sigma < 1$. Let the double sequence $(\alpha m + \beta n)^\sigma$, $m = 1, 2, \dots$, $n = 1, 2, \dots$, be reordered to an ordinary sequence x_n, $n = 1, 2, \dots$, in such a way that for every N the initial segment x_n, $n = 1, 2, \dots, N^2$, coincide with $(\alpha m + \beta n)^\sigma$, $m, n = 1, 2, \dots, N$. Then the sequence

$$x_n \bmod 1$$

is

$$\text{u.d.}$$

Notes: [KN, p. 25, Exer. 2.30]. This follows directly from 2.6.16.

Related sequences: 2.12.7

2.15.3. The sequence

$$\alpha n + \beta n^\sigma \bmod 1, \quad \text{with } \beta \neq 0,\ 0 < \sigma < 1/2,$$

is

u.d.

and for its discrepancy we have

$$D_N \ll N^{-\sigma/2}.$$

NOTES: (I) This was proved by Y. Ohkubo (1999) using his result 2.6.26.
(II) What concerns the lower bound for D_N K. Goto and Y. Ohkubo (2004) proved that if α is irrational with bounded partial quotients, $\beta > 0$, and $0 < \sigma < 1$ then

$$D_N \geq cN^{(\sigma-3)/4}$$

for all N with a positive constant c.
(III) For the logarithmic discrepancy of $\alpha n + \beta n^{1-\delta}$ mod 1, with $0 < \delta < 1$, see 2.12.31, Th.2.12.31.1.

K. GOTO – Y. OHKUBO: *Lower bounds for the discrepancy of some sequences*, Math. Slovaca **54** (2004), no. 5, 487–502 (MR2114620 (2005k:11153); Zbl. 1108.11054).
Y. OHKUBO: *Notes on Erdős – Turán inequality*, J. Austral. Math. Soc. A **67** (1999), no. 1, 51–57 (MR1699155 (2000d:11100); Zbl. 0940.11029).

2.15.4. For the sequence

$$\alpha\sqrt{n} \bmod 1$$

define

$$ND_N^+(\alpha) = \sup_{0\leq x<1} \left(\sum_{n=1}^{N} c_{[0,x)}(\{\alpha\sqrt{n}\}) - Nx \right),$$

$$ND_N^-(\alpha) = \sup_{0\leq x<1} \left(Nx - \sum_{n=1}^{N} c_{[0,x)}(\{\alpha\sqrt{n}\}) \right).$$

$\big($Evidently, $D_N^*(\alpha) = \max\left(D_N^+(\alpha), D_N^-(\alpha)\right)$ and $D_N(\alpha) = D_N^+(\alpha) + D_N^-(\alpha)\big)$.
If $\alpha^2 \notin \mathbb{Q}$, $\alpha > 0$, then

$$\begin{aligned}
\limsup_{N\to\infty} \sqrt{N} D_N^+(\alpha) &= \limsup_{N\to\infty} \sqrt{N} D_N^-(\alpha) = \limsup_{N\to\infty} \sqrt{N} D_N^*(\alpha) \\
&= \lim_{N\to\infty} \sqrt{N} D_N(\alpha) = \frac{1}{4\alpha}, \\
\liminf_{N\to\infty} \sqrt{N} D_N^+(\alpha) &= \liminf_{N\to\infty} \sqrt{N} D_N^-(\alpha) = 0
\end{aligned}$$

and

$$\liminf_{N\to\infty}\sqrt{N}D_N^*(\alpha)=\frac{1}{8\alpha}.$$

NOTES: This was proved by J. Schoißengeier (1981). The case $\alpha^2\in\mathbb{Q}$ was investiagted by C. Baxa and J. Schoißengeier (1998). They described a method how to calculate $\limsup_{N\to\infty}\sqrt{N}D_N^+(\alpha)$ and $\limsup_{N\to\infty}\sqrt{N}D_N^-(\alpha)$ and thus also $\limsup_{N\to\infty}\sqrt{N}D_N^*(\alpha)$. An analogous result for $\limsup_{N\to\infty}\sqrt{N}D_N(\alpha)$ can be found in C. Baxa ([a]1998) and for $\liminf_{N\to\infty}\sqrt{N}D_N^+(\alpha)$ in Baxa (1998). E.g. if $\alpha=\sqrt{\frac{q}{p}}$ then he proved

$$\liminf_{N\to\infty}\sqrt{N}D_N^+\left(\sqrt{\frac{q}{p}}\right)=\begin{cases}\frac{1}{\sqrt{p}}, & \text{if } q=1,\\ \left(1+\frac{1}{8p}\right)\frac{1}{\sqrt{2p}}, & \text{if } q=2,\\ \left(\frac{3}{2}+\frac{1}{8p}\right)\frac{1}{\sqrt{3p}}, & \text{if } q=3 \text{ and } p\equiv 2 \pmod 3,\\ \frac{1}{\sqrt{3p}}, & \text{if } q=3 \text{ and } p\equiv 1 \pmod 3.\end{cases}$$

RELATED SEQUENCES: 2.15.1

C. BAXA: *On the discrepancy of the sequence* $(\alpha\sqrt{n})$*. II*, Arch. Math. (Basel) **70** (1998), no. 5, 366–370 (MR1612590 (99f:11096); Zbl. 0905.11033).
[a] C. BAXA: *Some remarks on the discrepancy of the sequence* $(\alpha\sqrt{n})$, Acta Math. Inf. Univ. Ostraviensis **6** (1998), no. 1, 27–30 (MR1822511 (2002a:11088); Zbl. 1024.11053).
C. BAXA – J. SCHOISSENGEIER: *On the discrepancy of the sequence* $(\alpha\sqrt{n})$, J. Lond. Math. Soc. (2) **57** (1998), no. 3, 529–544 (MR1659825 (99k:11118); Zbl. 0938.11041).
J. SCHOISSENGEIER: *On the discrepancy of sequences* (αn^σ), Acta Math. Acad. Sci. Hungar. **38** (1981), 29–43 (MR0634563 (83i:10067); Zbl. 0484.10032).

2.15.5. Let $k\ge 2$ be an integer. Then the block sequence $X_n^{(k)}$ with

$$X_n^{(k)}=\left(\sqrt[k]{\frac{n}{1}},\sqrt[k]{\frac{n}{2}},\dots,\sqrt[k]{\frac{n}{n}}\right)\bmod 1,\quad n=1,2,\dots,$$

has the a.d.f.

$$g_k(x)=\sum_{n=1}^{\infty}\frac{1}{n^k}+\frac{(-1)^{k-1}}{(k-1)!}\frac{\mathrm{d}^k}{\mathrm{d}x}\log\Gamma(x+1).$$

I.J. SCHOENBERG: *Über die asymptotische Verteilung reeller Zahlen* mod 1, Math. Z. **28** (1928), 171–199 (MR1544950; JFM 54.0212.02).

~~**2.15.6.** If $c>1$ and $2<p<q-1$ then the discrepancy of the finite sequence~~

$$(cN^p-n^p)^{1/q}\bmod 1,\quad n=1,2,\dots,N,$$

satisfies

$$D_N = \mathcal{O}(N^{-1/q}) \quad \text{and} \quad \limsup_{N\to\infty} N^{1/q} D_N > 0.$$

RELATED SEQUENCES: 3.10.7

W.–G. NOWAK: *Die Diskrepanz der Doppelfolgen* $(cN^p - n^p)^{1/q}$ *und einige Verallgemeinerungen*, Österreich. Akad. Wiss. Math.–Natur. Kl. Sitzungsber. II **187** (1978), no. 8–10, 383–409 (MR0548968 (80m:10029); Zbl. 0411.10025).

2.15.7. Given real numbers $a > 0$, $b \geq 0$ and an α with $0 < \alpha < 1$, let

$$x_n = (an + b)^\alpha \bmod 1$$

and

$$y_n = \frac{\{x_n + n\lambda\} + \{x_n - n\lambda\}}{2}.$$

Then the a.d.f. of y_n exists for every real λ. In particular:

1. If λ is irrational, then

$$g(x) = \begin{cases} 2x^2, & \text{if } 0 \leq x \leq 1/2, \\ 1 - 2(1-x)^2, & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

2. If λ is rational and 2λ is an integer, then y_n is u.d.
3. If λ is rational and 2λ is not integral, then y_n is not u.d., but it is dense in $[0, 1]$.

NOTES: A.M. Ostrowski (1980) proved this result for more general u.d. sequences x_n mod 1.

A.M. OSTROWSKI: *On the distribution function of certain sequences* (mod 1), Acta Arith. **37** (1980), 85–104 (MR0598867 (82d:10073); Zbl. 0372.10036).

2.16 Sequences involving the integer part function

2.16.1. Let θ and α be non–zero real numbers and let

$$x_n = \alpha[\theta n] \bmod 1.$$

(i) If θ is rational then the sequence x_n is

u.d.

for all irrational α.

(ii) If θ is irrational then the sequence x_n is

u.d.

if and only if 1, θ, and $\alpha\theta$ are linearly independent over the rationals.

NOTES: (I) [KN, p. 310, Th. 1.8] and [KN, p. 318, Notes]. The results may be traced to D.L. Carlson (1971) who also studied sequences $\alpha[P(n)]$ with a polynomial $P(x)$. For a proof of (ii) cf. D.P. Parent (1984, p. 254, Exer. 5.28).
(II) The integer sequence $[\alpha n+\beta]$, $n=1,2,\dots$, is called the **Beatty sequence**.

S. BEATTY: *Problem 3173*, Amer. Math. Monthly **33** (1926), no. 3, 159 (solution: *ibid.* **34** (1927), no. 3, 159). (MR1520888; JFM 53.0198.06).
D.L. CARLSON: *Good sequences of integers*, Ph.D. Thesis, Univ. of Colorado, 1971 (MR2621141).
D.P. PARENT: *Exercises in Number Theory*, Problem Books in Mathematics, Springer Verlag, New York, 1984 (MR0759342 (86f:11002); Zbl. 0536.10001). (French original: *Exercices de théorie des nombres*, Gauthier – Villars, Paris, 1978 (MR0485646 (**58** #5471); Zbl. 0387.10001)).

2.16.2. The sequence

$$[\alpha n]\gamma n \bmod 1$$

is

u.d.

if and only if either
(i) $\alpha^2 \notin \mathbb{Q}$ and γ is irrational, or
(ii) $\alpha^2 \in \mathbb{Q}$ but γ is rationally independent of $1, \alpha$.
NOTES: I.J. Håland (1993, Prop. 5.3).

I.J. HÅLAND: *Uniform distribution of generalized polynomial*, J. Number Theory **45** (1993), 327–366 (MR1247389 (94i:11053); Zbl. 0797.11064).

2.16.3. If $\alpha \notin \mathbb{Q}$ and $0 \neq \beta \in \mathbb{R}$ then the sequence

$$\alpha[\beta n]^2 \bmod 1$$

is

u.d.

NOTES: [DT, p. 104, Coroll. 1.114]: Since αn^2 has empty spectrum and $[\beta n]$ is almost periodic, we can apply 2.4.2 proved by M. Mendès France (1973).

M. Mendès France: *Les suites à spectre vide et la répartition modulo 1*, J. Number Theory **5** (1973), 1–15 (MR0319909 (**47** #8450); Zbl. 0252.10033).

2.16.4. The sequence

$$[\alpha n][\beta n]\gamma \bmod 1$$

is

$$\text{u.d.}$$

if and only if either
(i) $\alpha/\beta \neq \sqrt{c}$ for all $c \in \mathbb{Q}^+$ and γ is irrational, or
(ii) $\alpha/\beta = \sqrt{c}$ for some $c \in \mathbb{Q}^+$ but γ is rationally independent of 1 and $\sqrt{c}$.
Notes: I.J. Håland (1993, Prop. 5.3). He proves the following examples:
(I) The sequence

$$[\sqrt{2}n]^2\sqrt{2} \bmod 1$$

is

$$\text{u.d.}$$

For an alternative proof he uses the u.d. of $(2\sqrt{2}n^2, \sqrt{2}n) \bmod 1$ (cf. 3.9.2).
(II) The sequence

$$2[\sqrt{2}n]^2\sqrt{2}n \bmod 1$$

is

$$\text{not u.d.}$$

and has the a.d.f.

$$g(x) = \sqrt{1-x}.$$

As basis for an alternative proof the author uses the observation made by I.Z. Ruzsa that

$$2[\sqrt{2}n]\sqrt{2}n \equiv 1 - \{\sqrt{2}n\}^2 \bmod 1.$$

(III) The sequence

$$[\sqrt{2}n][\sqrt{3}n]\sqrt{6} \bmod 1$$

is

$$\text{not u.d.}$$

I.J. Håland: *Uniform distribution of generalized polynomial*, J. Number Theory **45** (1993), 327–366 (MR1247389 (94i:11053); Zbl. 0797.11064).

2.16.5. If $\alpha_1, \dots, \alpha_k$, $k \geq 3$, are non–zero real numbers and γ is irrational then the sequence

$$[\alpha_1 n][\alpha_2 n]\dots[\alpha_k n]\gamma \bmod 1$$

is

u.d.

I.J. HÅLAND: *Uniform distribution of generalized polynomial of the product type*, Acta Arith. **67** (1994), 13–27 (MR1292518 (95g:11075); Zbl. 0805.11054).

2.16.6. The sequence

$$x_n = \alpha_1 n[\alpha_2 n \dots [\alpha_{k-1} n[\alpha_k n]] \dots] \bmod 1$$

is

u.d.

In the case that $\alpha_1 = \alpha_2 = \dots = \alpha_k = \alpha$ then the sequence x_n is u.d. if and only if α^k is irrational with k a prime.

I.J. HÅLAND – D.E. KNUTH: *Polynomials involving the floor function*, Math. Scand. **76** (1995), no. 2, 194–200 (MR1354576 (96f:11098); Zbl. 0843.11005).

2.16.7.

NOTES: Let $\frac{p_k}{q_k}$, $k \geq 0$, be the kth convergent of the irrational number θ. If λ is real then the θ is called λ–**admissible** if there exists a constant $c' = c'(\theta, \lambda)$ such that $q_{k+1} < c' q_k^{1+\lambda}$ for $k \geq 0$.

. . . .

If $c > 0$ is real then for any irrational θ the sequence

$$x_n = [n^c]\theta \bmod 1$$

is

u.d.

and if moreover the following inequalities $1 < c < \frac{3}{2}$, $0 \leq \lambda \leq vc - 3$ with $v = \frac{4}{3-2c}$ are fulfilled, and irrational number θ is λ–admissible, then we have

$$D_N = \mathcal{O}\left(\frac{\log N}{N^{\frac{1}{v}}}\right).$$

NOTES: This was proved by G.J. Rieger (1997, Th. 1,2) which showed that the above assumptions imply that $D_N \to 0$. He also writes that the referee pointed out that using an argument similar to Carlsom's one (see [KN, pp. 310–311]) it can be shown that the sequence x_n is u.d. for any real $c > 0$.

G.J. RIEGER: *On the integer part function and uniform distribution* mod 1, J. Number Theory **65** (1997), no. 1, 74–86 (MR1458203 (98e:11089); Zbl. 0886.11047).

2.16.8. If c, $1 < c < \frac{7}{6}$, and α, $0 < \alpha < 1$ are real numbers then the sequence

$$x_n = [n^c](\log n)^\alpha \bmod 1$$

is

u.d.

with discrepancy

$$D_N = \mathcal{O}\left(\frac{1}{(\log N)^{\frac{1-\alpha}{2}}}\right).$$

G.J. Rieger: *On the integer part function and uniform distribution* mod 1, J. Number Theory **65** (1997), no. 1, 74–86 (MR1458203 (98e:11089); Zbl. 0886.11047).

2.17 Exponential sequences

Notes: J.F. Koksma (1935) proved that the sequence $\lambda\theta^n \bmod 1$ with $\lambda \neq 0$ fixed is u.d. for almost all real $\theta > 1$. If we take $\lambda = 1$ then we get that the sequence $\theta^n \bmod 1$ is u.d. for almost [6] all real numbers $\theta > 1$. However, no explicit example of a real number θ is known for which this sequence is u.d. If $\theta > 1$ is fixed then H. Weyl (1916) proved that the sequence $\lambda\theta^n \bmod 1$ is u.d. for almost all real λ. A.D. Pollington (1983) proved that the Hausdorff dimension of the set of all $\lambda \in \mathbb{R}$ for which the sequence $\lambda\theta^n \bmod 1$ is nowhere dense is $\geq \frac{1}{2}$.

J.F. Koksma: *Ein mengentheoretischer Satz ueber die Gleichverteilung modulo Eins*, Compositio Math. **2** (1935), 250–258 (MR1556918; Zbl. 0012.01401; JFM 61.0205.01).
A.D. Pollington: *Sur les suites* $\{k\theta^n\}$, C. R. Acad. Sci. Paris Sér. I Math. **296** (1983), no. 23, 941–943 (MR0777581 (86i:11034); Zbl. 0528.10033).
E.W. Weisstein: *Power fractional parts*, Math World (`http://mathworld.wolfram.com/PowerFractionalParts.html`).
H. Weyl: *Über die Gleichverteilung von Zahlen mod. Eins*, Math. Ann. **77** (1916), 313–352 (JFM 46.0278.06).

2.17.1. Open problem. Characterize the distribution of the sequence

$$\left(\frac{3}{2}\right)^n \bmod 1.$$

[6] The silver ratio $\theta = 1 + \sqrt{2}$ and the golden ratio $\theta = (1 + \sqrt{5})/2$ are two exceptions, cf. E.W. Weisstein. More precisely, all PV (cf. 2.17.8) and Salem numbers (cf. 3.21.5) are also exceptions.

NOTES: The question seems to be difficult. Some of the most known related conjectures say that:
(i) $(3/2)^n \bmod 1$ is u.d. in $[0,1]$.
(ii) $(3/2)^n \bmod 1$ is dense in $[0,1]$.
(iii) $\limsup_{n\to\infty}\{(3/2)^n\} - \liminf_{n\to\infty}\{(3/2)^n\} > 1/2$ (T. Vijayaraghavan (1940)).
(iv) there exists no $0 \neq \xi \in \mathbb{R}$ such that $0 \le \{\xi(3/2)^n\} < 1/2$ for all $n = 0, 1, 2, \dots$ (K. Mahler (1968). (This Mahler conjecture is true, if the sequence $[\xi(3/2)^n]$, $n = 1, 2, \dots$, contains infinitely many odd numbers for each $\xi > 0$.)
(v) There is no $0 \neq \xi \in \mathbb{R}$ such that the closure of $\big\{\{\xi(3/2)^n\}\ ;\ n = 0, 1, 2, \dots\big\}$ is nowhere dense in $[0,1]$.
Some partial affirmative answers:
(I) L. Flatto, J.C. Lagarias and A.D. Pollington (1995) showed that for every $\xi > 0$ we have $\limsup_{n\to\infty}\{\xi(3/2)^n\} - \liminf_{n\to\infty}\{\xi(3/2)^n\} \ge 1/3$.
(II) G. Choquet (1980) proved the existence of infinitely many $\xi \in \mathbb{R}$ for which $1/19 \le \{\xi(3/2)^n\} \le 18/19$ for $n = 0, 1, 2, \dots$. Him is ascribed the conjecture (v).
(1) A. Dubickas (2006) proved that the sequence of fractional parts $\{\xi(3/2)^n\}$, $n = 1, 2, \dots$, has at least one limit point in the interval $[0.238117\dots, 0.761882\dots]$ of length $0.523764\dots$ for any $\xi \neq 0$. This immediately follows from:
(2) A. Dubickas (2006): If $\xi \neq 0$, then the sequence $\|\xi(3/2)^n\|$, $n = 1, 2, \dots$, has a limit point $\ge (3 - T(2/3))/12 = 0.238117\dots$ and a limit point $\le (1 + T(2/3))/4 = 0.285647\dots$, where $T(x) = \prod_{n=0}^{\infty}\big(1 - x^{2^n}\big)$.
(3) A. Dubickas (2007) derived from 2.17.4(V) that $\{\xi(-3/2)^n\}$ has a limit point ≤ 0.533547 and a limit point ≥ 0.466452.
(4) S. Akiyama, C. Frougny and J. Sakarovitch (2005) proved that there is a $\xi \neq 0$ such that $\|\xi(3/2)^n\| < 1/3$ for $n = 1, 2, \dots$.
(5) A. Pollington (1981) proved that there is a $\xi \neq 0$ such that $\|\xi(3/2)^n\| > 4/65$ for $n = 1, 2, 3, \dots$.
(III) R. Tijdeman (1972) showed that for every pair of integers k, m with $k \ge 2$ and $m \ge 1$ there exists $\xi \in [m, m+1)$ such that $0 \le \{\xi((2k+1)/2)^n\} \le 1/(2k-1)$ for $n = 0, 1, 2, \dots$.
(IV) O. Strauch (1997) proved that every distribution function $g(x)$ of $\xi(3/2)^n \bmod 1$ satisfies the functional equation

$$\begin{aligned} &g(x/2) + g((x+1)/2) - g(1/2) = \\ &\qquad = g(x/3) + g((x+1)/3) + g((x+2)/3) - g(1/3) - g(2/3). \quad (*) \end{aligned}$$

The following d.f.'s

$$g_1(x) = \begin{cases} 0, & \text{if } x \in [0, 2/6], \\ x - 1/3, & \text{if } x \in [2/6, 3/6], \\ 2x - 5/6, & \text{if } x \in [3/6, 5/6], \\ x, & \text{if } x \in [5/6, 1], \end{cases}$$

and

$$g_2(x) = \begin{cases} 0, & \text{if } x \in [0, 1/6], \\ 2x - 1/3, & \text{if } x \in [1/6, 3/12], \\ 4x - 5/6, & \text{if } x \in [3/12, 5/18], \\ 2x - 5/18, & \text{if } x \in [5/18, 2/6], \\ 7/18, & \text{if } x \in [2/6, 8/18], \\ x - 1/18, & \text{if } x \in [8/18, 3/6], \\ 8/18, & \text{if } x \in [3/6, 7/9], \\ 2x - 20/18, & \text{if } x \in [7/9, 5/6], \\ 4x - 50/18, & \text{if } x \in [5/6, 11/12], \\ 2x - 17/18, & \text{if } x \in [11/12, 17/18], \\ x, & \text{if } x \in [17/18, 1] \end{cases}$$

are non–trivial solutions of $(*)$. On the other hand, the d.f.

$$g_3(x) = \begin{cases} x, & \text{if } x \in [0, 2/3], \\ x^2 - (2/3)x + 2/3, & \text{if } x \in [2/3, 1] \end{cases}$$

is not a d.f. of $\xi(3/2)^n \bmod 1$ for any $\xi \in \mathbb{R}$ (cf. O. Strauch (1999, p. 126)).
Strauch (1997) also introduced the notion of a set of uniqueness for g. Here an $X \subset [0, 1]$ is said to be a **set of uniqueness** if g_1, g_2 are two d.f.'s of $\xi(3/2)^n \bmod 1$ with $g_1(x) = g_2(x)$ for $x \in X$ then $g_1(x) = g_2(x)$ for every $x \in [0, 1]$. He gives e.g. the following sets of uniqueness: $X = [0, 2/3]$, $X = [1/3, 1]$ or $X = [2/9, 1/3] \cup [1/2, 1]$.
(VI) The elements of the sequence $(3/2)^n$ appear in the Waring problem. Let

$$g(k) = \min\left\{s\ ;\ a = n_1^k + \cdots + n_s^k \text{ for all } a \in \mathbb{N} \text{ and suitable } n_i \in \mathbb{N}_0\right\}.$$

S. Pillai (1936) proved that if $k \geq 5$ and if we write $3^k = q2^k + r$ with $0 < r < 2^k$, then $g(k) = 2^k + \left[\left(\frac{3}{2}\right)^k\right] - 2$, provided that $r + q < 2^k$, i.e. $3^k - 2^k\left[\left(\frac{3}{2}\right)^k\right] < 2^k - \left[\left(\frac{3}{2}\right)^k\right]$.
F. Beukers (1981) has shown that $\|\left(\frac{3}{2}\right)^k\| > 2^{(-0.9)k}$ for all integers $k > 5000$, but this result is not sufficient to derive the above formula for $g(k)$.

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Quoted in: 2.17.1

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A. Dubickas: *On a sequence related to that of Thue-Morse and its applications*, Discrete Mathematics **307** (2007), no. 1, 1082–1093 (MR2292537 (2008b:11086); Zbl. 1113.11008).

L. Flatto – J.C. Lagarias – A.D. Pollington: *On the range of fractional parts* $\{\zeta(p/q)^n\}$, Acta Arith. **70** (1995), no. 2, 125–147 (MR1322557 (96a:11073); Zbl. 0821.11038).

K. Mahler: *An unsolved problem on the powers of* 3/2, J. Austral. Math. Soc. **8** (1968), 313–321 (MR0227109 (**37** #2694); Zbl. 0155.09501).
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O. Strauch: *On distribution functions of* $\zeta(3/2)^n$ mod 1, Acta Arith. **81** (1997), no. 1, 25–35 (MR1454153 (98c:11075); Zbl. 0882.11044).
O. Strauch: *Distribution of Sequences* (in Slovak), Mathematical Institute of the Slovak Academy of Sciences, DSc Thesis, Bratislava, Slovakia, 1999.
R. Tijdeman: *Note on Mahler's* 3/2*–problem*, Norske Vid. Selske. Skr. **16** (1972), 1–4 (Zbl. 0227.10025).
T. Vijayaraghavan: *On the fractional parts of the powers of a number. I*, J. London Math. Soc. **15** (1940), 159–160 (MR0002326 (2,33e); Zbl. 0027.16201).

2.17.2. Open problem. Characterize the distribution of the sequence

$$e^n \bmod 1, \quad n = 1, 2, \dots$$

2.17.3. Open problem. Characterize the distribution of the sequence

$$\pi^n \bmod 1, \quad n = 1, 2, \dots$$

2.17.4. Open problem. If $p > q > 1$ are coprime integers then distribution of the sequence

$$x_n = \left(\frac{p}{q}\right)^n \bmod 1, \quad n = 1, 2, \dots,$$

is a well–known and largely unsolved problem. Although it is conjectured that x_n is u.d., it is not even known if it is dense in $[0, 1]$. It is known that:
(I) x_n has an infinite number of points of accumulation,
(II) if ξ is a positive real number then

$$\limsup_{n\to\infty} \left\{\xi \left(\frac{p}{q}\right)^n\right\} - \liminf_{n\to\infty} \left\{\xi \left(\frac{p}{q}\right)^n\right\} \geq \frac{1}{p}.$$

Notes: (I) This was firstly proved by Ch. Pisot (1938), then by T. Vijayaraghavan (1940) and L. Rédei (1942). The density of x_n in $[0, 1]$ is a problem posed by Pisot and Vijayaraghavan.

(II) L. Flatto, J.C. Lagarias and A.D. Pollington (1995).
(III) The existence of an irrational limit point of x_n is also an open question. For its existence it is necessary that $\limsup_{n\to\infty} l(x_n) = \infty$, where $l(p/q)$ denotes the number of terms in the the continued fraction expansion for $p/q = [a_0; a_1, \dots, a_l]$. In this connection M. Mendès France (1971) conjectures that

$$\lim_{n\to\infty} l(x_n) = \infty.$$

This was proved by Y. Pourchet (unpublished) and G. Choquet (1981). See also M. Mendès France (1993).
(IV) A. Dubickas (2006): Let $T(x) = \prod_{n=0}^{\infty}(1 - x^{2^n})$, and $E(x) = \frac{1-(1-x)T(x)}{2x}$. If $\xi \neq 0$ and $p > q > 1$ are coprime integers, then the sequence $\|\xi(p/q)^n\|$, $n = 1, 2, \dots$, has a limit point $\geq E(q/p)/p$ and a limit point $\leq 1/2-(1-e(q/p))T(q/p)/2q$, where $e(q/p) = 1 - (q/p)$ if $p + q$ is even, and $e(q/p) = 1$ if $p + q$ is odd.
(V) A. Dubickas (2007): If $p > q > 1$ are two coprime integers and $\xi \neq 0$ a real number, then the sequence of fractional parts $\{\xi(-p/q)^n\}$, $n = 0, 1, 2, \dots$, has a limit point $\leq 1 - (1 - F(q/p))/q$, and a limit point $\geq (1 - F(q/p))/q$, where $F(x) = \prod_{k=1}^{\infty}(1 - x^{(2^k+(-1)^{k-1})/3})$.
(VI) S.D. Adhikari, P. Rath and N. Saradha (2005) proved that every d.f. $g(x)$ of $\{\xi(p/q)^n\}$ satisfies the functional equation
$\sum_{i=0}^{q-1} g\left(\frac{x+i}{q}\right) - \sum_{i=0}^{q-1} g\left(\frac{i}{q}\right) = \sum_{i=0}^{p-1} g\left(\frac{x+i}{p}\right) - \sum_{i=0}^{p-1} g\left(\frac{i}{p}\right)$.
(VII) S.D. Adhikari, P. Rath and N. Saradha (2005) generalized 2.17.1 (V) proving that every interval $I \subset [0, 1]$ of length $|I| = (p - 1)/q$ and every complement $[0, 1] \setminus [(i - 1)/p, i/p]$, $i = 1, 2, \dots, p$, are sets of uniqueness of d.f.'s of $\{\xi(p/q)^n\}$. In the second case, if $j/q \in [(i - 1)/p, i/p]$ for some $1 \leq j < q$ they assume $p \geq q^2 - q$.

S.D. Adhikari – P. Rath – N. Saradha: *On the set of uniqueness of a distribution function of* $\{\zeta(p/q)^n\}$, Acta Arith. **119** (2005), no. 4, 307–316 (MR2189064 (2006m:11112); Zbl. 1163.11333).
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M. Mendès France: *Quelques problèmes relatifs à la théorie des fractions continues limitées*, Séminaire de Théorie des Nombres, 1971–1972, Exp. No. 4, Univ. Bordeux I, Talence, 1972, 9 pp. (MR0389775 (**52** #10606); Zbl. 0278.10030).
M. Mendès France: *Remarks and problems on finite and periodic continued fractions*, Enseign. Math. (2) **39** (1993), no. 3–4 249–257 (MR1252067 (94i:11045); Zbl. 0808.11007).
Ch. Pisot: *La réparatition modulo 1 et les nombres algébraiques*, (French), Diss., Paris 1938, 44 pp. (Zbl. 0019.00703).
Ch. Pisot: *La réparatition modulo 1 et les nombres algébraiques*, Ann. Scuola norm. sup. Pisa, Sci. fis. mat. (2) **7** (1938), 205–248 (Identical with the previous item (JFM 64.0994.01)).
L. Rédei: *Zu einem Approximationssatz von Koksma*, Math. Z. **48** (1942), 500–502 (MR0008232 (4,266c); JFM 68.0083.03).

T. Vijayaraghavan: *On the fractional parts of the powers of a number. I*, J. London Math. Soc. **15** (1940), 159–160 (MR0002326 (2,33e); Zbl. 0027.16201).

2.17.5. Let $\theta = q^{\frac{1}{k}}$ be irrational, where k and $q \geq 2$ are integers. Then the set of limit points of the sequence

$$\theta^n \bmod 1, \quad n = 1, 2, \dots$$

is infinite.

T. Vijayaraghavan: *On decimals of irrational numbers*, Proc. Indian Acad. Sci., Sect. A **12** (1940), 20 (MR0002325 (2,33d); Zbl. 0025.30803).

2.17.6.

Notes: In this item, contrary to 1.8.1, we shall understand under the a.d.f. $g(x)$ of x_n the point–wise limit $\lim_{N\to\infty} \frac{A([0,x);N;x_n)}{N} = g(x)$ for every $x \in [0,1]$.

. . . .

Let $\theta > 1$ be a real number.

(I) There exists uncountably many ξ such that the sequence

$$\xi\theta^n \bmod 1, \quad n = 1, 2, \dots,$$

does not have the a.d.f.

(II) On the other hand, for an arbitrary d.f. $g(x)$ and for any sequence u_n of real numbers which satisfies $\lim_{n\to\infty}(u_{n+1} - u_n) = \infty$, there exists a real number θ such that the sequence

$$\theta^{u_n} \bmod 1, \quad n = 1, 2, \dots,$$

has

$$g(x) \text{ as its a.d.f.}$$

Notes: (I) H. Helson and J.–P. Kahane (1965).

(II) A. Zame (1967).

(III) F. Supnick, H.J. Cohen and J.F. Keston (1960) $\big($and also H. Ehlich (1961) and E.C. Posner (1962) by different methods$\big)$ solved the following two problems posed by Vijayaraghavan:

• If three different positive powers of θ are equal mod 1, e.g. $\theta^{n_1} = \theta^{n_2} = \theta^{n_3} \bmod 1$, then θ^{n_1}, θ^{n_2}, θ^{n_3} are integers.

• If two different powers of θ are equal mod 1 for infinitely many pairs of powers, then a positive integral power of θ is a rational integer.

H. Ehlich: *Die positiven Lösungen der Gleichung $y^a - [y^a] = y^b - [y^b] = y^c - [y^c]$*, Math. Z. **76** (1961), 1–4 (MR0122789 (**23** #A123); Zbl. 0099.02703).
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E.C. Posner: *Diophantine problems involving powers modulo one*, Illinois J. Math. **6** (1962), 251–263 (MR0137679 (**25** #1129); Zbl. 0107.04301).
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A. Zame: *The distribution of sequences modulo* 1, Canad. J. Math. **19** (1967), 697–709 (MR0217020 (**36** #115); Zbl. 0161.05001).

2.17.7. Let $\theta > 1$ be an algebraic integer such that all the conjugates of θ have modulus ≤ 1. If the modulus of some (and hence of all but one) conjugate of θ is unity, then the sequence

$$\theta^n \bmod 1, \quad n = 1, 2, \dots,$$

is

$$\text{dense in } [0,1], \text{ but not u.d.}$$

Notes:
(I) The real algebraic integer $\theta > 1$ is called a **Salem number** if all its conjugates lie inside or on the circumference of the unit circle and at least one of conjugates of θ lies on the circumference of the unit circle, see 3.21.5.
(II) It is well known that if θ is a Salem number of degree d, then d is even, $d \geq 4$ and $1/\theta$ is the only conjugate of θ with the modulus less than 1, while all the other conjugates are of modulus 1. Salem numbers are the only known concrete numbers whose powers are dense mod 1 in $[0,1]$.
(III) Toufik Zaimi (2006): Let θ be a Salem number, λ be a nonzero element of the field $\mathbb{Q}(\theta)$ and denote $\Delta = \limsup_{n\to\infty}\{\lambda\theta^n\} - \liminf_{n\to\infty}\{\lambda\theta^n\}$. Then
(i) $\Delta > 0$.
(ii) If λ is an algebraic integer, then $\Delta = 1$. Furthermore, for any $0 < t < 1$ there is an algebraic integer λ and a subinterval $I \subset [0,1]$ of length t such that the sequence $\{\lambda\theta^n\}$, $n = 1, 2, \dots$ has no limit point in I.
(iii) If $\theta - 1$ is a unit, then $\Delta \geq 1/L$, where L is the sum of the absolute values of the coefficients of the minimal polynomial of θ.
(iv) If $\theta - 1$ is not a unit, then $\inf_\lambda \Delta = 0$.
(IV) A. Dubickas ([a]2006): If θ is either a P.V. or a Salem number and $\lambda \neq 0$ and $\lambda \notin \mathbb{Q}(\theta)$, then $\Delta \geq 1/L$, where Δ and λ are defined as in (III).
(V) A. Dubickas (2006, Coroll. 3 of Th. 2): Let $d \geq 2$ be a positive integer. Suppose that $\alpha > 1$ is a root of the polynomial $x^d - x - 1$. Let ξ be an arbitrary positive number that lies outside the field $\mathbb{Q}(\alpha)$ if $d = 2$ or $d = 3$. Then the sequence $[\xi\alpha^n]$, $n = 1, 2, \dots$, contains infinitely many even numbers and infinitely many odd numbers. Thus α satisfies Mahler's conjecture (2.17.1 (iv)), i.e. $0 \leq \{\xi\alpha^n\} < 1/2$

does not holds for all $n = 1, 2, \dots$.
(VI) A. Dubickas' examples ([a]2006):
• If $\theta > 1$ is a root of $x^2 - 7x + 2$, then $\lim_{n\to\infty}\left\{\frac{2+3\theta}{4}\theta^n\right\} = \frac{1}{4}$.
• If $\theta > 1$ is a root of $x^3 - x - 1$, then the sequence $\{\zeta\theta^n\}$, $n = 1, 2, \dots$, does not have a limit for every $\zeta > 0$.
• If $\theta > 1$ is a root of $x^3 - x - 1$, then the set of limit points of the sequence $\{(2/3 + \theta/3)\theta^n\}$, $n = 1, 2, \dots$, is 0, 1/3, and 1.

[a] A. Dubickas: *On the limit points of the fractional parts of power of Pisot numbers*, Archivum Mathematicum (Brno) **42** (2006), 151–158 (MR2240352 (2007b:11167); Zbl. 1164.11026).
A. Dubickas: *Arithmetical properties of powers of algebraic numbers*, Bull. London Math. Soc. **38** (2006), no. 1, 70–80 (MR2201605 (2006i:11080); Zbl. 1164.11025).
Ch. Pisot – R. Salem: *Distribution modulo* 1 *of the powers of real numbers larger than* 1, Compositio Math. **16** (1964), 164–168 (MR0174547 (**30** #4748); Zbl. 0131.04804).
T. Zaïmi: *An arithmetical property of powers of Salem numbers*, J. Number Theory **120** (2006), 179–191 (MR2256803 (2007g:11080); Zbl. 1147.11037).

2.17.7.1 If λ is a Salem number of degree 4, then the sequence

$$n\lambda^n \bmod 1, \quad n = 1, 2, \dots$$

is

u.d.

Notes:
(I) D. Berend and G. Kolesnik (2011). They precisely proved: Let λ be a Salem number of degree 4 and $P(x)$ a nonconstant polynomial with integer coefficients. Then the sequence

$$(P(n)\lambda^n, P(n+1)\lambda^{n+1}, P(n+2)\lambda^{n+2}, P(n+3)\lambda^{n+3}) \bmod 1, n = 1, 2, \dots$$

is

u.d.

D. Berend – G. Kolesnik: *Complete uniform distribution of some oscillating sequences*, J. Ramanujan Math. Soc. **26** (2011), no. 2, 127–144 (MR2815328 (2012e:11134); Zbl. 1256.11041).

2.17.8.

Notes: A real algebraic integer $\theta > 1$ is called a **P.V. number (Pisot – Vijayaraghavan number)** if all its conjugates $\neq \theta$ lie strictly inside the unit circle.
. . . .
Let θ be a P.V. number. Then

$$\theta^n \bmod 1 \to 0 \quad \text{as } n \to \infty.$$

Notes: For the history of P.V. numbers consult D.W. Boyd (1983–84):
(I) A. Thue (1912) proved that θ is a P.V. number if and only if $\{\theta^n\} = \mathcal{O}(c^n)$ for some $0 < c < 1$.
(II) G.H. Hardy (1919) proved that if $\theta > 1$ is any algebraic number and $\lambda > 0$ is a real number so that $\{\lambda\theta^n\} = \mathcal{O}(c^n)$, $(0 < c < 1)$, then θ is a P.V. number. Hardy posed an interesting and still unanswered question of whether there is a transcendental numbers $\theta > 1$ for which a $\lambda > 0$ exists such that $\{\lambda\theta^n\} \to 0$.
(III) T. Vijayaraghavan (1941) proved that if $\theta > 1$ is an algebraic number and if θ^n, $n = 1, 3, \dots$, has only a finite set of limit points, then θ is a P.V. number.
(IV) Ch. Pisot (1937, [a]1937) proved that if $\theta > 1$ and $\lambda > 0$ are real numbers such that $\sum_{n=1}^{\infty}\{\lambda\theta^n\} < +\infty$, then θ is a P.V. number.
Pisot (1928) proved that if $\theta > 1$ and there exits a λ, $\frac{1}{\theta} \le \lambda \le 1$, such that $\sum_{n=1}^{\infty}\sin^2(\pi\lambda\theta^n) < +\infty$, then θ is a P.V. number.
(V). In (1946) Pisot proved the following generalization of (III): Let $\theta > 1$ and $\lambda > 0$ be real numbers. If θ is algebraic, then the set of limit points of $\{\lambda\theta^n\}$ is finite if and only if θ is a P.V. number and λ is an algebraic number from the field generated by θ. In this connection define $E(\theta) = \{\theta^n \bmod 1\,;\, n \in \mathbb{N}\}$ and $E'(\theta)$ is the **derived set** of $E(x)$ i.e. the set of all accumulation points of $E(x)$. Define $E^{(k)}(\theta)$ recursively as $E^{(k)}(\theta) = (E^{(k-1)}(\theta))'$. Pisot's (1946) result also states that if $\theta > 1$ is algebraic and $E''(\theta) = \emptyset$, then θ is a P.V. number. M. Mendès France (1993) asked whether $E^{(k)}(\theta) = \emptyset$ for some $k \in \mathbb{N}$, also implies that the algebraic number $\theta > 1$ is a P.V. number.
(VI) L. Rédei (1942, [a]1942) proved the following characterization: If θ is a real algebraic number with $|\theta| > 1$, then a necessary and sufficient condition that the sequence $\theta^n \bmod 1$ converges is that θ is an algebraic integer and that the absolute value of all its conjugates is less than 1. Moreover, if this conditions is satisfied, then $\lim_{n\to\infty}\theta^n \bmod 1 = 0$.
(VII) The set S of all P.V. numbers is closed (R. Salem (1944)). Two smallest elements of S are $1.324717\dots$, and $1.380277\dots$, the real roots of $x^3 - x - 1$, and $x^4 - x^3 - 1$, respectively. Both are isolated points of S and S contains no other point in the interval $(1, \sqrt{2}]$ (C.L. Siegel (1944)). The next one is $1.443269\dots$, the real root of $x^5 - x^4 - x^3 + x^2 - 1$ and $1.465571\dots$, the real root of $x^3 - x^2 - 1$. The smallest limit point of S is the root $\frac{(1+\sqrt{5})}{2} = 1.618033\dots$ of $x^2 - x - 1$, an isolated point of the derived set S' of S (J. Dufresnoy and Ch. Pisot (1952), (1953)). The smallest number S'' is 2.
(VIII) If $\theta \in S$, and $\varepsilon > 0$ is arbitrary then there are numbers λ in the field $\mathbb{Q}(\theta)$ such that $\|\lambda\theta^n\| \le \varepsilon$ for $n = 0, 1, 2, \dots$. On the other hand, if $\theta > 1$, $\lambda \ge 1$ are real numbers such that

$$\|\lambda\theta^n\| \le \big(2e\theta(\theta+1)(1+\log\lambda)\big)^{-1}, \quad n = 0, 1, 2, \dots,$$

then $\theta \in S$, $\deg(\theta) \le [\log\lambda] + 1$, and $\lambda \in \mathbb{Q}(\theta)$ (Ch. Pisot (1938)). This result is, in a certain sense, the best possible: Given any constant $c > 2e(1 + \log 2) = 9.24\dots$, there exists a real number $\lambda \ge 1$ and a transcendental θ as large as we wish such

that $\|\lambda\theta^n\| \le c(2e\theta(\theta+1)(1+\log\lambda))^{-1}$.
(IX) A necessary and sufficient condition for a real number $\theta > 1$ to be a rational integer is that $\|\theta^n\| \le \frac{1}{(\theta+1)(\theta+2)}$ for $n = 1, 2, \dots$ (M. Mignotte (1977)).
(X) Some criteria for P.V. numbers can be found in 2.6.22.
(XI) Let $\beta > 1$. Define the transformations $T_\beta(x) = \{\beta x\}$, $T_\beta^{(2)}(x) = T_\beta(T_\beta(x))$, etc., and denote

$$\mathrm{Per}(\beta) = \big\{x \in [0,1);\ \text{there exists } k \ge 1 \text{ such that } T_\beta^{(k)}(x) = x\big\}.$$

K. Schmidt (1980) proved that if $\mathrm{Per}(\beta) \supset \mathbb{Q} \cap [0,1)$, then β is a P.V. or a Salem number. Conversely, if β is a P.V. number, then $\mathrm{Per}(\beta) = \mathbb{Q}(\beta) \cap [0,1)$.
(XII) A polynomial over $\mathbb{Q}$ of degree ≥ 2 is said to be **reduced** if it has one positive root > 1 and all its other roots w satisfy $|w| < 1$ and $-1 < \Re(w) < 0$.
Let γ be an algebraic number with continued fraction expansion $\gamma = [a_0; a_1, a_2, \dots]$. Then there exists an effectively computable positive integer m_0 such that if $m \ge m_0$ and $\gamma = [a_0; a_1, a_2, \dots, a_m, \gamma_{m+1}]$, then γ_{m+1} is a positive root of a reduced polynomial (thus γ_{m+1} is a P.V. number). This was proved by A. Vincent in 1836, see the book by J.V. Uspensky (1948) and the paper by E. Bombieri and A.J. van der Poorten (1995).
(XIII) Dubickas (2006): Let θ be a P.V. number whose minimal polynomial $P(x)$ satisfies $P(1) \le -2$. Then

$$\lim_{n\to\infty}\left\{\frac{\theta^n}{P'(\theta)(\theta-1)}\right\} = \frac{1}{|P(1)|}.$$

E. Bombieri – A.J. van der Poorten: *Continued fractions of algebraic numbers*, in: *Computational algebra and number theory (Sydney, 1992)*, Math. Appl., 325, Kluwer Acad. Publ., Dordrecht, 1995, 137–152 (MR1344927 (96g:11079); Zbl. 0835.11025).
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Ch. Pisot: *La réparatition modulo 1 et les nombres algébraiques*, Ann. Scuola norm. sup. Pisa, Sci. fis. mat. (2) **7** (1938), 205–248 (Identical with the previous item (JFM 64.0994.01)).
Ch. Pisot: *Répartition* (mod 1) *des puissances successives des nombers réeles*, Comment. Math. Helv. **19** (1946), 135–160 (MR0017744 (8,194c); Zbl. 0063.06259).
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2.17.9. Assume that

- $f(x)$ is an arbitrary polynomial with integral coefficients, not identically zero,
- $\lambda > 1$ is an integer or a P.V. number, i.e. $\lambda^k = a_1\lambda^{k-1} + \cdots + a_k$, with $a_1, \dots, a_k$, $a_k \neq 0$, integers, $\lambda > 1$ and if $\lambda_2, \dots, \lambda_k$ are all conjugates of λ, then $\theta = \max_{2\le i\le k} |\lambda_i| < 1$,
- p_n, $n = 1, 2, \dots$, is an arbitrary increasing sequence of primes with $p_{n+1} = \mathcal{O}(p_n)$, and $|a_k| < p_1$,
- $\psi_n(i)$, $n = 1, 2, \dots$, satisfy the recurrence relations

$$\psi_n(i) = a_1\psi_n(i-1) + \cdots + a_k\psi_n(i-k)$$

  for $i = k+1, k+2, \dots$,
- τ_n, $n = 1, 2, \dots$, is an increasing sequence of positive integers such that

$$\psi_n(i + \tau_n) \equiv \psi_n(i) \pmod{p_n}, \quad \tau_n \equiv 0 \pmod{p_n},$$
$$\log \tau_{n+1} = o(\tau_n),$$

- the number of solutions of $\psi_n(i) \equiv 0 \pmod{p_n}$, for $i = 1, 2, \dots, \tau_n$, does not exceed τ_n/p_n
- t_n, $n = 1, 2, \dots$, is an arbitrary increasing sequence of positive integers such that $t_n \ge \tau_{n+1}$ and $\log t_n = \mathcal{O}(\log \tau_{n+1})$,

- define $k_{n+1} = k_n + \tau_n p_n t_n$ for $n = 1, 2, \dots$, with $k_1 = 0$,
- $\phi(i)$, $i = 1, 2, \dots$, is an arbitrary arithmetical function such that $\phi(i) \neq 0$ for all sufficiently large i and $\phi(i) = o(p_i)$.

The above assumptions imply that there exists a sequence γ_n such that $\psi_n(i) = \gamma_n \lambda^i + \mathcal{O}(p_n \theta^i)$, where $\gamma_n = \mathcal{O}(p_n)$. Let α be defined by the sum

$$\alpha = \sum_{i=1}^{\infty} \frac{\phi(i)\gamma_i}{p_i(\lambda^{\tau_i} - 1)} \left(\frac{1}{\lambda^{k_i}} - \frac{1}{\lambda^{k_{i+1}}} \right).$$

Then the sequence

$$x_n = \alpha \lambda^n f(n) \bmod 1$$

is

u.d.

NOTES: Theorem 1 of N.M. Korobov (1953). He also proved the following multi–dimensional generalizations:
Theorem 3: Assume additionally that

- $f_1(x), \dots, f_s(x)$ have integral coefficients and are linearly independent over $\mathbb{Z}$.

Then the sequence

$$\mathbf{x}_n = (\alpha \lambda^n f_1(n), \dots, \alpha \lambda^n f_s(n)) \bmod 1$$

is

u.d. in $[0, 1]^s$.

Theorem 2: Assume additionally that

- $\phi_1(i), \dots, \phi_s(i)$ are arithmetical functions such that for any s–tuple of integers $(m_1, \dots, m_s) \neq (0, \dots, 0)$ the relation $m_1\phi_1(i) + \cdots + m_s\phi_s(i) = 0$ holds only for finitely many i.

Let

$$\alpha_j = \sum_{i=1}^{\infty} \frac{\phi_j(i)\gamma_i}{p_i(\lambda^{\tau_i} - 1)} \left(\frac{1}{\lambda^{k_i}} - \frac{1}{\lambda^{k_{i+1}}} \right)$$

for $j = 1, 2, \dots, s$. Then the sequence

$$\mathbf{x}_n = (\alpha_1 \lambda^n f(n), \dots, \alpha_s \lambda^n f(n)) \bmod 1$$

is

u.d. in $[0, 1]^s$.

N.M. KOROBOV: *Multidimensional problems of the distribution of fractional parts*, (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. **17** (1953), 389–400 (MR0059321 (15,511a); Zbl. 0051.28603).

2.17.10. Let a and b be positive real numbers such that $\log_b a$ is an irrational number. Then the double sequence

$$\frac{a^n}{b^m}, \quad m,n = 1,2,\dots,$$

is

dense in $[0,\infty)$.

J. Bukor – M. Kmeťová – J.T. Tóth: *Note on ratio set of sets of natural numbers*, Acta Mathematica (Nitra) **2** (1995), 35–40.

$\boxed{\textbf{2.17.10.1}}$ If $p,q > 1$ are multiplicatively independent integers, i.e. they are not both integer powers of some integer, then for every irrational θ the double sequence

$$p^n q^m \theta \bmod 1, \quad m,n = 1,2,\dots,$$

is dense in $[0,1]$.

Notes:
(I) H. Furstenberg (1967).
(II) B. Kra (1999) extended this result as follows:
For positive integers $1 < p_i < q_i$, $i = 1,2,\dots,k$, assume that all pairs p_i, q_i are multiplicatively independent and $(p_i, q_i) \neq (p_j, q_j)$ for $i \neq j$. Then for distinct $\theta_1,\dots,\theta_k$ with at least one irrational θ_i the sequence

$$\sum_{i=1}^{K} p_i^n q_i^m \theta_i \bmod 1, \qquad m,n = 1,2,\dots, \tag{1}$$

is dense in $[0,1]$.
Also for irrational θ, multiplicatively independent integers $p,q > 1$ and any sequence x_n of real numbers, the sequence

$$p^n q^m \theta + x_n \bmod 1, \quad m,n = 1,2,\dots,$$

is dense in $[0,1]$.
(III) Berend in MR1487320 (99j:11079) reformulated Kra's result as follows:
Let p_i, q_i be integers and θ_i real, $i = 1,2,\dots,k$. If p_1 and q_1 are multiplicatively independent, θ_1 is irrational, and $(p_i,q_i) \neq (p_1,q_1)$ for $i \geq 2$, then the sequence (1) is dense in $[0,1]$.
(IV) R. Urban (2007) conjectured:
Let $k \in \mathbb{N}$ be fixed, and let λ_i, μ_i, for $1 \leq i \leq k$ be real algebraic numbers with absolute values greater than 1. Assume that the pairs λ_i, μ_i for $1 = 1,2,\dots,k$,

are multiplicatively independent (i.e. there does not exist non-zero integers m, n such that $\lambda_i^m = \mu_i^n$), and $(\lambda_i, \mu_i) \neq (\lambda_j, \mu_j)$ for $i \neq j$. Then for any real numbers $\theta_1, \dots, \theta_k$ with at least one $\theta_i \notin \mathbb{Q}\big(\cup_{i=1}^k \{\lambda_i, \mu_i\}\big)$ the double sequence

$$\sum_{i=1}^{k} \lambda_i^m \mu_i^n \theta_i \bmod 1, \quad m, n = 1, 2, \dots$$

is dense in $[0, 1]$.

Motivated by his conjecture he proved (Theorem 1.6):

Let λ_1, μ_1 and λ_2, μ_2 be two distinct pairs of multiplicatively independent real algebraic integers of degree 2 with absolute values greater than 1, such that the absolute values of their conjugates $\tilde\lambda_1, \tilde\mu_1, \tilde\lambda_2, \tilde\mu_2$ are also greater than 1. Let $\mu_1 = g_1(\lambda_1)$ for some $g_1 \in \mathbb{Z}[x]$ and $\mu_2 = g_2(\lambda_2)$ for some $g_2 \in \mathbb{Z}[x]$. Assume also that at least one element in each pair λ_i, μ_i has all its positive powers irrational. Further let there exist $k, l, k', l' \in \mathbb{N}$ such that

(a) $\min(|\lambda_2|^k|\mu_2|^l, |\tilde\lambda_2|^k|\tilde\mu_2|^l) > \max(|\lambda_1|^k|\mu_1|^l, |\tilde\lambda_1|^k|\tilde\mu_1|^l)$ and,

(b) $\min(|\lambda_1|^{k'}|\mu_1|^{l'}, |\tilde\lambda_1|^{k'}|\tilde\mu_1|^{l'}) > \max(|\lambda_2|^{k'}|\mu_2|^{l'}, |\tilde\lambda_2|^{k'}|\tilde\mu_2|^{l'})$.

Then for any real numbers θ_1, θ_2 with at least one $\theta_i \neq 0$ the sequence

$$\lambda_1^m \mu_1^n \theta_1 + \lambda_2^m \mu_2^n \theta_2 \bmod 1, \quad m, n = 1, 2, \dots$$

is dense in $[0, 1]$. For illustration

$$(\sqrt{23} + 1)^m (\sqrt{23} + 2)^n \theta_1 + (\sqrt{61} + 1)^m (\sqrt{61} - 6)^n \theta_2 \bmod 1, \quad m, n = 1, 2, \dots$$

is dense in $[0, 1]$, provided $(\theta_1, \theta_2) \neq (0, 0)$.

R. Urban noticed that (a) and (b) hold, when

$|\lambda_2| > |\tilde\lambda_2| > |\lambda_1| > |\tilde\lambda_1| > 1$ and $|\mu_1| > |\tilde\mu_1| > |\mu_2| > |\tilde\mu_2| > 1$.

He also noted that Theorem 1.6 can be extended to the case when not all of λ_i, μ_i are of degree 2, but if λ_i, μ_i are rational, then θ_i must be irrational. For example, for every $\theta_2 \neq 0$, the sequence

$$(3 + \sqrt{3})^m 2^n + 5^m 7^n \theta_2 \sqrt{2} \bmod 1, \quad m, n = 1, 2, \dots$$

is dense in $[0, 1]$

Related sequences: 2.8.3

H. Furstenberg: *Disjointness in ergodic theory, minimal sets and a problem in diophantine approximation*, Math. Systems Theory **1** (1967), no. 1, 1–49 (MR0213508 (**35** #4369); Zbl. 0146.28502).

B. Kra: *A generalization of Furstenberg's diophantine theorem*, Proc. Amer. Math. Soc. **127** (1999), no. 7 1951–1956 (MR1487320 (99j:11079); Zbl. 0921.11034)).

R. Urban: *On density modulo 1 of some expressions containing algebraic integers*, Acta Arith. **127** (2007), no. 3 217–229 (MR2310344 (2008c:11102); Zbl. 1118.11034).

2.17.10.2 Let $d, e \in \mathbb{N}$ and $\alpha, \beta \in [1, \infty)$. Then the sequence

$$\frac{\alpha^n n^d}{\beta^m m^e}, \quad n, m = 1, 2, \dots,$$

is

dense in $[0, \infty)$

if and only if one of the following three conditions holds:
(i) α and β are multiplicatively independent;
(ii) $\alpha, \beta > 1$ and $d \neq e$;
(iii) $\beta = 1$ and $e \neq 0$, or $\alpha = 1$ and $d \neq 0$.

F. Durand – M. Rigo: *Syndedicity and independent substitutions*, Adv. in Appl. Math. **42** (2009), 1–22 (MR2475310 (2010c:68133); Zbl. 1160.68028).

2.17.11. If $\gamma > 0$ then the sequence of individual blocks

$$A_n = (ne^{\gamma \frac{1}{n}}, ne^{\gamma \frac{2}{n}}, \dots, ne^{\gamma \frac{n}{n}}) \bmod 1$$

is

u.d.

with discrepancy satisfying

$$D_n \leq c(\gamma) n^{-\frac{1}{3}}.$$

Notes: This can be proved using the Erdős – Turán's inequality and van der Corput lemma. L.P. Usoľtsev (1999) proved that, for the following special L^2 discrepancy, we have

$$\int_0^1 \left(\frac{A([\lambda, \lambda + \tau) \bmod 1; A_n)}{n} - \tau \right)^2 \mathrm{d}\lambda = \mathcal{O}\left(\frac{\log n}{n} \right),$$

where τ is a constant which satisfies $0 < \tau \leq \gamma^2 e^{\gamma} < 1$ and the constant in $\mathcal{O}$ depends on γ.

L.P. Usoľtsev (Usoľcev): *On the distribution of a sequence of fractional parts of a slowly increasing exponential function*, (Russian), Mat. Zametki **65** (1999), no. 1, 148–152 (English translation: Math. Notes **65** (1999), no. 1–2, 124–127 (MR1708299 (2000i:11124); Zbl. 0988.11033)).

2.18 Normal numbers

NOTES: Recall that (cf. Th. 1.8.24.1) the **number α is normal in the base** q if and only if $\alpha q^n \bmod 1$ is u.d. The number α is called **absolutely normal** if it is normal in the base q for all integers $q \geq 2$. The sequence $\alpha q^n \bmod 1$ is also a Lehmer sequence because it satisfies the recurrence relation $x_{n+1} = qx_n \bmod 1$ with $x_0 = \alpha \in (0,1)$. It is u.d. for all integer $q > 1$ and almost all $\alpha \in (0,1)$.

2.18.1. Open problem. It is not known whether the following constants of general interest

$$e, \pi, \sqrt{2}, \log 2, \zeta(3), \zeta(5), \dots$$

are normal in the base 10. All are conjecturally absolutely normal.

NOTES: (I) Each of them resisted every attempts to prove this up to now, cf. [KN, p. 75, Notes].
(II) For instance, the sequence 0123456789 does not appear in the decimal representation of number π up to the 100 000th decimal place, cf. W. Sierpiński (1964, p. 276). For $\sqrt{2}$ cf. E. Borel (1950).
(III) Let $q \geq 2$ denote the scale basis. Let $r(x) = p(x)/q(x)$ be a rational function such that $p(x)$ and $q(x)$ are polynomials with integer coefficients, $p(x)$ is not identically zero, $q(n)$ does not vanish for all positive integer n, and $\deg p < \deg q$. Define the sequence x_n in $[0,1]$ by the recurrence relation

$$x_n = qx_{n-1} + r(n) \bmod 1$$

with $x_0 = 0$, and let $\alpha = \sum_{n=1}^{\infty} r(n)/q^n$. D.H. Bailey and R.E. Crandall (2001) proved:
• the u.d. of x_n implies the normality of α in the base q,
• x_n has a finite attractor if and only if α is rational,
They conjecture that x_n either has a finite attractor or else is u.d.
This results imply: any such α that is irrational is normal in the base q. Together with the facts that
• $\log 2 = \sum_{n=1}^{\infty} 1/(n2^n)$,
• $\pi = \sum_{n=1}^{\infty} r(n)/(16)^n$, where $r(n) = (120n^2 - 89n + 16)/(512n^4 - 1024n^3 + 712n^2 - 206n + 21)$,
• similar expressions are known for π^2 and $\zeta(3)$
this shows that $\log 2$, π, π^2, and $\zeta(3)$ are conditionally normal in the base 2.
(IV) By M.B. Levin(1999) found α with $D_N = O(N^{-1/2})$ as an answer to Korobov (1955) question to find a normal α with minimal discrepancy $D_N(x_n)$.
(V) J. Schiffer (1986) proved: Let $p(x)$ be a non-constant polynomial with rational coefficients, and let d_n, $n = 1, 2, \dots$, be a bounded sequence of rational numbers such that $p(n) + d_n$ is a positive integer for all $n \geq 1$. Then $D_N = O(\log^{-1})N$ for $\alpha = 0.(p(1) + d_1)(p(2) + d_2)\dots$. Moreover, if $p(x) \geq 1$ is a linear polynomial with rational coefficients, then the discrepancy of $\alpha = 0.[p(1)][p(2)]\dots$ satisfies $D_N \geq K/\log N$ for all N and a constant $K > 0$, i.e. such an α is a Champernowne

normal number.
(VI) N.M. Korobov (1966) found a normal number α with $D_N = O(N^{-2/3}\log^{4/3} N)$ if the base q is a prime number, and later M.B. Levin (1977) extended the construction to an arbitrary integral base q.
(VII) M.B. Levin (1999) constructed normal number α such that $D_N = O(N^{-1}\log^2 N)$.

D.H. Bailey – R.E. Crandall: *On the random character of fundamental constant expansions*, Experiment. Math. **10** (2001), no. 2, 175–190 (MR1837669 (2002h:11067); Zbl. 1047.11073).
E. Borel: *Sur les chiffres décimaux de $\sqrt{2}$ et divers problèmes de probabilités en chaine*, C. R. Acad. Sci. Paris **230** (1950), 591–593 (MR0034544 (11,605d); Zbl. 0035.08302).
N.M. Korobov: *Numbers with bounded quotient and their applications to questions of Diophantine approximation*, (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. **19** (1955), no. 5, 361–380 (MR0074464 (17,590a); Zbl. 0065.03202).
N.M. Korobov: *Distribution of fractional parts of exponential function*, (Russian), Vestn. Mosk. Univ., Ser. I **21** (1966), no. 4, 42–46 (MR0197435 (**33** #5600); Zbl. 0154.04801).
M.B. Levin: *On the distribution of fractional parts of the exponential function*, Soviet Math. (Izv. VUZ) **21** (1977), no. 11, 41–47 (translated from Izv. Vyssh. Uchebn. Zaved., Mat. (1977), no. 11(186), 50–57 (MR0506058 (**58** #21963); Zbl 0389.10037)).
M.B. Levin: *On the discrepancy estimate of normal numbers*, Acta Arith. **88** (1999), no. 2, 99–111 (MR1700240 (2000j:11115); Zbl. 0947.11023).
J. Schiffer: *Discrepancy of normal numbers*, Acta Arith. **47** (1986), no. 2, 175–186 (MR0867496 (88d:11072); Zbl. 0556.10036).
W. Sierpiński: *Elementary Theory of Numbers*, Monografie Matematyczne. Tom 42, Panstwowe Wydawnictwo Naukowe, Warszawa, 1964 (MR0175840 (**31** #116); Zbl. 0122.04402).

2.18.2. If α is irrational, then for any integer $q \geq 2$ the set of all limit points of the sequence

$$\alpha q^n \bmod 1$$

is infinite.

Notes:
(I) T. Vijayaraghavan (1940).
(II) A. Dubickas (2006[a]): Set $T(x) = \prod_{n=0}^{\infty}(1 - x^{2^n})$ and $E(x) = \frac{1-(1-x)T(x)}{2x}$. If ξ is an irrational number and $p > 1$ an integer, then the sequence $\|\xi p^n\|$, $n = 1, 2, \dots,$ has a limit point greater than or equal to $\xi_p = E(1/p)/p$, and a limit point smaller than or equal to $\hat{\xi}_p = e(1/p))T(1/p)/2$, where $e(1/p) = 1 - (1/p)$ if p is odd, and $e(1/p) = 1$ if p is even. Furthermore, both bounds are the best possible: in particular, ξ_p, $\hat{\xi}_p$ are irrational and $\|\xi_p p^n\| < \xi_p$, $\|\hat{\xi}_p p^n\| > \hat{\xi}_p$ for every $n = 1, 2, \dots$
(III) A. Dubickas (2007): For an integer $b \leq -2$ and any irrational ξ we have $\liminf_{n\to\infty}\{\xi b^n\} \leq F(-1/b))/q$ and $\limsup_{n\to\infty}\{\xi b^n\} \geq (1 - F(q/p))/q$, where $F(x) = \prod_{k=1}^{\infty}(1 - x^{(2^k+(-1)^{k-1})/3})$. He derived from this that:
(i) $\liminf_{n\to\infty}\{\xi(-2)^n\} < 0.211811$ and $\limsup_{n\to\infty}\{\xi(-2)^n\} > 0.788189$;
(ii) The sequence of integer part $[\xi(-2)^n]$, $n = 0, 1, 2, \dots,$ contains infinitely many numbers divisible by 3 and infinitely many numbers divisible by 4.

A. Dubickas: *On the distance from a rational power to the nearest integer*, J. Number Theory **117** (2006), 222–239 (MR2204744 (2006j:11096); Zbl. 1097.11035).
A. Dubickas: *On a sequence related to that of Thue-Morse and its applications*, Discrete Mathematics **307** (2007), no. 1, 1082–1093 (MR2292537 (2008b:11086); Zbl. 1113.11008).
T. Vijayaraghavan: *On decimals of irrational numbers*, Proc. Indian Acad. Sci., Sect. A **12** (1940), 20 (MR0002325 (2,33d); Zbl. 0025.30803).

2.18.3. Let a_n be an unbounded sequence of positive integers written in q–adic digit expansion, $q \geq 2$. Assume that $\alpha = 0.a_1a_2\dots$ be a normal number in the base q. Then the number

$$\alpha^* = 0.(c_1a_1)(c_2a_2)\dots,$$

where c_n is a bounded sequence of positive integers, and each c_na_n is written in q–adic digit expansion, is

normal in the base q.

Notes: J.–M. Dumont and A. Thomas (1986/87). P. Szüsz and B. Volkmann determined (subject to certain hypotheses) the set of all d.f.'s $G(\alpha^*q^n \bmod 1)$ from the knowledge of $G(\alpha q^n \bmod 1)$.

J.–M. Dumont – A. Thomas: *Une modification multiplicative des nombres g normaux*, Ann. Fac. Sci. Toulouse Math., (5) **8** (1986/87), 367–373 (MR0948760 (89h:11047); Zbl. 0642.10049).
P. Szüsz – B. Volkmann: *On numbers with given digit distributions*, Arch. Math. (Basel) **52** (1989), no. 3, 237–244 (MR0989878 (90h:11068); Zbl. 0648.10031).

2.18.4. Let a_n be an increasing sequence of positive integers written in q–adic digit expansion with $q \geq 2$ an integer such that $\#\{n \in \mathbb{N}; a_n \leq N\} \geq N^\theta$ for every $\theta < 1$ and all sufficiently large N. Then

$$\alpha = 0.a_1a_2\dots$$

is

normal in the base q.

A.H. Copeland – P. Erdős: *Note of normal numbers*, Bull. Amer. Math. Soc. **52** (1946), 857–860 (MR0017743 (8,194b); Zbl. 0063.00962).

2.18.5. Let b_n be an unbounded sequence of positive integers written q–adic digit expansion with $q \geq 2$ an integer. Let b_n^* be another sequence again expressed in the base q such that $b_n^* = b_n + i_n$, where the positive integers i_n satisfy $\log i_n = o(\log b_n)$. If $\alpha = 0.b_1b_2\dots$ is normal in the base q, then

$$\alpha^* = 0.b_1^*b_2^*\dots$$

is also

normal in the base of q.

B. VOLKMANN: *On modifying constructed normal numbers*, Ann. Fac. Sci. Toulouse Math. (5) **1** (1979), no. 3, 269–285 (MR0568150 (82a:10062); Zbl. 0429.10034).

2.18.6. Let p be an odd prime number and $q \geq 2$ an integer not divisible by p. Assume that a_n and b_n, $n = 1, 2, \dots,$ are two strictly increasing sequences of real numbers, which satisfy
(i) $b_{n-1} = o\,(a_n/n)$,
(ii) $a_n = o(\log b_n)$.
Let $\mathbb{O}$ be the ring generated by the set of all numbers x of the form

$$x = \prod_{n=1}^{\infty} \left(1 + \frac{\varepsilon_n}{P_n}\right),$$

where $\varepsilon_n \in \{1, 1\}$ is arbitrary and $P_n = p^{a_n} q^{b_n}$. Then
(iii) $\mathbb{O}$ is uncountable,
(iv) all non–zero numbers $x \in \mathbb{O}$ are normal in the base q,
(v) all $x \in \mathbb{O}$ are non–normal in the base pq.

NOTES: (I) G. Wagner (1995). He mentions as an example that $\sum_{i=1}^{\infty} 2^{-i}5^{-4^i}$ is normal in the base 5 but not in the base 10.
(II) The existence of real numbers which are normal in a given integer base $q \geq 2$ but non–normal in another integer base $h \neq q$, was first proved by J.W. Cassels (1959) for $q \neq 3^m$, $h = 3$ and independently by W.M. Schmidt (1960) who proved this for any pair (q, h) of bases, where $q, h \geq 2$ are multiplicatively independent integers (i.e. $q^m \neq h^n$, $m, n = 1, 2, \dots$).
(III) B. Volkmann (1984, 1985) proved: Let $q, h \geq 2$ be integers with $q^m \neq h^n$, $m, n = 1, 2, \dots,$ and let $V \subset \mathbb{R}^h$ be a closed connected set such that $0 \leq t_i \leq 1$ and $\sum_{i=0}^{h-1} t_i = 1$ for each $(t_0, \dots, t_{h-1}) \in V$. Then there exist (uncountably many) numbers α which are normal in the base q but whose digit frequency vectors[7] $\boldsymbol{\lambda}_N$ in base h have V as its set of limit points.

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[7] If $\alpha = a_0.a_1a_2\dots$ is the h–adic expansion of α, then $\boldsymbol{\lambda}_N = \left(\frac{A(0;N)}{N}, \dots, \frac{A(h-1;N)}{N}\right)$, where $A(j;N)$ is the number of those n, $n \leq N$, for which $a_n = j$.

J.W.S. CASSELS: *On a problem of Steinhaus about normal numbers*, Colloq. Math. **7** (1959), 95–101 (MR0113863 (**22** #4694); Zbl. 0090.26004).
W.M. SCHMIDT: *On normal numbers*, Pacific J. Math. **10** (1960), 661–672 (MR0117212 (**22** #7994); Zbl. 0093.05401).
B. VOLKMANN: *On the Cassels – Schmidt theorem. I*, Bull. Sci. Math. (2) **108** (1984), no. 3, 321–336 (MR0771916 (86g:11044); Zbl. 0541.10045).
B. VOLKMANN: *On the Cassels – Schmidt theorem. II*, Bull. Sci. Math. (2) **109** (1985), no. 2, 209–223 (MR0802533 (87c:11070); Zbl. 0563.10040).
G. WAGNER: *On rings of numbers which are normal to one base but non–normal to another*, J. Number Theory **54** (1995), no. 2, 211–231 (MR1354048 (96g:11093); Zbl. 0834.11032).

2.18.7. Let $f(x)=\alpha_0x^{\beta_0}+\alpha_1x^{\beta_1}+\cdots+\alpha_kx^{\beta_k}$ be a generalized polynomial where α's and β's are real numbers such that $\beta_0>\beta_1>\cdots>\beta_k\ge 0$. Assume that $f(x)\ge 1$ for $x\ge 1$ and that $q\ge 2$ is a fixed integer. Put

$$\alpha = 0.[f(1)][f(2)]\dots,$$

where the integer part $[f(n)]$ is represented in the q–adic digit expansion. Then α is

normal in the base q

and

$$D_N(\alpha q^n)=\mathcal{O}\left(\frac{1}{\log N}\right).$$

If $f(x)$ is a linear polynomial with rational coefficients and $f(n)\ge 1$ for $n=1,2,\dots$, then there exists a positive constant c such that

$$D_N(\alpha q^n)\ge\frac{c}{\log N}$$

for infinitely many N.

NOTES: (I) If $f(x)$ is a non–constant polynomial with rational coefficients all of whose values at $x=1,2,\dots,$ are positive integers then the normality of α in base 10 was proved by H. Davenport and P. Erdős (1952). However, they did not give explicit estimates for the discrepancy.
(II) J. Schoißengeier (1978) showed that $D_N=\mathcal{O}((\log\log N)^{4+\varepsilon}/\log N)$. J. Schiffer (1986) gave the best possible result $D_N=\mathcal{O}(1/\log N)$ (cf. [DT, p. 105, Th. 1.118–9]).
(III) If $f(x)$ is a generalized polynomial then the normality of α in the base q was studied by Y.–N. Nakai and I. Shiokawa who in the series of papers (1990, [a]1990, 1992) found the best possible discrepancy. They give the following examples $\alpha=0.1247912151822\dots$ with $f(x)=x^{\sqrt{2}}$, and $\alpha=0.151222355069\dots$ with $f(x)=\sqrt{2}x^2$.
(IV) The first classical example $\alpha_0=0.123456789101112\dots$ of a simple normal

number given by Champernowne (1933) is a special case of the above construction with $f(x) = x$ and $q = 10$. G. Pólya and G. Szegő (1964, p. 71, No. 166, 170) proved the u.d. of $\alpha_0 10^n \bmod 1$ without mentioning that α_0 is normal in the base 10. The normality of α_0 was also proved by S.S. Pillai (1940) and an elegant proof based on a weaker form of Th. 1.8.24 was given by A.G. Postnikov (1960).
(V) K. Mahler (1937) proved that α defined by an integer polynomial $f(x)$ is a transcendental number of the non–Liouville type.
(VI) Let a_n, $n = 1, 2, \dots$, be a strictly increasing sequence of positive integers represented in the decimal expansion and put $\alpha = 0.a_1a_2a_3\dots$. P. Martinez (2001) proved that if α is rational, then there exit a positive constant c and a real number $t > 1$ such that $a_n \geq ct^n$ for all n. E.g. $\alpha = 0.23571113\dots$ is irrational, cf. 2.18.8.

D.G. Champernowne: *The construction of decimals normal in the scale ten*, J. London Math. Soc., **8** (1933), 254–260 (JFM 59.0214.01; Zbl. 0007.33701).
H. Davenport – P. Erdős: *Note on normal decimals*, Canad. J. Math. **4** (1952), 58–63 (MR0047084 (13,825g); Zbl. 0046.04902).
K. Mahler: *Arithmetische Eigenschaften einer Klasse von Dezimalbrüchen*, Nederl. Akad. Wetensch. Proc. Ser. A **40** (1937), 421–428 (Zbl. 0017.05602; JFM 63.0156.01).
P. Martinez: *Some new irrational decimal fractions*, Amer. Math. Monthly **108** (2001), no. 3, 250–253 (MR1834705 (2002b:11096); Zbl. 1067.11506).
Y.–N. Nakai – I. Shiokawa: *A class of normal numbers*, Japan. J. Math. (N.S.) **16** (1990), no. 1, 17–29 (MR1064444 (91g:11081); Zbl. 0708.11037).
[a] Y.–N. Nakai – I. Shiokawa: *A class of normal numbers. II*, in: *Number Theory and Cryptography (Sydney, 1989)*, (J.H. Loxton ed.), Cambridge University Press, Cambridge, London Math. Soc. Lecture Note Ser., Vol. 154, 1990, pp. 204–210 (MR1055410 (91h:11074); Zbl. 0722.11040).
Y.–N. Nakai – I. Shiokawa: *Discrepancy estimates for a class of normal numbers*, Acta Arith. **62** (1992), no. 3, 271–284 (MR1197421 (94a:11113); Zbl. 0773.11050).
S.S. Pillai: *On normal numbers*, Proc. Indian Acad Sci., sec. A **12** (1940), 179–184 (MR0002324 (2,33c); Zbl. 0025.30802).
G. Pólya – G. Szegő: *Aufgaben und Lehrsätze aus der Analysis*, Vol. 1 & 2, 3rd corr. ed., Grundlehren d. math. Wiss., Band 19, Springer Verlag, Berlin, Göttingen, Heidelberg, New York, 1964 (MR0170985 (**30** #1219a); MR0170986 (**30** #1219b); Zbl. 0122.29704).
A.G. Postnikov: *Arithmetic modeling of random processes*, Trudy Math. Inst. Steklov. (Russian), **57** (1960), 1–84 (MR0148639 (**26** #6146); Zbl. 0106.12101).
J. Schiffer: *Discrepancy of normal numbers*, Acta Arith. **47** (1986), no. 2, 175–186 (MR0867496 (88d:11072); Zbl. 0556.10036).
J. Schoissengeier: *Über die Diskrepanz von Folgen* (ab^n), Österreich. Akad. Wiss. Math.–Natur. Kl. Abt. Sitzungsber. II **187** (1978), no. 4–7, 225–235 (MR0547935 (81a:10064); Zbl. 0417.10031).

2.18.8. Let $f(x)$ be a non–constant polynomial which takes positive integral values at all positive integers. The number

$$\alpha = 0.f(2)f(3)f(5)f(7)f(11)\dots,$$

where $f(p)$ is represented in the q–adic digit expansion and p runs through the primes, is

normal in the integral base q.

NOTES: Y.–N. Nakai and I. Shiokawa (1997). The normality of $\alpha = 0.235711\ldots$ with respect to base $q = 10$ was conjectured by D.G. Champernowne (1933) and proved by A.H. Copeland and P. Erdős (1946), cf. 2.18.4.

D.G. CHAMPERNOWNE: *The construction of decimals normal in the scale ten*, J. London Math. Soc., **8** (1933), 254–260 (JFM 59.0214.01; Zbl. 0007.33701).
A.H. COPELAND – P. ERDŐS: *Note of normal numbers*, Bull. Amer. Math. Soc. **52** (1946), 857–860 (MR0017743 (8,194b); Zbl. 0063.00962).
Y.–N. NAKAI – I. SHIOKAWA: *Normality of numbers generated by the values of polynomials at primes*, Acta Arith. **81** (1997), no. 4, 345–356 (MR1472814 (98h:11098); Zbl. 0881.11062).

2.18.9. The function $f(x)$ is said to have the **growth exponent** β if $\frac{\log f(x)}{\log x} \to \beta$ as $x \to \infty$. If both $f(x)$ and $f'(x)$ possess the growth exponents and the growth exponent of $f(x)$ is positive, then the number

$$\alpha = 0.\left[|f(a_1)|\right]\left[|f(a_2)|\right]\left[|f(a_3)|\right]\ldots$$

where $\left[|f(a_n)|\right]$ $\big($and $\left[|g(n)|\right]$ below$\big)$ are understood as the integer parts represented in the q–adic digit expansion with $q \geq 2$ an integer, is

normal in the base q

for every increasing sequence a_n of positive integers for which $\#\{n \in \mathbb{N};\, a_n \leq N\} \geq N^{1-\varepsilon}$ for all $N \geq N_0(\varepsilon)$ and all $\varepsilon > 0$. For example, if p_n denotes the nth prime and $g(n) = p_{p_n}^2$, then

$$\beta = 0.\left[|g(1)|\right]\left[|g(2)|\right]\left[|g(3)|\right]\ldots$$

is

normal in the base q.

P. SZÜSZ – B. VOLKMANN: *A combinatorial method for constructing normal numbers*, Forum Math. **6** (1994), no. 4, 399–414 (MR1277704 (95f:11053); Zbl. 0806.11034).

2.18.10. Let $\delta \in (0,1]$ and $f : [1,\infty) \to \mathbb{R}$ be a twice differentiable function such that for some constants c_1, c_2, c_3, c_4, c_5 and for all sufficiently large x we have
(i) $c_1 x^\delta < f(x) < c_2 x^\delta$,
(ii) $f'(x)$ is monotone and $c_3 x^{\delta-1} < f'(x) < c_4 x^{\delta-1}$,
(iii) $f''(x)$ is continuous and $|f''(x)| < c_5 x^{\delta-2}$.
Further, let d_n be a bounded sequence of real numbers such that $f(n) + d_n$ is a positive integer for all $n = 1, 2, \ldots$. Then the number

$$\alpha = 0.(f(1)+d_1)(f(2)+d_2)\ldots$$

with every $(f(n)+d_n)$ expressed in the base $q=10$ is

normal in this base

and for its discrepancy we have

$$D_N(\alpha 10^n \bmod 1) = \mathcal{O}\left(\frac{1}{\log N}\right).$$

NOTES: J. Schiffer (1986), who demonstrated the result on $\alpha = 0.[a][a2^\sigma][a3^\sigma]\dots$ for $a>0$ and $0<\sigma\le 1$.

J. SCHIFFER: *Discrepancy of normal numbers*, Acta Arith. **47** (1986), no. 2, 175–186 (MR0867496 (88d:11072); Zbl. 0556.10036).

2.18.11. Let $P(x)$ be a polynomial with real coefficients, $q\ge 2$ an integer and $\alpha\ne 0$ a real number. If $\alpha q^n \bmod 1$ is u.d. then also

$$\alpha q^n + P(n) \bmod 1, \quad n=1,2,\dots,$$

is

u.d.

NOTES: D.P. Parent (1984, p. 291, Solution 5.31).

D.P. PARENT: *Exercises in Number Theory*, Problem Books in Mathematics, Springer Verlag, New York, 1984 (MR0759342 (86f:11002); Zbl. 0536.10001). (French original: *Exercices de théorie des nombres*, Gauthier – Villars, Paris, 1978 (MR0485646 (**58** #5471); Zbl. 0387.10001)).

2.18.12. Let α be a normal number in the integral base $q\ge 2$. Then for every nonzero rational $\frac{a}{b}$ the product

$$\frac{a}{b}\alpha$$

is also

normal in the base q.

NOTES: M. Mendès France (1967). In D.P. Parent (1984, p. 254, Solution 5.30) a weaker result is proved, namely that $\frac{\alpha}{q-1}$ is normal in the base q.

M. MENDÈS FRANCE: *Nombres normaux. Applications aux fonctions pseudo–aléatoires*, J. Analyse Math. **20** (1967), 1–56 (MR0220683 (**36** #3735); Zbl. 0161.05002).
D.P. PARENT: *Exercises in Number Theory*, Problem Books in Mathematics, Springer Verlag, New York, 1984 (MR0759342 (86f:11002); Zbl. 0536.10001). (French original: *Exercices de théorie des nombres*, Gauthier – Villars, Paris, 1978 (MR0485646 (**58** #5471); Zbl. 0387.10001)).

2.18.13. Let p be an odd prime and q one of its primitive roots $\mod p^2$. Then the number

$$\alpha = \sum_{n=1}^{\infty} p^{-n} q^{-p^n}$$

is

a transcendental non–Liouville number, and
normal in the base q^k for each integer $k > 0$.

NOTES: R.G. Stoneham (1973, [a]1973). He also gave more general constructions of normal numbers.

R.G. STONEHAM: *On the uniform ε–distribution of residues within the periods of rational fractions with applications to normal numbers*, Acta Arith. **22** (1973), 371–389 (MR0318091 (**47** #6640); Zbl. 0276.10029).
[a] R.G. STONEHAM: *On absolute (j, ε)–normality in the rational fractions with applications to normal numbers*, Acta Arith. **22** (1972/73), 277–286 (MR0318072 (**47** #6621); Zbl. 0276.10028).

2.18.14. Let λ_n and μ_n, $n = 0, 1, 2, \dots$, be two increasing sequences of positive integers and p, q, are coprime such that $\mu_n \geq p^{\lambda_n}$ for all $n = 0, 1, 2, \dots$. Then the number

$$\alpha = \sum_{n=0}^{\infty} p^{-\lambda_n} q^{-\mu_n}$$

is

normal in the base q.

NOTES: A.N. Korobov (1990). As an example he gives the normal number $\alpha = \sum_{n=0}^{\infty} p^{-2^n} q^{-p^{2^n}}$.

A.N. KOROBOV: *Continued fraction expansions of some normal numbers*, (Russian), Mat. Zametki **47** (1990), no. 2, 28–33, 158 (Zbl. 0689.10059). (English translation: Math. Notes **47** (1990), no. 1–2, 128–132 (MR1048540 (91c:11044); Zbl. 0704.11019)).

2.18.15. Let $q \geq 2$ be an integer and $x_n \bmod 1$ a completely u.d. sequence. Then the number

$$\alpha = \sum_{n=1}^{\infty} \frac{\left[q\{x_n\}\right]}{q^n}$$

is

normal in the base q.

NOTES: N.M. Korobov (1948). M. Mendès France (1967) proved that the set E of real numbers α which can be written in the form $\alpha = \sum_{n=1}^{\infty} \frac{[q\{f(n)\}]}{q^n}$, where $f(n)$ runs through all real polynomials, has Hausdorff dimension 0 and that it contains no normal numbers.

RELATED SEQUENCES: For a multi–dimensional variant see 3.2.4.

N.M. KOROBOV: *On functions with uniformly distributed fractional parts*, (Russian), Dokl. Akad. Nauk SSSR **62** (1948), 21–22 (MR0027012 (10,235e); Zbl. 0031.11501).
M. MENDÈS FRANCE: *Nombres normaux. Applications aux fonctions pseudo–aléatoires*, J. Analyse Math. **20** (1967), 1–56 (MR0220683 (**36** #3735); Zbl. 0161.05002).

2.18.16. If α is a non–zero real number and $q \geq 2$ an integer then the sequence

$$\alpha q^n \bmod 1$$

has a.d.f. $g(x)$ if and only if

$$\int_0^1 f(x)\,\mathrm{d}g(x) = \int_0^1 f(qx)\,\mathrm{d}g(x)$$

for every continuous $f(x)$ which is defined on $[0,1]$, cf. 2.17.1(IV).

I.I. PJATECKIĬ–ŠAPIRO (I.I. ŠAPIRO – PJATECKIĬ): *On the laws of distribution of the fractional parts of an exponential function* (Russian), Izv. Akad. Nauk SSSR, Ser. Mat., Ser. Mat. **15** (1951), 47–52 (MR0043145 (13,213d); Zbl. 0042.04902).

$\boxed{\textbf{2.18.16.1}}$ Every distribution function $g(x)$ of αq^n mod 1 with integer $q > 1$ satisfies the functional equation

$$g(x) = \sum_{i=0}^{n-1}(g((x+i)/q) - g(i/q)).$$

I.I. PJATECKIĬ–ŠAPIRO (I.I. ŠAPIRO – PJATECKIĬ): *On the laws of distribution of the fractional parts of an exponential function* (Russian), Izv. Akad. Nauk SSSR, Ser. Mat., Ser. Mat. **15** (1951), 47–52 (MR0043145 (13,213d); Zbl. 0042.04902).

2.18.17. Let α be a non–zero real and $q \geq 2$ be an integer. If the sequence

$$x_n = \alpha q^n \bmod 1$$

has absolutely continuous a.d.f. $g(x)$, then $g(x) = x$ and thus the sequence x_n is

u.d.

I.I. Pjateckiĭ–Šapiro (I.I. Šapiro – Pjateckiĭ): *On the laws of distribution of the fractional parts of an exponential function* (Russian), Izv. Akad. Nauk SSSR, Ser. Mat., Ser. Mat. **15** (1951), 47–52 (MR0043145 (13,213d); Zbl. 0042.04902).

2.18.18. Let α be a non–zero real number and $q \geq 2$ be an integer. If the sequence

$$x_n = \alpha q^n \bmod 1$$

is u.d., then, for every integer $k \geq 1$, the subsequence

$$x_{kn} = \alpha q^{kn}, \quad n = 1, 2, \ldots,$$

is also

u.d.

In other words, α is normal in the base q if and only if α is normal in the base q^k.

Notes: I.I. Šapiro – Pjateckiĭ (1951), another proof can be found in [KN, p. 72, Th. 8.2].

I.I. Pjateckiĭ–Šapiro (I.I. Šapiro – Pjateckiĭ): *On the laws of distribution of the fractional parts of an exponential function* (Russian), Izv. Akad. Nauk SSSR, Ser. Mat., Ser. Mat. **15** (1951), 47–52 (MR0043145 (13,213d); Zbl. 0042.04902).

2.18.19. Let α be a real number, $q \geq 2$ an integer and

$$x_n = \alpha q^n \bmod 1.$$

If there exist two positive constants c and σ such that, for every subinterval $I \subset [0, 1]$ with $|I| > 0$, we have

$$\limsup_{N \to \infty} \frac{A_N(I; N; x_n)}{N} < c|I| \left(1 + \log \frac{1}{|I|}\right)^{\sigma},$$

then the sequence x_n is

u.d.

Notes: A.G. Postnikov (1952) who extended in this way an earlier result of I.I. Šapiro – Pjateckiĭ (1951) in which the right–hand side has the form $c|I|$ and which can be used to prove the normality of the Champernowne sequence, cf. 2.18.7. The Šapiro – Pjateckiĭ's result is reproduced in [KN, p. 71, Lemma 8.1] in the form: If for any non–negative continuous function f on $[0,1]$ we have

$$\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(\{\alpha q^n\}) \le c\int_0^1 f(x)\,\mathrm{d}x,$$

then α is normal in the base q.

Related sequences: 2.8.7, 3.11.5.

A.G. Postnikov: *On distribution of the fractional parts of the exponential function*, Dokl. Akad. Nauk. SSSR (N.S.) (Russian), **86** (1952), 473–476 (MR0050637 (14,359d); Zbl. 0047.05202).
I.I. Pjateckiĭ–Šapiro (I.I. Šapiro – Pjateckiĭ): *On the laws of distribution of the fractional parts of an exponential function* (Russian), Izv. Akad. Nauk SSSR, Ser. Mat., Ser. Mat. **15** (1951), 47–52 (MR0043145 (13,213d); Zbl. 0042.04902).

2.18.20. Let α be normal in the integer base $q \ge 2$ and

$$x_n = \alpha q^n \bmod 1.$$

If $f : [0,1] \to \mathbb{R}$ is non–constant and Riemann integrable, then the sequence

$$y_n = f(x_1) + f(x_2) + \cdots + f(x_n) \bmod 1$$

is

u.d.

Notes: P. Liardet (1981). He illustrated this results taking

$$y_n = \sum_{i=0}^{n-1} \sin(\pi\alpha q^i) \bmod 1$$

which is u.d. for any α normal in the base q.

P. Liardet: *Propriétés génériques de processus croisés*, Israel J. Math. **39** (1981), no. 4, 303–325 (MR0636899 (84k:22009); Zbl. 0472.28013).

2.18.21. Let $f \ge 1$ be an integer and $\theta = \frac{f+\sqrt{f^2+4}}{2}$. The **generalized Fibonacci sequence** is defined by

$$F_{k+1} = fF_k + F_{k-1}, \quad F_0 = 1, \quad F_1 = f, \quad k = 1, 2, \dots .$$

Every positive integer n can be uniquely expressed in the form (generalized Zeckendorf expansion)

$$n = \sum_{k=0}^{L(n)} a_k(n)F_k,$$

where $F_{L(n)} \le n < F_{L(n)+1}$, $0 \le a_k(n) \le f$, $a_0(n) \le f-1$, and $a_k(n) = f$ implies $a_{k-1}(n) = 0$.
Let $a_{L(n)} \dots a_0(n)$ be the string of digits of n in this expansion. Let $b_1b_2\dots$ be the concatenation of these strings for all positive integers n when n are written successively in the natural (i.e. increasing) order. The real number

$$\beta = \sum_{k=1}^{\infty} b_k\theta^{-k}$$

is called **Champernowne number in the base** θ and the sequence

$$\beta\theta^n \bmod 1$$

is

$$\text{u.d.}$$

i.e. β is normal in the base θ (cf. the def. 1.8.24 Note (IV)).

P.J. Grabner: *On digits expansions with respect to second order linear recurring sequences*, in: *Number–theoretic analysis (Vienna, 1988–89)*, Lecture Notes in Math., 1452, Springer, Berlin, 1990, 58–64 (MR1084638 (92d:11078); Zbl. 0721.11027).

2.18.22. Let r_n, $n = 1, 2, \dots$, be the sequence of all non–reduced fractions $\frac{a}{b} \in (0,1)$ ordered with respect to their increasing denominator, i.e. the first group (containing only one term) is formed by fractions with denominator 2, then follows the group of rational numbers with denominator 3, then with denominator 4, etc. The terms within each group are ordered according to their increasing numerators, i.e.

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{2}{6}, \dots.$$

Now let $r_n = [0; a_{n,1}, a_{n,2}, \dots, a_{n,l(n)}]$ be the continued fraction expansion of r_n with $a_{n,l(n)} \ne 1$, and let

$$\alpha = [0; a_{1,1}, a_{2,1}, a_{3,1}, a_{3,2}, a_{4,1}, \dots, a_{n,1}, a_{n,2}, \dots, a_{n,l(n)}, \dots](= [0; a_1, a_2, \dots])$$

i.e. the partial quotients a_i of α are obtained by concatenation of the partial quotients of $r_1, r_2, r_3, \dots$ successively in the given order (a Champernowne's type expansion). Then the sequence

$$\alpha_n = [0; a_{n+1}, a_{n+2}], \quad n = 1, 2, \dots,$$

has the Gaussian a.d.f.

$$g(x) = \frac{\log(1+x)}{\log 2},$$

i.e. α is continued fraction normal (for the def. cf. p. 1 – 37).

R. Adler – M. Keane – M. Smorodinsky: *A construction of normal number for the continued fraction transformation*, J. Number Theory **13** (1981), no. 1, 95–105 (MR0602450 (82k:10070); Zbl. 0448.10050).

2.19 Sequences involving primes

See also: 2.14.6, 2.18.8, 2.18.9

2.19.1. The sequence

$$p_n\theta \bmod 1, \text{ where } p_n \text{ is the } n\text{th prime and } \theta \text{ is irrational}$$

is

u.d.

Notes: I.M. Vinogradov (1937, 1948), cf. [KN, p. 22]. For u.d. of polynomial sequences $q(p_n)$ mod 1 see 2.19.4.

I.M. Vinogradov: *The representation of odd numbers as a sum of three primes*, (Russian), Dokl. Akad. Nauk SSSR **15** (1937), 291–294 (Zbl 0016.29101).
I.M. Vinogradov: *On an estimate of trigonometric sums with prime numbers* (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. **12** (1948), 225–248 (MR0029418 (10,599b); Zbl. 0033.16401).

2.19.2. If α is non–integral then the sequence

$$p_n^\alpha \bmod 1,$$

is

u.d.

and if $\alpha > 1$ then

$$\pi(N) D^*_{\pi(N)} < N^{1-\delta},$$

for $N > C(\alpha)$, where $\delta = (15000\alpha^2)^{-1}$ and as usual

$$\pi(N)D^*_{\pi(N)} = \sup_{x\in[0,1]} \left| \sum_{\substack{p_n\le N \\ \{p_n^\alpha\}<x}} 1 - \pi(N)x \right|.$$

NOTES: (I) The u.d. of θp_n^α mod 1 for $0 < \alpha < 1$ and $\theta > 0$ was first proved by I.M. Vinogradov (1940). He proved $\pi(N)D^*_{\pi(N)} = \mathcal{O}(N^{1+\varepsilon}\Delta)$, where $\Delta = (\theta N^{\alpha-1} + \theta^{-1}N^{-\alpha} + N^{-2\alpha/3})^{1/5}$.
(II) The u.d. of p_n^α mod 1 with $\alpha > 1$ and α non–integral was proved by I.M. Vinogradov in (1948) with $\pi(N)D^*_{\pi(N)} = \mathcal{O}(N^{1-\delta})$. In (1959) he found $\delta = (34.10^6)^{-1}$ for $\alpha \ge 6$ satisfying $\|\alpha\| \ge 1/3^\alpha$.
(III) I.E. Stux (1974) proved for the extremal discrepancy the estimate $\pi(N)D_{\pi(N)} = \mathcal{O}\left(\frac{N\log\log N}{\log^2 N}\right)$ for $0 < \alpha < 1$. A result by I.I. Pjateckiĭ – Šapiro (1953) implies $\pi(N)D^*_{\pi(N)} = \mathcal{O}(N^{1-\delta})$ for $\frac{2}{3} \le \alpha < 1$. D. Wolke (1975) reproved the u.d. of p_n^α for $0 < \alpha < 1$.
(IV) D. Leitmann (1976) recovered $\pi(N)D^*_{\pi(N)} = \mathcal{O}(N^{1-\delta})$ for $\alpha > 1$, and α non–integral, by a modification of the method used by I.I. Pjateckiĭ – Šapiro (1953).
(V) A. Balog (1983) proved that $\pi(N)D^*_{\pi(N)} = \mathcal{O}(N^{\frac{1+\alpha}{2}+\varepsilon})$ for $\frac{1}{2} \le \alpha < 1$, more precisely that $\mathcal{O}(N^{\frac{1+\alpha}{2}}\omega^2\log^8 N + \gamma N\omega^{-1}\log^{-1} N)$, where $0 \le \gamma \le 1$ and $1 \le \omega \le N^{\frac{1}{25}}$. For $\alpha = \frac{1}{2}$ this gives $\pi(N)D^*_{\pi(N)} = \mathcal{O}(N^{\frac{4}{5}+\varepsilon})$ for every $\varepsilon > 0$. The same result was reproved by G. Harman (1983) using sieve methods. A very interesting consequence says, if $\varepsilon > 0$ then $\left\{\sqrt{p}\right\} < p^{-\frac{1}{4}+\varepsilon}$ holds for infinitely many primes p. (The well–known conjecture H, which claims that there are infinitely many primes of the form n^2+1, is equivalent to $\left\{\sqrt{p}\right\} < p^{-1/2}$ for infinitely many primes p.)
(VI) R.C. Baker and G. Kolesnik (1985) found the sharpest $\delta = (15000\alpha^2)^{-1}$ for large α at present. For $\alpha = \frac{3}{2}$ they proved $\pi(N)D^*_{\pi(N)} = \mathcal{O}(N^{\frac{157}{168}+\varepsilon})$.
(VII) Recently, X. Cao and W. Zhai (1999) showed that the estimate $\pi(N)D^*_{\pi(N)} = \mathcal{O}(N^{1-\delta+\varepsilon})$ holds for $\frac{5}{3} \le \alpha < 3$, $\alpha \ne 2$, where

$$\delta = \delta(\alpha) = \begin{cases} 1/26, & \text{if } 5/3 \le \alpha \le 45/26, \\ (5-2\alpha)/40, & \text{if } 45/26 < \alpha \le 2.1, \alpha \ne 2, \\ 1/50, & \text{if } 2.1 < \alpha \le 317/150, \\ (9-3\alpha)/133, & \text{if } 317/150 < \alpha \le 347/160, \\ (5-\alpha)/151, & \text{if } 347/160 < \alpha \le 129/56, \\ (3-\alpha)/39, & \text{if } 129/56 < \alpha < 3. \end{cases}$$

(VIII) S.A. Gritsenko (1986) proved that if $1 < c \le 2$ and $\varepsilon > 0$ then there exists an $N_0(\varepsilon)$ such that the asymptotic formula $A([0,1/2);\pi(N);x_n) = \frac{\pi(N)}{2} + \mathcal{O}(R)$ holds

for the sequence $x_n = \frac{1}{2}p_n^{1/c}$ mod 1 and for $N \geq N_0(\varepsilon)$, where $R = N^{1/2+1/2c+\varepsilon}$ if $1 < c \leq 4/3$, and $R = N^{1-1/2c+(\sqrt{3/c}-1)^2+\varepsilon}$ if $4/3 < c \leq 2$.
(IX) A new method how to prove the u.d. of θp_n^α mod 1 ($\theta \neq 0, \alpha \in (0,1)$) can be found in J. Schoißengeier (1979).

Related sequences: 2.15.1

R.C. Baker – G. Kolesnik: *On distribution of p^α modulo one*, J. Reine Angew. Math. **356** (1985), 174–193 (MR0779381 (86m:11053); Zbl. 0546.10027).
A. Balog: *On the fractional part of p^θ*, Arch. Math. **40** (1983), 434–440 (MR0707732 (85e:11063); Zbl. 0517.10038).
X. Cao – W. Zhai: *On the distribution of p^α modulo one*, J. Théor. Nombres Bordeaux **11** (1999), no. 2, 407–423 (MR1745887 (2001a:11121); Zbl. 0988.11027).
S.A. Gritsenko: *A problem of I. M. Vinogradov*, Mat. Zametki **39** (1986), no. 5, 625–640 (MR0850799 (87g:11082); Zbl. 0612.10029).
G. Harman: *On the distribution of $\sqrt{p}$ modulo one*, Mathematika **30** (1983), 104–116 (MR0720954 (85e:11051); Zbl. 0504.10019).
D. Leitmann: *On the uniform distribution of some sequences*, J. London Math. Soc. **14** (1976), 430–432 (MR0432566 (**55** #5554); Zbl. 0343.10025).
I.I. Pjateckiĭ–Šapiro: *On the distribution of prime numbers in sequences of the form* $[f(n)]$ (Russian), Mat. Sb. (N.S.) **33(75)** (1953), 559–566 (MR0059302 (15,507e); Zbl. 0053.02702).
J. Schoissengeier: *The connection between the zeros of the ζ-function and sequences $(g(p))$, p prime* mod 1, Monatsh. Math. **87** (1979), no. 1, 21–52 (MR0528875 (80g:10054); Zbl. 0401.10046).
I.E. Stux: *On the uniform distribution of prime powers*, Comm. Pure Appl. Math. **27** (1974), 729–740 (MR0366844 (**51** #3090); Zbl. 0301.10039).
I.M. Vinogradov: *A general property of prime numbers distribution*, (Russian), Mat. Sbornik (N.S.) **7(49)** (1940), 365–372 (MR0002361 (2,40a); Zbl. 0024.01503).
I.M. Vinogradov: *On an estimate of trigonometric sums with prime numbers* (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. **12** (1948), 225–248 (MR0029418 (10,599b); Zbl. 0033.16401).
I.M. Vinogradov: *Estimate of a prime–number trigonometric sum* (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. **23** (1959), 157–164 (MR0106881 (**21** #5611); Zbl. 0088.03902).
D. Wolke: *Zur Gleichverteilung einiger Zahlenfolgen*, Math. Z. **142** (1975), 181–184 (MR0371839 (**51** #8056); Zbl. 0286.10018).

2.19.3. If $\theta > 0$ then the sequence

$$\theta p_n^{3/2} \bmod 1$$

is

u.d.

and

$$\pi(N)D_{\pi(N)} = \mathcal{O}(N^{1+\varepsilon-(1/56)}),$$

where $\varepsilon > 0$ is arbitrarily small.

Notes: The proof in E.P. Golubeva and O.M. Fomenko (1979) uses the method developed by I.M. Vinogradov (1940).

E.P. Golubeva – O.M. Fomenko: *On the distribution of the sequence $bp^{3/2}$ modulo 1*, in: *Analytic number theory and the theory of functions, 2*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **99** (1979), 31–39 (MR0566506 (81f:10061); Zbl. 0437.10017).
I.M. Vinogradov: *A general property of prime numbers distribution*, (Russian), Mat. Sbornik (N.S.) **7(49)** (1940), 365–372 (MR0002361 (2,40a); Zbl. 0024.01503).

2.19.4. Let $q(x)$ be a polynomial with real coefficients and let p_n, $n = 1, 2, \dots$, be the increasing sequence of all primes. Then the sequence

$$q(p_n) \bmod 1$$

is

$$\text{u.d.}$$

if and only if the polynomial $q(x) - q(0)$ has at least one irrational coefficient.
Notes: (I) This was implicitly proved by I.M. Vinogradov (1946, 1947, 1948) (see MR **48**#2087 by H. Niederreiter), and in full generally by G. Rhin (1973). Vinogradov (1946, 1948) proved that

$$\pi(N)D_{\pi(N)} = \mathcal{O}(N^{1-\rho}),$$

when in the polynomial $q(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_0$ there exits a coefficient a_i, $1 \le i \le k$, having the Diophantine approximation $|a_i - (A/Q)| \le 1/(QN^{i/2})$ with $0 < Q \le N^{i/2}$, where $\rho = 0.04/(k^2(\log k + 2))$ if $Q > N^{0.25}$ and $\rho = 0.36 \log Q/(k^2(\log(k^2/\log Q) + 4))$ if $Q \le N^{0.25}$.
(II) The sequence $q(p_n) \bmod 1$ is u.d. also when p_n runs over primes in an arithmetical progression. I. Allakov (2003) studied the discrepancy D_M of the finite sequence $\alpha p_n^k \bmod 1$ with integer $k \ge 2$, $p_n \equiv b \bmod B$, $n = 1, 2, \dots, M$, and $M = \pi(N; B, b)$ denoting the number of primes $\le N$ in the arithmetical progression $nB + b$ with $\gcd(b, B) = 1$. If $|\alpha q - a| < 1/q$ and $\gcd(a, q) = 1$ then he proved that

$$MD_M \ll \frac{N}{B}\left(\Delta \log q + \left(\frac{qN}{\delta}\right)^{\varepsilon}\left(\frac{d^k}{q} + \frac{B}{N}\right)^{1/2^{k-1}}\right),$$

where $B^2 \le N$, $d = \gcd(q, B)$ and

$$\Delta = \left(\frac{N}{\delta B}\right)^{\varepsilon}\left(\frac{d^{2k-1}}{q} + \frac{B}{\sqrt{N}} + q\delta\left(\frac{B}{Nd}\right)^k\right)^{1/2^{2k-2}}.$$

Here $\varepsilon > 0$ is arbitrary.
Related sequences: 2.14.1, 2.19.5

I. Allakov: *On the distribution of fractional parts of a sequence $\{\alpha p^k\}$ with prime arguments in an arithmetic progression*, (Russian), in: *Proceeding of the V International Conference "Algebra and Number Theory: Modern Problems and Applications", (Tula 2003)*, Chebyshevskiĭ Sb., **4**, (2003), no. 2(6), 30–37 (MR2038590 (2004m:11119); Zbl. 1116.11059).

G. Rhin: *Sur la répartition modulo 1 des suites $f(p)$*, Acta Arith. **23** (1973), 217–248 (MR0323731 (**48** #2087); Zbl. 0264.10026).
I.M. Vinogradov: *A general distribution law for the fractional parts of values of a polynomial with the variable running over the primes*, (Russian), Dokl. Akad. Nauk SSSR **51** (1946), no. 7, 491–492 (MR0016371 (8,6b); Zbl. 0061.08803).
I.M. Vinogradov: *The Method of Trigonometrical Sums in the Theory of Numbers*, (Russian), Trav. Inst. Math. Stekloff, Vol. 23, (1947) (MR0029417 (10,599a); Zbl. 0041.37002) *Translated, revised and annotated by K.F. Roth and A. Davenport*, Interscience Publishers, London, New York, 1954 (MR0062183 (15,941b); Zbl. 0055.27504).
I.M. Vinogradov: *On an estimate of trigonometric sums with prime numbers* (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. **12** (1948), 225–248 (MR0029418 (10,599b); Zbl. 0033.16401).

2.19.5. Let p_n, $n = 1, 2, \ldots$, be the increasing sequence of all primes. Let $q(x)$ be a polynomial of degree $h \geq 1$ with real coefficients and with positive leading coefficient, and let c be a positive real number.
(I) If hc is not an integer, then the sequence

$$q^c(p_n) \bmod 1$$

is

u.d.

(II) If hc is an integer, there exists a polynomial $r(x)$ of degree hc with real coefficients and a function ψ analytic in a neighbourhood of 0 such that $\psi(0) = 0$ and $q^c(x) = r(x) + \psi(1/x)$. Then the sequence

$$q^c(p_n) \bmod 1$$

is

u.d.

if and only if $r(x) - r(0)$ has at least one irrational coefficient.
Notes: This is an extension of 2.19.4.

Related sequences: 2.14.1, 2.19.4

P. Toffin: *Condition suffisantes d'équirépartition modulo 1 de suites $(f(n))_{n\in N}$ et $(f(p_n))_{n\in N}$*, Acta Arith. **32** (1977), no. 4, 365–385 (MR0447137 (**56** #5452); Zbl. 0351.10023).

2.19.6. Let $r(x) = \frac{f(x)}{g(x)}$, where $f(x), g(x) \in \mathbb{Z}[x]$ are coprime, and denote by $P_{r,q}$ the set of all primes $p \leq q$ such that $g(p) \not\equiv 0 \pmod q$. If $r(x)$ is not a linear polynomial, then the sequence of blocks

$$A_q = \left(\frac{r(p) \pmod q}{q} \right)_{p \in P_{r,q}}$$

is

u.d.

C. Cobeli – M. Vâjâitu – A. Zaharescu: *Equidistribution of rational functions of primes* mod q, J. Ramanujan Math. Soc. **16** (2001), no. 1, 63–73 (MR1824884 (2002b:11102); Zbl. 1007.11049).

2.19.7. The sequence

$$(\log p_n)^\sigma \bmod 1, \quad n = 1, 2, \dots,$$

is for $\sigma > 1$

u.d.,

but for $\sigma = 1$ it is

not u.d. (a consequence of 2.2.8).

Notes: Y.–H. Too (1992). This is a special case of 2.19.11 with $f(x) = (\log x)^\sigma$.

Related sequences: 2.3.6, 2.12.1, 2.19.8.

Y.–H. Too: *On the uniform distribution modulo one of some log–like sequences*, Proc. Japan Acad. Ser. A, Math. Sci. **68** (1992), no. 9, 269–272 (MR1202630 (94a:11114); Zbl. 0777.11027).

2.19.7.1 Let p_n, $n = 1, 2, \dots$, be the increasing sequence of all primes. The sequence

$$\log p_n \bmod 1, \quad n = 1, 2, \dots,$$

has the same d.f.s as $\log n \bmod 1$, i.e.

$$G(\log p_n \bmod 1) = G(\log n \bmod 1).$$

Also, for every $i = 1, 2, \dots,$

$$G(\log(p_n \log^{(i)} p_n) \bmod 1) = G(\log n \bmod 1).$$

Notes: Y. Ohkubo (2011). It follows from his theorem 2.19.14.1.

Y. Ohkubo: *On sequences involving primes*, Unif. Distrib. Theory **6** (2011), no. 2, 221–238 (MR2904049; Zbl. 1313.11090)

$\boxed{\mathbf{2.19.7.2}}$ Let b be a numeration base, $\log_b x$ be the logarithm in base b, p_n be the nth prime number and P_n be the product of the first n prime numbers (the so called primorial number).
Then the sequence

$$x_n = \log_b P_n \bmod 1, \quad n = 1, 2, \dots$$

is u.d.

with the discrepancy

$$D_N(x_n) \le C_b \frac{(\log\log N)^{1/2}}{(\log N)^{1/9}},$$

where the constant C_b is depends on b.
(I) The authors use the interpretation that the u.d. of x_n means that the sequence P_n satisfies the strong Benford's law, cf. 2.12.1.1.

B. Massé – D. Schneider: *The mantissa distribution of the primorial numbers*, Acta Arith. **163** (2014), no. 1, 45–58 (MR3194056; Zbl. 1298.11074).

2.19.8. Let p_n be the nth prime, and c_i, $i = 0, 1, 2, \dots, k-1$, be real numbers with $\sum_{i=0}^{k-1} c_i \neq 0$. The sequence

$$\sum_{i=0}^{k-1} c_i \log p_{n+i} \bmod 1, \quad n = 1, 2, \dots,$$

has the same distribution functions as the sequence

$$\left(\sum_{i=0}^{k-1} c_i\right) \log p_n \bmod 1, \quad n = 1, 2, \dots,$$

and they are

$$G\,(c \log p_n \bmod 1) = \left\{ \frac{e^{x/c} - 1}{e^{1/c} - 1} e^{-u/c} + (e^{\min(x/c, u/c)} - 1) e^{-u/c} : u \in [0, 1] \right\}, \tag{1}$$

where $c = \sum_{i=0}^{k-1} c_i$.
Notes: (I) A. Wintner (1935) has shown that $x_n = \log p_n \bmod 1$ is not u.d. A proof can be found in D.P. Parent (1984, pp. 282–283, Solut. 5.19). S. Akiyama (1996,

1998) proved that x_n is not almost u.d., i.e. $x \notin G(x_n)$.
(II) R.E. Whitney (1972) generalized the result from 2.12.1(V) and proved that $\log p_n$ mod 1 is u.d. with respect to the logarithmic weighted means, i.e.

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n}\right)^{-1}\sum_{n=1}^{N}\frac{c_{[0,x)}(\{\log p_n\})}{n}=x$$

for all $x\in[0,1]$, i.e. p_n is a weak Benford sequence (see 2.12.26).
(III) D.I.A. Cohen and T.M. Katz (1984) have shown the u.d. of $\log p_n$ mod 1 with respect to the zeta distribution, i.e. (see 1.8.7)

$$\lim_{\alpha\to1^+}\frac{1}{\zeta(\alpha)}\sum_{n=1}^{\infty}\frac{c_{[0,x)}(\{\log p_n\})}{n^\alpha}=x$$

for all $x\in[0,1]$.
(IV) The complete solution (1) is given by Y. Ohkubo (2011). It follows from an estimate of S. Akiyama (1998) that

$$\lim_{n\to\infty}\left(\sum_{i=0}^{k-1}c_i\log p_{n+i}-\left(\sum_{i=0}^{k-1}c_i\right)\log p_n\right)=0,$$

which implies $G\left(\sum_{i=0}^{k-1}c_i\log p_{n+i} \bmod 1\right)=G\left(\left(\sum_{i=0}^{k-1}c_i\right)\log p_n \bmod 1\right)$ and then Ohkubo used his theorem in 2.19.14.1.
(V) Since $x_n=\log p_n$ mod 1 is not u.d., the sequence of primes p_n, $n=1,2,\dots$, is not a strong Benford sequence (see 2.12.26), but we can solve the first digit problem as follows: Express all primes p_n in the base q. Let $K=k_1.q^{r-1}+k_2.q^{r-2}+\dots+k_{r-1}.q+k_r=k_1k_2\dots k_r$, $k_1\neq0$, $0\le k_i\le q-1$, $i=1,2,\dots,r$, be considered as an r-consecutive block of digits in base q. Similarly as the result of A.I. Pavlov (1981) in 2.12.1.(X), the result of Y. Ohkubo (2011) in 2.19.7.1. implies

$$\liminf_{N\to\infty}\frac{\#\{n\le N;\ \text{first } r\text{-digits of } p_n=K\}}{N}=\frac{1}{(q-1)K},$$
$$\limsup_{N\to\infty}\frac{\#\{n\le N;\ \text{first } r\text{-digits of } p_n=K\}}{N}=\frac{q}{(q-1)(K+1)}.$$

Related sequences: 2.3.6, 2.12.1, 2.19.7.

S. Akiyama: *A remark on almost uniform distribution modulo 1*, in: *Analytic number theory (Japanese) (Kyoto, 1994)*, Sūrikaisekikenkyūsho Kōkyūroku No. 958, 1996, pp. 49–55 (MR1467998 (99b:11081); Zbl. 0958.11507).
S. Akiyama: *Almost uniform distribution modulo 1 and the distribution of primes*, Acta Math. Hungar. **78** (1998), no. 1–2, 39–44 (MR1604062 (99b:1108); Zbl. 0902.110273).
Quoted in: 2.19.8
D.I.A. Cohen – T.M. Katz: *Prime numbers and the first digit phenomenon*, J. Number Theory **18** (1984), 261–268 (MR0746863 (85j:11014); Zbl. 0549.10040).

D.P. PARENT: *Exercises in Number Theory*, Problem Books in Mathematics, Springer Verlag, New York, 1984 (MR0759342 (86f:11002); Zbl. 0536.10001). (French original: *Exercices de théorie des nombres*, Gauthier – Villars, Paris, 1978 (MR0485646 (**58** #5471); Zbl. 0387.10001)).
Y. OHKUBO: *On sequences involving primes*, Unif. Distrib. Theory **6** (2011), no. 2, 221–238 (MR2904049; Zbl. 1313.11090)
A.I. PAVLOV: *On the distribution of fractional parts and F.Benford's law*, Izv. Aka. Nauk SSSR Ser. Mat. (Russian), **45** (1981), no. 4, 760–774 (MR0631437 (83m:10093); Zbl. 0481.10049).
R.E. WHITNEY: *Initial digits for the sequence of primes*, Amer. Math. Monthly **79** (1972), no. 2, 150–152 (MR0304337 (**46** #3472); Zbl. 0227.10047).
A. WINTNER: *On the cyclical distribution of the logarithms of the prime numbers*, Quart. J. Math. Oxford (1) **6** (1935), 65–68 (Zbl. 0011.14904).

2.19.9. If α, β, $\beta \neq 0$, are real numbers then the sequence

$$\alpha p_n + \beta \log p_n \bmod 1$$

is

u.d.

with respect to the logarithmically weighted means and has logarithmic discrepancy

$$L_N \leq c(\beta)(\log N)^{-1}$$

NOTES: R.C. Baker and G. Harman (1990) applied 1.10.7.2 to prove this.

RELATED SEQUENCES: 2.12.31

R.C. BAKER – G. HARMAN: *Sequences with bounded logarithmic discrepancy*, Math. Proc. Cambridge Philos. Soc. **107** (1990), no. 2, 213–225 (MR1027775 (91d:11091); Zbl. 0705.11040).

$\boxed{\textbf{2.19.9.1}}$ For every irrational θ the sequence

$$p_n\theta + \log p_n, \quad n = 1, 2, \dots,$$

is

u.d. mod 1.

NOTES: Y. Ohkubo (2011). The result follows from the fact that every u.d. sequence x_n mod 1 is statistically independent with $\log p_n$ mod 1, where p_n, $n = 1, 2, \dots,$ is the increasing sequence of all primes (see 2.3.6.2).

Y. OHKUBO: *On sequences involving primes*, Unif. Distrib. Theory **6** (2011), no. 2, 221–238 (MR2904049; Zbl. 1313.11090)

2.19.10. If $s_q(p_n)$ denotes (cf. 2.9.1) the sum of the q–adic digits of the nth prime p_n in its q–adic digit expansion, then

$$s_q(p_n)\theta \bmod 1$$

is

$$\text{u.d. for every irrational } \theta.$$

NOTES: Ch. Mauduit and J. Rivat (2005) filled the gap in the proof in M. Olivier (1971).

CH. MAUDUIT – J. RIVAT: *Propriétés q-multiplicatives de la suite* $\lfloor n^c \rfloor$, $c > 1$, Acta Arith. **118** (2005), no. 2 187–203 (MR2141049 (2006e:11151); Zbl. 1082.11058)).
M. OLIVIER: *Sur le développement en base g des nombres premiers*, C.R. Acad. Sci. Paris Sér. A–B **272** (1971), A937–A939 (MR0277492 (**43** #3225); Zbl. 0215.35801).

2.19.11. Let $f : [a, \infty) \to (0, \infty)$ with $a > 0$. Then the fulfilment of any of the following blocks of assumptions, denoted as (I), (II), and (III), implies that the sequence

$$\alpha f(p_n) \bmod 1, \quad n = n_0, n_0 + 1, n_0 + 2, \dots,$$

is

$$\text{u.d. for every non–zero real number } \alpha.$$

The corresponding discrepancies are different as given below.

(I) (i1) f be a differentiable function,
(i2) $f'(x) \log x$ be monotone for a sufficiently large x,
(i3) $\lim_{x\to\infty} x f'(x) = \infty$,
(i4) $f(x) = o((\log x)^\varepsilon)$ for some $\varepsilon > 0$.
Then

$$D_N \ll \sqrt{\frac{f(p_N)}{(\log p_N)^\varepsilon}} + \max\left(\frac{1}{N}, \frac{1}{p_N f'(p_N)}\right).$$

(II) (ii1) f be a twice differentiable function with $f' > 0$,
(ii2) $\lim_{x\to\infty} x^2 f''(x) = \infty$,
(ii3) $(\log x)^2 f''(x)$ be non–increasing in x for a sufficiently large x,
(ii4) $f(x) = o((\log x)^\varepsilon)$ for some $\varepsilon > 0$.
Then

$$D_N \ll \sqrt{\frac{f(p_N)}{(\log p_N)^\varepsilon}} + \sqrt{\frac{1}{p_N^2 f''(p_N)}}.$$

(III) (iii1) f be a twice differentiable function with $f' > 0$,
(iii2) $\lim_{x\to\infty} x^2 f''(x) = -\infty$,
(iii3) $(\log x)^2 f''(x)$ and $x(\log x)^2 f''(x)$ both be non–increasing for a sufficiently large x,
(iii4) $f(x) = o((\log x)^\varepsilon)$ for some $\varepsilon > 0$.
Then

$$D_N \ll \sqrt{\frac{f(p_N)}{(\log p_N)^\varepsilon}} + \sqrt{\frac{-1}{p_N^2 f''(p_N)}} + \frac{-1}{p_N^2(\log p_N) f''(p_N)}.$$

NOTES: (I) Y.–H. Too (1992, Th. 1) proved this result motivated by the results previously proved by K. Goto and T. Kano (1985). In (1992) they proved a related result subject to the following changes of the assumptions
(i1) f is a continuously differentiable function, and
(i3) $\lim_{x\to\infty} x|f'(x)| = \infty$.
Moreover, the condition (i4) is formulated with $\varepsilon > 1$, but a closer check of the proof shows that the weaker hypothesis $\varepsilon > 0$ is actually used.
(II) This result was proved by Y.–H. Too (1992, Th. 2). However, in K. Goto and T. Kano (1992) (and in Goto and Kano (1985)) again a similar result is stated under the condition that
(ii1) f is continuously differentiable and $f'(x) > 0$ and $f''(x) > 0$,
but without the condition (ii3) (and again under the assumption $\varepsilon > 1$ in (ii4) (cf. notes in 2.19.11)).
(III) This is Theorem 3 from Y.–H. Too (1992). In K. Goto and T. Kano (1985, 1992) a similar result is claimed under the following changes of the assumptions
(iii1) f is twice differentiable with $f \to \infty$, $f' > 0$ and $f'' > 0$,
(iii3) $(\log x)^2 f''(x)$ is increasing
Cf. also comments concerning ε in (I) and (II). In Goto and Kano (1992) the following result is proved:
Proposition . *Let $f(x)$ be a twice differentiable function with $f' > 0$ and $f'' < 0$. If $x^2 f''(x) \to -\infty$, then $xf'x \to \infty$. If $x(\log x)^2 f''(x)$ is increasing, then $(\log x)f'(x)$ is monotone. Moreover, $f'(x)\log x$ is decreasing or increasing according to if $f'(x)$ tends to zero or to a positive constant.*

K. GOTO – T. KANO: *Uniform distribution of some special sequences*, Proc. Japan Acad. Ser. A Math. Sci. **61** (1985), no. 3, 83–86 (MR0796473 (87a:11069); Zbl. 0573.10023).
K. GOTÔ – T. KANO: *Remarks to our former paper "Uniform distribution of some special sequences"*, Proc. Japan Acad. Ser. A Math. Sci. **68** (1992), no. 10, 348–350 (MR1202648 (94a:11111); Zbl. 0777.11026).
Y.–H. TOO: *On the uniform distribution modulo one of some log–like sequences*, Proc. Japan Acad. Ser. A, Math. Sci. **68** (1992), no. 9, 269–272 (MR1202630 (94a:11114); Zbl. 0777.11027).

2.19.12. Let f be an entire function assuming real values on the real axis. Suppose that $|f(z)| \le \exp(\log|z|)^\alpha$ for ($|z| \to +\infty$), where $1 \le \alpha < 4/3$, and suppose that $f - f(0)$ is not a polynomial with rational coefficients. Then the sequence

$$f(p_n) \bmod 1$$

is

$$\text{u.d.}$$

NOTES: R.C. Baker (1984) improving a result of G. Rhin (1975) with $1 \le \alpha < 7/6$. Previously, Rhin (1973) worked with the growth condition $\log\log M(r)/\log\log r < 5/4$, where $M(r) = \sup_{|z|=r}|f(z)|$. M.A. Wodzak (1994) extended Baker's result to the primes in a fixed arithmetic progression.

RELATED SEQUENCES: 2.6.21

R.C. BAKER: *Entire functions and uniform distribution modulo one*, Proc. London Math. Soc. (3) **49** (1984), no. 1, 87–110 (MR0743372 (86h:11055); Zbl. 0508.10023).
G. RHIN: *Répartition modulo 1 de $f(p_n)$ quand f est une série entière*, *Séminaire Delange-Pisot-Poitou (14e année: 1972/73), Théorie des nombres, Fasc. 2, Exp. No. 20*, Secrétariat Mathématique, Paris, 2 pp. (MR0404160 (**53** #7963); Zbl. 0327.10052).
G. RHIN: *Répartition modulo 1 de $f(p_n)$ quand f est une série entière*, in: *Actes Colloq. Marseille – Luminy 1974*, Lecture Notes in Math., Vol. 475, Springer Verlag, Berlin, 1975, pp. 176–244 (MR0392857 (**52** #13670); Zbl. 0305.10046).
M.A. WODZAK: *Primes in arithmetic progression and uniform distribution*, Proc. Amer. Math. Soc. **122** (1994), no. 1, 313–315 (MR1233985 (94k:11084); Zbl. 0816.11042).

2.19.13. Let

- $P(x)$ be a polynomial of degree ≥ 1,
- m be a positive integer,
- $h(x)$ be a periodic function with period 1, k times continuously differentiable with k sufficiently large,
- $h^{(i)}(x)$ has only finitely many zeros in $[0,1]$, for every $i \le k$,
- $|h^{(i)}(x)| + |h^{(i+1)}(x)| + \cdots + |h^{(i+m)}(x)| \ge c > 0$ for some absolute constant $c > 0$, every $i \le k - m$ and all x.

Then the sequence

$$P(p_n)h(p_n\alpha) \bmod 1$$

is

$$\text{u.d.}$$

for every non–Liouville number α (with p_n running over the set of primes).

NOTES: D. Berend, M.D. Boshernitzan and G. Kolesnik (2002, Th. 2.3). The authors noted that the required size of k depends on the degree of $P(x)$, on m and on α (precisely, on u, $0 < u < 1$, for which $\liminf_{q\to\infty} q^{1/u}\|\alpha q\| > 0$).

D. Berend – M.D. Boshernitzan – G. Kolesnik: *Distribution modulo* 1 *of some oscillating sequences. III*, Acta Math. Hungar. **95** (2002), no. 1–2, 1–20 (MR1906205 (2003h:11085); Zbl. 0997.11058).

2.19.14. Let $f(t)$ be a periodic real function with period 1 such that
- $f(t)$ is continuous except for a finite number of points in the interval $[0,1]$,
- $f(t)$ satisfies the Lipschiz condition in each of its intervals of continuity,
- $\int_0^1 f(t)\,\mathrm{d}t = 0$.

Let $g \geq 2$ be a fixed integer and $h(p)$ be an integral valued function at prime arguments p such that
- $h(p) \to \infty$ for $p \to \infty$,
- $h(p) \leq \frac{\log p}{2\log g}$.

Under these assumption the block sequence A_p with

$$A_p = \left(\frac{1}{\sqrt{h(p)}} \sum_{k=0}^{h(p)-1} f\left(\frac{ig^k}{p}\right)\ ;\ i = 0, 1, \dots, p-1 \right),$$

where p runs over the primes with $\gcd(g,p) = 1$ has the a.d.f. $g(x)$ in the interval $(-\infty, \infty)$. In addition to that the limit

$$\sigma^2 = \lim_{p\to\infty} \frac{1}{p} \sum_{i=0}^{p-1} \frac{1}{h(p)} \left(\sum_{k=1}^{h(p)-1} f\left(\frac{ig^k}{p}\right)\right)^2$$

exists, and
- if $\sigma \neq 0$ then $g(x)$ is the so–called **normal d.f.**, i.e.

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2\sigma^2}}\,\mathrm{d}u,$$

- if $\sigma = 0$, then $g(x)$ is the step d.f. $g(x) = c_0(x)$.

Notes: L.P. Usoľcev (1961) proved in this way a discrete analogue to a theorem of M. Kac (1946).

M. Kac: *On the distribution of values of sums of the type* $\sum f(2^k t)$, Ann. of Math. (2) **47** (1946), 33–49 (MR0015548 (7,436f); Zbl. 0063.03091).
L.P. Usoľtsev (Usoľcev): *An analogue of the Fortet – Kac theorem*, (Russian), Dokl. Akad. Nauk SSSR **137** (1961), 1315–1318 (MR0147466 (**26** #4982); Zbl. 0211.49104).

2.19.14.1 Let p_n, $n = 1, 2, \dots$, be the increasing sequence of all primes. Let the real-valued function $f(x)$ be strictly increasing for $x \geq 1$ and let $f^{-1}(x)$ be its inverse function. Assume that
(i) $\lim_{k\to\infty} f^{-1}(k+1) - f^{-1}(k) = \infty$,
(ii) $\lim_{k\to\infty} \frac{f^{-1}(k+u_k)}{f^{-1}(k)} = \psi(u)$ for every sequence $u_k \in [0,1]$ for which $\lim_{k\to\infty} u_k = u$, where this limit defines the function $\psi : [0,1] \to [1, \psi(1)]$,
(iii) $\psi(1) > 1$.
Then

$$G(f(p_n) \bmod 1) = \\ \left\{ g_u(x) = \frac{\min(\psi(x), \psi(u)) - 1}{\psi(u)} + \frac{1}{\psi(u)} \frac{\psi(x) - 1}{\psi(1) - 1}; u \in [0,1] \right\}.$$

The lower d.f. $\underline{g}(x)$ and the upper d.g. $\overline{g}(x)$ of $f(p_n)$ mod 1 are

$$\underline{g}(x) = \frac{\psi(x) - 1}{\psi(1) - 1}, \quad \overline{g}(x) = 1 - \frac{1}{\psi(x)}(1 - \underline{g}(x).)$$

Here

$$\underline{g}(x) = g_0(x) = g_1(x) \in G(f(p_n) \bmod 1)$$

and

$$\overline{g}(x) = g_x(x) \notin G(f(p_n) \bmod 1).$$

If

$$F_N(x) = \frac{\#\{n \leq N; f(p_n) \bmod 1 \in [0, x)\}}{N}$$

denotes the step d.f. and if $f(p_{N_i})$ mod 1 is a subsequence of the sequence $f(p_n)$ mod 1 such that $f(p_{N_i})$ mod 1 $\to u$ then $F_{N_i}(x) \to g_u(x)$ for every $x \in [0,1]$.

NOTES:
(I) Y. Ohkubo (2011).
(II) Compare with Theorem 2.6.18.1.

Y. OHKUBO: *On sequences involving primes*, Unif. Distrib. Theory **6** (2011), no. 2, 221–238 (MR2904049; Zbl. 1313.11090)

2.19.15. Let p_n be the increasing sequence of the all primes. Then the double sequence

$$\frac{p_m}{p_n}, \quad m,n = 1,2,\dots,$$

is

$$\text{dense in } [0,\infty).$$

NOTES: (I) According to W. Sierpiński (1964, p. 155) this result was proved by A. Schinzel.
(II) Independently, J. Smítal (1971) proved that $\frac{p_m}{p_n+1}$ is everywhere dense in $[0,\infty)$.
(III) A well–known conjecture (cf. P. Ribenboim (1988, p. 297)) says that he double sequence

$$\frac{p_m+1}{p_n+1}, \quad m,n = 1,2,\dots,$$

contains all positive rationals.
(IV) If the number of prime-twins $p_n^{(2)}, p_n^{(2)}+2 \le x$ asymptotically equals $cx/\log^2 x$ (cf. Hardy and Wright (1954, p. 412)), then T. Šalát (1969) proved that the double sequence

$$\frac{p_m^{(2)}}{p_n^{(2)}}, \qquad m,n = 1,2,\dots,$$

is dense in $[0,\infty)$, cf. 2.22.2.

G.H. HARDY – E.M. WRIGHT: *An Introduction to the Theory of Numbers*, 3nd edition ed., Clarendon Press, Oxford, 1954 (MR0067125 (16,673c); Zbl. 0058.03301).
P. RIBENBOIM: *The Book of Prime Number Records*, Springer Verlag, New York, 1988 (MR0931080 (89e:11052); Zbl. 0642.10001).
T. ŠALÁT: *On ratio sets of sets of natural numbers*, Acta Arith. **15** (1968/69), 273–278 (MR0242756 (**39** #4083); Zbl. 0177.07001).J. SMÍTAL: *Remarks on ratio sets of sets of natural numbers*, Acta Fac. Rerum Nat. Univ. Comenian. Math. **25** (1971), 93–99 (MR0374079 (**51** #10279); Zbl. 0228.10036).
W. SIERPIŃSKI: *Elementary Theory of Numbers*, Monografie Matematyczne. Tom 42, Panstwowe Wydawnictwo Naukowe, Warszawa, 1964 (MR0175840 (**31** #116); Zbl. 0122.04402).

2.19.16. Let p_n be the nth prime. Then the sequence of blocks

$$X_n = \left(\frac{2}{p_n}, \frac{3}{p_n}, \dots, \frac{p_{n-1}}{p_n}, \frac{p_n}{p_n}\right)$$

is

$$\text{u.d.}$$

and thus also the block sequence $\omega = (X_n)_{n=1}^{\infty}$ is u.d. (cf. 2.3.14).

NOTES: This example by O. Strauch and J.T. Tóth (2001) generalizes 2.19.15. The u.d. implies the following interesting limit

$$\lim_{n\to\infty} \frac{1}{n^2 p_n} \sum_{i,j=1}^{n} |p_i - p_j| = \frac{1}{3}.$$

O. STRAUCH – J.T. TÓTH: *Distribution functions of ratio sequences*, Publ. Math. (Debrecen) **58** (2001), 751–778 (MR1828725 (2002h:11068); Zbl. 0980.11031).

2.19.17. Let p_n be the increasing sequence of all primes and q be a given integer. Then the sequence

$$x_n = \frac{p_n}{q} \bmod 1, \quad n = 1, 2, \dots, \pi(N),$$

has discrepancy

$$D_{\pi(N)} = \mathcal{O}\left(N^{\varepsilon} \log N \left(\sqrt{\frac{1}{q} + \frac{q}{N}} + \frac{1}{N^{\frac{1}{6}}}\right)\right).$$

Consequently, if q and $\frac{N}{q}$ are large enough the distribution of x_n, $n = 1, 2, \dots, \pi(N)$, is (cf. 1.8.23)

asymptotically u.d.

I.M. VINOGRADOV: *An elementary proof of a theorem from the theory of prime numbers* (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. **17** (1953), 3–12 (MR0061622 (15,855f); Zbl. 0053.02703).

2.19.18. Let p_n be the increasing sequence of the all primes and α, β given positive real numbers. Then the double sequences

$$\frac{p_m^{\alpha}}{p_n^{\beta}}, \quad \frac{p_m^{\alpha}}{p_n^{p_n}}, \quad n, m = 1, 2, \dots,$$

are

dense in $[0, \infty)$,

but

$$\frac{p_m^{p_m}}{p_n^{p_n}}, \quad n, m = 1, 2, \dots,$$

is

not dense.

NOTES: J.T. Tóth and L. Zsilinszky (1995). In the joint paper J. Bukor, P. Erdős, T. Šalát and J.T. Tóth (1997, Th. 2.1) the following generalization is given:
Let α_n, $n = 1, 2, \dots$, be the sequence of positive real numbers such that

(i) $\alpha_n = \mathcal{O}(n^{3/8})$,
(ii) $\alpha_{n+1} - \alpha_n = \mathcal{O}(n^{-\varepsilon})$ for some $\varepsilon > 0$.
Then the double sequence

$$\frac{p_m^{\alpha_m}}{p_n^{\alpha_n}}, \quad n, m = 1, 2, \dots,$$

is

dense in $[0, \infty)$.

RELATED SEQUENCES: 2.19.15, 2.19.16, 2.22.4.

J. BUKOR – P. ERDŐS – T. ŠALÁT – J.T. TÓTH: *Remarks on the (R)–density of sets of numbers, II*, Math. Slovaca **47** (1997), no. 5, 517–526 (MR1635220 (99e:11013); Zbl. 939.11005).
J.T. TÓTH – L. ZSILINSKY: *On density of ratio sets of powers of primes*, Nieuw Arch. Wisk. (4) **13** (1995), no. 2, 205–208 (MR1345571 (96e:11013); Zbl. 0837.11009).

2.19.19. If p_n is the increasing sequence of all primes then the sequence

$$x_n = \frac{p_n}{n} \bmod 1.$$

has the same set of d.f.'s as $\log(n \log n) \bmod 1$, which in turn has the same set of d.f.'s as $\log n \bmod 1$ (see 2.12.16).

NOTES: O. Strauch and O. Blažeková (2003). They used an old result of M. Cipolla (1902) (cf. P. Ribenboim (1995, p. 249)) that

$$p_n = n \log n + n \log\log n - n + o\left(\frac{n \log\log n}{\log n}\right)$$

and then they applied 2.3.3.
Note that x_n is a subsequence of $\frac{n}{\pi(n)} \bmod 1$ from 2.20.12. These two sequences were introduced at the Number Theory Seminar of Prof. T. Šalát (Bratislava).

M. CIPOLLA: *La determinazione assintotica dell' n^{imo} numero primo*, Napoli Rend. 3 **8** (1902), 132–166 (JFM 33.0214.04).
P. RIBENBOIM: *The New Book of Prime Numbers Records*, Springer–Verlag, New York, 1996 (MR1377060 (96k:11112); Zbl. 0856.11001).
O. STRAUCH – O. BLAŽEKOVÁ: *Distribution of the sequence p_n/n mod 1*, Math. Institute, Slovak Acad. Sci., Bratislava, Slovak Republic, 2003, 15 pp.

$\boxed{\textbf{2.19.19.1}}$ For every irrational θ the sequence

$$p_n\theta + \frac{p_n}{n}, \quad n = 1, 2, \dots,$$

is

$$\text{u.d. mod } 1.$$

NOTES: Y. Ohkubo (2011). This follows from his result 2.3.6.3 and from u.d. of $p_n\theta$ mod 1.

Y. OHKUBO: *On sequences involving primes*, Unif. Distrib. Theory **6** (2011), no. 2, 221–238 (MR2904049; Zbl. 1313.11090)

2.19.19.2

Let $P(n)$ denote the largest prime factor of n.
Let $f(u) = g(\log u)$ where $g : [1, \infty) \to \mathbb{R}$ is a differentiable function.
Let $R(x) := \pi(x) - \mathrm{li}(x)$ be the error function in the Primer Number Theorem.
Further assume that
(i) $vg'(v)$ is increasing and tends to infinity;
(ii) $\lim_{y\to\infty} \int_y^{y^{1+d}} \frac{|R(u)|}{u} |f'(u)| \,\mathrm{d}u = 0$ for any given real number $d > 0$.
Then the sequence

$$f(P(n)) \bmod 1, \quad n = 1, 2, \ldots$$

is u.d.

J.-M. DE KONICK – I. KÁTAI: *The uniform distribution* mod 1 *of sequences involving the largest prime factor function*, Šiauliai Math. Semin. **8(16)** (2013), 117–129 (MR3145622; Zbl 1303.11087).

2.20 Sequences involving number–theoretical functions

See also: 2.3.23, 2.12.27

NOTES: An **arithmetical function** is a function with is defined on the positive integers, i.e. it is a sequence.
An **additive function** is an arithmetical function $f(n)$ which satisfies

$$f(mn) = f(m) + f(n)$$

whenever m and n are coprime integers.
A **strongly additive function** is an additive function which also satisfies

$$f(p^m) = f(p)$$

for every prime power p^m, $m \geq 1$.
A **completely additive function** satisfies

$$f(mn) = f(m) + f(n)$$

for every pair of integers m and n.
A **multiplicative function** is an arithmetical function $g(n)$ which satisfies

$$g(mn) = g(m)g(n)$$

whenever m and n are coprime integers.
A **strongly multiplicative function** is a multiplicative function which also satisfies

$$g(p^m) = g(p)$$

for every prime power p^m, $m \geq 1$.
A **completely multiplicative function** is an arithmetical function $g(n)$ which satisfies

$$g(mn) = g(m)g(n)$$

for every pair of positive integers.
See P.D.T.A. Elliott (1980, p. xv – xvi).

P.D.T.A. Elliott: *Probabilistic Number Theory II. Central Limit Theorems*, Grundlehren der mathematischen Wissenschaften, Vol. 240, Springer Verlag, New York, Heidelberg, Berlin, 1980 (MR0551361 (82h:10002a); Zbl. 0431.10030).

2.20.1. In order that for a given real valued additive function f the sequence

$$f(n) \bmod 1$$

is

$$\text{u.d.},$$

it is both necessary and sufficient that on of the following conditions holds:
(I) for each positive integer k the series

$$\sum_p \frac{\|kf(p) - \tau \log p\|^2}{p}$$

diverges for every real number τ,
(II) its spectrum (cf. 2.4.1 for the def.) is empty, i.e.

$$\mathrm{sp}(f(n)) = \emptyset.$$

NOTES: (I) Cf. the monograph by P.D.T.A. Elliott (1979, p. 284, Th. 8.1).
(II) H. Daboussi and M. Mendès France (1974/1975). They proved that if f is a real additive function then either $\mathrm{sp}(f(n)) = \emptyset$ or $\mathrm{sp}(f(n)) = \mathbb{Q}$ mod 1, and that $f(n)$ mod 1 is u.d. if and only if $\mathrm{sp}(f(n))$ contains no rational number. Consequently, (cf. 2.4.2) if $f(n)$ mod 1 is u.d. then $f(k_n)$ mod 1 is u.d. for any almost periodic sequence k_n.

H. DABOUSSI – M. MENDÈS FRANCE: *Spectrum, almost periodicity and equidistribution modulo 1*, Studia Sci. Math. Hungar. **9** (1974/1975), 173–180 (MR0374066 (**51** #10266); Zbl. 0321.10043).
P.D.T.A. ELLIOTT: *Probabilistic Number Theory I. Mean–value Theorems*, Grundlehren der mathematischen Wissenschaften, Vol. 239, Springer Verlag, New York, Heidelberg, Berlin, 1979 (MR0551361 (82h:10002a); Zbl. 0431.10029).

2.20.2. In order that for an additive function f the sequence

$$f(n) \bmod 1$$

possesses the limiting distribution (i.e. is has the a.d.f.)

$$g(x) \neq x,$$

it is both necessary and sufficient that for some positive integer k the series

$$\sum_p \frac{\|kf(p)\|^2}{p}, \qquad \sum_p \frac{\|kf(p)\|\mathrm{sign}\big((1/2) - \{kf(p)\}\big)}{p}$$

converge. When this condition is satisfied the limit law is continuous if and only if the series

$$\sum_{\|mf(p)\|\neq 0} \frac{1}{p}$$

diverges for every positive integer m.
NOTES: Cf. the monograph by P.D.T.A. Elliott (1979, p. 284, Th. 8.2).

P.D.T.A. ELLIOTT: *Probabilistic Number Theory I. Mean–value Theorems*, Grundlehren der mathematischen Wissenschaften, Vol. 239, Springer Verlag, New York, Heidelberg, Berlin, 1979 (MR0551361 (82h:10002a); Zbl. 0431.10029).

2.20.3. Erdős – Wintner theorem (1939). Let f be an additive arithmetical function. In order that the sequence

$$f(n), \quad n = 1, 2, \dots,$$

possesses the a.d.f.

$$g(x) = \lim_{N\to\infty} \frac{1}{N}\#\{n \le N\ ;\ f(n) < x\} \text{ a.e.}$$

defined on $(-\infty, \infty)$, it is both necessary and sufficient that the series

$$\sum_{|f(p)|>\alpha} \frac{1}{p}, \qquad \sum_{|f(p)|\le\alpha} \frac{f(p)}{p}, \qquad \sum_{|f(p)|\le\alpha} \frac{f^2(p)}{p}$$

converge for some $\alpha > 0$. When this condition is satisfied, the characteristic function $\widehat{g}(t) = \int_{-\infty}^{\infty} e^{itx}\,\mathrm{d}g(x)$ of $g(x)$ may be represented in the form

$$\widehat{g}(t) = \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \sum_{m=1}^{\infty} \frac{e^{itf(p^m)}}{p^m}\right),$$

where the product is taken over all prime numbers. The limiting distribution of the sequence will be continuous if and only if the series

$$\sum_{f(p)\ne 0} \frac{1}{p}$$

diverges.

NOTES: Cf. the monographs by P.D.T.A. Elliott (1979, p. 187, Th. 5.1) and by G. Tenenbaum (1990, p. 358, Th. 1). Elliott (1973, cf. 1979, p. 269, Th. 7.7) also proved: The additive function $f(n)$ possesses the a.d.f. $g(x)$ over $(-\infty, \infty)$ with a finite mean and variance if and only if

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(n) \quad \text{exists, and} \qquad \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} (f(n))^2 < \infty.$$

P.D.T.A. ELLIOTT: *On additive functions whose limiting distributions possess a finite mean and variance*, Pacif. J. Math. **48** (1973), 47–55 (MR0357359 (**50** #9827); Zbl. 0271.10047).

P.D.T.A. ELLIOTT: *Probabilistic Number Theory I. Mean–value Theorems*, Grundlehren der mathematischen Wissenschaften, Vol. 239, Springer Verlag, New York, Heidelberg, Berlin, 1979 (MR0551361 (82h:10002a); Zbl. 0431.10029).

G. TENENBAUM: *Introduction à la théorie analytique et probabiliste des nombres*, Institut Elie Cartan, Vol. 13, Université de Nancy, Nancy, 1990. (second edition: Société de France, Paris, 1995 (MR1366197 (97e:11005a); Zbl. 0880.11001)). (English translation: Studies in Advanced Mathematics, Vol. 46, Cambridge Univ. Press, Cambridge, 1995 (MR1342300 (97e:11005b); Zbl. 0880.11001)).

2.20.4. Let f be an additive arithmetical function. In order that the sequence of differences

$$f(n+1)-f(n),\quad n=1,2,\dots,$$

possesses the a.d.f.

$$g(x)=\lim_{N\to\infty}\frac{1}{N}\#\{n\le N\,;\,f(n+1)-f(n)<x\}\text{ a.e.}$$

defined on $(-\infty,\infty)$, it is both necessary and sufficient that there exists a real number λ such that for the function $h(n)=f(n)-\lambda\log n$ we have

$$\sum_{|h(p)|\le 1}\frac{|h(p)|^2}{p}<\infty,\qquad \sum_{|h(p)|>1}\frac{1}{p}<\infty.$$

When these conditions are satisfied then the characteristic function $\widehat{g}(t)=\int_{-\infty}^{\infty}e^{itx}\,\mathrm{d}g(x)$ of $g(x)$ is given by

$$\widehat{g}(t)=\prod_p\left(1-\frac{2}{p}+2\left(1-\frac{1}{p}\right)\Re\left(\sum_{m=1}^{\infty}\frac{e^{ith(p^m)}}{p^m}\right)\right),$$

where the product is taken over all prime numbers.

NOTES: This analogue of Erdős – Wintner theorem (cf. 2.20.3) was proved by A. Hildebrand (1988). As an application he proves the following conjecture of Erdős from 1946: If $f(n+1)-f(n)\to 0$ as $n\to\infty$ over a set of density one, then $f(n)=\lambda\log n$ for some λ.

A. HILDEBRAND: *An Erdős – Wintner theorem for differences of additive functions*, Trans. Amer. Math. Soc. **310** (1988), no. 1, 257–276 (MR0965752 (90a:11099); Zbl. 0707.11057).

2.20.5. In order that for a real valued multiplicative arithmetical function $f(n)$ the sequence

$$f(n),\quad n=1,2,\dots,$$

possesses the a.d.f.

$$g(x)\text{ defined on }(-\infty,\infty)$$

it is both necessary and sufficient that the three series

$$\sum_{f(p)=0}\frac{1}{p},\qquad \sum_{f(p)\ne 0}\frac{\psi(\log|f(p)|)}{p},\qquad \sum_{f(p)\ne 0}\frac{(\psi(\log|f(p)|))^2}{p}$$

converge (here p denotes primes and $\psi(y) = y$ if $|y| \le 1$ and $\psi(y) = 1$ if $|y| > 1$).
When these conditions are satisfies, the a.d.f. $g(x)$ is symmetric if and only if $f(2^k) = -1$ for every integer k, or the series

$$\sum_{f(p)<0} \frac{1}{p}$$

diverges $\big($here for $g(x)$ to be symmetric means that if both $\pm x$ are continuity points of $g(x)$, then $1 - g(x) = g(-x)\big)$.
The a.d.f. $g(x)$ will be continuous if and only if $f(n)$ is never zero and the series

$$\sum_{|f(p)|\ne 1} \frac{1}{p}$$

diverges.
NOTES: The Bakštis, Galambos, Levin, Timofeev and Tuljaganov theorem, cf. P.D.T.A. Elliott (1979, p. 280, Th. 7.11) and as application cf. 2.20.11 Note (IX).

A. BAKŠTIS: *Limit laws of a distribution of multiplicative arithmetic function. I*, (Russian), Litevsk. Mat. Sb. **8** (1968), no. 1, 5–20 (MR0251000 (**40** #4231)).
P.D.T.A. ELLIOTT: *Probabilistic Number Theory I. Mean–value Theorems*, Grundlehren der mathematischen Wissenschaften, Vol. 239, Springer Verlag, New York, Heidelberg, Berlin, 1979 (MR0551361 (82h:10002a); Zbl. 0431.10029).
J. GALAMBOS: *On the distribution of strongly multiplicative functions*, Bull. London Math. Soc. **3** (1971), 307–312 (MR0291106 (**45** #200); Zbl. 0228.10032).
B.V. LEVIN – N.M. TIMOFEEV – S.T. TULIAGONOV: *Distribution of values of multiplicative functions*, (Russian), Litevsk. Mat. Sb. **13** (1973), no. 1, 87–100, 232 (MR0314790 (**47** #3340); Zbl. 0257.10024).

2.20.6. Let f be an arithmetical function. Suppose that for every $\varepsilon > 0$ there exits a sequence $a_\varepsilon(n)$, $n = 1, 2, \dots$, of positive integers such that
(i) $\lim_{\varepsilon\to 0} \limsup_{T\to\infty} \overline{d}(\{n \in \mathbb{N}\,;\, a_\varepsilon(n) > T\}) = 0$,
(ii) $\lim_{\varepsilon\to 0} \overline{d}(\{n \in \mathbb{N}\,;\, |f(n) - f(a_\varepsilon(n))| > \varepsilon\}) = 0$,
(iii) the asymptotic density $d(\{n \in \mathbb{N}\,;\, a_\varepsilon(n) = k\})$ exists for every $k \ge 1$.
Then the sequence

$$f(n), n = 1, 2, \dots,$$

has the a.d.f.

$$g(x), \quad x \in (-\infty, \infty).$$

NOTES: Cf. the monograph G. Tenenbaum (1990, p. 317, Th. 2).

G. Tenenbaum: *Introduction à la théorie analytique et probabiliste des nombres*, Institut Elie Cartan, Vol. 13, Université de Nancy, Nancy, 1990. (second edition: Société de France, Paris, 1995 (MR1366197 (97e:11005a); Zbl. 0880.11001)). (English translation: Studies in Advanced Mathematics, Vol. 46, Cambridge Univ. Press, Cambridge, 1995 (MR1342300 (97e:11005b); Zbl. 0880.11001)).

2.20.7. Kubilius – Shapiro theorem. Let $f : \mathbb{N} \to \mathbb{R}$ be an additive real valued function, not identically vanishing. Put

$$A(f,N) = \sum_{p^n \le N} \frac{f(p^n)}{p^n},$$

$$B(f,N) = \left(\sum_{p^n \le N} \frac{(f(p^n))^2}{p^n} \right)^{\frac{1}{2}},$$

where p runs over all prime numbers. Define the sequence A_N, $N = 1, 2, \dots,$ of blocks by

$$A_N = \left(\frac{f(1) - A(f,N)}{B(f,N)}, \frac{f(2) - A(f,N)}{B(f,N)}, \dots, \frac{f(N) - A(f,N)}{B(f,N)} \right).$$

Then, the sequence A_N has on $(-\infty, \infty)$ the Gaussian a.d.f.

$$g(x) = \tfrac{1}{2\pi} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, \mathrm{d}t,$$

i.e.

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ n \le N \,;\, \frac{f(n) - A(f,N)}{B(f,N)} < x \right\} = g(x)$$

for all $x \in (-\infty, \infty)$.

Notes: (I) This is a generalization of the **Erdős – Kac theorem:** For every $x \in (-\infty, \infty)$

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ n \le N \,;\, \frac{\omega(n) - \log\log N}{\sqrt{\log\log N}} \le x \right\} = g(x),$$

where $\omega(n)$ is the number of distinct prime divisors of n, cf. A. Hildebrand (1987).
(II) The same holds for the function $\Omega(n)$, the number of prime divisors of n, cf. P.D.T.A. Elliott (1980, p. 26). A further example (p. 30):

$$\lim_{N\to\infty} \frac{1}{\pi(N)} \# \left\{ p \le N \,;\, \frac{\omega(p+1) - \log\log N}{\sqrt{\log\log N}} \le x \right\} = g(x).$$

(III) For the history of the Erdős – Kac theorem, see Elliott (1980, Chapt. 12).

P.D.T.A. Elliott: *Probabilistic Number Theory II. Central Limit Theorems*, Grundlehren der mathematischen Wissenschaften, Vol. 240, Springer Verlag, New York, Heidelberg, Berlin, 1980 (MR0551361 (82h:10002a); Zbl. 0431.10030).
A. Hildebrand: *Recent progress in probabilistic number theory*, Astérique **no. 147–148** (1987), 95–106, 343 (MR0891422 (88g:11051); Zbl. 0624.10045).

2.20.8. Let $f(n)$ be an arithmetical function which satisfies
(i) $f(n) = \sum_{d|n} \Phi(d)$,
(ii) $\sum_{d=1}^{\infty} \frac{|\Phi(d)|}{d} < \infty$
for some arithmetical function Φ. Then the sequence

$$f(n), \quad n = 1, 2, \dots,$$

has the a.d.f.

$$g(x)$$

defined on $(-\infty, \infty)$.

Notes: A.G. Postnikov (1971, p. 219, Th. 6b). This is a consequence of a more general theorem proved by Ju.V. Prochorov (cf. Postnikov (1971, p. 216, Th. 6a)): Let B^1 be the class of all arithmetical functions such that for every $\varepsilon > 0$ there exists an arithmetical periodic function $f_t(n)$ with period t such that

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |f(n) - f_t(n)| \le \varepsilon.$$

For every $f \in B^1$, the sequence $f(n)$ has the a.d.f. on $(-\infty, \infty)$. Note that every arithmetical function $f(n)$ which satisfies (i) and (ii) belongs to class B^1 (cf. Postnikov (1971, p. 202, Lemma)).

A.G. Postnikov: *Introduction to Analytic Number Theory*, (Russian), Izd. Nauka, Moscow, 1971 (MR0434932 (**55** #7895); Zbl. 0231.10001). (for the English translation see (MR0932727 (89a:11001); Zbl. 0641.10001)).

2.20.9. Denote by $\sigma(n)$ the sum of the positive divisors of n. Then the sequence

$$\frac{\sigma(n)}{n}$$

has continuous a.d.f.

$$g(x) \text{ defined on } [1, \infty)$$

and for its discrepancy we have

$$D_N^* = \mathcal{O}\left(\frac{\log\log N}{\log N \log\log\log N}\right).$$

NOTES: (I) This is a result of P. Erdős (1974), cf. P.D.T.A. Elliott (1979, p. 203, Lemma 5.8).
(II) Erdős (1974) claims that combining his result with the method of H.G. Diamond (1973) one can prove that $D_N = \mathcal{O}(1/\log N)$. This strong result does not seem to be immediately available in such a manner, so that this assertion remains an open conjecture, see P.D.T.A. Elliott (1979, p. 219).
(III) F. Luca ([a]2003) proved that, if $M_n = 2^n - 1$ is the nth Mersenne number then the subsequence $\sigma(M_n)/M_n$ is dense in $[1,\infty)$ and it has the a.d.f. (preprint [b]).

H.G. DIAMOND: *The distribution of values of Euler's phi function*, in: *Analytic Number Theory (Proceedings of a conference at the St. Louis Univ., St. Louis, Mo., 1972)*, Proc. Sympos. Pure Math., 24, Amer. Math. Soc., Providence, 1973, pp. 63–75 (MR0337835 (**49** #2604); Zbl. 0273.10036).
P.D.T.A. ELLIOTT: *Probabilistic Number Theory I. Mean–value Theorems*, Grundlehren der mathematischen Wissenschaften, Vol. 239, Springer Verlag, New York, Heidelberg, Berlin, 1979 (MR0551361 (82h:10002a); Zbl. 0431.10029).
P. ERDŐS: *On the distribution of numbers of the form $\sigma(n)/n$ and on some related questions*, Pacific J. Math. **52** (1974), 59–65 (MR0354601 (**50** #7079); Zbl. 0291.10040).
[a] F. LUCA: *On the sum divisors of the Mersenne numbers*, Math. Slovaca **53** (2003), no. 5, 457–466 (MR2038513 (2005a:11151); Zbl. 1053.11529).
[b] F. LUCA: *Some mean values related to average multiplicative orders of elements in finite fields*, Ramanujan J. **9** (2005), no. 1–2, 33–44 (MR2166376 (2006i:11111); Zbl. 1155.11344).

2.20.10. If $\lambda(n)$ denotes the **universal exponent of** n then the sequence

$$\frac{n}{\lambda(n)} \bmod 1$$

is

dense in $[0,1]$.

NOTES: J. Bukor and B. László (2000). The **universal exponent** mod n (or the Carmichael function) is the least number $\lambda(n)$ such that $n|a^{\lambda(n)}-1$ for every integer a with $\gcd(a,n)=1$. If $n = 2^{\alpha_0}p_1^{\alpha_1}\dots p_k^{\alpha_k}$ is the factorization of the positive integer n into different prime factors, then

$$\lambda(n) = \operatorname{lcm}[\lambda(2^{\alpha_0}), \varphi(p_1^{\alpha_1}), \dots, \varphi(p_k^{\alpha_k})],$$

where $\lambda(2)=1$, $\lambda(2^2)=2$ and $\lambda(2^\alpha)=2^{\alpha-2}$ for $\alpha=3,4,\dots$, cf. W. Sierpiński (1964, p. 246).

J. Bukor – B. László: *On the density of the set* $\{n/\lambda(n)\,;\,n\in\mathbb{N}\}$, (Slovak), Acta Mathematica (Nitra) **4** (2000), 73–78.
W. Sierpiński: *Elementary Theory of Numbers*, Monografie Matematyczne. Tom 42, Panstwowe Wydawnictwo Naukowe, Warszawa, 1964 (MR0175840 (**31** #116); Zbl. 0122.04402).

2.20.11. If φ is the Euler totient function then the sequence

$$\frac{\varphi(n)}{n},\quad n=1,2,3,\dots,$$

has in $[0,1]$ singular a.d.f.

$$g_0(x).$$

Notes: (I) I.J. Schoenberg (1928, 1936) proved that this sequence has continuous and strictly increasing a.d.f.
(II) P. Erdős (1939) showed that this a.d.f. is singular. Here a function is **singular**, if it is continuous, strictly monotone and has vanishing derivative almost everywhere on the interval of its definition.
(III) H. Davenport (1933) proved

$$g_0(x)=\sum_{n=1}^{\infty}S_n,\ \text{where } S_n=\frac{1}{a_n}-\sum_{i<n}\frac{1}{[a_i,a_n]}+\sum_{i<j<n}\frac{1}{[a_i,a_j,a_n]}-\dots,$$

where $[a,b]$ is the least common multiple of a and b, and $a_1,\dots,a_n,\dots$ is the sequence of the all positive integers a_i with $\varphi(a_i)/a_i<x$ and there is no $d|a_i$ such that $\varphi(d)/d<x$.
(IV) A.S. (1967) proved that

$$\frac{A([0,x);N;\varphi(n)/n)}{N}=g_0(x)+\mathcal{O}\left(\frac{1}{\log\log N}\right).$$

Proofs and other results by M.M. Tjan and I. Iljasov can be found in the monograph A.G. Postnikov (1971, Chap. 4, Par. 4.8). Tjan (1963) noted that $D_N=\mathcal{O}(1/\log\log\log N)$ and that if f is defined on $[0,1]$ and has the Lipschitz jth derivative here (i.e. $|f^{(j)}(x)-f^{(j)}(y)|<c|x-y|$ for $x,y\in[0,1]$ and a suitable constant $c>0$), then

$$\frac{1}{N}\sum_{n=1}^{N}f\left(\frac{\varphi(n)}{n}\right)=\int_0^1 f(x)\,\mathrm{d}g_0(x)+\mathcal{O}\left(\frac{c(\log\log N)^{j+1}}{\left(\log\frac{cN}{M}\right)^{j+1}}\right),$$

where $M=\max_{x\in[0,1]}|f(x)|$.
(V) O. Strauch (1996) proved that

$$\int_0^1 g_0^2(x)\,\mathrm{d}x=1-\frac{6}{\pi^2}-\frac{1}{2}\lim_{N\to\infty}\frac{1}{N^2}\sum_{m,n=1}^{N}\left|\frac{\varphi(m)}{m}-\frac{\varphi(n)}{n}\right|$$

and also the estimates

$$\frac{2}{\pi^4} \le \lim_{N\to\infty} \frac{1}{N^2} \sum_{m,n=1}^{N} \left| \frac{\varphi(m)}{m} - \frac{\varphi(n)}{n} \right| \le 2\frac{6}{\pi^2}\left(1 - \frac{6}{\pi^2}\right).$$

(VI) F. Luca ([a]2003) proved that, if $M_n = 2^n - 1$ is the nth Mersenne number then the subsequence

$$\frac{\varphi(M_n)}{M_n}, n = 1, 2, \dots$$

is dense in $[0, 1]$ and has the a.d.f. (preprint [b]).
(VI') F. Luca and I.E. Shparlinski (2007) proved the existence of the moments

$$\frac{1}{N} \sum_{n=0}^{N-1} \left(\frac{\varphi(F_n)}{F_n} \right)^k = \Gamma_k + O_k\left(\frac{(\log N)^k}{N} \right)$$

for all $k = 1, 2, \dots$ with some positive constant Γ_k. Thus the sequence

$$\frac{\varphi(F_n)}{F_n}, \quad n = 0, 1, 2, \dots$$

has an a.d.f. Luca in ([a]2003) also proved that $\varphi(F_n)/F_n$ is dense in $[0, 1]$, cf. also F. Luca, V.J. Mejía Huguet and F. Nicolae (2009).
(VII) A formula for the a.d.f. of $n/\varphi(n) \bmod 1$ can be found using 2.3.4.
(VIII) P.D.T.A. Elliott (1979, p. 219) wrote: "From a value–distribution point of view, the behaviour of the sequences $n/\sigma(n)$ and $\varphi(n)/n$ is similar."
(IX) If $\mu(n)$ denotes the Mőbius' function, then the sequence

$$\frac{\mu(n)\varphi(n)}{n}, \quad n = 1, 2, \dots,$$

has continuous symmetric a.d.f.

$$g(x) \text{ defined in } [-1, 1].$$

In this case $g(x)$ is symmetric means that $1 - g(x) = g(-x)$ for $x \in [-1, 1]$. Cf. P.D.T.A. Elliott (1979, p. 282).
(X) W. Schwarz (1962) (cf. A.G. Postnikov (1971, p. 267)) proved: Let $f(x)$ be a polynomial with integer coefficients having non-zero discriminant. Assume that the g.c.d. of the coefficients of $f(x)$ is 1 and $f(n) > 0$ for $n = 1, 2, \dots$. Let $L(d)$ denote the number of solutions of $f(n) \equiv 0 \pmod d$. Then

$$\frac{1}{N} \sum_{n=1}^{N} \frac{\varphi(f(n))}{f(n)} = \prod_{\substack{p=2\\ p-\text{prime}}}^{\infty} \left(1 - \frac{L(p)}{p^2}\right) + \mathcal{O}(\log^c N),$$

where $c > 0$ is a constant.
(XI) For $x_n = \varphi(n)/n$ define the step d.f.

$$F_{k,k+N}(x) = \frac{\#\{n \in (k, k+N]; x_n \in [0, x)\}}{N}.$$

(i) P. Erdős (1946) proved: If $\frac{\log\log\log k}{N} \to 0$ as $N \to \infty$, then $F_{k,k+N}(x) \to g_0(x)$ for $x \in [0,1]$.
(ii) For the proof of (i) he used that $\left(\frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n(t))}{n(t)}\right)^s - \frac{1}{N}\sum_{n=1}^N\left(\frac{\varphi(n)}{n}\right)^s\right) \to 0$, where $n(t) = \prod_{p|n,p\le t} p$, p runs over primes and $t = N$.
(XII) V. Baláž, P. Liardet and O. Strauch (2010) proved:
(i) Necessary and sufficient condition: For any two sequences of N's and k's of positive sequences, $N \to \infty$, we have $F_{(k,k+N]}(x) \to g_0(x)$, for every $x \in [0,1]$, if and only if, for every $s = 1, 2, \dots$ we have $\frac{1}{N}\sum_{k<n\le k+N}\sum_{N<d|n}\Phi(d) \to 0$, with $\Phi(d) = \prod_{p|d}\left(\left(1-\frac{1}{p}\right)^s - 1\right)$ for a squarefree d and $\Phi(d) = 0$ otherwise, where p runs over primes (cf. A.G. Postnikov (1971, p. 360)). In quantitative form:

$$\frac{1}{N}\sum_{k<n\le k+N}\sum_{N<d|n}\Phi(d) = \frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n)}{n}\right)^s - \frac{1}{N}\sum_{n=1}^N\left(\frac{\varphi(n)}{n}\right)^s + O\left(\frac{3^s(1+\log N)^s}{N}\right).$$

(ii) A quantitative form of Erdős' (XI)(ii): For all positive integers k, N and $t = N$ we have

$$\frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n(t))}{n(t)}\right)^s = \frac{1}{N}\sum_{n=1}^N\left(\frac{\varphi(n)}{n}\right)^s + O\left(\frac{3^s(1+\log N)^s}{N}\right)$$

for $s = 1, 2, \dots$.
(iii) This implies that every d.f. $g(x)$ for which $F_{(k,k+N]} \to g(x)$ on $(0,1)$ satisfies

$$\int_0^1 x^s \,\mathrm{d}g(x) \le \int_0^1 x^s \,\mathrm{d}g_0(x),$$

for every $s = 1, 2, \dots$.
(iv) Using the Chinese remainder theorem we can find a sequence of intervals $(k, k+N]$ such that $F_{(k,k+N]}(x) \to c_0(x)$, where d.f. $c_0(x)$ has a step 1 at $x = 0$.
(v) If $F_{(k,k+N]}(x) \to g(x)$ for all $x \in (0,1)$, then $g_0(x) \le g(x)$.
(XI) A. Schinzel and Y. Wang (1958) proved that for any given $(\alpha_1, \alpha_2, \dots, \alpha_{N-1}) \in [0,\infty)^{N-1}$ we can find a sequence of k such that

$$\left(\frac{\varphi(k+2)}{\varphi(k+1)}, \frac{\varphi(k+3)}{\varphi(k+2)}, \dots, \frac{\varphi(k+N)}{\varphi(k+N-1)}\right) \to (\alpha_1, \alpha_2, \dots, \alpha_{N-1}).$$

If a subsequence of k's is such that $\frac{\varphi(k+1)}{k+1} \to \alpha$ then

$$\left(\frac{\varphi(k+1)}{k+1}, \frac{\varphi(k+2)}{k+2}, \dots, \frac{\varphi(k+N)}{k+N}\right) \to (\alpha, \alpha\alpha_1, \alpha\alpha_1\alpha_2, \dots, \alpha\alpha_1\alpha_2\dots\alpha_{N-1}).$$

Given an arbitrary d.f. $\tilde{g}(x)$, there exists a sequence $\alpha_n \in (0,\infty)$, $n = 1,2,\dots$, such that $\alpha_1\alpha_2\dots\alpha_n \in (0,1)$ for every $n = 1,2,\dots$, and moreover the sequence $\alpha_1\alpha_2\dots\alpha_n$, $n = 1,2,\dots$, has a.d.f. $\tilde{g}(x)$. Then there exists $\alpha \in (0,1]$ and a sequence of intervals $(k, k+N]$ such that $F_{(k,k+N]}(x) \to g(x)$ and

$$g(x) = \begin{cases} \tilde{g}\left(\frac{x}{\alpha}\right) & \text{if } x \in [0,\alpha), \\ 1 & \text{if } x \in [\alpha, 1] \end{cases}$$

for $x \in (0,1)$.

Open problem. Describe the distribution of the sequence

$$\left(\frac{\varphi(n)}{n}, \frac{\varphi(n+1)}{n+1}\right), \quad n = 1,2,\dots$$

V. Baláž –P. Liardet – O. Strauch: *Distribution functions of the sequence $\varphi(M)/M, M \in (K, K+N]$ as K, N go to infinity*, INTEGERS **10** (2010), 705–732 (MR2799188; Zbl. 1216.11090).
H. Davenport: *Über numeri abundantes*, Sitzungsber. Preuss. Acad., Phys.–Math. Kl. **27** (1933), 830–837 (Zbl. 0008.19701).
P.D.T.A. Elliott: *Probabilistic Number Theory I. Mean–value Theorems*, Grundlehren der mathematischen Wissenschaften, Vol. 239, Springer Verlag, New York, Heidelberg, Berlin, 1979 (MR0551361 (82h:10002a); Zbl. 0431.10029).
P. Erdős: *On the smoothness of the asymptotic distribution of additive arithmetical functions*, Amer. Journ. Math. **61** (1939), 722–725 (MR0000248 (1,41a); Zbl. 0022.01001, JFM 65.0165.02).
P. Erdős: *Some remarks about additive and multiplicative functions*, Bull. Amer. Math. Soc. **52** (1946), 527–537 (MR0016078 (7,507g); Zbl. 0061.07901).
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[a] F. Luca: *On the sum divisors of the Mersenne numbers*, Math. Slovaca **53** (2003), no. 5, 457–466 (MR2038513 (2005a:11151); Zbl. 1053.11529).
[b] F. Luca: *Some mean values related to average multiplicative orders of elements in finite fields*, Ramanujan J. **9** (2005), no. 1–2, 33–44 (MR2166376 (2006i:11111); Zbl. 1155.11344).
F. Luca – I.E. Shparlinski: *Arithmetic functions with linear recurrences*, J. Number Theory **125** (2007), 459–472 (MR2332599 (2008g:11157); Zbl. 1222.11117).
F. Luca – V.J. Mejía Huguet – F. Nicolae: *On the Euler function of Fibonacci numbers*, J. Integer Sequences **9** (2009), A09.6.6 (MR2544925 (2010h:11005); Zbl. 1201.11006).
A.G. Postnikov: *Introduction to Analytic Number Theory*, (Russian), Izd. Nauka, Moscow, 1971 (MR0434932 (**55** #7895); Zbl. 0231.10001). (for the English translation see (MR0932727 (89a:11001); Zbl. 0641.10001)).
A. Schinzel – Y. Wang: *A note on some properties of the functions $\phi(n)$, $\sigma(n)$ and $\theta(n)$*, Bull. Acad. Polon. Sci. Cl. III **4** (1956), 207–209 (MR0079024 (18,17c); Zbl. 0070.04201).
I.J. Schoenberg: *Über die asymptotische Verteilung reeller Zahlen* mod 1, Math. Z. **28** (1928), 171–199 (MR1544950; JFM 54.0212.02).
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O. Strauch: *Integral of the square of the asymptotic distribution function of $\phi(n)/n$*, Math. Institute, Slovak Acad. Sci., Bratislava, Slovak Republic, 1996, 7 pp.
M.M. Tjan: *Remainder terms in the problem of the distribution of values of two arithmetic functions*, (Russian), Dokl. Akad. Nauk SSSR **150** (1963), 998–1000 (MR0154845 (**27** #4789)).

2.20.12. Open problem. Riemann hypothesis implies that the sequence

$$\frac{n}{\pi(n)} \bmod 1, \quad n = 1, 2, \dots,$$

is not u.d. Find all its d.f.'s.

NOTES: (I) Under the Riemann hypothesis $\pi(x) = \mathrm{li}(x) + \mathcal{O}(\sqrt{x}\log x)$ which implies $\lim_{n\to\infty}(n/\pi(n)) - (n/\mathrm{li}(n)) = 0$ the sequences $n/\pi(n) \pmod 1$ and $n/\mathrm{li}(n) \pmod 1$ have the same d.f.'s if we prove the continuity of all d.f.'s of $n/\mathrm{li}(n)$ mod 1 at 0 and 1, cf. 2.3.3. Niederreiter's theorem 2.2.8 implies that the sequence $n/\pi(n)$ mod 1 is not u.d. (probably without the Riemann hypothesis).
(II) **Solution:** F. Luca (2006) (personal communication) noticed that not assuming the Riemann hypothesis it can be proved that the sequences $n/\pi(n)$ and $\log n$ have the same d.f.'s mod 1. This immediately follows from identities
$\left|\frac{n}{\pi(n)} - \frac{n}{\mathrm{li(n)}}\right| = \mathcal{O}((\log n)^2 \exp(-c\sqrt{\log n})) = o(1)$,
$\left|\frac{n}{\mathrm{li(n)}} - \frac{n}{f(n)}\right| = \mathcal{O}((\log n)^{-1}) = o(1)$, where $f(n) = \frac{n}{\log n} + \frac{n}{(\log n)^2}$,
$\frac{n}{f(n)} = \log(n) - 1 + o(1)$.

2.20.13. The sequence

$$t(n) = \sum_{d|n} \frac{1}{2^d}, \quad n = 1, 2, \dots,$$

has continuous a.d.f.

$$g(x) \text{ defined on } [1/2, 1].$$

NOTES: The function $t(n)$ was introduced by E.V. Novoselov (1960) in connection with his theory of polyadic numbers. M.M. Tjan (1963) claimed the existence of the a.d.f. of $t(n)$ (note that this fact also follows from the result 2.20.8 which was proved later). The continuity of $g(x)$ was proved by E.V. Novoselov (1964). B.M. Širokov (1973) studied the sequence $1 - t(n)$ (which actually is the polyadic norm of n) on $[0, 1/2]$ and he found an explicit form of its a.d.f. (in the proof and the formulation the functions of polyadic numbers were instrumental) and its discrepancy $D_N = \mathcal{O}\left(\frac{1}{\log\log N}\right)$.

E.V. NOVOSELOV: *Topological theory of divisibility of integers*, (Russian), Učen. Zap. Elabuž. Gos. Ped. Inst. **8** (1960), 3–23.(RŽ Mat. 1961#10A157).
E.V. NOVOSELOV: *A new method in probabilistic number theory*, (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 307–364 (MR0168544 (**29** #5805); Zbl. 0213.33502).

B.M. Shirokov: *The distribution of the values of a polyadic norm*, Sov. Math., Dokl. **14** (1973), 148–150 (translated from Doklad. Akad. Nauk SSSR **208** (1973), no. 3, 553–554). (MR0323745 (**48** #2101); Zbl. 0284.10021).
M.M. Tjan: *Remainder terms in the problem of the distribution of values of two arithmetic functions*, (Russian), Dokl. Akad. Nauk SSSR **150** (1963), 998–1000 (MR0154845 (**27** #4789)).

2.20.14. Define the strongly additive arithmetical function $f(n)$ by its values at primes p by

$$f(p) = \begin{cases} \dfrac{(-1)^{\frac{p-1}{2}}}{(\log\log p)^{3/4}}, & \text{if } p > e^e, \\ 0, & \text{otherwise.} \end{cases}$$

Then the sequence

$$f(n), \quad n = 1, 2, \dots,$$

has absolutely continuous a.d.f.

$$g(x) \text{ defined on } (-\infty, \infty).$$

Notes: P. Erdős (1939), cf. P.D.T.A. Elliott (1979, pp. 219–220).

P.D.T.A. Elliott: *Probabilistic Number Theory I. Mean–value Theorems*, Grundlehren der mathematischen Wissenschaften, Vol. 239, Springer Verlag, New York, Heidelberg, Berlin, 1979 (MR0551361 (82h:10002a); Zbl. 0431.10029).
P. Erdős: *On the smoothness of the asymptotic distribution of additive arithmetical functions*, Amer. Journ. Math. **61** (1939), 722–725 (MR0000248 (1,41a); Zbl. 0022.01001, JFM 65.0165.02).

2.20.15. Given a sequence q_n of pairwise coprime positive integers, let a_n denote the increasing sequence of all integers which can be written as a product of distinct elements of q_n. Let $f(n)$ be a positive multiplicative function such that one of the following four groups of conditions holds:

(I) only one of the following series is divergent $\sum_{k=1}^{\infty}(f(q_k) - 1)$, $\sum_{k=1}^{\infty}(f(q_k) - 1)^2$,

(II) $f(q_k) > 1$ for every k, (resp. $f(q_k) < 1$ for every k), $\lim_{k\to\infty} f(q_k) = 1$, the series $\sum_{k=1}^{\infty} (f(q_k) - 1)$ diverges,

(III) the series $\sum_{k=1}^{\infty}(f(a_k) - 1)$ is convergent but not absolutely,

(IV) $f(q_1) \geq f(q_2) \geq f(q_3) \geq \cdots > 1$, the series $\sum_{k=1}^{\infty} (f(q_k) - 1)$ converges, $f(q_n) \leq \prod_{k=1}^{\infty} f(q_{n+k})$ for every $n = 1, 2 \dots$.

Then the sequence $f(a_n)$ is dense in the interval (A, B), where
in (I) $(A, B) = (\inf f(a_n), \sup f(a_n))$,

in (II) $(A, B) = (1, \infty)$ (resp. $(A, B) = (0, 1)$),
in (III) $(A, B) = (0, \infty)$, and
in (IV) $(A, B) = (1, \prod_{k=1}^{\infty} f(q_{n+k}))$.

NOTES: Š. Porubský (1979) where he extended the results of J. Mináč (1978) on density of sequences 2.20.9 and 2.20.11. Porubský (1979) illustrated the result (II) by

$$\frac{\varphi_\alpha(a_n)}{a_n^\alpha}, \qquad \frac{a_n^\alpha}{\sigma_\alpha(a_n)}, \qquad \frac{\varphi_\alpha(a_n)}{\sigma_\alpha(a_n)}, \qquad \frac{a^\alpha d(a_n)}{2\sigma_\alpha(a_n)},$$

where

$$\varphi_\alpha(a) = a^\alpha \prod_{p|a} \left(1 - \frac{1}{p^\alpha}\right), \qquad \sigma_\alpha(a) = \sum_{d|a} d^\alpha, \qquad d(a) = \sum_{d|a} 1$$

with $\alpha \in (0, 1]$. The result of (III) he applied to functions

$$\prod_{p|n} \left(1 - \frac{\chi(p)}{p}\right), \qquad \prod_{p|n} \left(1 + \frac{\chi(p)}{p}\right),$$

where χ is a real non–principal character modulo k. The result of (IV) is applied in Porubský and J.T. Tóth (1999) to the sequence

$$\frac{\sigma(a_n)\phi(a_n)}{a_n^2}$$

which is

dense in $[6/\pi^2, 1]$.

RELATED SEQUENCES: 2.20.16, 2.20.17.

J. MINÁČ: *On the density of values of some arithmetical functions*, (Slovak), Matematické obzory **12** (1978), 41–45.
Š. PORUBSKÝ: *Über die Dichtigkeit der Werte Multiplikativer Funktionen*, Math. Slovaca **29** (1979), 69–72 (MR0561779 (81a:10066); Zbl. 0403.10002).
Š. PORUBSKÝ – J.T. TÓTH: *Topological density of values of arithmetical functions*, Preprint, 1999, 8 pp.

2.20.16. Let a_n be the increasing sequence of all squarefree positive integers and let $f : \mathbb{N} \to [0, \infty)$ be a strictly increasing unbounded function with $\sum_{i=1}^{\infty} 1/f(p_i) = \infty$, where p_i is the ith prime. If $c > 0$ and

$$\sigma_f(n) = \sum_{d|n} f(d), \quad \phi_f^c(n) = f(n) \prod_{p|n} \left(1 - \frac{c}{f(p)}\right)$$

then all of the following sequences

$$\frac{\sigma_f(a_n)}{f(a_n)}, \quad \frac{f(a_n)}{\phi_f^c(a_n)}, \quad \frac{\sigma_f(a_n)}{\phi_f^c(a_n)}$$

are

$$\text{dense in the interval } [1, \infty).$$

J. Fulier – J.T. Tóth: *On certain dense sets*, Acta Mathematatica (Nitra) **2** (1995), 23–28.

2.20.16.1 Let $\nu(n)$ be a completely multiplicative arithmetic function which satisfies the conditions
(i) $|\nu(p)| \le \nu$ for some positive number ν and every prime p,
(ii) $\sum_{d\le x} \mu(d)\nu(d) \ll x(\log x)^{-A}$ for every positive A, where the implied constant depends only on ν and A.
Define arithmetic function ϕ by $\phi(m) = m\prod_{p|m}\left(1-\frac{\nu(p)}{p}\right)$.
Then, if the number $\alpha = \frac{1}{2}\prod_{p}\left(1-\frac{\nu(p)}{p^2}\right)$ is irrational, the sequence

$$\frac{1}{n}\sum_{m\le n}\phi(m) \bmod 1, \quad n = 1, 2, \dots$$

is

$$\text{u.d.}$$

Notes: J.-M. Deshouillers – H. Iwaniec (2008). By their comment, $\alpha = \frac{3}{\pi^2}$ for the classical Euler totient function $\varphi(n)$ and therefore the sequence

$$\frac{\varphi(1)+\cdots+\varphi(n)}{n} \bmod 1, \quad n = 1, 2, \dots$$

is

$$\text{u.d.}$$

This answers in affirmative an open problem posed by F. Luca (2007), see Unsolved Problems (2009).

J.-M. Deshouillers – H. Iwaniec: *On the distribution modulo one of the mean values of some arithmetical functions*, Unif. Distrib. Theory **3** (2008), no. 1, 111–124 (MR2471293 (2009k:11158); Zbl. 1174.11077).
F. Luca: *Section 1.11, Open problem 6*, in: *Unsolved Problems Section on the home-page of the journal Uniform Distribution Theory*, (O. Strauch ed.), `http://udt.mat.savba.sk/udt_unsolv.htm`, 2006, 1–84 pp. (Last update: June 29, 2011).

2.20.16.2 Let $\nu(n)$ be a completely multiplicative function such that
(i) $-\nu \le \nu(p) < \min\{p, \nu\}$ for some positive ν and every prime p,

(ii) there exist real numbers β and λ such that

$$\prod_{p\le n}\left(1-\frac{\nu(p)}{p}\right)=\beta(\log n)^{-\lambda}\left(1+O\left(\frac{1}{\log n}\right)\right),$$

where the implied constant depends only on ν.
Define a strongly multiplicative function ϕ by $\phi(m)=m\prod_{p|m}\left(1-\frac{\nu(p)}{p}\right)$, and we let $\alpha=\frac{1}{e}\prod_p\left(1-\frac{\nu(p)}{p}\right)^{\frac{1}{p}}$. Then if α is irrational, the sequence

$$\left(\prod_{m\le n}\phi(m)\right)^{\frac{1}{n}} \bmod 1,\quad n=1,2,\dots, \tag{1}$$

is

u.d.

(iii) If α is rational and ν takes only algebraic values, then the sequence (1) is not u.d.

NOTES: J.-M. Deshouillers – H. Iwaniec (2008).
(I) As noticed by authors, the arithmetic character of the corresponding

$$\alpha=\frac{1}{e}\prod_p\left(1-\frac{1}{p}\right)^{\frac{1}{p}}$$

for the classical Euler totient function $\varphi(n)$ is an **open problem**.
(II) This constant is very likely to be irrational: Richard Bumby showed that if α is rational, then its denominator has at least 20 decimal digits.
(III) A special case of (iii) shows that if the constant α is rational, then the sequence

$$\left(\prod_{m\le n}\varphi(m)\right)^{\frac{1}{n}} \bmod 1,\quad n=1,2,\dots,$$

is not u.d. This gives a conditional answer to an open problem posed by F. Luca (2007).

J.-M. Deshouillers – H. Iwaniec: *On the distribution modulo one of the mean values of some arithmetical functions*, Unif. Distrib. Theory **3** (2008), no. 1, 111–124 (MR2471293 (2009k:11158); Zbl. 1174.11077).
F. Luca: *Section 1.11, Open problem 6*, in: *Unsolved Problems Section on the home-page of the journal Uniform Distribution Theory*, (O. Strauch ed.), `http://udt.mat.savba.sk/udt_unsolv.htm`, 2006, 1–84 pp. (Last update: June 29, 2011).

2.20.16.3 Let $p_a(n)$ be the arithmetic mean of the distinct prime factors of n and $p_A(n)$ the arithmetic mean of all its prime factors, i.e. then the sequences

$$p_a(n) = \frac{1}{\omega(n)} \sum_{p|n} p, \quad p_A(n) = \frac{1}{\Omega(n)} \sum_{\substack{p^a|n \\ p^a>1}} p.$$

Then the sequences $p_a(n) \bmod 1$ and $p_A(n) \bmod 1$, $n = 1, 2, \dots$, are u.d.

NOTES:

W.D. BANKS – M.Z. GARAEV – F. LUCA – I.E. SHPARLINSKI: *Uniform distribution of the fractional part of the average prime factor*, Forum Math. **17** (2005), no. 6, 885–903 (MR2195712 (2007g:11093); Zbl. 1088.11062).

2.20.16.4 Define the geometric means of prime factors of n by

$$p_g(n) = \Big(\prod_{p|n} p\Big)^{1/\omega(n)}, \quad p_G(n) = n^{1/\Omega(n)}.$$

Then the sequences $p_g(n) \bmod 1$ and $p_G(n) \bmod 1$, $n = 1, 2, \dots$, are u.d.

F. LUCA – I.E. SHPARLINSKI: *On the distribution modulo* 1 *of the geometric mean prime divisor*, Bol. Soc. Mat. Mex. **12** (2006), no. 2, 155–163.(MR2292980; Zbl. 1145.11061).

2.20.16.5 If

$$p_h(n) = \frac{\omega(n)}{\sum_{p|n} \frac{1}{p}}, \quad p_H(n) = \frac{\Omega(n)}{\sum_{\substack{p^a|n \\ p^a>1}} \frac{1}{p}},$$

then the sequences $p_h(n) \bmod 1$ and $p_H(n) \bmod 1$, $n = 1, 2, \dots$, are u.d.

NOTES: I. Kátai and F. Luca (2009) proved in Theorem 1 the following more general result: Let $g(n)$ be an additive function such that $g(p) < c_1/p$ and $0 < g(p^a) < c_2$ for all primes p and all positive integers a with some positive constants c_1 and c_2. Let

$$\nu(n) = \frac{\omega(n)}{g(n)} \qquad \text{and} \qquad \rho(n) = \frac{\omega(n+1)}{g(n)}.$$

Then

(i) $\nu(n)$ is uniformly distributed modulo 1;

(ii) $\rho(n)$ is uniformly distributed modulo 1.

The same holds when the function $\omega(n)$ is replaced by $\Omega(n)$.
The authors noted that their Theorem 1 can be applied to functions $g(n)=\sum_{p|n}1/p$, $g(n)=\sum_{\substack{p^a|n\\p>1}}1/p$, $g(n)=\log(n/\phi(n))$ and $g(n)=\log(\sigma(n)/n)$. From there they deduces, in particular, that the sequence of harmonic means of the prime factors of n is u.d. modulo 1.

I. Kátai – F. Luca: *Uniform distribution modulo 1 of the harmonic prime factor of an integer*, Unif. Distrib. Theory **4** (2009), no. 2, 115–132 (MR2591845 (2011c:11130); Zbl. 1249.11086).

2.20.16.6 Let, as usual, $\omega(n)$ and $d(n)$ denote the number of prime divisors and the total number of divisors of n, and a is a fixed integer. Then the sequences of the fractional parts of the ratios

$$\frac{n}{\omega(n)},\quad \frac{n}{a^{\omega(n)}},\quad \frac{n}{d(n)},\quad \frac{n}{a^{d(n)}}$$

are

u.d.

in the unit interval $[0,1]$.

F. Luca – I.E. Shparlinski: *Uniform distribution of some ratios involving the number of prime and integer divisors*, Unif. Distrib. Theory **1** (2006), no. 1, 15–26 (MR2314264 (2008c:11133); Zbl. 1147.11057).
F. Luca – I.E. Shparlinski: *Errata to "Uniform distribution of some ratios involving the number of prime and integer divisors", UDT 1 (2006), 15–26*, Unif. Distrib. Theory **6** (2011), no. 2, p. 83 (MR2904040; Zbl. 1313.11106).

2.20.17. Let q_n be an increasing sequence of pairwise coprime positive integers and a_n is the increasing sequence of the all $m\in\mathbb{N}$ for which $q_i|m$ and $q_i^2\nmid m$. Let $f(n)$ be a positive arithmetical function such that
(i) $\prod_{j=1}^k f(q_{n_j})\le f\left(\prod_{j=1}^k q_{n_j}\right)\le f(q_{n_k+1})\prod_{j=1}^k f(q_{n_j})$ holds for every $\{q_{n_1},\dots,q_{n_k}\}$,
(ii) $\lim_{n\to\infty}f(q_n)=1$ and $f(q_n)>1$ for every n.
Then

$$f(a_n),\quad n=1,2,\dots,$$

is

$$\text{dense in }\left[1,\prod_{j=1}^\infty f(q_{n_j})\right].$$

Š. Porubský – J.T. Tóth: *On density of values of some multiplicative functions*, Preprint, 1997, 4 pp.

2.20.18. Let $\mathrm{ord}_p(n) = \alpha$ for $p^\alpha \| n$. If p stands for a prime then the sequence

$$\log p \frac{\mathrm{ord}_p(n)}{\log n}, \quad n = 2, 3, \dots,$$

is dense in $[0, 1]$ and has the a.d.f.

$$c_0(x),$$

and for its discrepancy we have

$$D_N^{(2)} = \mathcal{O}\left(\sqrt{\frac{1}{N} \sum_{n=2}^{N+1} \frac{1}{\log^2 n}} \right).$$

Notes: Cf. T. Šalát (1994). Another proof and the discrepancy were given by O. Strauch (1991).

T. Šalát: *On the function $a_p, p^{a_p(n)} \| n (n > 1)$*, Math. Slovaca **44** (1994), no. 2, 143–151 (MR1282531 (95c:11008); Zbl. 0798.11003).
O. Strauch: *On statistical convergence of bounded sequences*, Math. Institute, Slovak Acad. Sci., Bratislava, Slovak Republic, 1991, 10 pp.

2.20.19. Let $H(n) = \max(\alpha_1, \dots, \alpha_k)$ and $h(n) = \min(\alpha_1, \dots, \alpha_k)$ for $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$. Then the sequence

$$\log 2 \frac{H(n)}{\log n}, \quad n = 2, 3, \dots,$$

is dense in $[0, 1]$ and has the a.d.f.

$$c_0(x).$$

The same is also true for the sequence

$$\log 2 \frac{h(n)}{\log n}, \quad n = 2, 3, \dots.$$

A. Schinzel – T. Šalát: *Remarks on maximum and minimum exponents in factoring*, Math. Slovaca **44** (1994), no. 5, 505–514 (MR1338424 (96f:11017a); Zbl. 0821.11004).

2.20.20. Let $p_1 < p_2 < p_3 < \dots$ be an infinite sequence of pairwise coprime numbers. Then the infinite sequence

$$x_n = \sum_{k=1}^{\infty} \frac{r(k,n)}{p_1 p_2 \dots p_k}, \quad n = 1, 2, \dots,$$

where $n \equiv r(k,n) \bmod p_k$ with $0 \le r(k,n) < p_k$, is

u.d. in $[0,1]$.

NOTES: (I) T.A. Bick and J. Coffey (1991) also proved that this sequence has the property D defined by D. Maharam (1965). Here, a one–to–one sequence $x_n \in [0,1]$ is called a D–**sequence** if:

(i) To each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $X \subset \mathbb{N}$ and $\underline{d}(X) > 1 - \delta$ then $|\overline{\{x_n\,;\,n \in X\}}| > 1 - \varepsilon$. Here $\underline{d}$ is the lower asymptotic density (cf. p. 1 – 3), $\overline{A}$ is the closure of the set A and $|A|$ is its Lebesgue measure.

(ii) There exists a sequence of sets $X_n \subset \mathbb{N}$ such that $X_1 \subset X_2 \subset X_3 \subset \dots$, $\underline{d}(X_n) \to 1$, and whenever $m_1 < m_2 < m_3 < \dots$ with $m_k \in X_n$, then both of the following statements hold:
- The subsequence x_{m_k} converges if and only if the subsequence x_{m_k+1} converges.
- For each $n \in \mathbb{N}_0$, $\lim_{k\to\infty} x_{m_k} = x_n$ if and only if $\lim_{k\to\infty} x_{m_k+1} = x_{n+1}$.

(II) Note that the series $\sum_{k=1}^{\infty} \frac{r(k,n)}{p_1 p_2 \dots p_k}$ is the Cantor series of x_n.

T.A. BICK – J. COFFEY: *A class of example of D–sequences*, Ergodic Theory Dyn. Syst. **11** (1991), no. 1, 1–6 (MR1101080 (92d:11080); Zbl. 0717.28010).
D. MAHARAM: *On orbits under ergodic measure–preserving transformations*, Trans. Amer. Math. Soc. **119** (1965), 51–66 (MR0180653 (**31** #4884); Zbl. 0146.28601).

2.20.21. The sequence

$$\omega(n)\theta \bmod 1,$$

where $\omega(n)$ denotes the number of distinct prime divisors of n and θ is irrational is

u.d.

NOTES: P. Erdős (1946), H. Delange (1958), cf. [KN, p. 22].

H. DELANGE: *On some arithmetical functions*, Illinois J. Math. **2** (1958), 81–87 (MR0095809 (**20** #2310); Zbl. 0079.27302).
P. ERDŐS: *On the distribution function of additive functions*, Ann. of Math. (2) **47** (1946), 1–20 (MR0015424 (7,416c); Zbl. 0061.07902).

2.20.22. The sequence

$$\Omega(n)\theta \bmod 1,$$

where $\Omega(n)$ stands for the number of prime factors of n counted with multiplicities and θ is irrational, is

$$\text{u.d.}$$

NOTES: H. Delange (1958), cf. [DT, p. 100].

H. DELANGE: *On some arithmetical functions*, Illinois J. Math. **2** (1958), 81–87 (MR0095809 (**20** #2310); Zbl. 0079.27302).

2.20.23. Let $\omega_E(n)$ denote the number of distinct prime divisors of the positive integer n which belong to a set E of prime numbers, and let $\Omega_E(n)$ be the total number of prime divisors which belong to E. Assume that there is a number $\alpha \geq 0$ such that the number of integers in E which do not exceed x is equal to $\alpha\frac{x}{\log x} + o\left(\frac{x}{\log x}\right)$ as $x \to \infty$ if $\alpha > 0$, and if $\alpha = 0$ then $\sum_{p\in E}\frac{1}{p} = +\infty$. If θ is an irrational number then the sequences

$$\omega_E(n)\theta \bmod 1 \qquad \text{and} \qquad \Omega_E(n)\theta \bmod 1$$

are

$$\text{u.d.}$$

NOTES: H. Delange ([b]1958) generalized in this way his previous result in ([a]1958), cf. also ([c]1958).

RELATED SEQUENCES: 2.20.21,2.20.22.

[a] H. DELANGE: *On some arithmetical functions*, Illinois J. Math. **2** (1958), 81–87 (MR0095809 (**20** #2310); Zbl. 0079.27302).
[b] H. DELANGE: *Sur certain functions arithmétiques*, C. R. Acad. Sci. Paris **246** (1958), 514–517 (MR0095810 (**20** #2311); Zbl. 0079.06703).
[c] H. DELANGE: *Sur la distribution de certains entieres* , C. R. Acad. Sci. Paris **246** (1958), 2205–2207 (MR0095811 (**20** #2312); Zbl. 0081.04201).

2.20.24. Let f be a real valued function and let for sufficiently large x the following conditions are fulfilled
(i) $xf'(x)$ is monotonic,
(ii) $\log^{-\beta} x \ll |xf'(x)| \ll \log^{\gamma} x$, where $\max(\beta,\gamma) < \log\frac{4}{3}$.
Then the block sequence A_n defined by

$$A_n = (f(d) \bmod 1)_{d|n,d>0},$$

(i.e. d runs through the positive divisor of n) is (cf. p. 1 – 32)

generalized u.d.

NOTES: (I) This means the u.d. over a subsequence of n's possessing the asymptotic density 1 (for the def. cf. p. 1 – 32), or equivalently, the sequence $f(n)$ mod 1 for $n = 1, 2, \dots,$ is u.d. on the divisors (cf. 1.8.26).
(II) R.R. Hall (1976). He conjectures that A_n is generalized u.d. if
- $f(d) = (\log d)^\alpha$ with $\alpha > 0$ (i.e. that the correct condition in (ii) is simply $\beta < 1$),
- $f(d) = (\log\log d)^\alpha$ with $\alpha > 1$ (this would be best possible since for $\log\log d$ the result does not hold)

(III) The case
- $f(d) = \log d$

was studied in Hall (1974/75, 1975, 1975/76) and P. Erdős and R.R. Hall (1974), A_n is generalized u.d. In Hall (1974/75) it is proved that for every $\lambda < \frac{1}{2}$ there exists a subsequence of n's of asymptotic density 1 with extremal discrepancy satisfying

$$D(A_n) < \frac{1}{d(n)^\lambda}.$$

(IV) In the case
- $f(d)$ is an additive function

I. Kátai (1976) proved that $D(A_n) \to 0$ over a subsequence of asymptotic density 1 if and only if

$$\sum_{p-\text{prime}} \frac{\|2mf(p)\|^2}{p} = \infty$$

for $m = 1, 2, \dots.$ Here $\|x\| = \min(\{x\}, 1 - \{x\})$.
(V) Hall (1981) replaced condition (ii) by a weaker one with β and γ running over a specified convex subset of the rectangle $0 < \beta < 1$, $0 \le \gamma < \log 2$ and he proved that A_n is generalized u.d. for
- $f(d) = (\log d)^\alpha$, where $0 < \alpha < 1 + \log 2$.

(VI) Hall (1975/76) also studied

$$d(A_n) = \inf\{\| \log d_1 - \log d_2\| \ ; \ d_1, d_2 | n, d_1 \ne d_2\}$$

and he proved that for every $\varepsilon > 0$ there exist a subsequence of n's of asymptotic density 1 such that

$$3^{-(1+\varepsilon)\log\log n} < d(A_n) < 3^{-(1-\varepsilon)\log\log n}.$$

P. Erdős – R.R. Hall: *Some distribution problems concerning the divisors of integers*, Acta Arith. **26** (1974/75), 175–188 (MR0354592 (**50** #7070); Zbl. 0272.10021).
R.R. Hall: *The divisors of integers. I*, Acta Arith. **26** (1974/75), 41–46 (MR0347765 (**50** #266); Zbl. 0272.10019).

R.R. Hall: *Sums of imaginary powers of the divisors*, J. London Math. Society, II. Ser., **9** (1975), 571–580 (MR0364131 (**51** #386); Zbl. 0308.10037).
R.R. Hall: *The divisors of integers.II*, Acta Arith. **28** (1975/76), no. 2, 129–135 (MR0384719 (**52** #5592); Zbl. 0272.10020).
R.R. Hall: *The distribution of $f(d)$* (mod 1), Acta Arith. **31** (1976), no. 1, 91–97 (MR0432565 (**55** #5553); Zbl. 0343.10036).
R.R. Hall: *The divisor density of integer sequences*, J. London Math. Soc. **24** (2) (1981), no. 1, 41–53 (MR0623669 (82h:10068); Zbl. 0469.10035).
I. Kátai: *Distribution* mod 1 *on additive functions on the set of divisor*, Acta Arith. **30** (1976), no. 2, 209–212 (MR0417083 (**54** #5144); Zbl. 0295.10043).

2.20.24.1 Define

$$F_N(x) = \frac{1}{d(n)} \sum_{\substack{d|n \\ d \le n^x}} 1,$$

where $d(n)$ is the total numbers of divisors of n. Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F_n(x) = \frac{2}{\pi} \arcsin\sqrt{x} + \mathcal{O}\left(\frac{1}{\sqrt{\log N}}\right)$$

uniformly for N and $x \in [0,1]$.
Notes: G. Tenenbaum (1995, Th. 7, p. 207).

G. Tenenbaum: *Introduction à la théorie analytique et probabiliste des nombres*, Institut Elie Cartan, Vol. 13, Université de Nancy, Nancy, 1990. (second edition: Société de France, Paris, 1995 (MR1366197 (97e:11005a); Zbl. 0880.11001)). (English translation: Studies in Advanced Mathematics, Vol. 46, Cambridge Univ. Press, Cambridge, 1995 (MR1342300 (97e:11005b); Zbl. 0880.11001)).

2.20.25. Let $\rho(n) = \beta(n) + i\gamma(n)$ be the sequence of the non–trivial zeros of the Riemann zeta function ζ in the upper half of the critical strip, ordered by $0 < \gamma(1) \le \gamma(2) \le \dots$. If α is a non–zero real number then the sequence

$$\alpha\gamma(n) \bmod 1$$

is

u.d.

If $\alpha = \frac{\log z}{2\pi}$ with an integer $z \ge 2$ then

$$D_N = \mathcal{O}\left(\frac{\log z}{\log\log\gamma(N)}\right).$$

If the Riemann hypothesis is assumed, then

$$D_N = \mathcal{O}\left(\frac{\log z}{\log \gamma(N)}\right).$$

NOTES: (I) u.d. of $\alpha\gamma(n)$ mod 1 under the assumption that the Riemann hypothesis is true was noted by H.A. Rademacher, cf. (1974, p.455). P.D.T.A. Elliot (1972) noticed that this result can be established unconditionally. For a proof cf. E. Hlawka (1984, pp. 122–123) and the discrepancy estimate is proved in E. Hlawka (1975).
(II) Let $N(T)$ denote the number of these zeros for which $0 < \gamma(n) \le T$. A. Fujii (1976) proved that if $\varepsilon > 0$, $T > T_0(\varepsilon)$ and $(\log T)^{-1} \le t \le T^{\frac{1}{3}}$ then the star discrepancy of the sequence

$$\frac{\gamma(n)}{t} \bmod 1, \qquad n = 1, 2, \dots, N(T),$$

satisfies

$$D^*_{N(T)} = \mathcal{O}\left(\frac{1}{(t \log T)^{1-\varepsilon}}\right).$$

(III) Fujii (1978) proved that the sequence $\gamma(n)$ is u.d. mod Δ (for the def. see 1.5) for the subdivision $\Delta = (z_n)_{n=1}^{\infty}$, where $z_1 = z_2 = 0$ and $z_n = bn(\log n)^{a-1}$ and $a > 0$ and $b > 0$.

RELATED SEQUENCES: For the multi–dimensional case cf. 3.7.10, and for generalization to Dirichlet series cf. 2.20.27.

P.D.T.A. ELLIOTT: *The Riemann zeta function and coin tossing*, J. Reine Angew. Math. **254** (1972), 100–109 (MR0313206 (**47** #1761); Zbl. 0241.10025).
A. FUJII: *On the zeros of Dirichlet L–functions, III*, Trans. Amer. Math. Soc. **219** (1976), 347–349 (MR0418410 (81g:10056a); Zbl. 0336.10034).
A. FUJII: *On the uniformity of the distribution of the zeros of the Riemann zeta function*, J. Reine Angew. Math. **302** (1978), 167–185 (MR0511699 (80g:10053); Zbl. 0376.10029).
E. HLAWKA: *Über die Gleichverteilung gewisser Folgen, welche mit den Nullstellen der Zetafunktionen zusammenhäangen*, Österreich. Akad. Wiss. Math.–Natur. Kl. Sitzungsber. II **184** (1975), no. 8–10, 459–471 (MR0453661 (**56** #11921); Zbl. 0354.10031).
E. HLAWKA: *The Theory of Uniform Distribution*, A B Academic Publishers, Berkhamsted, 1984 (translation of the original German edition Hlawka (1979)) (MR0750652 (85f:11056); Zbl. 0563.10001).
H.A. RADEMACHER: *Collected Papers of Hans Rademacher*, Vol. II, Mathematicians of our times 4, The MIT Press, Cambridge (Mass.), London (England), 1974 (MR0505096 (**58** #21343b); Zbl. 0311.01023).

2.20.26. Montgomery – Odlyzko law (GUE conjecture). Let $\rho(n) = \beta(n) + i\gamma(n)$ be the sequence of the non–trivial zeros of the Riemann zeta function ζ ordered by $0 < \gamma(1) \le \gamma(2) \le \dots$. Assume the truth of the Riemann hypothesis, i.e. $\beta(n) = \frac{1}{2}$ for $n = 1, 2, \dots$. Renormalize $\gamma(n)$ by

$$\widehat{\gamma}(n) = \frac{\gamma(n) \log \gamma(n)}{2\pi}$$

and for $x \in [0, \infty)$ put

$$\widetilde{F}_N(x) = \frac{1}{N} \#\{(n,k)\,;\, 1 \le n \le N, k > 0, \widehat{\gamma}(n+k) - \widehat{\gamma}(n) \in [0,x)\},$$
$$r_2(x) = 1 - \left(\frac{\sin \pi x}{\pi x}\right)^2$$

The **Montgomery − Odlyzko law** conjectures that

$$\lim_{N\to\infty} \widetilde{F}_N(x) = \int_0^x r_2(t)\,\mathrm{d}t \quad \text{for all } x \in [0,\infty).$$

NOTES: (I) This conjecture appeared in H.L. Montgomery (1973) and has been extensively tested numerically by A.M. Odlyzko (1987, 1992). The density function $r_2(x)$ is called a **pair correlation function**.
(II) The conjecture claims that the consecutive spacing of the zeros of the zeta function is statistical identical with the consecutive spacing of the eigenvalues of the **Gaussian unitary ensemble** (GUE) matrices. Here GUE consists of $N \times N$ random complex Hermitian matrices of the form $\mathbf{A} = (a_{j,k})$, where

$$a_{j,k} = \begin{cases} \sqrt{2}\sigma_{j,j}, & \text{for } j = k, \\ \sigma_{j,k} + i\eta_{j,k}, & \text{for } j < k, \\ \overline{a}_{j,k} = \sigma_{j,k} - i\eta_{j,k}, & \text{otherwise}, \end{cases}$$

with $\sigma_{j,k}$ and $\eta_{j,k}$ being independent standard normal variables. The eigenvalues of these matrices are real $\lambda(1) \le \lambda(2) \le \cdots \le \lambda(N)$ and are renormalized to $\widehat{\lambda}(1) \le \widehat{\lambda}(2) \le \cdots \le \widehat{\lambda}(N)$ in such a way that $\widehat{\lambda}(n+1) - \widehat{\lambda}(n) = (\lambda(n+1) - \lambda(n))\sqrt{4N - \lambda(n)^2}/2\pi$ and

$$\lim_{N\to\infty} E\big(\#\{(n,k)\,;\, 1 \le n \le N, k > 0, \widehat{\lambda}(n+k) - \widehat{\lambda}(n) \in [0,x)\}\big) = \int_0^x r_2(t)\,\mathrm{d}t,$$

where E stands for the expected value.
(III) Since $N(T) := \#\{n \in \mathbb{N}\,;\, \gamma(n) \le T\} \sim (T \log T)/2\pi$, then $\gamma(n) \sim 2\pi n/(\log n)$, and consequently the mean value of $\widehat{\gamma}(n+1) - \widehat{\gamma}(n)$ is 1, what is the background for the renormalization. Similarly for $\widehat{\lambda}(n)$.
(IV) Another formulation of the GUE conjecture states that if f is continuous then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1 \le m \ne n \le N} f(\widehat{\gamma}(m) - \widehat{\gamma}(n)) = \int_{-\infty}^{\infty} f(x) r_2(x)\,\mathrm{d}x,$$

cf. N.M. Katz and P. Sarnak (1999).

(V) A further version says that

$$\frac{2\pi}{T\log T}\sum_{\substack{T\le\gamma(m),\gamma(n)\le 2T\\ m\ne n}} f((\log T/2\pi)(\gamma(m)-\gamma(n)))w(\gamma(m)-\gamma(n))\to$$

$$\to\int_{-\infty}^{\infty} f(x)r_2(x)\,\mathrm{d}x,\qquad \text{as } T\to\infty$$

where $w(x)=4/(4+x^2)$. Montgomery (1973) proved this for continuous L^1 functions f which support of their Fourier transform $\widehat{f}(t)=\int_{-\infty}^{\infty} f(x)e^{-2\pi ixt}\,\mathrm{d}x$ is contained in $(-1,1)$, cf. D.A. Hejhal (1994).
(VI) Hejhal (1994) reformulated the GUE3 conjecture as the limit

$$\frac{2\pi}{T\log T}\sum_{\substack{T\le\gamma(n_i)\le 2T\\ n_1,n_2,n_3 \text{ are distinct}}} f\big((\log T/2\pi)\big(\gamma(n_1)-\gamma(n_2)\big),(\log T/2\pi)\big(\gamma(n_1)-$$

$$-\gamma(n_3)\big)\big)w\big(\gamma(n_1),\gamma(n_2),\gamma(n_3)\big)\to\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)w(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

where f is a suitable explicitly given positive function, and w is an explicitly given determinant of a 3×3 matrix depending on $(\sin\pi x)/\pi x$. He proved this for continuous L^1 functions f having the support in the hexagon $|x|+|y|+|x+y|<2$.
(VII) If $s\ge 2$ the GUEs conjecture can be restated as the limit (cf. D.W. Farmer (1995))

$$\frac{2\pi}{T\log T}\sideset{}{'}\sum_{0<\gamma(n_1),\dots,\gamma(n_s)<T} f((\log T/2\pi)\gamma(n_1),\dots,(\log T/2\pi)\gamma(n_s))\to$$

$$\to\int_{\mathbb{R}^s} f(\mathbf{x})W_s(\mathbf{x})\delta(\overline{\mathbf{x}})\,\mathrm{d}x,\quad \text{as } T\to\infty$$

for a general class of test functions f possessing some reasonable properties. Here $W_s(\mathbf{x})=\det(K(x_i-x_j))$, $K(t)=\sin(\pi t)/\pi t$, $\mathbf{x}=(x_1,\dots,x_s)$, $\overline{\mathbf{x}}=(x_1+\dots+x_s)/s$ and δ is the Dirac δ–function, and the prime in the sum indicates that the summation runs over distinct n_i.
(VIII) F. Dyson also rediscovered the functions $r_2(x)$ as eigenvalues of certain matrices when studying the energy levels of an atomic nucleus predicates.

D.W. Farmer: *Mean values of ζ'/ζ and the Gaussian unitary ensemble hypothesis*, Internat. Math. Res. Notices (1995), no. 2, 71–82 (electronic). (MR1317644 (96g:11109); Zbl. 0829.11043).
A.D. Hejhal: *On the triple correlation of zeros of the zeta function*, Internat. Math. Res. Notices (1994), no. 7, 10 pp. (electronic).(MR1283025 (96d:11093); Zbl. 0813.11048).
N.M. Katz – P. Sarnak: *Zeroes of zeta functions and symmetry*, Bull. Amer. Math. Soc. **36** (1999), no. 1, 1–26 (MR1640151 (2000f:11114); Zbl. 0921.11047).
H.L. Montgomery: *The pair correlation of zeros of the zeta function*, in: *Analytic number theory (Proc. Sympos. Pure Math., Vol. XXIV, St. Louis Univ., St. Louis, Mo., 1972)*, Amer. Math. Soc., Providence, R.I., 1973, pp. 181–193 (MR0337821 (**49** #2590); Zbl. 0268.10023).

A.M. Odlyzko: *On the distribution of spacings between zeros of the zeta function*, Math. Comp. **48** (1987), no. 177, 273–308 (MR0866115 (88d:11082); Zbl. 0615.10049).
A.M. Odlyzko: *The* 10^{20}*–th zero of the Riemann zeta function and* 175 *million of its neighbours*, A.T.T., 1992, 120 pp. (Preprint).

2.20.27. Let χ be a primitive Dirichlet character modulo q. Let $0 < \gamma_1 \le \gamma_2 \le \dots$ denote the sequence of positive imaginary parts of the zeros of the Dirichlet L-function $L(s,\chi)$ counted with multiplicity. If $\alpha \neq 0$ then the sequence

$$\alpha\gamma_n \bmod 1$$

is

$$\text{u.d.}$$

A. Fujii: *On the zeros of Dirichlet L–functions, IV*, J. Reine Angew. Math. **286(287)** (1976), 139–143 (MR0436639 (81g:10056b); Zbl. 0332.10027).
J. Kaczorowski: *The k–function in multiplicative number theory, II. Uniform distribution of zeta zeros*, Acta Arith. **56** (1990), no. 3, 213–225 (MR1083000 (91m:11068a); Zbl. 0716.11040).

2.20.28. Let $\rho(n) = \beta(n) + i\gamma(n)$ be the sequence of the non–trivial zeros of the Riemann zeta function ζ ordered by $0 < \gamma(1) \le \gamma(2) \le \dots$. If the Riemann hypothesis holds, i.e. if $\beta(n) = 1/2$ for $n = 1, 2, \dots$, then for $0 < b < 6/5$ and any positive α the sequence

$$\frac{b\gamma(n)}{2\pi}\log\frac{b\gamma(n)}{2\pi e\alpha} \bmod 1$$

is

$$\text{u.d.}$$

Notes: This was proved by A. Fujii (1996, p. 54). He conjectured that the same conclusion is true for any positive b.

A. Fujii: *An additive theory of the zeros of the Riemann zeta function*, Commen. Math. Univ. St. Paul. **45** (1996), no. 1, 49–116 (MR1388606 (97k:11125); Zbl. 0863.11050).

2.20.29. Let $\rho(n) = \beta(n)+i\gamma(n)$ be the sequence of the non–trivial zeros of the Riemann zeta function ζ ordered by $0 < \gamma(1) \le \gamma(2) \le \dots$. Let $x(n)$ be the double sequence $\gamma(i) + \gamma(j)$, $i, j = 1, 2, \dots$, arranged according to their magnitude. Then the sequence

$$x(n) \bmod 1$$

is

$$\text{u.d.}$$

NOTES: This was proved by A. Fujii (1996, Cor. 3). He writes: We understand that the "multiplicity" of $\gamma(i)+\gamma(j)$ is at least 2 for $i \neq j$. So the above arrangement is with the "multiplicities". We expect that the "multiplicity" of $\gamma(i)+\gamma(j)$ for $i \neq j$ is exactly 2 as we have already stated above.

A. FUJII: *An additive theory of the zeros of the Riemann zeta function*, Commen. Math. Univ. St. Paul. **45** (1996), no. 1, 49–116 (MR1388606 (97k:11125); Zbl. 0863.11050).

2.20.30. Let $v(n)$ be the Farey sequence of the reduced rational numbers in $[0,1)$ ordered by increasing denominators (cf. 2.23.4). Let

$$s(d,c) = \sum_{\nu=0}^{c-1} \left(\left(\frac{\nu}{c}\right)\right)\left(\left(\frac{\nu d}{c}\right)\right),$$

where c, d are integers, $c > 0$, and

$$((x)) = \begin{cases} 0, & \text{if } x \text{ is an integer,} \\ x - [x] - \frac{1}{2}, & \text{otherwise} \end{cases}$$

is the **Dedekind sum**. Since $s(ad, ac) = s(d,c)$, we can write $s(d/c)$ for $s(d,c)$. Then for any non–zero real number α, the sequence

$$\alpha s(v(n)) \bmod 1$$

is

$$\text{u.d.}$$

I. VARDI: *A relation between Dedekind sums and Kloosterman sums*, Duke Math. J. **55** (1987), 189–197 (MR0883669 (89d:11066); Zbl. 0623.10025).

2.20.31. The Kloosterman sum $K(q,a)$ is defined by

$$K(q,a) = \sum_{\substack{b \in F_q \\ b \neq 0}} \chi(b + ab^{-1}),$$

where χ is a fixed non–trivial additive character of the finite field $\mathbb{F}_q$ of order q. Then the sequence of blocks

$$A_q = \left(\frac{K(q,1)}{2\sqrt{q}}, \frac{K(q,2)}{2\sqrt{q}}, \dots, \frac{K(q,q-1)}{2\sqrt{q}} \right)$$

lies in the interval $[-1, 1]$ and has in this interval the a.d.f.

$$g(x) = \frac{2}{\pi}\int_{-1}^{x}\sqrt{1-t^2}\,\mathrm{d}t$$

and for its discrepancies we have

$$D^*_{q-1} < 10q^{-1/4} \quad \text{and} \quad D_{q-1} < 20q^{-1/4}.$$

NOTES: It is known that $K(q, a)$ is always real and a classical bound of A. Weil (1948) says $|K(q, a)| \le 2\sqrt{q}$. The form of $g(x)$ was found by V.M. Katz (1988) and discrepancy bounds were given by H. Niederreiter (1991) (cf. D.S. Mitrinović, J. Sándor and J. Crstici (1996, p. 415)).

N.M. KATZ: *Gauss Sums, Kloosterman Sums, and Monodromy Groups*, Ann. of Math. Stud., Vol. 116, Princeton Univ. Press, Princeton, NJ, 1988 (MR0955052 (91a:11028); Zbl. 0675.14004).
D.S. MITRINOVIĆ – J. SÁNDOR – J. CRSTICI: *Handbook of Number Theory*, Mathematics and its Applications, Vol. 351, Kluwer Academic Publishers Group, Dordrecht, Boston, London, 1996 (MR1374329 (97f:11001); Zbl. 0862.11001).
H. NIEDERREITER: *The distribution of values of Kloosterman sums*, Arch. Math.(Basel) **56** (1991), no. 3, 270–277 (MR1091880 (92b:11057); Zbl. 0752.11055).
A. WEIL: *On some exponential sums*, Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 204–207 (MR0027006 (10,234e); Zbl. 0032.26102).

2.20.32. The classical Kloosterman sums $S(a, b; c)$ are trigonometric sums of the form

$$S(a, b; c) = \sum_{\substack{1\le x,y\le c\\ xy\equiv 1 \,(\mathrm{mod}\, c)}} e^{2\pi i c^{-1}(ax+by)},$$

where a, b, c are integers with $c > 0$. If $c = p$, a prime, then the optimal estimate $|S(a, b; p)| \le 2\sqrt{p}$ was proved by A. Weil in 1941. This implies that for an integer a prime to p there is a unique $\theta_{p,a} \in [0, \pi]$ such that $S(a, 1; p) = 2\sqrt{p}\cos\theta_{p,a}$. The sequence of blocks

$$A_p = (\theta_{p,1}, \theta_{p,a_2}, \dots, \theta_{p,a_{p-1}}),$$

where $1 < a_2 < \cdots < a_{p-1}$, $(p, a_i) = 1$, has in $[0, \pi]$ the a.d.f. with density

$$h(x) = \frac{2\sin^2 x}{\pi}$$

as $p \to \infty$. The same holds for prime powers $q_n = p_n^{\alpha_n}$, A_{q_n}, $q_n \to \infty$.
NOTES:
(I) N.M. Katz (1988).

(II) A considerably easier proof gives A. Adolphson (1989).
(III) $h(x)$ is the density of the so-called Sato-Tate measure on $[0, \pi]$.
(IV) It seems that nothing is known about the distribution of the sequence $\theta_{p,a}$ for fixed a and $p \to \infty$. S.A. Stepanov (1971) conjectured (the so-called $\sin^2 \theta$ **conjecture**) that the limit distribution is again $h(x)$, cf. T.A. Springer (2000), and for a numerical test cf. N.M. Glazunov (1983).

RELATED SEQUENCES: 2.20.39.2.

A. ADOLPHSON: *On the distribution of angles of Kloosterman sums*, J. Reine Angew. Math. **395** (1989), 214–222 (MR0983069 (90k:11109); Zbl. 0682.40002).
N.M. GLAZUNOV: *Equidistribution of values of Kloosterman sums*, (Russian), Dokl. Akad. Nauk. Ukrain. SSR Ser. A (1983), no. 2, 9–12 (MR0694613 (84h:10052); Zbl. 0515.10034; (L05–506)).
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T.A. SPRINGER: *H.D. Kloosterman and his work*, Notices Amer. Math. Soc. **47** (2000), no. 8, 862–867 (MR1776104 (2001d:01036); Zbl. 1040.01007).

2.20.33. Let $g(x)$ be an increasing function such that $g(x) \geq \log\log x$ for $x \geq x_0 > 0$ and

$$\lim_{x\to\infty} \frac{g(x)}{\log x} = 0.$$

Put

$$N(T) = \#\{n \leq T\,;\, \forall_{p|n} \log p \leq g(n), p \text{ a prime}\}$$

for $T > 0$. Let s be a fixed integer and $f(x) = a_r x^r + \cdots + a_m x^m + \cdots + a_t x^t$ be a polynomial with $1 \leq r < \cdots < m < \cdots < t \leq s$ and

$$a_m = \frac{a}{q} + \frac{\theta}{q^2}, \qquad (a, q) = 1,\ |\theta| \leq 1,$$

where q satisfies

$$T^{\varepsilon m} \leq q T^{(1-\varepsilon)m}$$

for some $0 < \varepsilon < 1/2$. Then for the discrepancy of the finite sequence

$$f(n) \quad \text{for } n = 1, 2, \ldots, [T] \text{ such that } \forall_{p|n} \log p \leq g(n),$$

we have

$$D_N = \mathcal{O}\left(N^{1-\frac{\gamma}{k}}\right),$$

where

$$N = N(T) \sim T e^{-\frac{1}{2}\sqrt{\log T}\log\log T},$$

with $k = r + \cdots + m + \cdots + t$, and $\gamma = \gamma(\varepsilon) > 0$.

NOTES: This was proved by A.A. Karacuba (1975). As an example he also shows that for the polynomial $f(x) = ax^r + \sqrt{2}x^s$ with $1 \le r < s$ we have $D_N = \mathcal{O}\left(N^{1-\frac{\gamma_1}{s}}\right)$.

A.A. KARACUBA (A.A. KARATSUBA): *Some arithmetical problems with numbers having small prime divisors*, (Russian), Acta Arith. **27** (1975), 489–492 (MR0366830 (**51** #3076); Zbl. 0303.10037).

2.20.34. Let m, a, b be fixed integers such that $m \ge m_1 > 0$ and $\gcd(a, m) = 1$. Let n be an integer with $\gcd(n, m) = 1$, and n^* denote the positive integer $1 \le n^* < m$ which satisfies $nn^* \equiv 1 \pmod m$.

(I) Let ε be a fixed sufficiently small positive real number with $\varepsilon < 0.001$. If N is such that $m^\varepsilon \le N \le m$ then the finite sequence

$$x_n = \frac{an^* + bn}{m} \bmod 1 \quad \text{with} \quad 1 \le n \le N, \quad \text{and} \quad \gcd(n, m) = 1,$$

has the following property: If $M = \sum_{1 \le n \le N, (n,m)=1} 1$ is the number of all terms of x_n with $n \le N$ and $A([\alpha, \beta); M; x_n)$ is the counting function defined in 1.2 then for any subinterval $[\alpha, \beta) \subset [0, 1]$ we have

$$A([\alpha, \beta); M; x_n) = (\beta - \alpha)\frac{\varphi(m)}{m} X(1 + \mathcal{O}((\log m)^{-c_1}))$$

for some $c_1 = c_1(\varepsilon) > 0$.

(II) If N satisfies $1 \le N \le m^{4/7}$ and $[\alpha, \beta) \subset [0, 1]$ then we have

$$A([\alpha, \beta); M; x_n) \ge \frac{cN}{(\log N)^{3.5}} \left((\beta - \alpha) - e^{-\frac{\log^3 N}{320 \log^2 m}}\right),$$

where $c > 0$ is an absolute constant.

(III) Given $a_1 \ge 7$ and N satisfying $e^{a_1 \log^{2/3} m} \le N \le m^{4/7}$, and an integer k such that $m^{\frac{1}{2k-1} + \frac{1}{4k-1}} \le N < m^{\frac{1}{2k-3} + \frac{1}{4k-3}}$, let

$$4X = m^{\frac{1}{2k-1}},\ 4Y = m^{\frac{1}{4k-1}},\ X_1 = 2X,\ Y_1 = 2Y,\ N_1 = Nm^{-\frac{1}{2k-1} - \frac{1}{4k-1}}.$$

If A is the set of indices $n \le N$ of the form $n = rpq$ where p, q are primes, r is 1 or a prime such that $X < p \le X_1$, $Y < q \le Y_1$, $1 \le r \le N_1$, and $A([\alpha, \beta); A; x_n) = \#\{n \in A\,;\, x_n \in [\alpha, \beta)\}$, then we have

$$A([\alpha, \beta); A; x_n) = (\beta - \alpha)|A| + \mathcal{O}(R)$$

with $R = (4k)^{180k} N^{1-\frac{1}{320k^2}}$.

NOTES:

(I) A.A. Karacuba (1996). With n replaced by the nth prime p_n in the finite sequence

$$x_n = \frac{ap_n^* + bp_n}{m} \bmod 1, \quad 1 \le p_n \le X, (p_n, m) = 1,$$

he gives

$$A([\alpha,\beta); M; x_n) = (\beta - \alpha)\pi(X)\big(1 + \mathcal{O}((\log m)^{-c_2})\big)$$

for some $c_2 = c_2(\varepsilon)$.

(II,III) A.A. Karacuba (1997).

A.A. Karacuba (A.A. Karatsuba): *Sums of fractional parts of functions of a special type*, (Russian), Dokl. Akad. Nauk **349** (1996), no. 3, 302 (MR1440998 (98f:11072); Zbl. 0918.11038). (English translation Dokl. Math. **54** (1996), no. 1, 541).

A.A. Karacuba (A.A. Karatsuba): *Analogues of incomplete Kloosterman sums and their applications*, (Russian), Tatra Mt. Math. Publ. **11** (1997), 89–120 (MR1475508 (98j:11062); Zbl. 0978.11037).

2.20.35. Let $a_1 = 1 < a_2 < \cdots < a_{\varphi(n)}$, $0 < a_i < n$, be the sequence of all integers coprime to n and define a_i^* by the congruence $a_i a_i^* \equiv 1 \pmod n$. Then the sequence of blocks

$$A_n = \left(\left| \frac{a_1}{n} - \frac{a_1^*}{n} \right|, \left| \frac{a_2}{n} - \frac{a_2^*}{n} \right|, \dots, \left| \frac{a_{\varphi(n)}}{n} - \frac{a_{\varphi(n)}^*}{n} \right|, \right), \quad n = 1, 2, \dots,$$

has the a.d.f.

$$g(x) = 2x - x^2,$$

and for its star discrepancy we have

$$D_{\varphi(n)}^* \le 4 \left(\frac{3}{2}\right)^2 17 \, \frac{d^2(n)\sqrt{n}}{\varphi(n)} (\log \varphi(n))^2$$

for every $n \ge 8$.

NOTES: The problem of finding the a.d.f. of A_n was formulated as an open problem in W. Zhang (1995). However the solution directly follows from the fact that the block sequence 3.7.2

$$A_n = \left(\left(\frac{a_1}{n}, \frac{a_1^*}{n} \right), \left(\frac{a_2}{n}, \frac{a_2^*}{n} \right), \dots, \left(\frac{a_{\varphi(n)}}{n}, \frac{a_{\varphi(n)}^*}{n} \right) \right), \quad n = 1, 2, \dots,$$

is u.d. and that $\iint_{\substack{|u-v|<x\\(u,v)\in[0,1]^2}} 1.\,\mathrm{d}u\,\mathrm{d}v = 2x - x^2$. Zhang also found the estimates for the even moments

$$\frac{1}{\varphi(n)}\sum_{i=1}^{\varphi(n)}\left|\frac{a_i}{n}-\frac{a_i^*}{n}\right|^{2k} = \frac{1}{(2k+1)(k+1)} + \mathcal{O}\left(\frac{4^k\sqrt{n}d^2(n)}{\varphi(n)}(\log n)^2\right),$$

where $\varphi(n)$ is the Euler function, $d(n)$ is the divisor function and the $\mathcal{O}$–constant is absolute. Note that for the odd moment $2k+1$ the leading term is again $\int_0^1 x^{2k+1}\,\mathrm{d}(2x-x^2) = \frac{1}{(2k+3)(k+1)}$. Zhang found this for any real $k \ge 0$ in (1997) but without the factor 4^k in the $\mathcal{O}$-term. Using the theory of u.d., especially the Koksma – Hlawka inequality, O. Strauch, M. Paštéka and G. Grekos (2003) proved the error term

$$\left|\frac{1}{\varphi(n)}\sum_{i=1}^{\varphi(n)}\left|\frac{a_i}{n}-\frac{a_i^*}{n}\right|^{K} - \int_0^1\int_0^1 |x-y|^K\,\mathrm{d}x\,\mathrm{d}y\right| \le V(|x-y|^K)D^*_{\varphi(n)},$$

which is independent on K for $K = 1, 2, \dots$. Here for the Hardy – Krause variation we have $V(|x-y|^K) = 4$ and for the star discrepancy (cf. 3.7.2)

$$D^*_{\varphi(n)}\left(\left(\frac{a_i}{n},\frac{a_i^*}{n}\right)\right) = \mathcal{O}\left(\frac{d(n)\sqrt{n}}{\varphi(n)}(\log\varphi(n))^2\right).$$

In 1996 Zhang also found the a.d.f. $g(x) = 2x - x^2$ and for the star discrepancy of A_n with respect to $g(x)$ he proved

$$D^*_{\varphi(n)} = \mathcal{O}\left(\frac{d^2(n)\sqrt{n}}{\varphi(n)}(\log n)^3\right).$$

In (1997) he improved this estimate with $(\log n)^3$ replaced by $(\log n)^2$. On the other hand, the estimate containing $(\log\varphi(n))^2$ follows from 2.3.20.

Related sequences: 2.20.36

O. Strauch – M. Paštéka – G. Grekos: *Kloosterman's uniformly distributed sequence*, J. Number Theory **103** (2003), no. 1, 1–15 (MR2008062 (2004j:11081); Zbl. 1049.11083).
W. Zhang: *On the difference between an integer and its inverse modulo n*, J. Number Theory **52** (1995), no. 1, 1–6 (MR1331760 (96f:11123); Zbl. 0826.11002).
W. Zhang: *On the distribution of inverse modulo n*, J. Number Theory **61** (1996), no. 2, 301–310 (MR1423056 (98g:11109); Zbl. 0874.11006).
W. Zhang: *Some estimates of trigonometric sums and their applications*, Acta Math. Hungarica **76** (1997), no. 1–2, 17–30 (MR1459767 (99b:11093); Zbl. 0906.11043).

2.20.36. Let p be an odd prime and k a positive integer. Let $a_1 = 1 < a_2 < \cdots < a_{p-1}$, $0 < a_i < p$, be the sequence of all integers coprime to p and define a_i^* by the congruence $a_i a_i^* \equiv 1 \pmod p$. Then the sequence of blocks

$$A_p = \left(\left| \left\{ \frac{(a_1)^k}{p} \right\} - \left\{ \frac{(a_1^*)^k}{p} \right\} \right|, \left| \left\{ \frac{(a_2)^k}{p} \right\} - \left\{ \frac{(a_2^*)^k}{p} \right\} \right|, \dots, \left| \left\{ \frac{(a_{p-1})^k}{p} \right\} - \left\{ \frac{(a_{p-1}^*)^k}{p} \right\} \right| \right)$$

with $n = 1, 2, \dots$, has for $p \to \infty$ the a.d.f.

$$g(x) = 2x - x^2$$

and for its star discrepancy there holds

$$D_{p-1}^* = \mathcal{O}\left(\frac{(\log p)^2}{\sqrt{p}} \right),$$

where the $\mathcal{O}$–constant depends only on k.

RELATED SEQUENCES: 2.20.35

W. ZHANG: *On the distribution of inverse modulo p*, Acta Arith. **100** (2001), no. 2, 189–194 (MR1864154 (2002j:11115); Zbl. 0997.11077).

2.20.37. Let α_1, β_1, α_2, and β_2 be real numbers such that $0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < 1$. If N is a positive integer then let S_N be the set of pairs (p, q) of coprime positive integers which satisfy $\alpha_1 N < p < \beta_1 N < \alpha_2 N < q < \beta_2 N$. If (p, q) is a couple of integers let $p_1/q_1, \dots, p_r/q_r = p/q$ be the sequence of the successive convergents of the continued fraction expansion of p/q. In particular, $(x, y) = (q_{r-1}, p_{r-1})$ is a solution of the equation $px - qy = \pm 1$. Then the sequence of individual block

$$A_N = \left\{ \frac{q_{r-1}}{q} \; ; \; (p, q) \in S_N \right\}, \quad N = 1, 2, \dots,$$

is

u.d. which respect to the interval $[0, 1]$.

NOTES: The u.d. of A_N was proved in E.I. Dinaburg and Ya.G. Sinaĭ (1990) where they noticed that quantitative estimates can be proved using Kloosterman's sum. A quantitative estimate was given by A. Fujii (1992) and G.J. Rieger (1993). The two–dimensional generalization was given by D.I. Dolgopyat (1994), cf. 3.7.5.

RELATED SEQUENCES: 2.20.38

E.I. Dinaburg – Ya.G. Sinaĭ: *The statistics of the solutions of the integer equation $ax - by = \pm 1$*, (Russian), Funkts. Anal. Prilozh. **24** (1990), no. 3, 1–8,96 (English translation: Funct. Anal. Appl. **24** (1990), no. 3, 165–171). (MR1082025 (91m:11056); Zbl. 0712.11018).
D.I. Dolgopyat: *On the distribution of the minimal solution of a linear diophantine equation with random coefficients*, (Russian), Funkts. Anal. Prilozh. **28** (1994), no. 3, 22–34, 95 (English translation: Funct. Anal. Appl. **28** (1994), no. 3, 168–177 (MR1308389 (96b:11111); Zbl. 0824.11046)).
A. Fujii: *On a problem of Dinaburg and Sinaĭ*, Proc. Japan. Acad. Ser. A Math. Sci. **68** (1992), no. 7, 198–203 (MR1193181 (93i:11092); Zbl. 0779.11032).
G.J. Rieger: *Über die Gleichung $ad - bc = 1$ und Gleichverteilung*, Math. Nachr. **162** (1993), 139–143 (MR1239581 (94m:11092); Zbl. 0820.11013).

2.20.38. Let c and d be positive integers. For $x > 0$ define the block

$$A_x = \left\{ \frac{d^{-1} \pmod c}{c} \; ; \; 0 < c \le x, 0 < d \le x, \gcd(c,d) = 1 \right\}.$$

Then the sequence of blocks A_x with $x \to \infty$ is

u.d.

and for the discrepancy of the individual block A_x we have

$$D_x = \mathcal{O}(x^{\varepsilon - 1/2})$$

where ε is an arbitrarily small positive number.

Notes: u.d. was proved by E.I. Dinaburg and Ya.G. Sinaĭ (1990). G.J. Rieger (1993) proved the estimate $D_x = \mathcal{O}(x^{-1/4} \log^3 x)$ using estimates for Kloosterman sums and noticed that the result can be improved using their better estimates. This was done independently by A. Fujii (1992), cf. also MR 94m:11092.

Related sequences: 2.20.37

E.I. Dinaburg – Ya.G. Sinaĭ: *The statistics of the solutions of the integer equation $ax - by = \pm 1$*, (Russian), Funkts. Anal. Prilozh. **24** (1990), no. 3, 1–8,96 (English translation: Funct. Anal. Appl. **24** (1990), no. 3, 165–171). (MR1082025 (91m:11056); Zbl. 0712.11018).
A. Fujii: *On a problem of Dinaburg and Sinaĭ*, Proc. Japan. Acad. Ser. A Math. Sci. **68** (1992), no. 7, 198–203 (MR1193181 (93i:11092); Zbl. 0779.11032).
G.J. Rieger: *Über die Gleichung $ad - bc = 1$ und Gleichverteilung*, Math. Nachr. **162** (1993), 139–143 (MR1239581 (94m:11092); Zbl. 0820.11013).

2.20.39. Let $h(-n)$ denote the class number of the quadratic number field $\mathbb{Q}(\sqrt{-n})$. Then the sequence

$$\frac{\pi h(-n)}{2\sqrt{n}}, \qquad n = 1, 2, \dots,$$

has, in $[0, \infty)$, the a.d.f.

$$g(x)$$

which characteristic function (for the def. see 1.6) is

$$f(t) = \sum_{k=0}^{\infty} \frac{r(k)}{k!}(it)^k,$$

where

$$r(k) = \sum_{n=1,2\nmid n}^{\infty} \frac{\varphi(n)\tau_k(n^2)}{n^3}$$

S. Chowla – P. Erdős: *A theorem of distribution of values of L–functions*, J. Indian Math. Soc. (N.S.) **15** (1951), 11–18 (MR0044566 (**13**,439a); Zbl. 0043.04602).

2.20.39.1 **Open problem:** Characterize the distribution of the sequence

$$B_{2n} \bmod 1 \quad n = 1, 2, \dots.$$

where B_n denotes the n-th Bernoulli number.

Notes: (I) By von Staudt-Clausen formula

$$B_{2n} = A_{2n} - \sum_{(p-1)|2n} \frac{1}{p},$$

where p runs over primes and A_{2n} are suitable integers.
(II) The distribution of the fractional parts of B_{2n} was studied by P. Erdős and S.S. Wagstaff Jr. (1980). They proved that $\sum_{(p-1)|2n} \frac{1}{p}$ is everywhere dense in $[5/6, \infty)$ (F. Luca's comment on the problem).

P. Erdős – S.S. Wagstaff, Jr.: *The fractional parts of the Bernoulli numbers*, Illinois J. Math. **24** (1980), no. 1, 104–112 (MR0550654 (81c:10064); Zbl. 0405.10011).

2.20.39.2 The Ramanujan tau function $\tau(n)$ is defined by

$$\Delta(z) = e^{2\pi iz} \prod_{n=1}^{\infty} \left(1 - e^{2\pi iz}\right)^{24} = \sum_{n=1}^{\infty} \tau(n) e^{2\pi inz},$$

where $\Im z > 0$.

When p is a prime, write

$$\tau(p) = 2p^{11/2} \cos\theta_p.$$

A conjecture of Ramanujan claims that θ_p is real. Assuming the truth of this conjecture, Sato and Tate **conjectured** that the sequence

$$\theta_{p_n}, \quad n = 1, 2, \dots,$$

has the a.d.f. in $[0, \pi]$ with the density

$$h(x) = \frac{2}{\pi}(\sin x)^2.$$

Here p_n is the increasing sequence of all primes.

NOTES: D.H. Lehmer (1970) reports on a test of this conjecture for the primes $< 10^4$.

RELATED SEQUENCES: 2.20.32

D.H. LEHMER: *Note on the distribution of Ramanujan's tau function*, Math. Comp. **24** (1970), 741–743 (MR0274401 (**43** #166); Zbl. 0214.30601).

2.21 Sequences involving special functions

2.21.1. Let α_n, $n = 1, 2, \dots$, denote the sequence of positive zeros of the Bessel function $J_0(x)$ ordered in the increasing order and set

$$A_N = \left(\frac{\alpha_1}{\alpha_N}, \frac{\alpha_2}{\alpha_N}, \dots, \frac{\alpha_N}{\alpha_N}\right).$$

Then for the finite sequence A_N we have

$$D_N = \mathcal{O}\left(\frac{1}{N}\right)$$

and thus the block sequence $(A_n)_{n=1}^{\infty}$ is

u.d.

NOTES: R.F. Tichy (1998) answers in this way a question posed by F.J. Schnitzer.

R.F. TICHY: *Three examples of triangular arrays with optimal discrepancy and linear recurrences*, in: *Applications of Fibonacci Numbers (The Seventh International Research Conference, Graz, 1996)*, Vol. 7, (G.E. Bergum, A.N. Philippou and A.F. Horadam eds.), 1998, Kluwer Acad. Publ., Dordrecht, Boston, London, pp. 415–423 (MR1638468; Zbl. 0942.11036).

2.21.1.1 Let $x \in [0,1) \setminus \mathbb{Q}$ and $\frac{p_n}{q_n}$ be the nth regular continued fraction convergent of x, $n \geq 0$. The approximation coefficient $\Theta_n = \Theta_n(x)$ is defined by

$$\Theta_n = q_n^2 \left| x - \frac{p_n}{q_n} \right| \quad \text{for} \quad n \geq 0.$$

Then for almost all x and all $z \in [0,1]$, the limit

$$\lim_{N\to\infty} \frac{1}{N} \#\{1 \leq n \leq N \mid \Theta_n(x) \leq z\}$$

exists and equals the d.f. $g(z)$ defined by

$$g(z) = \begin{cases} \dfrac{z}{\log 2} & \text{if } 0 \leq z \leq \frac{1}{2}, \\ \dfrac{1 - z + \log 2z}{\log 2} & \text{if } \frac{1}{2} \leq z \leq 1. \end{cases}$$

NOTES: See C. Kraaikamp and I. Smeets (2010, p. 18): In the early 1980s it is was conjectured by H.W. Lenstra. A version of this conjecture had been formulated by W. Doeblin (1940) before. In 1983 W. Bosma *et al.* (1983) proved the Doeblin-Lenstra-conjecture for regular continued fractions and Nakada's α-expansions for $\alpha \in \left[\frac{1}{2}, 1\right]$.

C. KRAAIKAMP – I. SMEETS: *Approximation results for α-Rosen fractions*, Unif. Distrib. Theory **5** (2010), no. 2, 15–53 (MR2608015 (2011d:11189); Zbl. 1249.11083)
W. BOSMA – H. JAGER – F. WIEDIJK: *Some metrical observations on the approximation by continued fractions*, Nederl. Akad. Wetensch. Indag. Math. **45** (1983), no. 3, 281–299 (MR0718069 (85f:11059); Zbl. 0519.10043).
W. DOEBLIN: *Remarques sur la théorie métrique des fractions continues*, Compositio Math. **7** (1940), 353–371 (MR0002732 (2,107e); Zbl. 0022.37001).

2.22 Sequences of rational numbers

2.22.1. Let a_n be a given strictly increasing sequence of positive integers and define

$$A_n = \left(\frac{1}{a_n}, \frac{2}{a_n}, \dots, \frac{a_n}{a_n} \right)$$

and let $\omega = (A_n)_{n=1}^{\infty}$ to be the block sequence formed from these blocks. The sequence A_n of individual blocks is

u.d.

for any such a_n and the compound sequence ω is

u.d.

if and only if

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_1+\cdots+a_n}=0. \qquad (*)$$

Let $N=k+\sum_{i=1}^n a_i$, where $0\le k\le a_{n+1}$. Then

$$\begin{aligned}N^2D_N^{(2)} =&\frac{1}{4}n^2+\frac{1}{12}\sum_{i,j=1}^n\frac{(a_i,a_j)^2}{a_ia_j}+\frac{k^2}{a_{n+1}^2}\left(\frac{1}{3}k^2+\frac{1}{2}k+\frac{1}{6}\right)+\\ &+\frac{k}{a_{n+1}}\left(-\frac{2}{3}k^2-\frac{1}{2}k+\frac{1}{6}\right)+\frac{1}{3}k^2+\frac{1}{2}kn+\frac{1}{6}k\sum_{i=1}^n\frac{1}{a_i}+\\ &+2\int_0^{k/a_{n+1}}\{xa_{n+1}\}\left(\sum_{i=1}^n\{xa_i\}\right)\mathrm{d}x-\\ &-2a_{n+1}\int_0^{k/a_{n+1}}x\left(\sum_{i=1}^n\{xa_i\}\right)\mathrm{d}x-2k\int_{k/a_{n+1}}^1\left(\sum_{i=1}^n\{xa_i\}\right)\mathrm{d}x.\end{aligned}$$

For $k=0$ we have

$$N^2D_N^{(2)}=\frac{1}{4}n^2+\frac{1}{12}\sum_{i,j=1}^n\frac{(a_i,a_j)^2}{a_ia_j}=\frac{1}{4}n^2+\frac{1}{2\pi^2}\sum_{h=1}^\infty\frac{1}{h^2}\left(\sum_{\substack{i=1\\ a_i|h}}^n a_i\right)^2.$$

NOTES: (I) The block sequence of this type was investigated by S. Knapowski (1957) who proved the sufficiency of $(*)$ for the u.d. The necessity of $(*)$ with a complete theory of u.d. is given in Š. Porubský, T. Šalát and O. Strauch (1990). They also proved, if a_n is strictly increasing then:

(i) $\lim_{n\to\infty}a_{n-1}/a_n=1$ implies $(*)$, and in the opposite direction $(*)$ implies $\limsup_{n\to\infty}a_{n-1}/a_n=1$,

(ii) $a_n=o(n^2)$ implies $(*)$,

(iii) positive upper asymptotic density of a_n implies $(*)$,

(iv) $(*)$ implies $\lim_{n\to\infty}n^{-1}\log(a_1+\cdots+a_n)=0$,

(v) if a subsequence a_{k_n} satisfies $(*)$, then also a_n does,

(vi) if two strictly increasing sequences a_n and b_n satisfy $(*)$, then also a_n+b_n and the convolution $a_1b_n+a_2b_{n-1}+\cdots+a_1b_n$ satisfy $(*)$,

(vii) if $a_n=\mathcal{O}(n^{3/2})$ and $b_n=o(n^{3/2})$ then a_nb_n satisfies $(*)$,

(viii) Let $p(x)$ be a polynomial with integer coefficients with a positive leading coefficient. Then a_n and $p(n)a_n$ satisfy simultaneously condition $(*)$.

(ix) Let a_n be a linear recurring sequence with the characteristic polynomial $Q(x)$. Then a_n satisfies $(*)$ if and only if a) all roots of $Q(x)$ are roots of 1, b) $Q(1)=0$ and the multiplicity of 1 is ≥ 2 and it is strongly greater than a multiplicity of any other root of $Q(x)$.

(x) Consequently an increasing linear recurring sequence a_n satisfies $(*)$ if and only if $\lim_{n\to\infty} a_{n-1}/a_n = 1$.

(II) The result for L^2 discrepancy in the case $k = 0$ can be found in O. Strauch (1989) and for general k in Š. Porubský, T. Šalát and O. Strauch (1990). Note that the integrals can be computed using the following formulas

$$\begin{aligned}\int_0^{k/b} \{xb\}\{xa\}\,\mathrm{d}x =& \frac{1}{b}\left(\frac{ak}{3b} - \frac{k}{2ba}(a-1)\left(\frac{2a}{3}+\frac{1}{6}\right)+\right.\\ &\left.+\sum_{s=0}^{a-1}\sum_{i=0}^{k-1}\frac{2s+1}{2a^2}\left\{\frac{s+ia}{b}\right\}\right), \quad 0\le k\le b,\\ \int_0^t x\{ax\}\,\mathrm{d}x =& \frac{t^2}{4}+\frac{t}{12a}-\frac{\{ta\}^3}{2a^2}+\frac{t\{ta\}^2}{2a}-\frac{t\{ta\}}{2a}+\frac{\{ta\}^2}{4a^2}-\frac{\{ta\}}{12a^2},\\ \int_0^t \{xa\}\,\mathrm{d}x =& \frac{t}{2}+\frac{\{ta\}^2}{2a}+\frac{\{ta\}}{2a}.\end{aligned}$$

(III) B. Jessen (1934) proved that if a_n is a strictly increasing sequence of positive integers such that $a_n|a_{n+1}$ for $n = 1, 2, \dots$, and f is a Lebesgue integrable function on $[0, 1]$, then the limit

$$\lim_{n\to\infty}\frac{1}{a_n}\sum_{j=1}^{a_n} f\left(\left\{x+\frac{j}{a_n}\right\}\right) = \int_0^1 f(t)\,\mathrm{d}t$$

holds almost everywhere with respect to the Lebesgue measure. In the opposite direction, R.C. Baker (1976) proved that if an increasing sequence of positive integers a_n satisfies the following two conditions

(i) $\liminf_{n\to\infty} \frac{\log a_n}{n} = 0$, and

(ii) $\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\frac{A_n}{a_n} > 0$, where A_n is the number of fractions j/a_n with $0 < j < a_n$ that are not equal to i/a_m for some integer i and $m < n$,

then there exists a Lebesgue integrable function f on $[0, 1]$ such that

$$\limsup_{n\to\infty}\frac{1}{a_n}\sum_{j=1}^{a_n} f\left(\left\{x+\frac{j}{a_n}\right\}\right) = \infty$$

for almost all x. Baker (1976) noted that (ii) is probable superfluous and he proved that if $a_n = \mathcal{O}(e^{\sqrt{n}(\log n)^{-7/2-\varepsilon}})$ for some $\varepsilon > 0$ then the lim sup above equals ∞.

(IV) Let $F^{(1)}_{N_1}(x)$ and $F^{(2)}_{N_2}(x)$ be step d.f.'s of sequences $x_1, \dots, x_{N_1}$ and $y_1, \dots, y_{N_2}$, respectively. The integral formula 4.2(I) gives

$$\int_0^1 (F^{(1)}_{N_1}(x)-x)(F^{(2)}_{N_2}(x)-x)\,\mathrm{d}x = \int_0^1\int_0^1 -\frac{|x-y|}{2}\,\mathrm{d}(F^{(1)}_{N_1}(x)-x)\,\mathrm{d}(F^{(2)}_{N_2}(y)-y).$$

Applying this to block sequences

$$\left(\frac{0}{a},\frac{1}{a},\dots,\frac{a-1}{a}\right),\quad \left(\frac{0}{b},\frac{1}{b},\dots,\frac{b-1}{b}\right),$$

where a and b are positive integers (with $N_1 = a$ and $N_2 = b$) and using the Franel–Kluyver's integral

$$\int_0^1 \left(\{ax\}-\frac{1}{2}\right)\left(\{bx\}-\frac{1}{2}\right)\mathrm{d}x = \frac{1}{12}\frac{(\gcd(a,b))^2}{ab}$$

O. Strauch (1989) proved: For every positive integers a, b and X we have

$$\frac{1}{12}\frac{(\gcd(a,b))^2}{ab} = \sum_{k=2}^{\infty}\frac{(2k)!}{2(2k-1)(2^k.k!)^2}\times$$
$$\times\left(\sum_{\substack{r,s=1\\2\le r+s\le k}}^{k}\frac{1}{X^{2(r+s)-2}}\binom{2(r+s)}{2r}\frac{B_{2r}}{a^{2r-1}}\cdot\frac{B_{2s}}{b^{2s-1}}\cdot\frac{(-2)}{2(r+s)(2(r+s)-1)}\cdot\right.$$
$$\left.\cdot\left((-1)^{r+s-1}\binom{k}{r+s-1}-(-1)^k 2^{2k-2(r+s)+2}\binom{k}{2k-2(r+s)+2}\right)\right),$$

where B_r is the rth Bernoulli number and for the binomial coefficients we take $\binom{m}{n}=0$ if $n<0$ or $n>m$. The remainder $\sum_{k=K+1}^{\infty}$ of the infinite series on the right hand side does not exceed

$$\sum_{k=K+1}^{\infty} \le 9\frac{X}{\sqrt{K}}\min\{a,b\}.$$

R.C. Baker: *Riemann sums and Lebesgue integrals*, Quart. J. Math. Oxford Ser. (2) **27** (1976), no. 106, 191–198 (MR0409395 **53** #13150; Zbl. 0333.10033).
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J.C. Kluyver: *An analytical expression for the greatest common divisor of two integers*, Proc. Royal Acad. Amsterdam **V, II** (1903), 658–662 (= *Eene analytische uitdrukking voor den grootsten gemeenen deeler van twee geheele getallen*, (Dutch), Amst. Versl. **11** (1903), 782–786 JFM 34.0214.04).
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Š. Porubský – T. Šalát – O. Strauch: *On a class of uniform distributed sequences*, Math. Slovaca **40** (1990), 143–170 (MR1094770 (92d:11076); Zbl. 0735.11034).
O. Strauch: *Some applications of Franel – Kluyver's integral, II*, Math. Slovaca **39** (1989), 127–140 (MR1018254 (90j:11079); Zbl. 0671.10002).

2.22.2. Ratio sequences. For an increasing sequence of positive integers x_n let $\underline{d}(x_n)$, and $\overline{d}(x_n)$ denote the lower and upper asymptotic density of x_n, resp., and $d(x_n)(=\underline{d}(x_n)=\overline{d}(x_n))$ its asymptotic density if it exists, cf. p. 1 – 3 . The double sequence, called the **ratio sequence of** x_n,

$$\frac{x_m}{x_n}, \qquad m,n=1,2,\dots,$$

is

everywhere dense in $[0,\infty)$

assuming that one of the following conditions holds:
(i) $d(x_n)>0$,
(ii) $\overline{d}(x_n)=1$,
(iii) $\underline{d}(x_n)+\overline{d}(x_n)\geq 1$,
(iv) $\underline{d}(x_n)\geq 1/2$,
(v) $A([0,x);x_n)\sim\frac{cx}{\log^\alpha x}$, where $c>0,\alpha>0$ are constant, $A([0,x);x_n)=\#\{n\in\mathbb{N}\,;\,x_n\in[0,x)\}$, and $\sim$ denotes the asymptotically equivalence (i.e. the ratio of the left and the right–hand side tends to 1 as $x\to\infty$).

Notes: (I) (i), (ii) and (v) were proved by T. Šalát (1969), for (iii) see O. Strauch and J.T. Tóth (1998) and (iv) follows from (iii).
(II) Strauch and Tóth (1998, Th. 2) proved that if the interval $(\alpha,\beta)\subset[0,1]$ has an empty intersection with $\frac{x_m}{x_n}$ for $m,n=1,2,\dots$, then

$$\underline{d}(x_n)\leq\frac{\alpha}{\beta}\min(1-\overline{d}(x_n),\overline{d}(x_n)), \qquad \overline{d}(x_n)\leq 1-(\beta-\alpha).$$

S. Konyagin (personal communication) improved the inequality to

$$\overline{d}(x_n)\leq\frac{1-\beta}{1-\alpha\beta}.$$

(III) In O. Strauch and J.T. Tóth (2001) the ratio sequence $\frac{x_m}{x_n}$, $m,n=1,2,\dots$, is ordered to a block sequence X_n, $n=1,2,\dots$, with blocks

$$X_n=\left(\frac{x_1}{x_n},\frac{x_2}{x_n},\dots,\frac{x_n}{x_n}\right),$$

Related sequences: 2.22.6, 2.22.7, 2.22.8, 2.22.5.1.

(IV) J.T. Tóth, L. Mišík and F. Filip (2004) introduced the dispersion $\tilde{d}(X_n)$ of a block X_n defining

$$\tilde{d}(X_n)=\max\left(\frac{x_1}{x_n},\frac{x_2-x_1}{x_n},\frac{x_3-x_2}{x_n},\dots,\frac{x_n-x_{n-1}}{x_n}\right).$$

Given a sequence x_n of positive integers, define its dispersion by

$$\tilde{d} = \liminf_{n\to\infty} \tilde{d}(X_n),$$

where $X_n - \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right)$. Contrary to the classical dispersion (cf. 1.10.11), $\tilde{d} = 0$ does not characterize the everywhere density of the ratio sequence x_m/x_n, $m, n = 1, 2, \dots$. They proved that for every $\alpha \in [0, 1/2]$ there exists an increasing sequence of positive integers x_n, $n = 1, 2, \dots$, such that $\tilde{d} = \alpha$ and the double sequence x_m/x_n, $m, n = 1, 2, \dots$, is everywhere dense in $[0, \infty)$. For a better estimate of $\tilde{d}$ consult F. Filip and J.T. Tóth (2005).

F. Filip – J.T. Tóth: *On estimation of dispersions of certain dense block sequences*, Tatra Mt. Math. Publ. **31** (2005), 65–74 (MR2208788 (2006k:11014); Zbl. 1150.11338)
T. Šalát: *On ratio sets of sets of natural numbers*, Acta Arith. **15** (1968/69), 273–278 (MR0242756 (**39** #4083); Zbl. 0177.07001).
O. Strauch – J.T. Tóth: *Asymptotic density of $A \subset \mathbb{N}$ and density of the ratio set $R(A)$*, Acta Arith. **87** (1998), no. 1, 67–78 (correction *ibid.* **103** (2002), no. 2, 191–200). (MR1659159 (99k:11020); Zbl. 0923.11027).
O. Strauch – J.T. Tóth: *Distribution functions of ratio sequences*, Publ. Math. (Debrecen) **58** (2001), 751–778 (MR1828725 (2002h:11068); Zbl. 0980.11031).
J.T. Tóth – L. Mišík – F. Filip: *On some properties of dispersion of block sequences of positive integers*, Math. Slovaca **54** (2004), no. 5.453–464 (MR2114616 (2005k:11014); Zbl. 1108.11017)

2.22.3. If x_n and y_n are two increasing sequences of positive integers then the double sequence (again called ratio sequence of x_n and y_n)

$$\frac{x_m}{y_n}, \qquad m, n = 1, 2, \dots,$$

is

everywhere dense in $[0, \infty)$

(which is clearly equivalent to the everywhere density of $\frac{y_m}{x_n}$, $m, n = 1, 2, \dots$, in $[0, \infty)$) assuming that one of the following conditions holds:
(i) $d(x_n) > 0$,
(ii) $x_{n+1}/x_n \to 1$,
(iii) $\underline{d}(x_n) > 0$ and $\underline{d}(x_n) + \overline{d}(y_n) \geq 1$.
Notes: (i) was proved by T. Šalát (1971) and (ii) and (iii) were proved by J. Bukor and J.T. Tóth (2003). They also proved a converse to (iii): For any two positive real numbers γ and δ with $\gamma + \delta < 1$ there exist two sequences x_n and y_n such that $\underline{d}(x_n) = \gamma$, $\overline{d}(y_n) = \delta$ and which ratio sequence x_m/y_n, $m, n = 1, 2, \dots$, is not everywhere dense in $[0, \infty)$.

J. Bukor – J.T. Tóth: *On some criteria for the density of the ratio sets of positive integers*, JP J. Algebra Number Theory Appl. **3** (2003), no. 2, 277–287 (MR1999166 (2004e:11009); Zbl. 1043.11009).
T. Šalát: *Quotientbasen und (R)–dichte Mengen*, Acta Arith. **19** (1971), 63–78 (MR0292788 (**45** #1870); Zbl. 0218.10071).

2.22.4. If p and q be two coprime positive integers and

$$x_n = p^n, \quad y_n = q^n, \qquad \text{for } n = 1, 2, \dots,$$

then the double sequence

$$\frac{x_m}{y_n}, \quad m, n = 1, 2, \dots,$$

is

$$\text{dense in } [0, \infty).$$

J. Smítal: *Remarks on ratio sets of sets of natural numbers*, Acta Fac. Rerum Nat. Univ. Comenian. Math. **25** (1971), 93–99 (MR0374079 (**51** #10279); Zbl. 0228.10036).

2.22.5. Let L and M be two non–zero coprime integers with $L - 4M \neq 0$ and moreover let α and β be the roots of the quadratic equation $x^2 - \sqrt{L}x + M = 0$ such that α/β is not a root of 1. The n**th Lehmer number** l_n corresponding to the pair (L, M) is defined by

$$l_n = \begin{cases} \dfrac{\alpha^n - \beta^n}{\alpha - \beta}, & \text{if } n \equiv 1 \pmod{2}, \\ \dfrac{\alpha^n - \beta^n}{\alpha^2 - \beta^2}, & \text{otherwise.} \end{cases}$$

Then the double sequence

$$\frac{l_m}{l_n}, \qquad m, n = 1, 2, \dots,$$

is

$$\text{dense in } [0, \infty).$$

F. Luca – Š. Porubský: *The multiplicative group generated by the Lehmer numbers*, Fibonacci Quart. **41** (2003), no. 2, 122–132 (MR1990520 (2004c:11016); Zbl. 1044.11008).

2.22.5.1 Let x_n, $n = 1, 2, \dots$ be an increasing sequence of positive integers, $\underline{d}(x_n)$ be the lower and $\overline{d}(x_n)$ the upper asymptotic density of x_n, $n = 1, 2, \dots$, and $X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right)$. As in 1.8.23, let $G(X_n)$ be the set of all d.f.'s of the block sequence X_n, $n = 1, 2, \dots$, i.e. the set of the all possible weak limits $g(x)$, where

$$\frac{\#\{i \le n_k; x_i/x_{n_k} < x\}}{n_k} \to g(x), \text{ as } k \to \infty.$$

$G(X_n)$ has the following properties:

(i) If $g(x) \in G(X_n)$ is increasing and continuous at $x = \beta$ and $g(\beta) > 0$, then there exists $1 \le \alpha < \infty$ such that $\alpha g(x\beta) \in G(X_n)$. If every d.f. of $G(X_n)$ is continuous at 1, then $\alpha = 1/g(\beta)$.

(ii) Let the all d.f.'s in $G(X_n)$ be continuous at 0 and $c_1(x) \notin G(X_n)$. Then for every $\tilde{g}(x) \in G(X_n)$ and every $1 \le \alpha < \infty$ there exists $g(x) \in G(X_n)$ and $0 < \beta \le 1$ such that $\tilde{g}(x) = \alpha g(x\beta)$ a.e.

(iii) Let the all d.f.'s in $G(X_n)$ be continuous at 1. Then all d.f.'s in $G(X_n)$ are continuous on $(0, 1]$, i.e. the only possible discontinuity is at 0.

(iv) If $\underline{d}(x_n) > 0$, then for every $g(x) \in G(X_n)$ we have

$$(\underline{d}(x_n)/\overline{d}(x_n)).x \le g(x) \le (\overline{d}(x_n)/\underline{d}(x_n)).x$$

for every $x \in [0, 1]$. Consequently, if $\underline{d}(x_n) = \overline{d}(x_n) > 0$ then the block sequence X_n, $n = 1, 2, \dots$, is u.d.

(v) If $\underline{d}(x_n) > 0$, then every $g(x) \in G(X_n)$ is continuous on $[0, 1]$.

(vi) If $\underline{d}(x_n) > 0$, then there exists $g(x) \in G(X_n)$ such that $g(x) \ge x$ for every $x \in [0, 1]$.

(vii) If $\overline{d}(x_n) > 0$, then there exists $g(x) \in G(X_n)$ such that $g(x) \le x$ for every $x \in [0, 1]$.

(viii) Let $G(X_n)$ be a singleton, i.e. $G(X_n) = \{g(x)\}$. Then either $g(x) = c_0(x)$ for $x \in [0, 1]$, or $g(x) = x^\lambda$ for some $0 < \lambda \le 1$ and $x \in [0, 1]$. Moreover, if $\overline{d}(x_n) > 0$, then $g(x) = x$.

(ix) $\max_{g \in G(X_n)} \int_0^1 g(x)\,\mathrm{d}x \ge \frac{1}{2}$.

(x) Let every d.f. $g(x) \in G(X_n)$ be a constant over a fixed interval $(u, v) \subset [0, 1]$ (the values of the functions may be distinct). If $\underline{d}(x_n) > 0$ then every d.f. in $G(X_n)$ is constant over infinitely many subintervals of $[0, 1]$.

(xi) There exists an increasing sequence x_n, $n = 1, 2, \dots$, of positive integers such that $G(X_n) = \{h_\alpha(x); \alpha \in [0, 1]\}$, where $h_\alpha(x) = \alpha$, $x \in (0, 1)$ is a constant d.f.

(xii) There exists an increasing sequence x_n, $n = 1, 2, \dots$, of positive integers such that

$$c_1(x) \in G(X_n) \text{ but } c_0(x) \notin G(X_n)$$

where $c_0(x)$ and $c_1(x)$ are one–jump d.f.'s with the jump of height 1 at $x = 0$ and $x = 1$, respectively.

(xiii) There exists an increasing sequence x_n, $n = 1, 2, \dots$, of positive integers such that $G(X_n)$ is non-connected.

(xiv) $G(X_n) = \{x^\lambda\}$ if and only if $\lim_{n\to\infty}(x_{k.n}/x_n) = k^{1/\lambda}$ for every $k = 1, 2, \dots$. Here as in (viii) we have $0 < \lambda \le 1$.

(xv) For every increasing integer sequence x_n, $n = 1, 2, \dots$, there exists $g(x) \in G(X_n)$ such that $g(x) \ge x$ for all $x \in [0, 1]$. This extend (vi).

(xvi) If $\underline{d}(x_n) > 0$, then all d.f.s $g(x) \in G(X_n)$ bounded by $h_1(x) \le g(x) \le h_2(x)$, where

$$h_1(x) = \begin{cases} x\frac{\underline{d}}{\overline{d}} & \text{if } x \in \left[0, \frac{1-\overline{d}}{1-\underline{d}}\right], \\ \frac{\underline{d}}{\frac{1}{x}-(1-\underline{d})} & \text{otherwise,} \end{cases}$$

$$h_2(x) = \min\left(x\frac{\overline{d}}{\underline{d}}, 1\right),$$

where $h_1(x)$, $h_2(x)$ are best possible.

(xvii) If $\underline{d}(x_n) > 0$, then for every $g(x) \in G(X_n)$ we have

$$0 \le \frac{g(y) - g(x)}{y - x} \le \frac{1}{d_g}$$

for $x < y$, $x, y \in [0, 1]$. Here $d_g = \lim_{k\to\infty} \frac{n_k}{x_{n_k}}$ if $\lim_{k\to\infty} F(X_{n_k}, x) = g(x)$.

(xviii) For every increasing sequence $x_1 < x_2 < \dots$ of positive integers with $0 < \underline{d} \le \overline{d}$ we have

1^0 $\frac{1}{2}\frac{\underline{d}}{\overline{d}} \le \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n}$,

2^0 $\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n} \le \frac{1}{2} + \frac{1}{2}\left(\frac{1-\min(\sqrt{\underline{d}},\overline{d})}{1-\underline{d}}\right)\left(1 - \frac{\underline{d}}{\min(\sqrt{\underline{d}},\overline{d})}\right)$.

(xix) Let H be a nonempty set of d.f.s defined on $[0, 1]$. Then there exists an integer sequence $1 \le x_1 < x_2 < \dots$ such that $H \subset G(X_n)$.

(xx) If $\underline{d}(x_n) > 0$, then the lower d.f. $\underline{g}(x)$ and the upper d.f. $\overline{g}(x)$ satisfy

$$\underline{g}(x).\underline{g}(y) \le \underline{g}(x.y) \le \overline{g}(x.y) \le \overline{g}(x).\overline{g}(y)$$

for every $x, y \in (0, 1)$.

NOTES: The properties (i)–(x) can be found in O. Strauch and J.T. Tóth (2001, 2002) (xi), (xiii) in G. Grekos and O. Strauch (2004), (xii) was found by L. Mišík (2004, personal communication) and (xiv) is in F. Filip and J.T. Tóth (2006). The properties (xv)–(xx) are from V. Baláž, L. Mišík, O. Strauch and J.T. Tóth ([a] 2013). For concrete examples, see 2.22.6, 2.22.7, 2.22.8.

[a]V. BALÁŽ – L. MIŠÍK – O. STRAUCH – J.T. TÓTH: *Distribution functions of ratio sequences, III*, Publ. Math. Debrecen **82** (2013), no. 3–4.511–529 (MR3066427; Zbl. 1274.11118).
[b]V. BALÁŽ – L. MIŠÍK – O. STRAUCH – J.T. TÓTH: *Distribution functions of ratio sequences, IV*, Periodica Math. Hungarica **66** (2013), no. 1.1–22 (MR3018198; Zbl. 1274.11119).
G. GREKOS – O. STRAUCH: *Distribution functions of ratio sequences, II*, Unif. Distrib. Theory **2** (2007), no. 1, 53–77 (MR2318532 (2008g:11125); Zbl. 1183.11042).
F. FILIP – J.T. TÓTH: *Distribution functions of ratio sequences*, 2006 (preprint).
O. STRAUCH – J.T. TÓTH: *Distribution functions of ratio sequences*, Publ. Math. (Debrecen) **58** (2001), 751–778 (MR1828725 (2002h:11068); Zbl. 0980.11031).
O. STRAUCH – J.T. TÓTH: *Corrigendum to Theorem 5 of the paper "Asymptotic density of $A \subset \mathbb{N}$ and density of the ratio set $R(A)$" (Acta Arith. 87 (1998), 67–78)*, Acta Arith. **103.2** (2002), 191–200 (MR1904872 (2003f:11015); Zbl. 0923.11027).

2.22.6. Let γ, δ, and a be given real numbers such that $1 \le \gamma < \delta \le a$. If x_n is the increasing sequence of all integers lying in the intervals

$$(\gamma,\delta),(\gamma a,\delta a),\dots,(\gamma a^k,\delta a^k),\dots,$$

then define the sequence of blocks

$$X_n = \left(\frac{x_1}{x_n},\frac{x_2}{x_n},\dots,\frac{x_n}{x_n}\right).$$

The set $G(X_n)$ of all d.f.'s of the sequence of individual blocks X_n can be parameterized in the form

$$G(X_n) = \{g_t(x)\,;\,t\in[0,1]\},$$

where $g_t(x)$ is the function which is constant over the intervals $\frac{(\delta,a\gamma)}{a^{i+1}(t\delta+(1-t)\gamma)}$, $i = 0,1,2,\dots,$ (here we use the shorthand notation for intervals where we write $(xz,yz) = (x,y)z$ and $\left(\frac{x}{z},\frac{y}{z}\right) = \frac{(x,y)}{z}$) attaining the values

$$g_t(x) = \frac{1}{a^i(1+t(a-1))} \quad \text{if } x \in \frac{(\delta,a\gamma)}{a^{i+1}(t\delta+(1-t)\gamma)},\ \text{and } i = 0,1,2,\dots,$$

while over the complementary intervals its derivative is constant

$$g_t'(x) = \frac{t\delta+(1-t)\gamma}{(\delta-\gamma)(\frac{1}{a-1}+t)} \quad \text{if } x \in \frac{(\gamma,\delta)}{a^{i+1}(t\delta+(1-t)\gamma)},\ \text{and } i = 0,1,2,\dots,$$
$$\text{and } x \in \left(\frac{\gamma}{t\delta+(1-t)\gamma},1\right)$$

NOTES: This example can be found in O. Strauch and J.T. Tóth (2001). Note the following interesting properties of the above functions $g_t(x)$:

(i) as already mentioned, every $g \in G(X_n)$ is constant over infinitely many intervals, i.e. $g'(x) = 0$ here, while over the infinitely many complementary intervals its derivative is also constant $g'(x) = d$. Moreover, this constant d satisfies the inequalities

$$\frac{1}{\overline{d}} \le d \le \frac{1}{\underline{d}}$$

where the lower $\underline{d}$ and upper $\overline{d}$ asymptotic density of x_n can also be given explicitly

$$\underline{d}(x_n) = \frac{(\delta - \gamma)}{\gamma(a-1)}, \qquad \overline{d}(x_n) = \frac{(\delta-\gamma)a}{\delta(a-1)}.$$

(ii) The graph of every $g \in G(X_n)$ lies in the union of squares

$$\left[\frac{1}{a}, 1\right] \times \left[\frac{1}{a}, 1\right] \bigcup \left[\frac{1}{a^2}, \frac{1}{a}\right] \times \left[\frac{1}{a^2}, \frac{1}{a}\right] \bigcup \dots$$

Moreover, the graph of g in the square $\left[\frac{1}{a^k}, \frac{1}{a^{k-1}}\right] \times \left[\frac{1}{a^k}, \frac{1}{a^{k-1}}\right]$ is similar to the graph of g in $\left[\frac{1}{a^{k+1}}, \frac{1}{a^k}\right] \times \left[\frac{1}{a^{k+1}}, \frac{1}{a^k}\right]$ with the coefficient of similarity $\frac{1}{a}$. Therefore if we use the above parameterized form of $g_t(x)$ we can write

$$g_t(x) = \frac{g_t(a^i x)}{a^i} \quad \text{for all } x \in \left(\frac{1}{a^{i+1}}, \frac{1}{a^i}\right) \text{ and } i = 0, 1, 2, \dots,$$

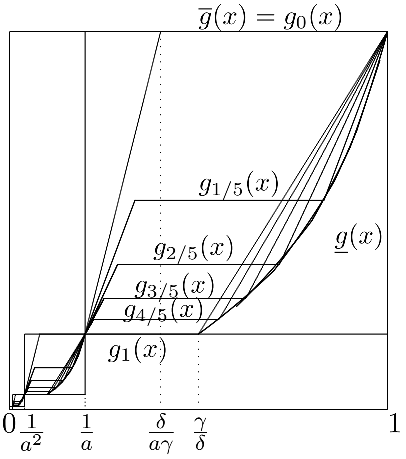
and consequently, the graphs of $g \in G(X_n)$ are completely determined by their branches in $\left[\frac{1}{a}, 1\right] \times \left[\frac{1}{a}, 1\right]$.

(iii) There follows from the graphs of $g \in G(X_n)$ that $G(X_n)$ is connected and the upper distribution function $\overline{g}(x)$ belongs to $G(X_n)$ since $\overline{g}(x) = g_0(x) \in G(X_n)$, while for the lower distribution function we have $\underline{g}(x) \notin G(X_n)$. The graph of $\underline{g}(x)$ on $\left[\frac{1}{a}, 1\right] \times \left[\frac{1}{a}, 1\right]$ is given by

$$\underline{g}(x) = \begin{cases} \dfrac{1}{a}, & \text{if } x \in \left[\dfrac{1}{a}, \dfrac{\gamma}{\delta}\right], \\ \left(1 + \dfrac{1}{\underline{d}}\left(\dfrac{1}{x} - 1\right)\right)^{-1}, & \text{if } \left[\frac{\gamma}{\delta}, 1\right]. \end{cases}$$

(iv) We also have

$$G(X_n) = \left\{ \frac{g_0(x\beta)}{g_0(\beta)} \; ; \; \beta \in \left[\frac{1}{a}, \frac{\delta}{a\gamma}\right] \right\}.$$



$G(X_n)$ with $a = 5, \gamma = 1, \delta = 2$

RELATED SEQUENCES: 2.22.7

O. STRAUCH – J.T. TÓTH: *Distribution functions of ratio sequences*, Publ. Math. (Debrecen) **58** (2001), 751–778 (MR1828725 (2002h:11068); Zbl. 0980.11031).

2.22.7. Let $k_0 < k_1 < k_2 < \dots$ be an increasing sequence of positive integers, n_0 and m_0 be two positive integers and γ, δ and a be real numbers which satisfy
(i) $(k_s - k_{s-1}) \to \infty$ as $s \to \infty$,
(ii) $0 < \gamma < \delta$, $a > 1$, $n_0 \le m_0$ and $\frac{1}{a^{n_0}} \le \frac{\gamma}{\delta}$.
(In what follows, the interval of the form $(\gamma\lambda, \delta\lambda)$ will be written in the abbreviated form $(\gamma, \delta)\lambda$). Let x_n be the increasing sequence of all integers lying in the intervals

$$(\gamma, \delta)a^{k_s m_0 n_0 + j n_0}, \quad \text{for } 0 \le j < (k_{s+1} - k_s)m_0, \text{ and } s = 0, 2, 4, \dots,$$
$$(\gamma, \delta)a^{k_s m_0 n_0 + j m_0}, \quad \text{for } 0 \le j < (k_{s+1} - k_s)n_0, \text{ and } s = 1, 3, 5, \dots.$$

Note that the terms of both interval sequences mutually interchange in blocks and the turning points are the intervals of the form $(\gamma, \delta)(a^{n_0 m_0})^{k_s}$.
For this x_n define the sequence of blocks

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n} \right),$$

and put

$$I(n_0,t)=\left(\frac{\delta}{a^{n_0}},\gamma\right)\frac{1}{t\gamma+(1-t)\delta}\,,$$
$$I(m_0,t)=\left(\frac{\delta}{a^{m_0}},\gamma\right)\frac{1}{t\gamma+(1-t)\delta}\,,$$
$$I(t)=(\gamma,\delta)\frac{1}{t\gamma+(1-t)\delta}\,.$$

The set $G(X_n)$ of all d.f.'s of X_n has the structure

$$\begin{aligned}G(X_n)=&\{g_{n_0,j,t}(x)\,;\ j=0,1,\dots,t\in[0,1]\}\\ &\bigcup\{g_{m_0,j,t}(x)\,;\ j=0,1,\dots,t\in[0,1]\}.\end{aligned}$$

Here the d.f. $g_{n_0,j,t}(x)$ is constant in intervals

$$I(n_0,t),\frac{I(n_0,t)}{a^{n_0}},\dots,\frac{I(n_0,t)}{(a^{n_0})^{j-1}},\frac{I(m_0,t)}{(a^{n_0})^{j}},\frac{I(m_0,t)}{(a^{n_0})^{j}(a^{m_0})},\frac{I(m_0,t)}{(a^{n_0})^{j}(a^{m_0})^2},\dots$$

and in the complementary intervals in $[0,1]$

$$\left(\frac{\gamma}{t\gamma+(1-t)\delta}\,,\,1\right),\frac{I(t)}{a^{n_0}},\frac{I(t)}{(a^{n_0})^2},\dots,\frac{I(t)}{(a^{n_0})^{j}},\frac{I(t)}{(a^{n_0})^{j}(a_{m_0})},\frac{I(t)}{(a^{n_0})^{j}(a_{m_0})^2},\dots$$

its derivative is constant

$$g'_{n_0,j,t}(x)=\frac{1}{d_t},$$

with d_t satisfying

$$\underline{d}(x_n)\le d_t\le\overline{d}(x_n),$$

where

$$\underline{d}(x_n)=\frac{(\delta-\gamma)}{\gamma}\cdot\frac{1}{a^{m_0}-1},\qquad\overline{d}(x_n)=\frac{(\delta-\gamma)}{\delta}\cdot\frac{a^{n_0}}{a^{n_0}-1},$$

and

$$d_t=\frac{\delta-\gamma}{t\gamma+(1-t)\delta}\left(1-t+\frac{1}{a^{n_0}-1}-\frac{1}{(a^{n_0})^j}\left(\frac{1}{a^{n_0}-1}-\frac{1}{a^{m_0}-1}\right)\right).$$

These properties completely characterize the d.f. $g_{n_0,j,t}(x)$. The d.f. $g_{m_0,j,t}(x)$ can be defined in a similar way replacing n_0 by m_0 in the above intervals and in the derivative.

Notes: (I) O. Strauch and J.T. Tóth (2002). They also proved: Let X be the complement of the set of all limit points of x_m/x_n in $[0,1]$, then

$$X = \left(\bigcap_{j=0}^{\infty} B(n_0,j)\right) \bigcap \left(\bigcap_{j=0}^{\infty} B(m_0,j)\right),$$

where

$$I(n_0) = \left(\frac{\delta}{\gamma a^{n_0}}, \frac{\gamma}{\delta}\right), \qquad I(m_0) = \left(\frac{\delta}{\gamma a^{m_0}}, \frac{\gamma}{\delta}\right)$$

$$\begin{aligned} B(n_0,j) =& I(n_0) \cup I(n_0)\frac{1}{a^{n_0}} \cup \cdots \cup I(n_0)\frac{1}{(a^{n_0})^{j-1}} \cup \\ & \cup \left(I(m_0) \cup I(m_0)\frac{1}{a^{m_0}} \cup I(m_0)\frac{1}{(a^{m_0})^2} \cup I(m_0)\frac{1}{(a^{m_0})^3} \cup \dots \right) \frac{1}{(a^{n_0})^j}, \\ B(m_0,j) =& I(m_0) \cup I(m_0)\frac{1}{a^{m_0}} \cup \cdots \cup I(m_0)\frac{1}{(a^{m_0})^{j-1}} \cup \\ & \cup \left(I(n_0) \cup I(n_0)\frac{1}{a^{n_0}} \cup I(n_0)\frac{1}{(a^{n_0})^2} \cup I(n_0)\frac{1}{(a^{n_0})^3} \cup \dots \right) \frac{1}{(a^{m_0})^j}. \end{aligned}$$

Thus, in all cases we have $X \supset I(n_0)$. If we assume additionally that
(iii) $1 < n_0 < m_0$, $\gcd(n_0, m_0) = 1$,
(iv) $\dfrac{1}{a^{n_0}} < \left(\dfrac{\gamma}{\delta}\right)^2$,
(v) $\left(\dfrac{\gamma}{\delta}\right)^2 \le \dfrac{a^{n_0}}{a^{m_0}}$, $\left(\dfrac{\gamma}{\delta}\right)^2 \le \dfrac{a^{m_0}}{a^{2n_0}}$,
(vi) $\left(\dfrac{\gamma}{\delta}\right)^2 \le \dfrac{(a^{n_0})^{\left[\frac{m_0 k}{n_0}\right]+1}}{(a^{m_0})^{k+1}}$, $\left(\dfrac{\gamma}{\delta}\right)^2 \le \dfrac{(a^{m_0})^k}{(a^{n_0})^{\left[\frac{m_0 k}{n_0}\right]+1}}$, for $k = 1, 2, \dots, n_0 - 2$,

then we have

$$X = I(n_0) \neq \emptyset.$$

The assumptions (i) – (vi) are satisfied e.g. if $k_s = s^2$, $\gamma = 1$, $\delta = 2$,$a = 2$, $n_0 = 3$, and $m_0 = 4$, in which case $X = \left(\frac{1}{2^2}, \frac{1}{2}\right)$.
(II) Strauch and Tóth (2002) also proved that for every increasing sequence x_n of positive integers we have:
Theorem 2.22.7.1. *Suppose that $\underline{d}(x_n) > 0$. If there exists an interval $(u,v) \subset [0,1]$ such that every $g \in G(X_n)$ is constant over (u,v) (different d.f.'s may attain distinct values over (u,v)), then every $g \in G(X_n)$ is constant over infinitely many intervals and the sequence of values of g at their endpoints increases.*

Related sequences: 2.22.6, 2.22.8

O. Strauch – J.T. Tóth: *Corrigendum to Theorem 5 of the paper "Asymptotic density of $A \subset \mathbb{N}$ and density of the ratio set $R(A)$" (Acta Arith. 87 (1998), 67–78)*, Acta Arith. **103.2** (2002), 191–200 (MR1904872 (2003f:11015); Zbl. 0923.11027).

2.22.8. Put $k_s = s$, for $s = 0, 1, 2, \dots$, in 2.22.7, i.e. x_n is the sequence of all integers belonging to the intervals

$$\begin{aligned}
&(\gamma,\delta)(a^{n_0})^0, \quad (\gamma,\delta)(a^{n_0})^1, \quad \dots, (\gamma,\delta)(a^{n_0})^{m_0-1},\\
&(\gamma,\delta)(a^{m_0})^{n_0}, \quad (\gamma,\delta)(a^{m_0})^{n_0+1}, \quad \dots, (\gamma,\delta)(a^{m_0})^{2n_0-1},\\
&(\gamma,\delta)(a^{n_0})^{2m_0}, (\gamma,\delta)(a^{n_0})^{2m_0+1}, \dots, (\gamma,\delta)(a^{n_0})^{3m_0-1},\\
&(\gamma,\delta)(a^{m_0})^{3n_0}, (\gamma,\delta)(a^{m_0})^{3n_0+1}, \dots .
\end{aligned}$$

Let

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right)$$

and

$$\begin{aligned}
I(n_0,t) &= \left(\frac{\delta}{a^{n_0}}, \gamma\right)\frac{1}{t\gamma + (1-t)\delta},\\
I(m_0,t) &= \left(\frac{\delta}{a^{m_0}}, \gamma\right)\frac{1}{t\gamma + (1-t)\delta},\\
I(t) &= (\gamma,\delta)\frac{1}{t\gamma + (1-t)\delta}.
\end{aligned}$$

The set $G(X_n)$ of all d.f.'s of X_n has the structure

$$\begin{aligned}
G(X_n) = &\{g_{n_0,j,t}(x) \,;\, j = 0, 1, \dots, m_0 - 1, t \in [0,1]\}\\
\cup &\{g_{m_0,j,t}(x) \,;\, j = 0, 1, \dots, n_0 - 1, t \in [0,1]\},
\end{aligned}$$

where the d.f. $g_{n_0,j,t}(x)$ is constant in the following intervals

$$I(n_0,t), I(n_0,t)\frac{1}{a^{n_0}}, \dots, I(n_0,t)\frac{1}{(a^{n_0})^{j-1}},$$

$$I(m_0,t)\frac{1}{(a^{n_0})^j}, I(m_0,t)\frac{1}{(a^{n_0})^j a^{m_0}}, \dots, I(m_0,t)\frac{1}{(a^{n_0})^j (a^{m_0})^{n_0-1}},$$

$$I(n_0,t)\frac{1}{(a^{n_0})^j (a^{m_0 n_0})}, I(n_0,t)\frac{1}{(a^{n_0})^j (a^{m_0 n_0}) a^{n_0}}, \dots,$$

$$I(n_0,t)\frac{1}{(a^{n_0})^j (a^{m_0 n_0})(a^{n_0})^{m_0-1}},$$

$$I(m_0,t)\frac{1}{(a^{n_0})^j (a^{2m_0 n_0})}, \dots$$

and in the complementary intervals in $[0,1]$

$$\left(\frac{\gamma}{t\gamma+(1-t)\delta},1\right),$$

$$I(t)\frac{1}{(a^{n_0})},I(t)\frac{1}{(a^{n_0})^2},\dots,$$

$$I(t)\frac{1}{(a^{n_0})^j},I(t)\frac{1}{(a^{n_0})^j(a^{m_0})},I(t)\frac{1}{(a^{n_0})^j(a^{m_0})^2},\dots,$$

$$I(t)\frac{1}{(a^{n_0})^j a^{m_0n_0}},I(t)\frac{1}{(a^{n_0})^j a^{m_0n_0}(a^{n_0})},I(t)\frac{1}{(a^{n_0})^j a^{m_0n_0}(a^{n_0})^2},\dots,$$

$$I(t)\frac{1}{(a^{n_0})^j a^{2m_0n_0}},\dots$$

it has constant derivative

$$g'_{n_0,j,t}(x)=\frac{1}{d_t},$$

with $\underline{d}(x_n)\le d_t\le\overline{d}(x_n)$, where

$$d_t=\frac{\delta-\gamma}{t\gamma+(1-t)\delta}\Bigg((1-t)+\frac{1}{a^{n_0}-1}-$$

$$-\frac{1}{(a^{n_0})^j}\cdot\frac{a^{m_0n_0}}{a^{m_0n_0}+1}\left(\frac{1}{a^{n_0}-1}-\frac{1}{a^{m_0}-1}\right)\Bigg).$$

and

$$\underline{d}(x_n)=\frac{(\delta-\gamma)}{\gamma}\left(\frac{1}{a^{n_0}-1}-\frac{a^{m_0n_0}}{a^{m_0n_0}+1}\left(\frac{1}{a^{n_0}-1}-\frac{1}{a^{m_0}-1}\right)\right),$$

$$\overline{d}(x_n)=\frac{(\delta-\gamma)}{\delta}\left(1+\frac{1}{a^{m_0}-1}+\frac{a^{m_0n_0}}{a^{m_0n_0}+1}\left(\frac{1}{a^{n_0}-1}-\frac{1}{a^{m_0}-1}\right)\right).$$

These properties completely characterize the d.f. $g_{n_0,j,t}(x)$. The d.f. $g_{m_0,j,t}(x)$ can be determined in a similar way replacing n_0 by m_0 in the above intervals and in the derivative.

NOTES: O. Strauch and J.T. Tóth (2002). They also proved that if X is the complement of the set of all limit points of x_m/x_n in $[0,1]$, then

$$X=\left(\bigcap_{j=0}^{m_0-1}B(n_0,j)\right)\bigcap\left(\bigcap_{j=0}^{n_0-1}B(m_0,j)\right),$$

where

$$
\begin{aligned}
B(n_0,j) =& I(n_0)\cup I(n_0)\frac{1}{a^{n_0}}\cup\dots\cup I(n_0)\frac{1}{(a^{n_0})^{j-1}}\cup\\
&\cup\left(A(m_0)\cup A(n_0)\frac{1}{a^{m_0n_0}}\cup A(m_0)\frac{1}{a^{2m_0n_0}}\cup A(n_0)\frac{1}{a^{3m_0n_0}}\cup\dots\right)\frac{1}{(a^{n_0})^j},\\
B(m_0,j) =& I(m_0)\cup I(m_0)\frac{1}{a^{m_0}}\cup\dots\cup I(m_0)\frac{1}{(a^{m_0})^{j-1}}\cup\\
&\cup\left(A(n_0)\cup A(m_0)\frac{1}{a^{m_0n_0}}\cup A(n_0)\frac{1}{a^{2m_0n_0}}\cup A(m_0)\frac{1}{a^{3m_0n_0}}\cup\dots\right)\frac{1}{(a^{m_0})^j}.
\end{aligned}
$$

Thus if $m_0 = n_0$ we have $X = \bigcup_{i=0}^{\infty}\frac{I(n_0)}{(a^{n_0})^i}$. The case $n_0 = m_0 = 1$ reduces to 2.22.6. If (i) – (vi) from 2.22.7 are satisfied we get

$$
\begin{aligned}
X =& I(n_0)\cup I(n_0)\frac{1}{a^{2m_0n_0}}\cup I(n_0)\frac{1}{a^{4m_0n_0}}\cup I(n_0)\frac{1}{a^{6m_0n_0}}\cup\dots\\
&\cup\left(I(n_0)\frac{1}{a^{2m_0n_0}}\cup I(n_0)\frac{1}{a^{4m_0n_0}}\cup I(n_0)\frac{1}{a^{6m_0n_0}}\cup\dots\right)a^{n_0}
\end{aligned}
$$

Related sequences: 2.22.6, 2.22.7

O. Strauch – J.T. Tóth: *Corrigendum to Theorem 5 of the paper "Asymptotic density of $A\subset\mathbb{N}$ and density of the ratio set $R(A)$" (Acta Arith. 87 (1998), 67–78)*, Acta Arith. **103.2** (2002), 191–200 (MR1904872 (2003f:11015); Zbl. 0923.11027).

2.22.9. Let x_n, $n = 1,2,\dots$, be an increasing sequence of positive integers for which there exists a sequence n_k, $k = 1,2,\dots$, of positive integers such that (as $k\to\infty$)

(i) $\frac{n_{k-1}}{n_k}\to 0$,

(ii) $\frac{n_k}{x_{n_k}}\to 0$,

(iii) $\frac{x_{n_{k-1}}}{x_{n_k}}\to 0$, and

(iv) $x_{n_k-i} = x_{n_k} - i$ for $i = 0,1,\dots,n_k-n_{k-1}-1$.

Then the sequence of blocks

$$X_n = \left(\frac{x_1}{x_n},\frac{x_2}{x_n},\dots,\frac{x_n}{x_n}\right)$$

has

$$G(X_n) = \{h_\alpha(x) \,;\, \alpha \in [0,1]\},$$

where $h_\alpha(x) = \alpha$, $x \in (0,1)$ is the constant d.f.

NOTES: Examples of such sequences are $n_k = 2^{k^2}$ and $x_{n_k} = 2^{(k+1)^2}$.

G. GREKOS – O. STRAUCH: *Distribution functions of ratio sequences, II*, Unif. Distrib. Theory **2** (2007), no. 1, 53–77 (MR2318532 (2008g:11125); Zbl. 1183.11042).

2.22.10. Let x_n, $n = 1, 2, \dots$, be an increasing sequence of positive integers which satisfies the following conditions

(i) if $n_k = (k+1)(k-1)!2^{\frac{k(k-1)}{2}}$ for $k = 1, 2, \dots$, then $x_{n_k} = (k+1)n_k$,

(ii) if $n'_k = k(k-2)!2^{\frac{k(k-1)}{2}}$ then $x_{n'_k} = k^2 n'_k$,

(iii) if $n = 2^i n_{k-1} + j$, $0 \le j < 2^i n_{k-1}$ and $0 \le i < k-1$ for $k = 1, 2, \dots$, then $x_n = x_{n_{k-1}}(i+1)2^i + (i+3)kj$ (i.e. $n \in [n_{k-1}, n'_k]$),

(iv) if $n \in [n'_k, n_k]$ for $k = 1, 2, \dots$, then $x_n = x_{n'_k} + n - n'_k$.

Then for the sequence of blocks

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n} \right)$$

we have

$$c_1(x) \in G(X_n) \text{ but } c_0(x) \notin G(X_n)$$

where $c_0(x)$ and $c_1(x)$ are one–jump d.f.'s with the jump of height 1 at $x = 0$ and $x = 1$, respectively.

NOTES: L. Mišík (2004, personal communication). A detailed description of $G(X_n)$ is open.

2.22.11. Let x_n and y_n, $n = 1, 2, \dots$, be two strictly increasing sequences of positive integers such that for the corresponding block sequences

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n} \right), \quad Y_n = \left(\frac{y_1}{y_n}, \frac{y_2}{y_n}, \dots, \frac{y_n}{y_n} \right),$$

we have $G(X_n) = \{g_1(x)\}$ and $G(Y_n) = \{g_2(x)\}$. Furthermore, let n_k, $k = 1, 2, \dots$, be an increasing sequence of positive integers such that $N_k = \sum_{i=1}^k n_i$ satisfies

$$\frac{N_{k-1}}{N_k} \to 0 \text{ as } k \to \infty \quad \left(\text{which is equivalent to } \frac{n_k}{N_k} \to 1 \right).$$

Let z_n be the increasing sequence of positive integers composed from blocks (where we use the shorthand notation $a(b,c,d,\dots) = (ab, ac, ad, \dots)$)

$$(x_1,\dots,x_{n_1}),\ x_{n_1}(y_1,\dots,y_{n_2}),\ x_{n_1}y_{n_2}(x_1,\dots,x_{n_3}),\dots.$$

Then the set of d.f.'s of sequence of blocks

$$Z_n = \left(\frac{z_1}{z_n}, \frac{z_2}{z_n}, \dots, \frac{z_n}{z_n}\right)$$

is

$$\begin{aligned} G(Z_n) = &\{g_1(x), g_2(x), c_0(x)\}\cup\\ &\cup\{g_1(xy_n)\,;\ n=1,2,\dots\}\cup\\ &\cup\{g_2(xx_n)\,;\ n=1,2,\dots\}\cup\\ &\cup\left\{\frac{1}{1+\alpha}\,c_0(x)+\frac{\alpha}{1+\alpha}\,g_1(x)\,;\ \alpha\in(0,\infty)\right\}\cup\\ &\cup\left\{\frac{1}{1+\alpha}\,c_0(x)+\frac{\alpha}{1+\alpha}\,g_2(x)\,;\ \alpha\in(0,\infty)\right\}. \end{aligned}$$

Notes: Consequently $G(Z_n)$ is not a connected set.

G. Grekos – O. Strauch: *Distribution functions of ratio sequences, II*, Unif. Distrib. Theory **2** (2007), no. 1, 53–77 (MR2318532 (2008g:11125); Zbl. 1183.11042).

2.22.12. Let $J(n)$ be a positive integer and X_n the following block

$$X_n = \left(\frac{n}{1}, \frac{n}{2}, \dots, \frac{n}{J(n)}\right) \bmod 1.$$

If $J(n)$ satisfies
(i) $\frac{J(n)}{n} \to 0$ as $n\to\infty$,
(ii) $\frac{J(n)}{n^\alpha} \to \infty$ as $n\to\infty$ for some $\alpha>0$,
then the sequence X_n of individual blocks is

u.d.

Notes: J. Isbell and S. Schanuel (1976). They also noted that the special case $\alpha = \frac{1}{2}$ was proved by Dirichlet (cf. L.E. Dickson (1934, Vol. I, p. 327)) and further that (i) is a necessary condition. The proof uses a special form of the following result proved by A. Walfisz (1932, Hilfssatz 6).

Theorem 2.22.12.1. *Let r, N be positive integers and $R = 2^{r-1}$, $R_1 = R(r+1)$. Let $t \ge (2N)^{r+3}$ and $0 < w \le 1$ be real numbers and M and M' integers such that $t^{\frac{1}{r+2}} \le M \le M' \le 2M \le 2Nt^{\frac{2}{r+3}}$. Then*

$$\sum_{j=M}^{M'} e^{i\frac{t}{j+w}} = \mathcal{O}(M^{1-\frac{1}{R}-\frac{1}{R_1}} t^{\frac{1}{R_1}} \log t).$$

L.E. Dickson: *History of the Theory of Numbers*, Vol. I, Carnegie Institution of Washington, Publication No. 256, 1919 (JFM 47.0100.04).
J. Isbell – S. Schanuel: *On the fractional parts of n/j, $j = o(n)$*, Proc. Amer. Math. Soc. **60** (1976), 65–67 (MR0429796 (**55** #2806); Zbl. 0341.10032).
A. Walfisz: *Über Gitterpunkte in mehrdimensionalen Ellipsoiden. IV*, Math. Z. **25** (1932), 212–229 (MR1545298; Zbl. 0004.10302).

2.22.13. The block sequence X_n, $n = 1, 2, \dots$, where

$$X_n = \left(\frac{n}{1}, \frac{n}{2}, \dots, \frac{n}{n}\right) \bmod 1$$

has the a.d.f.

$$g(x) = \int_0^1 \frac{1-t^x}{1-t}\,\mathrm{d}t = \sum_{n=1}^{\infty} \frac{x}{n(n+x)} = \gamma_0 + \frac{\Gamma'(1+x)}{\Gamma(1+x)}.$$

where γ_0 is Euler's constant.

Notes: This was proved by G. Pólya (cf. I.J. Schoenberg (1928)). The second expression for $g(x)$ follows from 2.3.4 and the third one from Ryshik and Gradstein (1957, p. 304, 6.352).

I.M. Ryshik – I.S. Gradstein: *Tables of Series, Products, and Integrals*, (German and English dual language edition), VEB Deutscher Verlag der Wissenschaften, Berlin, 1957 (translation from the Russian original Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951 (MR0112266 (**22** #3120))).
I.J. Schoenberg: *Über die asymptotische Verteilung reeller Zahlen* mod 1, Math. Z. **28** (1928), 171–199 (MR1544950; JFM 54.0212.02).

2.22.14. Let p_n/q_n be a sequence of rational numbers which is u.d. in $\mathbb{R}$. Then the sequence

$$\frac{p_n+1}{q_n} \bmod 1$$

is

u.d.

H. Niederreiter – J. Schoissengeier: *Almost periodic functions and uniform distribution* mod 1, J. Reine Angew. Math. **291** (1977), 189–203 (MR0437482 (**55** #10412); Zbl. 0338.10053).

2.22.15. If $N \in \mathbb{N}$ then for the finite sequence

$$\left(\frac{1}{2N}, \frac{3}{2N}, \dots, \frac{2N-1}{2N}\right)$$

we have

$$D_N^* = \frac{1}{2N}, \quad D_N^{(2)} = \frac{1}{12N^2}.$$

For the related finite sequence

$$\left(\frac{1}{N}, \frac{2}{N}, \dots, \frac{N}{N}\right)$$

we have

$$D_N = \frac{1}{N}.$$

NOTES: These two finite sequences have minimal possible discrepancies.

2.22.16. Let $m \geq 2$ and $y_0, y_1, \dots, y_{N-1}$ be integers. Then the discrepancy D_N of the finite sequence of fractional parts

$$\left\{\frac{y_0}{m}\right\}, \left\{\frac{y_1}{m}\right\}, \dots, \left\{\frac{y_{N-1}}{m}\right\}$$

satisfies

$$D_N \leq \frac{1}{m} + \sum_{\substack{-m/2<h\leq m/2\\ h\neq 0}} \frac{1}{m\sin(\pi|h|/m)} \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i h y_n/m} \right|.$$

The right hand hand can be simplified to

$$D_N \leq \frac{1}{m} + \sum_{h=1}^{[m/2]} \frac{1}{h} \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i h y_n/m} \right|.$$

NOTES: For the proof consult H. Niederreiter (1978, pp. 974–976).

H. NIEDERREITER: *Quasi–Monte Carlo methods and pseudo–random numbers*, Bull. Amer. Math. Soc. **84** (1978), no. 6, 957–1041 (MR0508447 (80d:65016); Zbl. 0404.65003).

2.22.17. If m is a positive integer then the sequence of blocks

$$X_n = \left(\frac{1}{m}\left\{\frac{mn}{1}\right\}, \frac{1}{m}\left\{\frac{mn}{2}\right\}, \frac{1}{m}\left\{\frac{mn}{3}\right\}, \dots, \frac{1}{m}\left\{\frac{mn}{n}\right\}\right).$$

has the a.d.f. $g(x)$ for $n \to \infty$ possessing the density

$$g'(x) = \begin{cases} m^2\psi''(mx+m), & \text{if } x \in [0, 1/m), \\ 0, & \text{if } x \in (1/m, 1], \end{cases}$$

where $\psi(x)$ is the classical psi–function

$$\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\log\Gamma(x).$$

M.R. Currie – E.H. Goins: *The fractional parts of* $\frac{N}{K}$, in: *Council for African American Researchers in the Mathematical Sciences, Vol. III (Baltimore, MD, 1997/Ann Arbor, MI, 1999)*, (A.F. Noël ed.), Contemp. Math., 275, Amer. Math. Soc., Providence, RI, 2001, pp. 13–31 (MR1827332 (2002b:11099); Zbl. 1010.11041).

2.22.18. A positive rational number $\frac{a}{b}$ is called a P–**rational** if the pair of integers a and b has given property P. If P is one of the following properties
(i) $a^2 + b^2$ is a square,
(ii) $a^2 + b^2$ is a cube,
(iii) $a^3 + b^3$ is a square,
then the set of all P-rationals is

dense in $[0, \infty)$.

Notes: (i) is given in L.H. Lange and D.E. Thoro (1964); (ii) and (iii) can be found in P. Schaefer (1965). He generalized the result of Lange and Thoro in the following simple way: Let $N(x, y)$ and $D(x, y)$ be polynomials in x and y with integral coefficients and which are homogeneous of the same degree. Suppose that $\frac{N(x,y)}{D(x,y)}$ is a P-rational for all positive integers x and y with $\frac{x}{y} \in I$, I an interval. Then the continuity of $f(t) = \frac{N(t,1)}{D(t,1)}$ for $t \in I$ implies the density of such P–rationals in $f(I)$. The above cases we get for
(i) $N(x, y) = x^2 - y^2$ and $D(x, y) = 2xy$,
(ii) $N(x, y) = x^3 - 3xy^2$ and $D(x, y) = 3x^2y - y^3$,
(iii) $N(x, y) = x(x^3 + y^3)$ and $D(x, y) = y(x^3 + y^3)$.

L.H. Lange – D.E. Thore: *The density of Pythagorean rationals*, Amer. Math. Monthly **71** (1964), no. 6, 664–665 (MR1532769).
P. Schaefer: *The density of certain classes rationals*, Amer. Math. Monthly **72** (1965), no. 8, 894–895 (MR0183688 (**32** #1168); Zbl. 0151.02604).

2.23 Sequences of reduced rational numbers

2.23.1. Given an infinite sequence q_n of positive integers, consider the sequence of blocks of reduced rational numbers with denominators q_n each block of which has the form

$$A_n = \left(\frac{1}{q_n}, \frac{a_2}{q_n}, \dots, \frac{a_{\varphi(q_n)}}{q_n}\right)$$

where $(a_i, q_n) = 1$ and φ stands for the Euler totient function. Let $\sigma = (A_n)_{n=1}^{\infty}$ denote the block sequence formed from these blocks. If $\lim_{n\to\infty} q_n = \infty$, then the sequence of individual blocks A_n is

u.d.

and the compound sequence $\sigma = (A_n)_{n=1}^{\infty}$ formed from these blocks is

u.d.

if and only if

$$\lim_{n\to\infty} \frac{\varphi(q_n)}{\sum_{i=1}^{n} \varphi(q_i)} = 0.$$

If N is of the type $N = \sum_{i=1}^{n} \varphi(q_i)$, then for the discrepancy of σ we have

$$N^2 D_N^{(2)} = \frac{1}{12} \sum_{i,j=1}^{n} \frac{2^{\omega(d_{ij})}}{q_{ij}} \prod_{\substack{p|q_i q_j \\ p\nmid d_{ij}}} (1-p) \prod_{\substack{p|d_{ij} \\ p\nmid q_{ij}}} \left(1 - \frac{1}{p}\right) \prod_{\substack{p|d_{ij} \\ p|q_{ij}}} \left(1 - \frac{p}{2}\left(1 + \frac{1}{p^2}\right)\right)$$

$$= \frac{1}{2\pi^2} \sum_{h=1}^{\infty} \frac{1}{h^2} \left| \sum_{i=1}^{n} \frac{\varphi(q_i)}{\varphi\left(\frac{q_i}{(h,q_i)}\right)} \mu\left(\frac{q_i}{(h,q_i)}\right) \right|^2 .$$

where

1. $d_{ij} = (q_i, q_j)$ is the greatest common divisor of numbers q_i and q_j,
2. $q_{ij} = \frac{q_i q_j}{d_{ij}^2}$,
3. p runs over the prime divisors,
4. $\omega(n) = \#\{p\,;\, p|n\}$,

5. μ is the Mőbius function.

For the extremal discrepancy we have

$$D_N \leq \frac{\sum_{i=1}^{n} 2^{\omega(q_i)}}{\sum_{i=1}^{n} \varphi(q_i)} \leq 2\sqrt{2}\sqrt{\frac{n}{N}}.$$

NOTES: In the case $n = 1$ (i.e. for one segment A_1) $D^{(2)}_{\varphi(q_1)}$ was implicitly given by E. Spence (1962) (he also proved formulas for sums $\sum_{i=1}^{\varphi(n)} ia_i$, $\sum_{i=1}^{\varphi(n)} i^2a_i$, $\sum_{i=1}^{\varphi(n)} ia_i^2$ and $\sum_{i,j=1}^{\varphi(n)}(a_{i+1} - a_i)^2$) and explicitly by H. Delange (1968). For the above general formula see O. Strauch (1987). The estimate for extremal discrepancy can be proved by elementary sieve type arguments (cf. O. Strauch (1997)).

H. DELANGE: *Sur la distribution des fractions irréducible de dénominateur n ou de dénominateur au plus égal à x*, in: *Hommage an Professeure Lucion Godeaux*, Centre Blege de Recherches Mathématiques, Librairie Universitaire, Louvain, 1968, pp. 75–89 (MR0238780 (**39** #144); Zbl. 0174.08401).

O. STRAUCH: *Some applications of Franel's integral, I*, Acta Math. Univ. Comenian. **50–51** (1987), 237–245 (MR0989416 (90d:11028); Zbl. 0667.10023).

O. STRAUCH: *A numerical integration method employing the Fibonacci numbers*, Grazer Math. Ber. **333** (1997), 19–33 (MR1640470 (99h:65038); Zbl. 0899.11037).

E. SPENCE: *Formulae for sums involving a reduced set of residues modulo n*, Proc. Edinburgh Math. Soc. (2) **13** (1962/63), 347–349 (MR0160755 (**28** #3966); Zbl. 0116.26802).

2.23.2. For a finite sequence

$$\left(\frac{1}{q}, \frac{a_2}{q}, \dots, \frac{a_{\varphi(q)}}{q}\right),$$

where $(a_i, q) = 1$ we have

$$\frac{2^{\omega(q)/2}}{\sqrt{12q\varphi(q)}} \leq D^*_{\varphi(q)} \leq \frac{2^{\omega(q)}}{\varphi(q)}.$$

RELATED SEQUENCES: 2.23.1

H.G. MEIJER – H. NIEDERREITER: *Équirépartition et théorie des nombres premiers*, in: *Répartition modulo 1 (Actes Colloq., Marseille – Luminy, 1974)*, Lecture Notes in Math., 475, Springer Verlag, Berlin, 1975, pp. 104–112 (MR0389819 (**52** #10649); Zbl. 0306.10032).

2.23.3. Let $1 = a_1 < a_2 < \cdots < a_{\varphi(n)} = n - 1$ be the integers coprime to n and A_n be the block

$$A_n = \left(\frac{a_2 - a_1}{n/\varphi(n)}, \frac{a_3 - a_2}{n/\varphi(n)}, \dots, \frac{a_{\varphi(n)} - a_{\varphi(n)-1}}{n/\varphi(n)}\right).$$

If the index n runs over such a set that $n/\varphi(n) \to \infty$ then the sequence A_n of individual blocks has in $[0,\infty)$ d.f. $g(x)$ of the form

$$g(x) = 1 - e^{-x}.$$

RELATED SEQUENCES: For a multi–dimensional version cf. 3.7.3.

CH. HOOLEY: *On the difference between consecutive numbers prime to n. II*, Publ. Math. Debrecen **12** (1965), 39–49 (MR0186641 (**32** #4099); Zbl. 0142.29201).

2.23.4. Let $\sigma = (A_n)_{n=1}^{\infty}$ be the block sequence described in 2.23.1 for $q_n = n$. It is also called the **Farey sequence** $v(n)$, $n = 1, 2, \dots$. Its initial segment $v(1), \dots, v(N)$ for N of the form $N = \sum_{i=1}^{n} \varphi(i)$ consists of the fractions

$$\mathbf{F}_n = \left\{ \frac{p}{q} \, ; \, 1 \le q \le n, \, 1 \le p \le q, \, \gcd(p,q) = 1 \right\}$$

ordered firstly by increasing q, and then for constant q by increasing p. If N is of the above form then

$$c_1 N^{-1/2} \le D_N^* \le c_2 N^{-1/2}$$

with positive absolute constants c_1 and c_2. More precisely

$$D_N^* = \frac{1}{n}$$

for any n which implies

$$D_N^* \sim \frac{\sqrt{3}}{\pi\sqrt{N}}.$$

The Riemann hypothesis is equivalent to

$$N D_N^{(2)} = \mathcal{O}(N^{1/2+\varepsilon})$$

for every $\varepsilon > 0$. Given any $\alpha \in [1/2, 1)$, let RH(α) denote the following statement: $\alpha = \sup\{\beta \, ; \, \rho = \beta + i\gamma, \zeta(\rho) = 0\}$. Let $f : [0,1] \to \mathbb{R}$ be absolutely continuous and let $f' \in L^p[0,1]$ for some $p \in (1,2]$. If RH(α) holds then for every $\varepsilon > 0$

$$\frac{1}{N} \sum_{q=1}^{n} \sum_{\substack{1 \le a < q \\ (a,q)=1}} f\left(\frac{a}{q}\right) - \int_0^1 f(x) \, \mathrm{d}x = \mathcal{O}_\varepsilon \left(\frac{n^{\max(\alpha, 1/p)+\varepsilon}}{N} \right).$$

NOTES: (I) With discrepancy estimates H. Niederreiterr (1973) improved a weaker estimate given by E.H. Neville (1949), who proved $D_N = \mathcal{O}\left(\frac{\log n}{n}\right)$.
(II) The exact result $D_N^* = 1/n$ was given by F. Dress (1999).
(III) The equivalence between the Riemann hypothesis and the order of discrepancy was proved by J. Franel (1924).
(IV) The dependence of the error term on RH(α) in the numerical integration for a rather wide class of functions given above was proved by P. Codeca and A. Perelli (1988).
(V) Previously, M. Mikolás (1949) proved that the Riemann hypothesis RH($\frac{1}{2}$) is equivalent to the error term $\mathcal{O}_\varepsilon\left(\frac{n^{\frac{1}{2}+\varepsilon}}{N}\right)$ for every one of the following classes of functions:

- $\sin \lambda x$,
- $\cos \lambda x$,
  in both cases if λ satisfies the conditions $0 < |\lambda| < 2\sqrt{\frac{5}{\zeta(3)+5\pi^{-2}}}$, $|\lambda| \neq \pi$,
- quadratic polynomials,
- cubic polynomials $a_0x^3 + a_1x^2 + a_2x + a_3$ with $a_1 \neq 3a_0/2$.

(VI) A similar result to (V) was proved by J. Kopřiva (1955) for some subsequences of Farey fractions.

RELATED SEQUENCES: 2.20.38, 2.23.2, 2.23.1

P. CODECA – A. PERELLI: *On the uniform distribution* mod 1 *of the Farey fractions and l^p space*, Math. Ann. **279** (1988), 413–422 (MR0922425 (89b:11065); Zbl. 0606.10041).
F. DRESS: *Discrépance des suites de Farey*, J. Théor. Nombres Bordeaux **11** (1999), no. 2, 345–367 (MR1745884 (2001c:11083); Zbl. 0981.11026).
J. FRANEL: *Les suites de Farey et le probleme des nombres premiers*, Nachr. Ges. Wiss. Göttingen Math.-Phys. Kl. (1924), 198–201 (JFM 50.0119.01).
J. KOPŘIVA: *Remark on the significance of the Farey series in number theory*, Publ. Fac. Sci. Univ. Masaryk (1955), 267–279 (MR0081315 (18,382a); Zbl. 0068.26701).
M. MIKOLÁS: *Farey series and their connection with the prime number problem. I*, Acta Univ. Szeged. Sect. Sci. Math. **13** (1949), 93–117 (MR0034802 (11,645a); Zbl. 0035.31402).
E.H. NEVILLE: *The structure of Farey series*, Proc. London Math. Soc. **51** (2) (1949), 132–144 (MR0029924 (10,681f); Zbl. 0034.17401).
H. NIEDERREITER: *The distribution of Farey points*, Math. Ann. **201** (1973), 341–345 (MR0332666 (**48** #10992); Zbl. 0248.10013).

2.23.5. For the following blocks of quadratic non–residues

$$A_n = \left(\frac{i}{n}\right)_{0<i<n,\, i^2\equiv -1 \,(\mathrm{mod}\, n)}$$

the block sequence $(A_n)_{n=1}^{\infty}$ is

u.d.

NOTES: This was proved by Hooley. A new proof of a weaker version was given by D. Hensley (1988).

D. HENSLEY: *A truncated Gauss – Kuz'min law*, Trans. Amer. Math. Soc. **360** (1988), 307–327 (MR0927693 (89g:11066); Zbl. 0645.10043).

2.23.6. Let q_n, $n = 1, 2, \dots$, be a one–to–one sequence of positive integers and let $(A_n)_{n=1}^{\infty}$ be the sequence composed from blocks

$$A_n = \left(\frac{1}{q_n}, \frac{a_2}{q_n}, \dots, \frac{a_{\varphi(q_n)}}{q_n}\right),$$

where $1 = a_1 < a_2 < a_3 < \cdots < a_{\varphi(q_n)} = q_n - 1$ are coprime to q_n. Then the block sequence

$$(A_n)_{n=1}^{\infty}$$

is

almost u.q.

(with respect to indices of the form $N = \sum_{i=1}^{n} \varphi(q_i)$, for the def. cf. 1.8.28) provided q_n is any of the following sequences:

(i) $\frac{\varphi(q_n)}{q_n} \geq c > 0$ for every n,

(ii) $\frac{\varphi(q_n)}{\varphi(q_{n+1})} \leq c < 1$ for all sufficiently large n,

(iii) $\sum_{i \neq j=1}^{\infty} \frac{4^{\omega(q_{ij})}}{\varphi(q_{ij})} < \infty$ where $q_{ij} = \frac{q_i q_j}{\gcd(q_i,q_j)^2}$,

(iv) $\sum_{n=1}^{\infty} \frac{\varphi(q_n)}{q_n} < \infty$,

(v) $(q_m, q_n) = 1$ for every $m \neq n$,

(vi) $\sum_{i,j=1}^{\infty} \frac{(\log q_{ij})^2}{q_{ij}} \frac{\varphi(q_i)}{q_i} \frac{\varphi(q_j)}{q_j} < \infty$,

(vii) $\sum_{n=1}^{\infty} \frac{(\log q_n)^2}{q_n^{2\varepsilon}} < \infty$ and $d_{ij} \leq (q_i q_j)^{\frac{1}{2}-\varepsilon}$ for some $\varepsilon > 0$ and every $i \neq j$ where $d_{ij} = \gcd(q_i, q_j)$,

(viii) The sequence $d_{ij} = \gcd(q_i, q_j)$, $i, j = 1, 2, \dots$, has only finitely many different terms,

(ix) $\frac{q_n}{q_{n+1}} \leq c < 1$ for every n,

(x) $\frac{\varphi(q_n)}{q_n} < Kn^{-\delta}$ for some $K, \delta > 0$ and $n = 1, 2, \dots$,

(xi) $q_n = n^k$, for $k \geq 2$.

(xii) $q_n = q^n$, $q_n = n!$, $q_n = 2^{2^n}$, $q_n = F_n$, $q_n = q^n - 1$, $q_n = q^n + 1$ (for every positive integer $q \geq 2$).

NOTES: (i) and (ii) can be found in Duffin and Schaeffer (1941).
(iii) and (iv) were proved by O. Strauch (1982, Th. 14 and 15).

(v) and (vi) are from Strauch (1983, Th. 7 and 2).
(vii) is from Strauch (1984, Th. 6).
(viii) is from Strauch (1986, Th. 8).
(ix), (x), (xi) are from G. Harman (1990, Th. 1).
(xii) is from Strauch (1986).
All of them satisfy the Duffin – Schaeffer conjecture, cf. 1.8.28, Note (VI).

R.J. Duffin – A.C. Schaeffer: *Khintchine's problem in metric diophantine approximation*, Duke Math. J. **8** (1941), 243–255 (MR0004859 (3,71c); Zbl. 0025.11002).
G. Harman: *Some cases of the Duffin and Schaeffer conjecture*, Quart. J. Math. Oxford Ser.(2) **41** (1990), no. 2, 395–404 (MR1081102 (92c:11073); Zbl. 0688.10046).
O. Strauch: *Duffin – Schaeffer conjecture and some new types of real sequences*, Acta Math. Univ. Comenian. **40–41** (1982), 233–265 (MR0686981 (84f:10065); Zbl. 0505.10026).
O. Strauch: *Some new criterions for sequences which satisfy Duffin – Schaeffer conjecture, I*, Acta Math. Univ. Comenian. **42–43** (1983), 87–95 (MR0740736 (86a:11031); Zbl. 0534.10045).
O. Strauch: *Some new criterions for sequences which satisfy Duffin – Schaeffer conjecture, II*, Acta Math. Univ. Comenian. **44–45** (1984), 55–65 (MR0775006 (86d:11059); Zbl. 0557.10038).
[a] O. Strauch: *Two properties of the sequence $n\alpha$* (mod 1), Acta Math. Univ. Comenian. **44–45** (1984), 67–73 (MR0775007 (86d:11057); Zbl. 0557.10027).
O. Strauch: *Some new criterions for sequences which satisfy Duffin – Schaeffer conjecture, III*, Acta Math. Univ. Comenian. **48–49** (1986), 37–50 (MR0885318 (88h:11053); Zbl. 0626.10046).
O. Strauch: *A numerical integration method employing the Fibonacci numbers*, Grazer Math. Ber. **333** (1997), 19–33 (MR1640470 (99h:65038); Zbl. 0899.11037).

2.23.7. Let a_n/b_n the sequence of the all reduced rationals for which $1 \le a_n < b_n$ and which all elements in the simple continued fraction expansion of a_n/b_n are $\le k$. If we order this sequence lexicographically first according to the magnitude of b_n and then according to the magnitude of a_n, then it has singular a.d.f.

D. Hensley: *The distribution of badly approximable rationals and continuants with bounded digits, II*, J. Number Theory **34** (1990), 293–334 (MR1049508 (91i:11094); Zbl. 0712.11036).

$\boxed{\textbf{2.23.7.1}}$ Let p be an odd prime, a an integer with $(a,p)=1$. Then for the extremal discrepancy $D_N = D(a,H,K,k)$ of the sequence

$$\frac{a}{p}\prod_{i=1}^{k} n_i! \mod 1, \qquad H+1 \le n_1,\dots,n_k \le H+K,$$

where H and K are integers with $0 \le H < H+K < p$ we have

$$\max_{1\le a\le p-1} D(a,H,K,k) \ll K^{-k/4+r/2-1+2^{-r}} p^{(k-2r+4)/8} (\log p)^{(k-2r+4)/4}$$

for any fixed integers k, r such that $k \ge 2r \ge 1$.

NOTES: (I) M.Z. Garaev, F. Luca and I.E. Shparlinski (2004; Th. 10).
(II) For $r = 1$ this implies that $\max_{1\leq a\leq p-1} D(a, 0, p-1, 3) = \mathcal{O}(p^{-1/8}(\log p)^{5/4})$ and also that for any $\varepsilon > 0$ we have $\max_{1\leq a\leq p-1} D(a, 0, p-1, 3) = 0(1)$ for $N \geq p^{5/6+\varepsilon}$. As a consequence we get that for any fixed $\varepsilon > 0$ the products of three factorials $n_1!n_2!n_3!$ with $\max\{n_1, n_2, n_3\} = \mathcal{O}(p^{5/6+\varepsilon})$ are u.d. modulo p.
(III) M.Z. Garaev, F. Luca and I.E. Shparlinski (2005; Th. 4): Let K, L, M and N be integers with $0 \leq K < K + M$ and $0 \leq L < L + M < p$. Then for any integers $t, s \geq 1$ we have

$$\max_{(a,p)=1} \left| \sum_{m=K+1}^{K+M} \sum_{n=L+1}^{L+N} e^{2\pi i a m! n!/p} \right| \ll M^{1-1/2k(t+1)} N^{1-1/2t(k+1)} p^{1/st}.$$

They noticed that using the Erdős – Turán inequality (Th. 1.9.0.8) we get form this result the essentially the same bound (up to an extra $\log p$ factor) for the discrepancy of the sequence

$$\frac{m!n!}{p} \bmod 1, \quad K + 1 \leq m \leq K + M, \quad L + 1 \leq n \leq L + N.$$

Consequently, the products of two factorials $m!n!$ with $\max\{m, n\} = \mathcal{O}(p^{1/2+\varepsilon})$ are u.d. modulo p.
(IV) The following phenomenon was found empirically (cf. R.K. Guy (1994; **F11**)): The sequence $1!, 2!, \dots, p!$ misses about p/e residue classes modulo p for large prime numbers p. C. Cobeli, M. Vâjâitu and A. Zaharescu (2000) showed that this is a general behavior of a randomly chosen sequence of p elements modulo p. They proved and that the map $n \mapsto n! \pmod p$ with sufficiently large p is far from being a permutation of at least one of the sets $\{1, 2, \dots, p-1\}$ or $\{2, 4, \dots, \lfloor (p-1)/4 \rfloor\}$ for it misses a positive proportion of residues in at least one of the these sets.
(V) M. Shub and S. Smale (1995) proved that the complexity of computing factorials is related to an algebraic version of the $NP \neq P$ problem. Q. Cheng (2003) found a subexponential upper bound ($\exp(c\sqrt{\log n \log\log n})$) for the ultimate complexity of $n!$ assuming a widely believed number–theoretic conjecture concerning smooth numbers in short interval. The best known current algorithm to compute $n!$ over $\mathbb{Z}$ or modulo n needs about $\sqrt{n}$ arithmetic operations (cf. Q. Cheng (2003) or P. Bürgisser, M. Clausen and M.A. Shokrollahi (1997)).

P. BÜRGISSER – M. CLAUSEN – M.A. SHOKROLLAHI: *Algebraic complexity theory. With the collaboration of Thomas Lickteig*, Grundlehren der Mathematischen Wissenschaften 315, Springer, Berlin, 1997 (MR1440179 (99c:68002); Zbl. 1087.68568).
Q. CHENG: *On the ultimate complexity of factorials*, in: *STACS 2003. 20th annual symposium of theoretical aspects on computer science*, (A. Helmut *et al.* eds.), Lect. Notes Comput. Sci., Vol. 2607, Springer, Berlin, 2003, 157–166 (MR2066589 (2005c:68065); Zbl. 1035.68056).
C. COBELI – M. VÂJÂITU – A. ZAHARESCU: *The sequence* $n!$ (mod p), J. Ramanujan Math. Soc. **15** (2000), no. 2, 135–154 (MR1754715 (2001g:11153); Zbl. 0962.11005).
M.Z. GARAEV – F. LUCA – I.E. SHPARLINSKI: *Character sums and congruences with* $n!$, Trans. Amer. Math. Soc. **356** (2004), no. 12, 5089–5102 (MR2084412 (2005f:11175); Zbl. 1060.11046).

M.Z. Garaev – F. Luca – I.E. Shparlinski: *Exponential sums and congruences with factorials*, J. Reine Angew. Math. **584** (2005), 29–44 (MR2155084 (2006c:11095); Zbl. 1071.11051).
R.K. Guy: *Unsolved Problems in Number Theory*, Problem Books in Mathematics. Unsolved Problems in Intuitive Mathematics, I., Second ed., Springer Verlag, New York, 1994; 3rd ed. 2004 (MR1299330 (96e:11002); Zbl. 0805.11001).
M. Shub – S. Smale: *On the intractability of Hilbert's Nullstellensatz and an algebraic version of "$NP \neq P$?"*, Duke Math. J. **81** (1995), no. 1, 47–54 (MR1381969 (97h:03067); Zbl. 0882.03040).

2.23.7.2 The centered version of the Euclidean algorithm which uses the least absolute remainder in each step of division $a = bq + r$, $-\frac{b}{2} < r \leqslant \frac{b}{2}$ leads to the so-called centered continued fraction expansion of a real number x of the form (see Perron (1954) p. 137 or Hensley (2006) p. 40)

$$x = \left[a_0; \frac{\varepsilon_1}{a_1}, \cdots, \frac{\varepsilon_l}{a_l}, \cdots\right] = a_0 + \cfrac{\varepsilon_1}{a_1 + \cfrac{\varepsilon_2}{a_2 + \cdots + \cfrac{\varepsilon_l}{a_l + \cdots}}}.$$

Here $a_0 \in \mathbb{Z}$, $\varepsilon_i = \pm 1$ and $a_j \geqslant 2$, $a_j + \varepsilon_{j+1} > 2$ for $j \geqslant 1$. For a rational x, if $a_s = 2$ is the last partial quotient, then we choose $\varepsilon_s = 1$ to ensure the uniqueness of the representation. Given a rational x, denote
$S(x) = a_0 + a_1 + \cdots + a_s$,
$\mathcal{Z}_n = \{x \in \mathbb{Q} \cap [0,1] : S(x) \leqslant n+1\}$.
$g(x) = \lim_{n\to\infty} \dfrac{\#\{\xi \in \mathcal{Z}_n : \xi < x\}}{\#\mathcal{Z}_n}, x \in [0,1]$.
Then
(i) $g(x) = a_0 - c\lambda\left(\dfrac{E_1}{\lambda^{A_1}} + \dfrac{E_2}{\lambda^{A_2}} + \cdots + \dfrac{E_j}{\lambda^{A_j}} + \cdots\right)$, where
$E_j = \prod_{1\leqslant i\leqslant j}(-\varepsilon_i)$, $A_j = \sum\limits_{0\leqslant i\leqslant j} a_i$, $c = 1/(\lambda-1)$, and λ is the unique real root of the equation $\lambda^3 - \lambda^2 - \lambda - 1 = 0$.
(ii) If the derivative (finite or infinite) of $g'(x)$ at $x \in [0,1]$ exists then either $g'(x) = 0$ or $g'(x) = \infty$.
(iii) We have $g(1-x) = 1 - \dfrac{g(x)}{\lambda}$ for $x \in [0, 1/2]$.
Notes: E. Zhabitskaya (2010).

O. Perron: *Die Lehre von den Kettenbrüchen. Bd I. Elementare Kettenbrüche. 3. erweiterte und verbesserte Aufl.*, (German), B. G. Teubner Verlagsgesellschaft, Stuttgart 1954 (MR0064172; Zbl 0056.05901).
D. Hensley: *Continued fractions*, World Scientific, Hackensack, NJ, 2006 (MR2351741 (2009a: 11019); Zbl 1161.11028).

E. Zhabitskaya: *Continued fractions with minimal remainders*, Unif. Distrib. Theory **5** (2010), no. 2, 55–78.(MR2651862 (2011e:11125); Zbl 1313.11081).

2.24 Recurring sequences

2.24.1.

NOTES: Let $f(x) = x^s - \sum_{j=0}^{s-1} a_j x^j$ be a polynomial with coefficients $a_0, a_1, \dots, a_{s-1}$, with roots $\lambda_1, \dots, \lambda_s$ and discriminant D. A **linear recurring sequence of order** s is given by a relation of the form

$$r_{n+s} = a_{s-1} r_{n+s-1} + \cdots + a_1 r_{n+1} + a_0 r_n, \quad n = 1, 2, \dots,$$

where the initial (real) values of $r_1, \dots, r_s$ are not all zero.

. . . .

If the coefficients $a_0, a_1, \dots, a_{s-1}$ are non–negative rational numbers, $a_0 \neq 0$, and all roots $\lambda_1, \dots, \lambda_s$ are real with distinct absolute values not equal to one, then for every positive starting points $r_1, \dots, r_s$, the sequence

$$r_n \bmod 1, \quad n = 1, 2, \dots,$$

is

u.d.

Under the assumption that

- $a_0, a_1, \dots, a_{s-1}$ are integers,
- $f(x)$ is irreducible over $\mathbb{Z}$ and with discriminant D,
- $q_1 > 1$ is an integer, $p \geq 3$ is a prime such that $p | f(q_1)$, $p \nmid a_0 D$ and the integers
- q_k, $k = 1, 2, \dots$, satisfy $f(q_k) \equiv 0 \pmod{p^k}$, $q_k \equiv q_{k-1} \pmod{p^{k-1}}$ and $q_k^{\tau_k} \equiv 1 \pmod{p^k}$ (i.e. τ_k is the exponent modulo p^k), and
- $n_1 = 0, n_{k+1} = n_k + [p^{k/2} k^2] \tau_k$, $k = 1, 2, \dots$,
- $|\lambda_i| \neq 1$, for $i = 1, \dots, s$, and $|\lambda_i| > 1$ for $i = 1, \dots, m$, and $|\lambda_i| < 1$ for $i = m+1, \dots, s$, and the initial values $r_1, \dots, r_s$ satisfy
- $r_i = \theta_1 \lambda_1^i + \cdots + \theta_m \lambda_m^i$, $i = 1, \dots, s$, where
- $\theta_i = \sum_{j=1}^{\infty} \lambda_i^{-n_j} p^{-j} \sum_{k=1}^{s} a_{k,j} \lambda_i^{k-1}$, $i = 1, \dots, m$, and the integers
- $a_{k,j} \in [0, p^j)$, $k = 1, \dots, s$, $j = 1, 2, \dots$, are such that the numbers
- $A_{i,J} = \sum_{j=1}^{J} \lambda_i^{n_J - n_j} p^{J-j} \sum_{k=1}^{s} a_{k,j} \lambda_i^{k-1}$ satisfy the congruences
- $A_{1,j} \lambda_1^i + \cdots + A_{s,j} \lambda_s^i \equiv q_j^i \pmod{p^j}$ for $i = 1, \dots, s$, and $j = 1, 2, \dots$,

then for the discrepancy of r_n we have

$$D_N^* = \mathcal{O}\left(\frac{(\log N)^{4/3}}{N^{2/3}}\right).$$

NOTES: u.d. has proved by L. Kuipers and J.–S. Shiue (1973). The discrepancy bound is given by M.B. Levin and I.E. Šparlinskiĭ (1979). They noted that if $m = 1$ then the root λ_1 is a P.V. number and the recurring sequence has the form $\theta_1\lambda_1^n \bmod 1$.

L. Kuipers – J.–S. Shiue: *Remark on a paper by Duncan and Brown on the sequence of logarithms of certain recursive sequences*, Fibonacci Quart. **11** (1973), no. 3, 292–294 (MR0332699 (**48** #11025); Zbl. 0269.10019).
M.B. Levin – I.E. Šparlinskiĭ: *Uniform distribution of fractional parts of recurrent sequences*, (Russian), Uspehi Mat. Nauk **34** (1979), no. 3(207), 203–204 (MR0542250 (80k:10046); Zbl. 0437.10016).

2.24.2. Open problem. Consider a linear recurring sequence

$$r_{n+s} = a_{s-1}r_{n+s-1} + \cdots + a_1 r_{n+1} + a_0 r_n$$

with integral coefficients $a_0, a_1, \dots, a_{s-1}$ and initial values $r_1, \dots, r_s$ not all zero. Characterize the real numbers θ for which the sequence

$$r_n\theta \bmod 1, \quad n = 1, 2, \dots,$$

is

u.d.

NOTES: P. Kiss and S. Molnár (1982) proved a necessary and sufficient conditions for numbers θ for which the sequence $r_n\theta \bmod 1$ has finitely many points of accumulation. If the characteristic polynomial $f(x) = x^s - \sum_{j=0}^{s-1} a_j x^j$ is the minimal polynomial of a P.V. number, then they gave a construction for uncountably many numbers θ such that the sequence $r_n\theta \bmod 1$ has infinitely many limit points but the sequence itself is not u.d.

P. Kiss – S. Molnár: *On distribution of linear recurrences modulo* 1, Studia Sci. Math. Hungar. **17** (1982), no. 1-4, 113–127 (MR0761529 (85k:11033); Zbl. 0548.10006).

2.24.3. Let r_n be a linear recurring sequence of order s which characteristic polynomial $x^s - \sum_{j=0}^{s-1} a_j x^j$ has two complex conjugate roots of maximum modulus. Then the sequence

$$x_n = \log_{10}|r_n| \bmod 1$$

is

$$\text{u.d.}$$

NOTES: S. Kanemitsu, K. Nagasaka, G. Rauzy and J.–S. Shiue (1988). P. Schatte (1988) independently found conditions under which $\log_{10}|r_n| \bmod 1$ is u.d. in the case when the sequence r_n satisfies the relation $r_{n+2} = ar_{n+1} + br_n$, $n = 0, 1, 2, \dots$, where a and b are real numbers and $x^2 = ax + b$ has two (conjugate) complex root.

S. KANEMITSU – K. NAGASAKA – G. RAUZY – J.–S. SHIUE: *On Benford's law: the first digit problem*, in: *Probability theory and mathematical statistics (Kyoto, 1986)*, Lecture Notes in Math., 1299, Springer Verlag, Berlin, New York, 1988, pp. 158–169 (MR0935987 (89d:11059); Zbl. 0642.10007).
P. SCHATTE: *On the uniform distribution of certain sequences and Benford's law*, Math. Nachr. **136** (1988), 271–273 (MR0952478 (89j:11075); Zbl. 0649.10044).

2.24.4. Let r_n be a sequence of real numbers which satisfy a second order linear recurrence relation

$$r_{n+2} = a_{n+2}r_{n+1} + b_{n+2}r_n, \quad n = 0, 1, 2, \dots,$$

where a_n and b_n are given real sequences with a common period p (i.e. $a_{n+p} = a_n$ and $b_{n+p} = b_n$). Let

$$A_p = \begin{vmatrix} a_2 & -1 & 0 & \cdots & 0 \\ b_3 & a_3 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & b_r & a_{r-1} & -1 \\ 0 & 0 & \cdots & b_r & a_r \end{vmatrix}, \qquad B_p = \begin{vmatrix} b_2 & 0 & 0 & \cdots & 0 \\ b_3 & a_3 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & b_r & a_{r-1} & -1 \\ 0 & 0 & \cdots & b_r & a_r \end{vmatrix}$$

and $E_p = (-1)^{p-1}b_1 \dots b_p$, $D_p = a_1A_p + b_1A_{p-1} + B_p$. Let
• $D_p^2 - 4E_p \ge 0$ and λ_1, λ_2 be real roots of $\lambda^2 = D_p\lambda + E_p$ with $|\lambda_1| \ge |\lambda_2|$. If $\log_{10}|\lambda_1|$ is irrational and $u_n \ne 0$ for $n \ge n_0$, then the sequence

$$x_n = \log_{10}|r_n| \bmod 1$$

is

$$\text{u.d.}$$

• $D_p^2 - 4E_p < 0$, $\beta = \sqrt{E_p}$ and $\cos 2\pi\gamma = D_p/2\beta$. If 1, $\log_{10}\beta$, and γ are linearly independent over the rationals and $u_n \ne 0$ for $n \ge n_0$, then the sequence

$$x_n = \log_{10}|r_n| \bmod 1$$

is

u.d.

NOTES: Thus x_n obeys the Benford's law, cf. 2.12.26. This result covers the case when $r_n = p_n$ or $r_n = q_n$, where p_n/q_n is the nth convergent of a quadratic irrational, cf. 2.12.27.

P. SCHATTE – K. NAGASAKA: *A note on Benford's law for second order linear recurrences with periodical coefficients*, Z. Anal. Anwend. **10** (1991), no. 2, 251–254 (MR1155374 (93b:11101); Zbl. 0754.11021).

2.24.5.

NOTES: If $r_{n+s} = a_{s-1}r_{n+s-1} + \cdots + a_1 r_{n+1} + a_0 r_n$ with a_i's real is a linear recurring sequence then

$$r_n = A_{\sigma-1}n^{\sigma-1}\beta^n + \cdots + A_0\beta^n + \mathcal{O}(\rho^n),$$

where β is the **dominating characteristic root** (i.e. the root of the characteristic equation with the maximal absolute value) of multiplicity σ of the characteristic polynomial $x^s - \sum_{j=0}^{s-1} a_j x^j$ and $0 < |\rho| < \beta$.

. . . .

Let $b > 1$ be an integer and let r_n, $r_n > 0$, be a linear recurring sequence of order s with positive dominating characteristic root β. Then the sequence

$$x_n = \log_b r_n \bmod 1$$

is

u.d.

in the following cases:

(i) If $\beta > 0$ has multiplicity 1 and $\log_b \beta$ is the irrational with bounded partial quotients in the continued fraction expansion. In this case

$$D_N(x_n) = \mathcal{O}\left(\frac{\log N}{N}\right).$$

(ii) If $\beta > 0$ has multiplicity 1 and $\log_b \beta$ is irrational. Moreover we have

$$D_N(x_n) = \mathcal{O}\left(\frac{1}{N^{1/\eta}}\right),$$

where $\eta \geq 1$ is some constant depending on the recurrence.

(iii) If β has multiplicity > 1 and $\log_b \beta$ is irrational. In this case

$$D_N(x_n) = \mathcal{O}\left(\frac{\log N}{N^{1/(1+\eta)}}\right),$$

where $\eta \geq 1$ is a constant depending on the recurrence (namely the approximation type of $\log_b \beta$).

NOTES: This was proved by R.F. Tichy (1998). If $r_{n+2} = r_{n+1} + r_n$, with r_1 and r_2 both positive, then the u.d. of $\log r_n$ mod 1 was proved by J.L. Brown, Jr. and R.L. Duncan (1972).

RELATED SEQUENCES: 2.12.21, 2.12.22.

J.L. BROWN, JR. – R.L. DUCAN: *Modulo one uniform distribution of certain Fibonacci–related sequences*, Fibonacci Quart. **10** (1972), no. 3, 277–280, 294 (MR0304291 (**46** #3426); Zbl. 0237.10033).
R.F. TICHY: *Three examples of triangular arrays with optimal discrepancy and linear recurrences*, in: *Applications of Fibonacci Numbers (The Seventh International Research Conference, Graz, 1996)*, Vol. 7, (G.E. Bergum, A.N. Philippou and A.F. Horadam eds.), 1998, Kluwer Acad. Publ., Dordrecht, Boston, London, pp. 415–423 (MR1638468; Zbl. 0942.11036).

2.24.6. Let r_n be a linear recurring sequence with positive elements and with positive dominating characteristic root $\beta \neq 1$. Suppose that $\log_b \beta$ is the irrational number with bounded partial quotients in the continued fraction expansion, where $b \neq 1$ is an arbitrary positive real number. Then for the discrepancy of the finite sequence

$$\frac{\log_b r_1}{\log_b r_N}, \frac{\log_b r_2}{\log_b r_N}, \dots, \frac{\log_b r_N}{\log_b r_N}$$

we have

$$D_N = \mathcal{O}\left(\frac{\log N}{N}\right).$$

NOTES: R.F. Tichy (1998). He noticed that if the multiplicity of β is 1 then the estimate for D_N can be improved to the optimal one $D_N = \mathcal{O}(1/N)$.

R.F. TICHY: *Three examples of triangular arrays with optimal discrepancy and linear recurrences*, in: *Applications of Fibonacci Numbers (The Seventh International Research Conference, Graz, 1996)*, Vol. 7, (G.E. Bergum, A.N. Philippou and A.F. Horadam eds.), 1998, Kluwer Acad. Publ., Dordrecht, Boston, London, pp. 415–423 (MR1638468; Zbl. 0942.11036).

2.24.7. Let u_n, $n = 0, 1, 2, \dots$, be a linear recurring sequence defined by $u_n = Au_{n-1} + Bu_{n-2}$ with non–zero real coefficients A, B, real initial values u_0, u_1 and negative discriminant $D = A^2 + 4B$. If the number

$$\theta = \frac{1}{\pi} \arctan \frac{\sqrt{-D}}{A}$$

is irrational, then the sequence

$$x_n = \frac{u_{n+1}}{u_n} \bmod 1$$

has the a.d.f.

$$g(x) = g_1(x - \{A/2\}) - g_1(-\{A/2\})$$

where

$$g_1(x) = x + \frac{1}{\pi}\arctan\frac{\sin(2\pi x)}{e^{\pi\sqrt{-D}} - \cos(2\pi x)}$$

and the star discrepancy D_N^* of x_n with respect to $g(x)$ satisfies

$$D_N^*(x_n, g) \le 2\sqrt{2}(-D)^{1/4}\sqrt{D_N(n\theta)} + 6D_N(n\theta),$$

where $D_N(n\theta)$ denotes the classical extremal discrepancy of the sequence $n\theta \bmod 1$.

NOTES: (I) P. Kiss and R.F. Tichy (1989). In the proof they transformed x_n to the form

$$\frac{u_{n+1}}{u_n} = c + d\tan(\pi(n\theta + \omega)),$$

where c, d, ω are real numbers which do not depend on n, cf. [DT, proof of Th. 1.143, p. 144].

(II) A. Pethő (1982) proved that θ is a rational number if and only if $A^2 = -kB$ for some $k = 1, 2, 3, 4$.

(III) S.H. Molnár (2003) generalized this result under the same assumptions on the linear recurring sequence u_n as follows: For every $k = \pm 1, \pm 2, \dots$ the sequence

$$x_n^{(k)} = \frac{u_{n+k}}{u_n} \bmod 1$$

has the a.d.f.

$$g^{(k)}(x) = g_1^{(k)}\big(x - \{c\}\big) - g_1^{(k)}\big(-\{c\}\big)$$

where

$$g_1^{(k)}(x) = x + \frac{1}{\pi}\arctan\frac{\sin(2\pi x)}{e^{2\pi|d|} - \cos(2\pi x)}$$

with

$$c = r^k\cos(k\pi\theta), \quad d = -r^k\sin(k\pi\theta), \quad \text{and } r = \left|\frac{A + i\sqrt{-D}}{2}\right|.$$

He also used the expression $u_{n+k}/u_n = c + d\tan(\pi(n\theta + \omega))$, and for discrepancy he proved the estimate

$$D_N^*\left(x_n^{(k)}, g^{(k)}\right) \le 4\sqrt{|r^k\sin(k\pi\theta)|}\sqrt{D_N(n\theta)} + 6D_N(n\theta).$$

Molnár (2003) also noted that x_n and $x_n^{(-1)}$ has the same a.d.f. if and only if $B = -1$ (in this case he called u_{n+1}/u_n reciprocal invariant). If discriminant D is positive and the initial values u_0, u_1 are integers then the above statement is true if and only if $B = 1$.

P. Kiss – R.F. Tichy: *A discrepancy problem with applications to linear recurrences. I*, Proc. Japan Acad. Ser. A Math. Sci. **65** (1989), no. 5, 135–138 (MR1011853 (90j:11060a); Zbl. 0692.10041).
S.H. Molnár: *Reciprocal invariant distributed sequences constructed by second order linear recurrences*, Acta Acad. Paedagog. Agriensis Sect. Mat. (N.S.) **30** (2003), 101–108 (MR2054719 (2005a:11112); Zbl. 1050.11013).
A. Pethő: *Perfect powers in second order linear recurrences*, J. Number Theory **15** (1982), no. 1, 5–13 (MR0666345 (84f:10024); Zbl. 0488.10009).

2.24.8. α–refinement. Let $0 < \alpha < 1$ be a real number. Consider the sequence $A_n = (x_{n,1}, \dots, x_{n,N_n})$, $n = 1, 2, \dots$, of blocks in the unit interval $[0, 1]$ defined inductively as follows (note that we can consider A_n as a partition of $[0, 1]$):

- $A_1 = (x_{1,1}, x_{1,2})$, where $x_{1,1} = 0$ and $x_{1,2} = 1$,
- A_{n+1} is an α–refinement of A_n, i.e. between any two consecutive elements $x_{n,i}, x_{n,i+1} \in A_n$ we insert

$$x = x_{n,i} + \alpha(x_{n,i+1} - x_{n,i}), \quad i = 1, 2, \dots, N_n - 1.$$

(In other words, each interval $[x_{n,i}, x_{n,i+1}]$ with consecutive $x_{n,i}, x_{n,i+1} \in A_n$ is decomposed into two subintervals $[x_{n,i}, x]$ and $[x, x_{n,i+1}]$ using points x given above).

Then the sequence of blocks A_n has the a.d.f

$$g_\alpha(x)$$

such that
(i) $g_\alpha(x)$ and $g_{\alpha'}(x)$ are singular to each other if $0 < \alpha < \alpha' < 1$,
(ii) $g_\alpha(x) = x$ if $\alpha = \frac{1}{2}$.

Notes: S. Kakutani (1976) classifies this as well–known.

For the definition when two given measures are **mutually singular** consult P. Billingsley (1986, p. 442).

Related sequences: 2.24.9

P. Billingsley: *Probability and Measure*, Wiley Series in Probability and Mathematical Statistics, Second ed., J. Wiley & Sons, Inc., New York, 1986 (MR0830424 (87f:60001); Zbl. 0649.60001).
S. Kakutani: *A problem of equidistribution on the unit interval* $[0, 1]$, in: *Measure Theory Oberwolfach 1975 (Proceedings of the Conference Held at Oberwolfach 15–20 June, 1975*, (A. Doldan and B. Eckmann eds.), Lecture Notes in Mathematics, 541, Springer Verlag, Berlin, Heidelberg, New York, 1976, pp. 369–375 (MR0457678 (**56** #15882); Zbl. 0363.60023).

2.24.9. α-maximal refinement. Let $0 < \alpha < 1$ be a real number. Define the sequence B_n of blocks in the unit interval $[0, 1]$ by induction:

- $B_1 = (x_{1,1}, x_{1,2})$, where $x_{1,1} = 0$ and $x_{1,2} = 1$,
- B_{n+1} is an α–maximal refinement of B_n, what means that we add to $B_n = (x_{n,1}, \dots, x_{n,N_n})$ all the points (in ascending order)

$$x = x_{n,i} + \alpha(x_{n,i+1} - x_{n,i}),$$

for all those i which satisfy

$$x_{n,i+1} - x_{n,i} = \max_{1 \le j \le N_n - 1} x_{n,j+1} - x_{n,j},$$

i.e. every maximal interval $[x_{n,i}, x_{n,i+1}]$ is decomposed into two subintervals $[x_{n,i}, x]$ and $[x, x_{n,i+1}]$.

Then for every $0 < \alpha < 1$ the sequence of blocks B_n is

u.d.

Related sequences: 2.24.8

S. Kakutani: *A problem of equidistribution on the unit interval* $[0,1]$, in: *Measure Theory Oberwolfach 1975 (Proceedings of the Conference Held at Oberwolfach 15–20 June, 1975*, (A. Doldan and B. Eckmann eds.), Lecture Notes in Mathematics, 541, Springer Verlag, Berlin, Heidelberg, New York, 1976, pp. 369–375 (MR0457678 (**56** #15882); Zbl. 0363.60023).

2.24.10. Open problem. Characterize the distribution properties of the so–called **strange recurring sequences** of the form

(i) $x_n = x_{n-[x_{n-1}]} + x_{n-[x_{n-2}]}$,

(ii) $x_n = x_{n-[x_{n-1}]} + x_{[x_{n-1}]}$,

(iii) $x_n = x_{[x_{n-2}]} + x_{n-[x_{n-2}]}$,

with real initial values x_1, x_2.

Notes: If $x_1 = x_2 = 1$ the sequence (i) was defined by D.R. Hofstadter (1979), (ii) was defined by J.H. Conway (1988) during one of his lectures and C.L. Mallows (1991) established the regular structure of (ii) and introduced the monotone sequence (iii).

D.R. Hofstadter: *Gödel, Escher, Bach: an External Golden Braid*, Basic Books, Inc., Publishers, New York, 1979 (MR0530196 (80j:03009); Zbl 0457.03001 reprint 1981).

C.L. Mallows: *Conway's challenge sequence*, Amer. Math. Monthly **98** (1991), no. 1, 5–20 (MR1083608 (92e:39007); Zbl. 0738.11014).

2.25 Pseudorandom Numbers Congruential Generators

Notes: There is no formal fully satisfactory definition of the pseudorandomness of a sequence x_n, and thus we have only a scale of tests which such a candidate sequence x_n should satisfy, cf. 1.8.22, [DT, pp. 424–430, 3.4.] and J.C. Lagarias (1990, 1992). Various statistical tests can be found in literature, see e.g. P. L'Ecuyer and P. Hellekalek (1998), G. Marsaglia (1996), H. Niederreiter (1978, 1992), A. Rukhin, J. Soto, J. Nechvatal, *et al.* (2001). As an illustration let us mention some **empirical statistical tests**: χ^2 test, uniformity test, gap test, run test, permutation test, correlation test, negative entropy, collision, empty boxes, cf. 2.26. Progress in the recent development in the pseudorandom number generation is described in H. Niederreiter and I.E. Shparlinski (2002).

D.E. Knuth: *Seminumerical Algorithms*, The Art of Computer Programming, Vol. 2, 2nd ed., Addison Wesley, Reading, MA, 1981 (First ed.: Reading, MA, 1969) (MR0286318 (**44** #3531); Zbl. 0477.65002).
J.C. Lagarias: *Pseudorandom number generators in cryptography and number theory*, in: *Cryptology and Computational Number Theory (Boulder, CO, 1989)*, (C. Pomerance ed.), Proc. Sympos. Appl. Math., 42, Amer. Math. Soc., Providence, RI, 1990, pp. 115–143 (MR1095554 (92f:11109); Zbl. 0747.94011).
J.C. Lagarias: *Pseudorandom numbers*, in: *Probability and Algorithms*, Nat. Acad. Press, Washington, D.C., 1992, pp. 65–85 (MR1194441; Zbl. 0766.65003).
P. L'Ecuyer – P. Hellekalek: *Random number generators: Selection criteria and testing*, in: *Random and Quasi–Random Point Sets*, (P. Hellekalek, G. Larcher eds.), Lecture Notes in Statist., 138, Springer Verlag, New York, Berlin, 1998, pp. 223–265 (MR1662843 (99m:65014); Zbl. 0915.65004).
G. Marsaglia: *DIEHARD: a Battery of Test of Randomness*, (electronic version: `http://stat.fsu.edu/~geo/diehard.html`).
H. Niederreiter: *Quasi–Monte Carlo methods and pseudo–random numbers*, Bull. Amer. Math. Soc. **84** (1978), no. 6, 957–1041 (MR0508447 (80d:65016); Zbl. 0404.65003).
H. Niederreiter: *Random Number Generation and Quasi–Monte Carlo Methods*, CBMS–NSF Regional Conference Series in Applied Mathematics, 63, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992, vi+241 pp. (MR1172997 (93h:65008); Zbl. 0761.65002).H. Niederreiter – I.E. Shparlinski: *Recent advances in the theory of nonlinear pseudorandom number generators*, in: *Monte Carlo and Quasi–Monte Carlo Methods 2000 (Proceedings of a Conference held at Hong Kong Baptist University, Hong Kong SAR, China, Nov. 27–Dec. 1, 2000*, (Kai–Tai Fang, F.J. Hickernell, H. Niederreiter eds.), Springer Verlag, Berlin, Heidelberg 2002, pp. 86–102 (MR1958848 (2003k:65005); Zbl. 1076.65008).
A. Rukhin – J. Soto – J. Nechvatal – M. Smid – E. Barker – S. Leigh – M. Levenson – M. Vangel – D. Banks – A. Heckert – J. Dray – S. Vo: *A Statistical Test Suite for Random and Pseudorandom Number Generators for Cryptographic Applications*, NIST Special Publication 800-22, (2000 with revision dated May 15, 2001). (`http://csrc.nist.gov/rng/SP800-22b.pdf`).

2.25.1. Linear congruential generator (LCG). The linear multiplicative congruential generator produces the sequence

$$x_n = \frac{y_n}{M}, \quad n = 0, 1, \dots,$$

where M is a large modulus and

$$y_n \equiv a y_{n-1} + c \pmod M \text{ with } 0 \le y_n \le M - 1,\ n = 1, \dots,$$

and $0 < y_0 \le M - 1$ is an initial seed.

(A) The sequence x_n, $n = 0, 1, 2, \ldots$, has the maximal period M if and only if $\gcd(c, M) = 1$, and $a \equiv 1 \pmod p$ for every prime divisor p of M, and if 4 divides M then also $a \equiv 1 \pmod 4$.
(B) If M is a prime number, a is a primitive root modulo M, $c = 0$, and $y_0 \neq 0$, then the sequence x_n has the period $M - 1$ and the s–dimensional sequence (s–dimensional serial test)

$$\mathbf{x}_n = (x_n, x_{n+1}, \ldots, x_{n+s-1}), \quad n = 0, 1, \ldots, M-2,$$

has discrepancy satisfying

$$D_{M-1} \leq \frac{s+1}{M-1} + \frac{M}{M-1} R(\mathbf{g}, M).$$

Here $\mathbf{g} = (1, a, a^2, \ldots, a^{s-1})$ and

$$R(\mathbf{g}, M) = \sum_{\substack{\mathbf{h}=(h_1,\ldots,h_s)\neq\mathbf{0}, -M/2<h_i\leq M/2 \\ \mathbf{h}\cdot\mathbf{g}\equiv 0 \pmod M}} \frac{1}{r(\mathbf{h}, M)},$$

with $r(\mathbf{h}, M) = \prod_{i=1}^{s} r(h_i, M)$, where

$$r(h_i, M) = \begin{cases} M \sin \frac{\pi |h_i|}{M}, & \text{if } h_i \neq 0, \\ r(0, M) = 1, & \text{otherwise.} \end{cases}$$

NOTES: (I) This generator was introduced by D.H. Lehmer (1951), cf. D.E. Knuth (1981, Chapt. 3) or H. Niederreiter (1992). Typical values for the modulus are $M = 2^{32}$, or the Mersenne prime $M = 2^{31} - 1$. The number $M = 2^{48}$ was also used.
(II) For a proof of (A) cf. Knuth (1981, § 3.2.1).
(III) Discrepancy bound (B) is from Niederreiter (1976, 1977).
(IV) In P. L'Ecuyer and P. Hellekalek (1998) a list of some LCG's with prime moduli M, $c = 0$ and a a primitive root modulo M can be found together with their classification as "good" and "bad" LCG's with respect to the spectral test, e.g. if $M = 2^{36} - 5$, then the number $a = 49865143810$ yields a good and $a = 102254510$ a bad LCG.

P. L'Ecuyer – P. Hellekalek: *Random number generators: Selection criteria and testing*, in: *Random and Quasi–Random Point Sets*, (P. Hellekalek, G. Larcher eds.), Lecture Notes in Statist., 138, Springer Verlag, New York, Berlin, 1998, pp. 223–265 (MR1662843 (99m:65014); Zbl. 0915.65004).
D.H. Lehmer: *Mathematical methods in large–scale computing units*, in: *Proc. 2nd Sympos. on Large–Scale Digital Calculating Machinery (Cambridge, Ma; 1949)*, Harvard University Press, Cambridge, Ma, 1951, pp. 141–146 (MR0044899 (13,495f); Zbl. 0045.40001).
D.E. Knuth: *Seminumerical Algorithms*, The Art of Computer Programming, Vol. 2, 2nd ed., Addison Wesley, Reading, MA, 1981 (First ed.: Reading, MA, 1969) (MR0286318 (**44** #3531); Zbl. 0477.65002).

H. Niederreiter: *Statistical independence of linear congruential pseudo–random numbers*, Bull. Amer. Math. Soc. **82** (1976), no. 6, 927–929 (MR0419395 (**54** #7416); Zbl. 0348.65005).
H. Niederreiter: *Pseudo–random numbers and optimal coefficients*, Advances in Math. **26** (1977), no. 2, 99–181 (MR0476679 (**57** #16238); Zbl. 0366.65004).
H. Niederreiter: *Random Number Generation and Quasi–Monte Carlo Methods*, CBMS–NSF Regional Conference Series in Applied Mathematics, 63, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992, vi+241 pp. (MR1172997 (93h:65008); Zbl. 0761.65002).

2.25.2. Linear feedback shift register generator. The shift register generator produces the sequence

$$x_n = \sum_{j=1}^{m} \frac{y_{mn+j}}{p^j}, \qquad \text{where } y_{n+k} \equiv \sum_{j=0}^{k-1} a_j y_{n+j} \pmod p\,,\ n = 0, 1, 2, \dots.$$

Let p be a prime and $2 \le m \le k$. Then

- the sequence y_n, $n = 0, 1, 2, \dots$, is called the **k–th order shift register sequence**,
- the polynomial $f(x) = x^k - a_{k-1}x^{k-1} - \cdots - a_0$ is called the **characteristic polynomial** of the shift register sequence y_n which is completely determined by the initial seeds $y_0, y_1, \dots, y_{k-1}$,
- y_n is purely periodic if its characteristic polynomial $f(x)$ satisfies $f(0) \neq 0$,
- if $f(x)$ is irreducible such that $f(0) \neq 0$ and $y_0, y_1, \dots, y_{k-1}$ are not all vanishing, then the length M of the minimal period of y_n is equal to the order of any root α of $f(x)$ in the multiplicative group $\mathbb{F}_q^*$ of non–zero elements of $\mathbb{F}_q$, where $q = p^k$,
- the maximal value of period $M = p^k - 1$ is achieved if and only if $f(x)$ is a primitive polynomial over $\mathbb{F}_p$ (i.e. a monic polynomial of degree k which root generates $\mathbb{F}_q^*$),
- the terms of the sequence x_n, $n = 0, 1, 2, \dots$, are called **digital k–step pseudorandom numbers**,
- x_n is purely periodic and its minimal period is $\frac{M}{\gcd(m,M)}$ thus $\gcd(m, M) = 1$ is usually assumed.
- The preferred choice of p is 2.

Notes: The author of the method is R.C. Tausworthe (1965). If $p = 2$ then he proved that the the mean value, variance and autocorrelation of the sequence x_n, $n = 1, \dots, 2^k - 1$, equal within the error of 2^{-k} to those of a sequence of uniform independent random variables in $[0, 1]$. Properties and proofs can be found in R. Lidl and H. Niederreiter (1986, Chapt. 7), H. Niederreiter (1984).

R. Lidl – H. Niederreiter: *Introduction to Finite Fields and their Applications*, Cambridge Univ. Press, Cambridge, 1986 (MR0860948 (88c:11073); Zbl. 0629.12016).

H. Niederreiter: *The performance of k–step pseudorandom number generators under the uniformity test*, SIAM J. Sci. Statist. Comput. **5** (1984), no. 4, 798–810 (MR0765207 (86f:65029); Zbl. 0557.65005).
R.C. Tausworthe: *Random numbers generator by linear recurrence modulo two*, Math. Comput. **19** (1965), 201–209 (MR0184406 (**32** #1878); Zbl. 0137.34804).

2.25.3. GFSR generator. Let p be a prime and y_n, $n = 0, 1, 2, \dots$ the kth order shift register sequence (cf. 2.25.2). Given an integer $m \geq 2$ and non–negative integers $k_1, \dots, k_m$ the terms of the sequence

$$x_n = \sum_{j=1}^{m} \frac{y_{n+k_j}}{p^j}, \quad n = 0, 1, 2, \dots,$$

are called **GFSR pseudorandom numbers**.
- x_n is purely periodic with the least period length M as given in 2.25.2.

Notes: GFSR (generalized feedbackshift register) sequences were introduced by T.G. Lewis and W.H. Payne (1973).

Related sequences: 2.25.2

H. Niederreiter: *Pseudorandom numbers generated from shift register sequence*, in: *Number–Theoretic Analysis (Seminar, Vienna 1988-89)*, (H. Hlawka – R.F. Tichy eds.), Lecture Notes in Math., 1452, Springer Verlag, Berlin, Heidelberg, 1990, pp. 165–177 (MR1084645 (92g:11082); Zbl. 0718.11034).
T.G. Lewis – W.H. Payne: *Generalized feedback shift register pseudorandom number algorithm*, J. Assoc. Comput. Mach. **20** (1973), 456–468 (Zbl. 0266.65009).

2.25.4. Recursive matrix method. Assume that
- $\mathbb{F}_p$ is the finite field of the prime order p and identify the elements of $\mathbb{F}_p$ with digits $\{0, 1, \dots, p-1\}$,
- s is a positive integer and $\mathbf{A}$ is a non–singular $s \times s$ matrix over $\mathbb{F}_p$,
- $\mathbf{y}_0$ is the initial row vector, $\mathbf{y}_0 \neq \mathbf{0}$,
- $\mathbf{y}_n = (y_{n,1}, \dots, y_{n,s})$, $n = 0, 1, 2, \dots$, is the sequence of row vectors in $\mathbb{F}_p^s$ defined by the recursion $\mathbf{y}_{n+1} = \mathbf{y}_n \cdot \mathbf{A}$.

Then the sequence

$$x_n = \sum_{j=1}^{s} \frac{y_{n,j}}{p^j}, \qquad n = 0, 1, 2, \dots,$$

has the maximal period $p^s - 1$ if and only if the polynomial $\det(x\mathbf{I} - \mathbf{A})$ of degree s (i.e. characteristic polynomial of $\mathbf{A}$) is primitive over $\mathbb{F}_p$.

NOTES: This method was introduced by H. Niederreiter (1993). For the star discrepancy of the j–dimensional sequence $\mathbf{z}_n = (x_n, \dots, x_{n+j-1})$ see H. Niederreiter (1995, 1996) and G. Larcher (1998).

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H. NIEDERREITER: *Factorization of polynomials and some linear–algebra problems over finite fields*, in: *Computational linear algebra in algebraic and related problems (Essen, 1992)*, Linear Algebra Appl. **192** (1993), 301–328 (MR1236747 (95b:11114); Zbl. 0845.11042).
H. NIEDERREITER: *The multiple recursive matrix method for pseudorandom number generation*, Finite Fields Appl. **1** (1995), no. 1, 3–30 (MR1334623 (96k:11103); Zbl. 0823.11041).
H. NIEDERREITER: *Improved bounds in the multiple–recursive matrix method for pseudorandom number and vector generation*, Finite Fields Appl. **2** (1996), no. 3, 225–240 (MR1398075 (97d:11120); Zbl. 0893.11031).
G. LARCHER: *A bound for the discrepancy of digital nets and its application to the analysis of certain pseudo–random number generators*, Acta Arith. **83** (1998), no. 1, 1–15 (MR1489563 (99j:11086); Zbl. 0885.11050).

2.25.5. Quadratic congruential generator. Let $M \geq 2$ be a large integer, called the modulus, and let $a, b, c \in \mathbb{Z}_M$ be three parameters and y_0 be the initial seed. The quadratic generator produces the sequence

$$x_n = \frac{y_n}{M}, \text{ where } y_{n+1} \equiv a y_n^2 + b y_n + c \pmod{M} \text{ and } 0 \leq y_n \leq M-1$$

of **quadratic congruential pseudorandom numbers**. They have the following properties:

(A) If M is odd, then x_n is purely periodic with the maximum possible period length M if and only if $a \equiv 0 \pmod{p}$, $b \equiv 1 \pmod{p}$, and $c \not\equiv 0 \pmod{p}$ for all primes p which divide the modulus M, and moreover $a \not\equiv 3c \pmod{9}$ if $9|M$.

(B) If $M = 2^\omega$ then the sequence x_n is purely periodic with the maximum possible period length M if and only if $a \equiv 0 \pmod{2}$, $b \equiv a+1 \pmod{4}$, and $c \equiv 1 \pmod{2}$. In this case the full period of x_n shows the best possible distribution in $[0,1]$.

(C) If $a \equiv 2 \pmod{4}$, $b \equiv 3 \pmod{4}$, $c \equiv 1 \pmod{2}$, and $M = 2^\omega$ then for the extremal discrepancy D_M of the sequence of pairs (x_n, x_{n+1}), $n = 0, \dots, M-1$, we get

$$\begin{aligned} D_M &< \frac{2\sqrt{2}+8}{7\pi^2}\frac{(\log M)^2}{\sqrt{M}} - 0.0791\frac{\log M}{\sqrt{M}} + \frac{0.3173}{\sqrt{M}} + \frac{4}{M}, \\ D_M &\geq \frac{1}{3(\pi+2)\sqrt{M}}. \end{aligned}$$

(D) If $M = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ with distinct primes $p_i \geq 3$ and integers $\alpha_i \geq 2$ for $i = 1, \dots, r$, then for the extremal discrepancy D_M of (x_n, x_{n+1}), $n = 0, 1, \dots, M-1$, we get

$$\begin{aligned} D_M <& \frac{1}{\sqrt{M}} \prod_{i=1}^{r} \left(1 + \frac{5}{4p_i^{\frac{3}{2}}}\right) \times \\ & \left(\frac{4}{\pi^2\sqrt{P}} \left(\log M + \frac{4\sqrt{3}\pi}{9} \log P\right) (\log M + 0.778) + \frac{16}{27}\sqrt{P}\right) + \frac{2}{M}, \\ D_M \geq& \frac{\sqrt{P}}{2(\pi+2)} \frac{1}{\sqrt{M}} \end{aligned}$$

where $P = p_1 \dots p_r$. This implies that the order of magnitude of D_M lies between $\sqrt{\frac{P}{M}}$ and $\left(\sqrt{P} + \frac{(\log M)^2}{\sqrt{P}}\right) \frac{1}{\sqrt{M}}$.

(E) If $a \equiv 2 \pmod 4$, $b \equiv 3 \pmod 4$, $c \equiv 1 \pmod 2$, and $M = 2^\omega$ then the extremal discrepancy $D_{\frac{M}{2}}$ of the sequence of points (x_{2n}, x_{2n+1}) with $n = 0, \dots, \frac{M}{2}$ satisfies

$$\begin{aligned} D_{\frac{M}{2}} <& \frac{2\sqrt{2}+8}{7\pi^2} \frac{(\log M)^2}{\sqrt{M}} - 0.0791 \frac{\log M}{\sqrt{M}} + \frac{0.3173}{\sqrt{M}} + \frac{4}{M}, \\ D_{\frac{M}{2}} \geq& \frac{2}{B(\pi+2)\sqrt{M}}, \qquad \text{where } B = \begin{cases} 1, & \text{if } y_0 \equiv (b+1)/4 \pmod 2, \\ 3, & \text{if } y_0 \equiv (b+5)/4 \pmod 2. \end{cases} \end{aligned}$$

(F) If $a \equiv 2 \pmod 4$, $b \equiv 3 \pmod 4$, $M = 2^\omega$, and if in order to emphasize the dependence of the quadratic generator on the parameter c we denote by $D_M(c)$ the extreme discrepancy of the sequence of (x_n, x_{n+2}), $n = 0, 1, \dots, M-1$, then

$$\frac{2}{M} \sum_{c \in \mathbb{Z}_M^*} D_M(c) < \frac{4\sqrt{2}+2}{7\pi^2} \frac{(\log M)^2}{\sqrt{M}} + 0.0977 \frac{\log M}{\sqrt{M}} - \frac{0.1753}{\sqrt{M}} + \frac{2}{M}.$$

If, in addition, the parameters $a \equiv 2 \pmod 4$, $b \equiv 3 \pmod 4$, and $c \equiv 1 \pmod 2$ also satisfy the relation $4ac \equiv (b-1)^2 - 28 + 2^{2\nu+4} \left(\bmod 2^{\omega-\nu+1}\right)$, where $\nu \geq 1$ and $\mu \in \{0, 1, 2\}$ are integers with $\omega = 3\nu + \mu + 2$, then

$$D_M(c) \geq \frac{2^{(\mu-1)/3}}{27(\pi+2)M^{1/3}}.$$

(G) If $a \equiv 2 \pmod 4$, $b \equiv 3 \pmod 4$, $M = 2^\omega$, and $D_M(c)$ is the extreme discrepancy of the sequence of triples (x_n, x_{n+1}, x_{n+2}), $n = 0, \dots, M-1$, then

$$\begin{aligned} \frac{2}{M} \sum_{c \in \mathbb{Z}_M^*} D_M(c) <& \frac{24\sqrt{2}+68}{31\pi^3} \frac{(\log M)^3}{\sqrt{M}} + 0.8427 \frac{(\log M)^2}{\sqrt{M}} \\ & + 2.0927 \frac{\log M}{\sqrt{M}} + \frac{1.6495}{\sqrt{M}} + \frac{3}{M}. \end{aligned}$$

If moreover $c \equiv 1 \pmod 2$ then

$$D_M(c) \geq \frac{1}{3(\pi+2)\sqrt{M}}.$$

If the parameters $a \equiv 2 \pmod 4$, $b \equiv 3 \pmod 4$ and $c \equiv 1 \pmod 2$ satisfy $4ac \equiv (b-1)^2 - 4 \pm 8 + 2^{2\nu+4} \left(\bmod 2^{\omega-\nu+1}\right)$, where $\nu \geq 1$ and $\mu \in \{0,1,2\}$ are integers with $\omega = 3\nu+\mu+2$, then

$$D_M(c) \geq \frac{2^{(\mu-1)/3}}{4(\pi^2+3\pi+3)M^{1/3}}.$$

As in the case (F) for (x_n, x_{n+2}), also in this case the congruence $4ac \equiv (b-1)^2 - 28 + 2^{2\nu+4} \left(\bmod 2^{\omega-\nu+1}\right)$ implies that

$$D_M(c) \geq \frac{2^{(\mu-1)/3}}{27(\pi+2)M^{1/3}}.$$

Notes: (I) The quadratic congruential method was coined by D.E. Knuth in 1969 in the first edition of his famous book (1981), cf. also Knuth (1981, p. 25).
(II) For the conditions for the maximum possible period consult Knuth (1981, pp. 34, 526).
(III) The discrepancies estimates for sequences of (x_n, x_{n+1}), (x_{2n}, x_{2n+1}), (x_n, x_{n+2}), (x_n, x_{n+1}, x_{n+2}) of full length modulo $M = 2^\omega$ were proved by J. Eichenauer–Herrmann and H. Niederreiter (1991, 1995) and by Eichenauer–Herrmann (1995a, 1995b, 1997). They are also summarized in J. Eichenauer–Herrmann, E. Herrmann and S. Wegenkittl (1998).
(IV) The estimates for D_M with respect to the composite modulus $M = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ were given by S. Strandt (1998). The author also gives a comparison of two sequences x_n and $z_n = x_n^{(1)} + \dots + x^{(r)}$, where x_n corresponds to the modulus $M = p_1^{\alpha_1} \dots p_r^{\alpha_r}$, $\alpha_i \geq 2$, $i = 1, \dots, r$, with parameters $a, b, c \pmod M$ and the initial seed y_0; and $x_n^{(i)}$ corresponds to the modulus $q_i = p_i^{\alpha_i}$ and the initial seed $y_0^{(i)}$ and parameters $a_i, b_i, c_i \pmod{q_i}$. Strandt (1998) proved that x_n and z_n are equal if and only if
(i) $a \equiv n_i a_i + \tau_i \frac{q_i}{2} \pmod{q_i}$,
(ii) $b \equiv b_i + \tau_i \frac{q_i}{2} \pmod{q_i}$,
(iii) $c \equiv m_i c_i \pmod{q_i}$,
(iv) $y_0 \equiv m_i y_0^{(i)} \pmod{q_i}$,
with $m_i = \frac{m}{q_i}$, $n_i \equiv m_i^{-1} \pmod{q_i}$, $\tau_i = 0$ if q_i is odd, and $\tau_i \in \{0,1\}$ if q_i is even, for $i = 1, \dots, r$.

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2.25.6. The discrete exponential generator produces the sequence

$$x_n = \frac{y_n}{M}, \text{ where } y_{n+1} \equiv g^{y_n} \pmod M \text{ and } 0 \le y_n \le M-1.$$

Here M is an odd prime, g a primitive root mod M and y_0 is the initial seed.
Notes: Let $M = p$ and q be primes with $q|(p-1)$ and let $g \in \mathbb{F}_p^*$ be of multiplicative order q. If $j \ge 2$ is an integer then for each j–dimensional vector $\mathbf{a} = (a_1, \dots, a_j) \in (\mathbb{F}_q)^j$ define

$$y_{\mathbf{a}}(n) = g^{a_1^{i_1} \dots a_j^{i_j}} \in \mathbb{F}_p,$$

where $n = i_1 \dots i_j$ is the 2–adic (or bit) representation of the integer n, $0 \le n \le 2^j - 1$, with amended extra leading zeros if necessary. The **Naor – Reingold generator** produces the sequence

$$x_n = \frac{y_{\mathbf{a}}(n)}{M}, \qquad n = 0, 1, \dots, 2^j - 1,$$

see also H. Niederreiter and I.E. Shparlinski (2002).

M. Naor – O. Reingold: *Number–theoretic construction of efficient pseudorandom functions*, in: *Proc. 38th IEEE Symp. on Foundations of Computer Science*, IEEE Computer Society Press, Los Alamitos, Calif., 1997, pp. 458–467 (Full version at `http://www.wisdom.weizmann.ac.il/%7Enaor/PAPERS/gdh_abs.html`).
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2.25.7. The power generator produces the sequence

$$x_n = \frac{y_n}{M}, \text{ where } y_{n+1} \equiv (y_n)^e \pmod{M} \text{ and } 0 \le y_n \le M-1, \quad n = 0, 1, \dots,$$

where $M \ge 2$ is a given modulus, y_0 is the initial seed such that $\gcd(y_0, M) = 1$ and e is a given exponent.

NOTES: If $M = p_1 p_2$, where p_1 and p_2 are distinct primes, and $\gcd(e, \varphi(M)) = 1$, then it is called the **RSA generator** and in the special case $e = 2$ it is called **Blum – Blum – Shub generator**, see J.C. Lagarias (1990) and H. Niederreiter and I.E. Shparlinski (2002). These generators are u.d. (i.e. y_n is u.d. in $\mathbb{Z}_M$) when the period $> M^{3/4+\delta}$ with a fixed $\delta > 0$, see J.B. Friedlander and I.E. Shparlinski (2001).

L. Blum – M. Blum – M. Shub: *A simple unpredicable pseudo–random number generator*, SIAM J. Computing **15** (1986), no. 2, 364–383 (MR0837589 (87k:65007); Zbl. 0602.65002).
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J.C. Lagarias: *Pseudorandom number generators in cryptography and number theory*, in: *Cryptology and Computational Number Theory (Boulder, CO, 1989)*, (C. Pomerance ed.), Proc. Sympos. Appl. Math., 42, Amer. Math. Soc., Providence, RI, 1990, pp. 115–143 (MR1095554 (92f:11109); Zbl. 0747.94011).
H. Niederreiter – I.E. Shparlinski: *Recent advances in the theory of nonlinear pseudorandom number generators*, in: *Monte Carlo and Quasi–Monte Carlo Methods 2000 (Proceedings of a Conference held at Hong Kong Baptist University, Hong Kong SAR, China, Nov. 27–Dec. 1, 2000*, (Kai–Tai Fang, F.J. Hickernell, H. Niederreiter eds.), Springer Verlag, Berlin, Heidelberg 2002, pp. 86–102 (MR1958848 (2003k:65005); Zbl. 1076.65008).

2.25.8. The inverse congruential generator produces the sequence

$$x_n = \frac{y_n}{M}, \text{ where } y_{n+1} \equiv a y_n^{-1} + b \pmod{M} \text{ and } 0 \le y_n \le M-1,$$

where M is a given modulus, a, b are parameters, y_0 is the initial seed and y^{-1} is defined by $y.y^{-1} \equiv 1 \pmod{M}$.

(I) Let $M = 2^\omega$ for some integer $\omega \ge 6$ and $a + b \equiv 1 \pmod 2$. Inverse congruential sequences y_n are purely periodic with the maximum possible period length $M/2$ if and only if $a \equiv 1 \pmod 4$ and $b \equiv 2 \pmod 4$. Then the terms of the sequence of pseudorandom numbers $x_0, x_1, \dots, x_{(M/2)-1}$ run over all rationals in $[0, 1)$ of the form $\frac{2i+1}{M}$ and hence shows a perfect equidistribution in $[0, 1)$.

If $a \equiv 1 \pmod 4$ and $b \equiv 2 \pmod 4$ then for the sequence of the overlapping pairs (x_n, x_{n+1}), $n = 0, 1, \dots, (M/2) - 1$, we have

$$D_{M/2} < \frac{8\sqrt{2}+4}{7\pi^2} \cdot \frac{(\log M)^2}{\sqrt{M}} - 0.4191 \frac{\log M}{\sqrt{M}} + 0.6328 \frac{1}{\sqrt{M}} + 8\frac{1}{M}$$

The same upper estimate is also true for discrepancy $D_{M/4}$ of non–overlapping pairs (x_{2n}, x_{2n+1}) with $n = 0, 1, \dots, (M/4) - 1$.
(II) If $M = p$ with p an odd prime, then the recurring sequence y_n can also be given by the formula

$$y_{n+1} \equiv a y_n^{p-2} + b \pmod p .$$

The corresponding output sequence

$$x_n = \frac{y_n}{p} \bmod 1, \quad n = 0, 1, \dots, N,$$

has the discrepancy

$$D_N \le \left(\sqrt{\sqrt{\frac{8}{3}} + 2} \cdot \frac{p^{\frac{1}{4}}}{\sqrt{N}} + \sqrt{\frac{3}{8}} \cdot \frac{\sqrt{p}}{N} \right) \left(\frac{4}{\pi^2} \log p + \frac{2}{5} \right) + \frac{1}{p}$$

for N which is less than the period of the sequence x_n. If the order of magnitude of N is at least $\sqrt{p}\,(\log p)^2$ then

$$D_N = \mathcal{O}\left(\frac{p^{1/4} \log p}{\sqrt{N}} \right).$$

NOTES: The inverse congruential method was introduced by J. Eichenauer–Herrmann, J. Lehn, and A. Topuzoğlu (1988) based on the suggestion of D.E. Knuth. The discrepancy bounds of pairs of elements was given by J. Eichenauer–Herrmann, cf. Eichenauer–Herrmann, E. Herrmann and S. Wegenkittl (1998). This is an improvement to the previous result of Eichenauer–Herrmann and Niederreiter (1993). In the case $M = p$ the discrepancy estimate was proved by H. Niederreiter and I.E. Shparlinski (2001) thereby improving the general estimate given in their previous paper (1999). Some results for the case $y_{n+1} \equiv y_n^e \pmod M$, $M = pl$, where p, l are different primes can be found in J.B. Friedlander, C. Pomerance and I.E. Shparlinski (2001).

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J.B. Friedlander – C. Pomerance – I.E. Shparlinski: *Period of the power generator and small values of Carmichael's function*, Math. Comp. **70** (2001), no. 236, 1591–1605 (MR1836921 (2002g:11112); Zbl. 1029.11043).
H. Niederreiter – I.E. Shparlinski: *On the distribution and lattice structure of nonlinear congruential pseudorandom numbers*, Finite Fields Appl. **5** (1999), no. 3, 246–253 (MR1702905 (2000i:11126); Zbl. 0942.11037).
H. Niederreiter – I.E. Shparlinski: *On the distribution of inverse congruential pseudorandom numbers in parts of the period*, Math. Comp. **70** (2001), no. 236, 1569–1574 (MR1836919 (2002e:11104); Zbl. 0983.11048).

2.25.9. Compound inverse congruential generator. Assume that

- $p_1, \dots, p_k$ are distinct primes,
- $a_i, b_i \in \mathbb{F}_{p_i}$, $a_i \neq 0$, $i = 1, 2, \dots, k$,
- $\psi_i(y) = a_i y^{-1} + b_i$, $\psi(0) = 0$,
- $y_{0,1}, \dots, y_{0,k}$ are initial seeds,
- $y_{n,i}$, $i = 1, 2, \dots, k$, $n = 0, 1, 2, \dots$, are sequences defined by recurrence relations $y_{n+1,i} = \psi_i(y_{n,i})$.

The output sequence is

$$x_n = \sum_{i=1}^{k} \frac{y_{n,i}}{p^i} \bmod 1, \quad n = 0, 1, 2, \dots.$$

If t_i denotes the period of the sequence $y_{n,i}$, $n = 0, 1, 2, \dots$, and $t_1, \dots, t_k$ are pairwise coprime, then the sequence x_n, $n = 0, 1, 2, \dots$, has the period $T = t_1 \dots t_k$. When the maximum possible period $T = P = p_1 \dots p_k$ is achieved then the discrepancy of the sequence

$$\mathbf{x}_n = (x_{sn}, x_{sn+1}, \dots, x_{sn+s-1}), \quad n = 0, 1, 2, \dots,$$

satisfies

$$D_N = \mathcal{O}\left(\frac{(\log P)^s}{\sqrt{N}}\right)$$

for every $1 \leq N \leq P$ and $1 \leq s < \min(p_1 \dots p_k)$, where the $\mathcal{O}$–constant depends on k. If the period satisfies $T \leq P$ and $1 \leq N \leq T$ then we have

$$D_N = \mathcal{O}\left(\frac{\sqrt{T} P^{1/4} (\log P)^{s+1}}{N}\right),$$

with the $\mathcal{O}$–constant depending on k.

Notes: This generator was introduced by J. Eichenauer–Herrmann (1994). The first discrepancy bound was given by J. Eichenauer–Herrmann and F. Emmerich (1996).

The second one was proved by H. Niederreiter (2001) thereby generalizing the previous result of H. Niederreiter and A. Winterhof (2001).

J. Eichenauer–Herrmann: *On generalized inverse congruential pseudorandom numbers*, Math. Comp. **63** (1994), no. 207, 293–299 (MR1242056 (94k:11088); Zbl. 0868.11035).
J. Eichenauer–Herrmann – F. Emmerich: *Compound inverse congruential pseudorandom numbers: an average-case analysis*, Math. Comp. **65** (1996), 215–225 (MR1322889 (96i:65005); Zbl. 0852.11041).
H. Niederreiter: *Design and analysis of nonlinear pseudorandom numbers generators*, in: *Monte Carlo Simulation*, (G.I. Schuëller and P.D. Spans eds.), A.A. Balkema Publishers, Rotterdam, 2001, pp. 3–9.
H. Niederreiter – A. Winterhof: *On the distribution of compound inverse congruential pseudorandom numbers*, Monatsh. Math. **132** (2001), no. 1, 35–48 (MR1825718 (2002g:11113); Zbl. 0983.11047).

2.25.10. The explicit inverse congruential generator is based on the formula

$$x_n = \frac{y_n}{M}, \text{ where } y_n \equiv (an)^{-1} \bmod M \text{ and } 0 \le y_n \le M-1,$$

with M an odd prime and the multiplier a coprime to M. The period length of x_n is M, and the inverse of an can be computed using the relation $y_n = (an)^{M-2}$.

Notes: This procedure was proposed J. Eichenauer–Herrmann (1992).

J. Eichenauer–Herrmann: *Inverse congruential pseudorandom numbers: A tutorial*, Int. Stat. Rev. **60** (1992), no. 3, 167–176 (Zbl. 0766.65002).

2.25.10.1 Digital explicit inversive sequences.

• Let $q = p^k$ with a prime number p and an integer $k \ge 1$;

• Let $\mathbb{F}_q$ denote the finite field of order q;

• Let $\{\beta_1, \dots, \beta_k\}$ be an ordered basis of $\mathbb{F}_q$ as a vector space over its prime subfield $\mathbb{F}_p$;

• $\mathbb{Z}_p = \{0, 1, \dots, p-1\}$ is the least nonnegative residue system modulo p;

• Define the sequence $\xi_n \in \mathbb{F}_q$, $n = 0, 1, \dots$, by $\xi_n := \sum_{l=1}^{k} n_l \beta_l$ if $n \equiv \sum_{l=1}^{k} n_l p^{l-1} \pmod q$ with $n_l \in \mathbb{Z}_p$ for $1 \le l \le k$.

• For $\varrho \in \mathbb{F}_q$, put $\overline{\varrho} := \varrho^{-1} \in \mathbb{F}_q$ if $\varrho \ne 0$ and $\overline{\varrho} := 0 \in \mathbb{F}_q$ if $\varrho = 0$.

Given $\alpha \in \mathbb{F}_q^*$ and $\delta \in \mathbb{F}_q$,

• define $\gamma_n := \overline{\alpha\xi_n + \delta} \in \mathbb{F}_q$ for $n = 0, 1, \dots$, (Note that the $\gamma_0, \gamma_1, \dots$ is periodic with least period q.)

- identify $\mathbb{F}_p$ with $\mathbb{Z}_p$ and write
- $\gamma_n = \sum_{l=1}^{k} c_{n,l}\,\beta_l$ for $n = 0, 1, \ldots$, with all $c_{n,l} \in \mathbb{F}_p = \mathbb{Z}_p$.

Then a digital explicit inversive sequence is defined by

$$z_n = \sum_{l=1}^{k} c_{n,l}p^{-l} \in [0,1) \quad \text{for } n = 0, 1, \ldots .$$

Note that the sequence $z_0, z_1, \ldots$ is periodic with least period q.

NOTES:

(I) The notion of the digital explicit inversive sequences was introduced by H. Niederreiter and A. Winterhof (2000).

(II) In the special case $k = 1$ we obtain an explicit inversive congruential sequence as introduced in J. Eichenauer-Herrmann (1993) and further studied in Niederreiter (1994).

J. EICHENAUER-HERRMANN: *Statistical independence of a new class of inversive congruential pseudorandom numbers*, Math. Comp. **60** (1993), 375–384 (MR1159168 (93d:65011); Zbl. 0795.65002).

H. NIEDERREITER: *On a new class of pseudorandom numbers for simulation methods*, (In: Stochastic programming: stability, numerical methods and applications (Gosen, 1992)), J. Comput. Appl. Math. **56** (1994), 159–167 (MR1338642 (96e:11101); Zbl. 0823.65010).

H. NIEDERREITER: *A discrepancy bound for hybrid sequences involving digital explicit inversive pseudorandom numbers*, Unif. Distrib. Theory **5** (2010), no. 1, 53–63 (MR2804662 (2012f:11143); Zbl. 1249.11074).

H. NIEDERREITER – A. WINTERHOF: *Incomplete exponential sums over finite fields and their applications to new inverse pseudorandom number generators*, Acta Arith. **XCIII** (2000), no. 4, 387–399 (MR1759483 (2001d:11120); Zbl. 0969.11040).

2.25.11. Compound cubic congruential generator. Let M_1 and M_2 be two distinct primes, a_1 and a_2 two positive integers less than M_1 and M_2, resp., and $y_{0,1}$, $y_{0,2}$ are integral initial seeds. If

$$y_{n,1} = a_1 y_{n-1,1}^3 + 1 \pmod{M_1},$$
$$y_{n,2} = a_2 y_{n-1,2}^3 + 1 \pmod{M_2},$$

then the numbers are generated by

$$x_n = \frac{y_{n,1}}{M_1} + \frac{y_{n,2}}{M_2} \bmod 1.$$

The maximal period length of x_n is M_1M_2 for a suitable choice of values in (M_1, a_1, M_2, a_2).

NOTES: This method was proposed by J. Eichenauer–Herrmann and E. Herrmann (1997). They also give pairs (M_1, a_1) with maximal possible period M_1 of $y_{n,1}$.

P. L'Ecuyer and P. Hellekalek (1998) computed a table of values (M_1, a_1, M_2, a_2) for which x_n has the maximal period $M_1 M_2$, e.g. $M_1 = 131063$, $a_1 = 110230$, $M_2 = 130859$, and $a_2 = 48249$.

J. Eichenauer–Herrmann – E. Herrmann: *Compound cubic congruential pseudorandom numbers*, Computing **59** (1997), 85–90 (MR1465312 (98g:11089); Zbl. 0880.65001).
P. L'Ecuyer – P. Hellekalek: *Random number generators: Selection criteria and testing*, in: *Random and Quasi–Random Point Sets*, (P. Hellekalek, G. Larcher eds.), Lecture Notes in Statist., 138, Springer Verlag, New York, Berlin, 1998, pp. 223–265 (MR1662843 (99m:65014); Zbl. 0915.65004).

2.26 Binary sequences

Under binary sequences we understand sequences attaining only two values, usually $x_n = 0 \vee 1$, or $x_n = -1 \vee 1$.

Notes: Let x_n, $n = 1, 2, \dots, N$, be a sequence such that $x_n = -1$ or $x_n = 1$.

(I) Ch. Mauduit and A. Sárközy (1997) introduced the following measures of pseudorandomness:

- The **well–distribution measure** of x_n, $n = 1, 2, \dots, N$,

$$W_N(x_n) = \max_{a,b,m} \left| \sum_{j=1}^{m} x_{a+jb} \right|,$$

  where the maximum is taken over all a, b, m such that $a \in \mathbb{Z}$, $b, m \in \mathbb{N}$ and $1 \le a + b \le a + mb \le N$.

- The **correlation measure of order** k

$$C_N^{(k)}(x_n) = \max_{M, d_1, \dots, d_k} \left| \sum_{n=1}^{M} x_{n+d_1} \dots x_{n+d_k} \right|,$$

  where the maximum is taken over all $0 \le d_1 < \dots < d_k$ and M such that $M + d_k \le N$.

- The **combined pseudorandom measure of order** k

$$Q_N^{(k)}(x_n) = \max_{\substack{a,b,m\\ d_1,\dots,d_k}} \left| \sum_{j=0}^{m} x_{a+jb+d_1} \dots x_{a+jb+d_k} \right|,$$

  where the maximum is taken over all a, b, m and $0 \le d_1 < \dots < d_k$ such that all the indices $a + jb + d_i$ belong to $\{1, 2, \dots, N\}$.

- The **normality measure of order** k

$$N_N^{(k)}(x_n) = \max_{\mathbf{x} \in \{-1,1\}^k} \max_{0 < M \le N+1-k} \left| \#\{0 \le n < M \,;\, (x_{n+1}, \dots, x_{n+k}) = \mathbf{x}\} - \frac{M}{2^k} \right|.$$

- The **normality measure**

$$N_N(x_n) = \max_{k \le (\log N)/\log 2} N_N^{(k)}(x_n).$$

(II) National Institute of Standards and Technology (U.S.A.) recommends in the book by A. Rukhin, J. Soto, J. Nechvatal, *et al.* (2000, revised 2001) the following 16 statistical random number generation tests:

1. The frequency (monobit) test.
2. Frequency test within a block.
3. The runs test.
4. Test for the longest–run–of–ones in a block.
5. The binary matrix rank test.
6. The discrete Fourier transform (spectral) test.
7. The non–overlapping template matching test.
8. The overlapping template matching test.
9. Maurer's "Universal statistical" test.
10. The Lempel – Ziv compression test.
11. The linear complexity test.
12. The serial test.
13. The approximate entropy test.
14. The cumulative sums (cusums) test.
15. The random excursions test.
16. The random excursion variant test.

Compare the discrepancies W_N, $C_N^{(k)}$, $Q_N^{(k)}$ and $N_N^{(k)}$ with some of the above statistical tests, e.g. with:

Frequency (monobit) test (2000, pp. 14–16, Par. 2.1): Let x_n, $n = 1, 2, \dots, N$, be a binary sequence such that $x_n = -1 \vee 1$ (if $x_n = 0 \vee 1$ we convert it using the rule $x_n \mapsto 2x_n - 1$). If $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, \mathrm{d}t$ denotes the **complementary error function** and if

$$\mathrm{erfc}\left(\frac{|\sum_{n=1}^N x_n|}{\sqrt{N}}\right) < 0.01$$

then we conclude that the sequence x_n, $n = 1, 2, \dots, N$, is non–random.

Frequency test within a block (2000, pp. 16–18, Par. 2.2): Let x_n, $n = 1, 2, \dots, N$, be a binary sequence, where $x_n = 0 \vee 1$. Partition x_n into $K = \left[\frac{N}{M}\right]$ non–overlapping M–terms blocks (discarding the unused terms) and compute

- $\pi_i = \dfrac{\sum_{j=1}^M x_{(i-1)M+j}}{M}$, $i = 1, 2, \dots, K$,
- $\chi^2 = 4M \sum_{i=1}^K \left(\pi_i - \frac{1}{2}\right)^2$.

If $\mathrm{igamc}(u, x) = \frac{1}{\Gamma(u)} \int_x^\infty e^{-t} t^{u-1}\, \mathrm{d}t$ denotes the **incomplete gamma function**, where $\Gamma(u) = \int_0^\infty e^{-t} t^{u-1}\, \mathrm{d}t$, and

$$\mathrm{igamc}\left(\frac{K}{2}, \frac{\chi^2}{2}\right) < 0.01$$

then we conclude that the sequence x_n, $n = 1, 2, \ldots, N$, is non–random.

Binary matrix rank test (2000, pp. 24–27, Par. 2.5): Let x_n, $n = 1, 2, \ldots, N$, be a binary sequence, where $x_n = 0 \vee 1$. Divide the sequence x_n sequentially into disjoint blocks with $M.Q$–terms thus obtaining $K = \left[\frac{N}{M.Q}\right]$ blocks in total (after discarding the unused terms).

- Collect the $M.Q$–terms blocks into $M \times Q$ matrices A_k, $k = 1, 2, \ldots, K$ (each row of the matrix A_k is filled successively with a Q–terms block of the original sequence x_n).
- Determine the binary (i.e. over $\mathbb{F}_2$) rank(A_k) for $k = 1, 2, \ldots, K$. If
  $F_M = \#\{k \le K \,;\, \mathrm{rank}(A_k) = M\}$, and
  $F_{M-1} = \#\{k \le K \,;\, \mathrm{rank}(A_k) = M - 1\}$ then compute

$$\chi^2 = \frac{(F_M - 0.2888K)^2}{0.2888K} + \frac{(F_{M-1} - 0.5776K)^2}{0.5776K} + \\ + \frac{(N - F_M - F_{M-1} - 0.1336K)^2}{0.1336K}.$$

If

$$e^{-\chi^2/2} < 0.001$$

then the conclusion is that the sequence x_n, $n = 1, 2, \ldots$, is non–random.

Discrete Fourier transform (spectral) test (2000, pp. 27–28, Par. 2.6): Let x_n, $n = 1, 2, \ldots, N$, be a binary sequence, where $x_n = -1 \vee 1$. Apply the discrete Fourier transform on x_n to obtain

$$f_j = \sum_{n=1}^{N} x_n e^{2\pi i (n-1)\frac{j}{N}}, \quad j = 0, 1, \ldots, N - 1.$$

Compute

- $N_1 = \#\{0 \le j \le N/2 \,;\, |f_j| < \sqrt{3N}\}$,
- $d = \dfrac{N_1 - 0.95(N/2)}{\sqrt{N(0.95)(0.05)/2}}$.

If

$$\mathrm{erfc}\left(\frac{|d|}{\sqrt{2}}\right) < 0.01$$

then conclude that the sequence x_n, $n = 1, 2, \ldots$, is non–random.

(III) As in the case of circle sequences (see 3.11) the randomness of infinite binary $-1 \vee 1$–sequences x_n can also be viewed from the point of pseudorandomness in the sense of Bertrandias or Bass.

Ch. Mauduit – A. Sárközy: *On finite pseudorandom binary sequences, I. Measure of pseudorandomness, the Legendre symbol*, Acta Arith. **82** (1997), no. 4, 365–377 (MR1483689 (99g:11095); Zbl. 0886.11048).

A. Rukhin – J. Soto – J. Nechvatal – M. Smid – E. Barker – S. Leigh – M. Levenson – M. Vangel – D. Banks – A. Heckert – J. Dray – S. Vo: *A Statistical Test Suite for Random and Pseudorandom Number Generators for Cryptographic Applications*, NIST Special Publication 800-22, (2000 with revision dated May 15, 2001). (`http://csrc.nist.gov/rng/SP800-22b.pdf`).

2.26.1. Binary Champernowne sequence. In 2.18.7 we replace the decimal representation of the consecutive integers by the dyadic one which gives the sequence y_n

$$1,1,0,1,1,1,0,0,1,0,1,1,1,0,1,1,1,1,0,0,0,\dots$$

and then let

$$x_n = \begin{cases} 1, & \text{if } y_n = 1 \\ -1, & \text{if } y_n = 0 \end{cases}$$

i.e. the initial segment of x_n is

$$1,1,-1,1,1,1,-1,-1,1,-1,1,1,1,-1,1,1,1,1,-1,-1,-1,\dots.$$

Then for the discrepancies we have

$$\begin{aligned} W_N(x_n) &> \frac{1}{32}\frac{N}{\log N}, \quad && \text{if } N \ge 2, \\ C_N^{(2)}(x_n) &> \frac{1}{48}N, && \text{if } N \ge 17. \end{aligned}$$

Ch. Mauduit – A. Sárközy: *On finite pseudorandom binary sequences. II. The Champernowne, Rudin – Shapiro, and Thue – Morse sequences, a further construction*, J. Number Theory **73** (1998), no. 2, 256–276 (MR1657960 (99m:11084); Zbl. 0916.11047).

2.26.2. The Thue – Morse sequence. For a positive integer n, let $s(n)$ denote the sum of digits in the dyadic representation of n, and let

$$x_n = (-1)^{s(n)}, \quad n = 0,1,\dots.$$

The sequence can also be defined by the recurrence relations

$$x_0 = 1, \quad x_{2n} = x_n, \quad x_{2n+1} = -x_n \qquad \text{for all } n = 0,1,\dots.$$

For the discrepancies we have

$$\begin{aligned} W_N(x_n) &\le 2(1+\sqrt{3})N^{\log 3/\log 4}, \quad && \text{if } N \in \mathbb{N}, \\ C_N^{(2)}(x_n) &\ge \frac{1}{12}N, && \text{if } N \ge 5. \end{aligned}$$

NOTES: (I) This sequence was repeatedly independently discovered by many authors, e.g. by E. Prouhet (1851), A. Thue (1906), H.M. Morse (1921) and others. The $0 \vee 1$ Thue – Morse sequence x_n is defined by the recursion

$$x_0 = 0, \quad x_{2n} = x_n, \text{ and } x_{2n+1} = 1 - x_n,$$

and its initial segment is

$$0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, \dots.$$

It can be generated by so called Thue – Morse automaton (cf. M. Hörnquist (1999, Chap. 2, p. 22) and J.–P. Allouche (2000)). K. Mahler (1929) proved that the related dyadic number

$$\alpha = 0.110100110010110\dots.$$

is transcendental and M. Queffélec (1998) showed that also the continued fraction expansion

$$\alpha' = [0; a, b, b, a, b, a, a, b, b, a, a, b, a, b, b, a, \dots].$$

represents a transcendental number, where $a \neq b$ are two integers ≥ 2, and the sequence of a's and b's is obtained from the $0 \vee 1$ Thue–Morse sequence by replacing 0's by a's and 1's by b's.

(II) The discrepancy bounds for W_N and $C_N^{(2)}$ were proved by Ch. Mauduit and A. Sárközy (1998).

(III) D.J. Newman (1969) proved that

$$\sum_{n=0,3|n}^{N-1} (-1)^{s(n)} > c.N^{\log 3/\log 4}$$

for some constant $c > 0$ and all N. Consequently

$$\#\{0 \leq n < N; 3|n \text{ and } 2|s(n)\} - \frac{N}{6} > c.N^{\log 3/\log 4}.$$

J.–P. ALLOUCHE: *Algebraic and analytic randomness*, in: *Noise, oscillators and algebraic randomness. From noise communication system to number theory. Lectures of a school, Chapelle des Bois, France, April 5–10, 1999*, Lect. Notes Phys. 550, 345–356 Springer, Berlin, 2000, (MR1861985 (2002i:68099); Zbl. 1035.68089).

M. HÖRNQUIST: *Aperiodically Ordered Structures in One Dimension*, Department of Physics and Measurement Technology, Linköping University, Ph.D. thesis in theoretical physics, Linköping, Sweden, 1999 (www.ifm.liu.se/˜micho/phd).

K. MAHLER: *Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen*, Math. Ann. **101** (1929), 342–366; *Corrigendum*, Math. Ann. **103** (1930), 532 (MR1512537 (MR1512635); JFM 55.0115.01 (JFM 56.0185.02)).

CH. MAUDUIT – A. SÁRKÖZY: *On finite pseudorandom binary sequences. II. The Champernowne, Rudin – Shapiro, and Thue – Morse sequences, a further construction*, J. Number Theory **73** (1998), no. 2, 256–276 (MR1657960 (99m:11084); Zbl. 0916.11047).

M. MORSE: *Recurrent geodesics on a surface of negative curvature*, Trans. Amer. Math. Soc. **22** (1921), 84–100 (MR1501161; JFM 48.0786.06).

D.J. NEWMAN: *On the number of binary digits in a multiple of three*, Proc. Amer. Math. Soc. **21** (1969), 719–721 (MR0244149 (**39** #5466); Zbl. 0194.35004).

E. Prouhet: *Mémoire sur quelques relations entre les puissances des nombres*, C. R. Acad. Sci. Paris Sér. I **33** (1851), 225.
M. Queffélec: *Transcedance des fractions continues de Thue–Morse*, J. Number Theory **73** (1998), 201–211 (MR1658023 (99j:11081); Zbl. 0920.11045).
A. Thue: *On infinite character series (Über unendliche Zeichenreihen)*, (Swedish & Norwegian), Norske vid. Selsk. Skr. Mat. Nat. Kl. (1906), no. 7, 22 p. (JFM 39.0283.01, JFM 37.0066.17).

2.26.3. The Rudin – Shapiro sequence. Define the pairs of polynomials $P_{2^n}(t)$, $Q_{2^n}(t)$, $n = 0, 1, \ldots$, of degree $2^n - 1$ by the recurrence relations:

$$\begin{aligned}
&P_1(t) = Q_1(t) = 1,\\
&P_{2^{n+1}}(t) = P_{2^n}(t) + t^{2^n} Q_{2^n}(t),\\
&Q_{2^{n+1}}(t) = P_{2^n}(t) - t^{2^n} Q_{2^n}(t).
\end{aligned}$$

The Rudin – Shapiro sequence is the sequence x_n, $n = 0, 1, \ldots$, formed by the coefficients in the expression

$$P_{2^n}(t) = \sum_{j=0}^{2^n-1} x_j t^j.$$

The sequence can alternatively be defined also by the recurrence relations

$$x_0 = 1, \quad x_{2n} = x_n, \quad x_{2n+1} = (-1)^n x_n \quad \text{for all } n = 0, 1, \ldots.$$

For the discrepancies we have

$$\begin{aligned}
&W_N(x_n) \le 2(2 + \sqrt{2})N^{1/2}, && \text{if } N \in \mathbb{N},\\
&C_N^{(2)}(x_n) > \frac{1}{6}N, && \text{if } N \ge 4
\end{aligned}$$

and

$$\sup_{\alpha\in[0,1)} \left| \textstyle\sum_{n=1}^{N} x_n e^{-2\pi i n\alpha} \right| \le (2 + \sqrt{2})\sqrt{N}.$$

Notes: See Ch. Mauduit and A. Sárközy (1998) for the discrepancies W_N, $C_N^{(2)}$ and W. Rudin (1959) for the supremum.

Ch. Mauduit – A. Sárközy: *On finite pseudorandom binary sequences. II. The Champernowne, Rudin – Shapiro, and Thue – Morse sequences, a further construction*, J. Number Theory **73** (1998), no. 2, 256–276 (MR1657960 (99m:11084); Zbl. 0916.11047).
W. Rudin: *Some theorems on Fourier coefficients*, Proc. Amer. Math. Soc. **10** (1959), 855–859 (MR0116184 (**22** #6979); Zbl. 0091.05706).

2.26.4. Paperfolding sequence. It is defined by the recurrence relation

$$x_0 = 0, \quad x_{4n} = 0, \quad x_{4n+2} = 1 \quad \text{and } x_{2n+1} = x_n.$$

The sequence x_n is not quasiperiodic, but Besicovitch almost periodic (see the def. 2.4.2) and its spectral measure (for the def. see 3.11) is discrete.

NOTES: M. Mendès France and A.J. van der Poorten (1981). The sequence x_n can also be generated by folding a sheet of paper in the left (0) and the right (1) direction. Its initial segment is

$$0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0, \dots.$$

J.H. Loxton and A.J. van der Poorten (1977) proved that $\sum_{n=0}^{\infty} x_n \alpha^n$ is transcendental, if α $(0 < |\alpha| < 1)$ is algebraic. A description of the paperfolding automaton can be found in M. Hörnquist (1999, Chap. 2, p. 27).

M. HÖRNQUIST: *Aperiodically Ordered Structures in One Dimension*, Department of Physics and Measurement Technology, Linköping University, Ph.D. thesis in theoretical physics, Linköping, Sveden, 1999 (www.ifm.liu.se/˜micho/phd).

J.H. LOXTON – A.J. VAN DER POORTEN: *A class of hypertranscendental functions*, Aequationes Math. **16** (1977), no. 1–2, 93–106 (MR0476659 (**57** #16218); Zbl. 0384.10014).

M. MENDÉS FRANCE – A.J. VAN DER POORTEN: *Arithmetic and analytic properties of paper folding sequences*, Bull. Austral. Math. Soc. **24** (1981), no. 1, 123–131 (MR0630789 (83b:10040); Zbl. 0451.10018).

2.26.5. Period–doubling sequence. It is generated by the recurrence

$$x_0 = 0, \quad x_{2n} = 0, \quad \text{and } x_{2n+1} = 1 - x_n.$$

NOTES: See M. Hörnquist (1999, Chap. 2, p. 24). The sequence starts with the segment

$$0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, \dots.$$

M. HÖRNQUIST: *Aperiodically Ordered Structures in One Dimension*, Department of Physics and Measurement Technology, Linköping University, Ph.D. thesis in theoretical physics, Linköping, Sveden, 1999 (www.ifm.liu.se/˜micho/phd).

2.26.6. Let p be a prime number, and let $g(x)$ be a permutation polynomial of $\mathbb{F}_p[x]$ (i.e. the associated polynomial function $g : c \to g(c)$ from $\mathbb{F}_p$ into $\mathbb{F}_p$ is a permutation of $\mathbb{F}_p$) of degree m such that the multiplicity of the (only) zero of $g(x)$ is odd. Define the sequence x_n, $n = 1, 2, \dots, p$, by

$$x_n = \begin{cases} \left(\frac{g(n)}{p}\right), & \text{if } g(n) \not\equiv 0 \pmod{p}, \\ 1, & \text{if } g(n) \equiv 0 \pmod{p}, \end{cases}$$

where $\left(\frac{n}{p}\right)$ is the Legendre symbol. Then for $k \in \mathbb{N}$, $k < p$, we have

$$Q_p^{(k)}(x_n) < 11km\sqrt{p}\log p.$$

NOTES: (I) Ch. Mauduit and A. Sárközy (1998, Th. 5). They illustrate the result by following classes of permutation polynomials:

- linear polynomials $ax + b \in \mathbb{F}_p[x]$.
- monomials x^k with $\gcd(k, p-1) = 1$.
- The Dickson polynomials $D_n(x, a)$ of the first kind of degree n, which are defined by

$$D_n(x, a) = \sum_{i=0}^{[n/2]} \frac{n}{n-i}\binom{n-i}{i}(-a)^i x^{n-2i}.$$

(II) If $g(x) = x$ then some discrepancy estimates can be found in (1997).

CH. MAUDUIT – A. SÁRKÖZY: *On finite pseudorandom binary sequences, I. Measure of pseudorandomness, the Legendre symbol*, Acta Arith. **82** (1997), no. 4, 365–377 (MR1483689 (99g:11095); Zbl. 0886.11048).
CH. MAUDUIT – A. SÁRKÖZY: *On finite pseudorandom binary sequences. II. The Champernowne, Rudin – Shapiro, and Thue – Morse sequences, a further construction*, J. Number Theory **73** (1998), no. 2, 256–276 (MR1657960 (99m:11084); Zbl. 0916.11047).

2.26.7. Let α be an irrational number, and k a positive integer. Define the sequence

$$x_n = \begin{cases} +1, & \text{if } \{n^k\alpha\} \in [0, 1/2), \\ -1, & \text{if } \{n^k\alpha\} \in [1/2, 1). \end{cases}$$

Assume that $k, l \in \mathbb{N}$, $k \geq 3$, $k \geq 2l + 1$, and that the partial quotients in the continued fraction expansion of $\alpha = [a_0; a_1, a_2, \dots]$ are bounded, say, $a_i \leq K \in \mathbb{N}$ for $i \geq 1$.

Define $\sigma^*(k)$ as follows: $\sigma^*(3) = 9$, $\sigma^*(4) = 20$, $\sigma^*(5) = 51$, $\sigma^*(6) = 116$, $\sigma^*(7) = 247$, $\sigma^*(8) = 422$, $\sigma^*(9) = 681$, $\sigma^*(10) = 1090$, $\sigma^*(11) = 1781$, and $\sigma^*(k) = 2k^2(2\log k + \log\log k + 3)$ for $k \geq 12$. Then for all $\varepsilon > 0$ there exists a number $N_0 = N_0(K, k, \varepsilon)$ such that if $N > N_0$, then

$$W_N(x_n) < N^{1-1/\sigma^*(k)+\varepsilon},$$
$$C_N^{(l)}(x_n) < N^{1-1/\sigma^*(k)+\varepsilon}$$

NOTES: (I) Ch. Mauduit and A. Sárközy ([a]2000). In the proof they used Erdős – Turán inequality 1.9.0.8. For the more general case in which the sequence $n^k\alpha$ is replaced by $n_k\alpha$, where the sequence n_k of positive integers increases, they proved the bound

$$W_N(x_n) \le \max_{a,b,m} mD_m(n_{a+bj}\alpha),$$

where D_m is the classical extremal discrepancy of m points $n_{a+b}\alpha, n_{a+b2}\alpha, \dots, n_{a+bm}\alpha$, all taken mod 1. Cf. W. Philipp and R.F. Tichy (2002) and H. Albrecher (2002).

(II) Mauduit and Sárközy (2000) generalized the results for sequence $n^2\alpha$ to $n^k\alpha$, they proved that

- $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N x_n x_{n+d} = 0$ for all fixed $d \in \mathbb{N}$,
- $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N x_n x_{n+d_1} x_{n+d_2} \dots x_{n+d_{2l}} = 0$ for all $l \in \mathbb{N}$ and positive integers $d_1 < d_2 < \dots < d_{2l}$.

(III) Erdős suggested to study the sequence

$$y_n = \begin{cases} +1, & \text{if } \{n^k\alpha\} < \{(n+1)^k\alpha\}, \\ -1, & \text{if } \{n^k\alpha\} > \{(n+1)^k\alpha\}. \end{cases}$$

Mauduit and Sárközy ([a]2000) found for the correlation

- $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N y_n y_{n+1} = -\frac{1}{3}$

and they suggest the study of the sequences

$$z_n = \begin{cases} +1, & \text{if } \{n\alpha\} < \{n^k\alpha\}, \\ -1, & \text{if } \{n\alpha\} > \{n^k\alpha\} \end{cases}$$

for $k \ge 2$, and

$$u_n = \begin{cases} +1, & \text{if } \{n^c\} \in [0, 1/2), \\ -1, & \text{if } \{n^c\} \in [1/2, 1) \end{cases}$$

for $c > 0$, $c \notin \mathbb{N}$.

H. Albrecher: *Metric distribution results for sequences* $(\{q_n\vec{\alpha}\})$, Math. Slovaca **52** (2002), no. 2, 195–206 (MR 2003h:11083; Zbl. 1005.11036).
Ch. Mauduit – A. Sárközy: *On finite pseudorandom binary sequences. V: On* $n\alpha$ *and* $(n^2\alpha)$ *sequences*, Monatsh. Math. **129** (2000), no. 3, 197–216 (MR1746759 (2002c:11088); Zbl. 0973.11076)).
[a] Ch. Mauduit – A. Sárközy: *On finite pseudorandom binary sequences. VI: On* $(n^k\alpha)$ *sequences*, Monatsh. Math. **130** (2000), no. 4, 281–298 (MR1785423 (2002c:11089); Zbl. 1011.11054).
W. Philipp – R. Tichy: *Metric theorems for distribution measures of pseudorandom sequences*, Monatsh. Math. **135** (2002), no. 4, 321–326 (MR1914808 (2003e:11083); Zbl. 1033.11039).

2.26.8. Open problem. Let $\theta = [0; a_1, a_2, \dots]$ be an irrational number in $[0, 1]$ given by its continued fraction expansion and let $p_n(\theta)/q_n(\theta)$, $n = 0, 1, 2, \dots$, be the corresponding sequence of its convergents. In the sequence

$$x_n = q_n(\theta) \pmod 2$$

find the frequency of each possible block $(\dots,0,\dots,1,\dots,0,\dots)$ of length s which occurs in x_n as $(x_{n+1},\dots,x_{n+s})$ for a special class of θ (e.g. with bounded a_i).

NOTES: R. Moeckel (1982) proved that, for almost all θ, the three possible blocks $(0,1)$, $(1,0)$ and $(1,1)$ of length $s=2$ occur in x_n with equal frequencies. The blocks of lengths $s=3$ and $s=4$ are investigated in V.N. Nolte (1990).

R. MOECKEL: *Geodesic on modular surfaces and continued fractions*, Ergodic Theory Dynamical Systems **2** (1982), no. 1, 69–83 (MR0684245 (84k:58176); Zbl. 0497.10007).
V.N. NOLTE: *Some probabilistic results on the convergents of continued fractions*, Indag. Math. (N.S.) **1** (1990), no. 3, 381–389 (MR1075886 (92b:11053); Zbl. 0713.11038).

3. Multi–dimensional sequences

3.1 Criteria and basic properties

3.1.1. The s–dimensional sequence

$$\mathbf{x}_n = (x_{n,1}, \dots, x_{n,s}) \bmod 1$$

is

u.d.

if and only if it satisfies any of the following conditions:

- $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{\mathbf{x}_n\}) = \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x}$ holds for all continuous functions $f : [0,1]^s \to \mathbb{R}$,
- $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \mathbf{h}\cdot\mathbf{x}_n} = 0$ holds for all $\mathbf{h} \in \mathbb{Z}^s$, $\mathbf{h} \neq \mathbf{0}$,
- the one–dimensional sequence $h_1x_{n,1} + \cdots + h_sx_{n,s} \bmod 1$, $n = 1, 2, \dots$, is u.d. for every integer vector $(h_1, \dots, h_s) \neq (0, \dots, 0)$,
- $\lim_{N\to\infty} D_N(\mathbf{x}_n) = 0$,
- $\lim_{N\to\infty} D_N^*(\mathbf{x}_n) = 0$,
- $\lim_{N\to\infty} D_N^{(2)}(\mathbf{x}_n) = 0$.

H. Weyl: *Über die Gleichverteilung von Zahlen mod. Eins*, Math. Ann. **77** (1916), 313–352 (JFM 46.0278.06).

3.1.2. If x_n is an infinite sequence in $[0,1)$ and s a positive integer, let

$$\mathbf{x}_n = (x_{n+1}, \dots, x_{n+s}).$$

If there is a constant c such that

$$\limsup_{N\to\infty} \frac{A(I_s; N; \mathbf{x}_n)}{N} \leq c|I_s|$$

for all $s \geq 1$ and every subinterval $I_s \subset [0,1]^s$, then the sequence x_n is

completely u.d.

A.G. Postnikov: *A test for a completely uniformly distributed sequence*, (Russian), Dokl. Akad. Nauk. SSSR **120** (1958), 973–975 (MR0101858 (**21** #665); Zbl. 0090.35504).

3.1.3. Let $f(x,y)$ be a twice continuously differentiable function defined on $[0,1]^2$. If (x_n,y_n) mod 1, $n=1,2,\dots$, is u.d. in $[0,1]^2$, then

$$\lim_{N\to\infty}\sum_{n=1}^{N}\left(f\left(\left\{x_n+\frac{1}{N}\right\},\left\{y_n+\frac{1}{N}\right\}\right)-f\left(\left\{x_n+\frac{1}{N}\right\},\{y_n\}\right)-\right.$$
$$\left.-f\left(\{x_n\},\left\{y_n+\frac{1}{N}\right\}\right)+f(\{x_n\},\{y_n\})\right)=$$
$$=f(1,1)-f(1,0)-f(0,1)+f(0,0).$$

Notes: R.F.Tichy (1982). If $f(x,y)$ is a three times continuously differentiable function, then

$$\left|\sum_{n=1}^{N}\left(f\left(\left\{x_n+\frac{1}{N}\right\},\left\{y_n+\frac{1}{N}\right\}\right)-\right.\right.$$
$$\left.-f\left(\left\{x_n+\frac{1}{N}\right\},\{y_n\}\right)-f\left(\{x_n\},\left\{y_n+\frac{1}{N}\right\}\right)+f(\{x_n\},\{y_n\})\right)-$$
$$\left.-\big(f(1,1)-f(1,0)-f(0,1)+f(0,0)\big)\right|\le C(f)D_N.$$

Here D_N denotes the discrepancy of (x_n,y_n) mod 1 and $C(f)$ is a constant which depends only on f, and which can be explicitly given using the Koksma – Hlawka inequality. The results also remain true for weighted means.

Related sequences: 2.2.20

R.F. Tichy: *Einige Beiträge zur Gleichverteilung modulo Eins*, Anz. Österreich. Akad. Wiss. Math.–Natur. Kl. **119** (1982), no. 1, 9–13 (MR0688688 (84e:10061); Zbl. 0495.10030).

3.2 General operations with sequences

3.2.1. If the sequence $(x_{n,1}/2,\dots,x_{n,s}/2)$ mod 1 is u.d., then the sequence

$$\left((-1)^{[x_{n,1}]}x_{n,1},\dots,(-1)^{[x_{n,s}]}x_{n,s}\right)\ \text{mod}\ 1$$

is

u.d.

P.J. Grabner – O. Strauch – R.F. Tichy: *Maldistribution in higher dimension*, Math. Panon. **8** (1997), no. 2, 215–223 (MR1476099 (99a:11094); Zbl. 0923.11110).

$\boxed{\textbf{3.2.1.1}}$ Let $(m_1,\dots,m_s)\in\mathbb{N}^s$ and $x_{n,i}\in\mathbb{R}$ for $n\ge 0$ and $1\le i\le s$. If the sequence

$$\big(\{x_{n,1}/m_1\},\dots,\{x_{n,s}/m_s\}\big),\quad n=0,1,2,\dots,$$

is u.d. in $[0,1)^s$, then the sequence

$$\big([x_{n,1}],\dots,[x_{n,s}]\big),\quad n=0,1,2,\dots,$$

is u.d. modulo $(m_1,\dots,m_s)$.
Notes:

H. Niederreiter: *On a class of sequences of lattice points*, J. Number Theory **4** (1972), 477–502 (MR0306144 (**46** #5271); Zbl. 0244.10036).

3.2.2. Let M_i, $i=1,2,\dots$, be a sequence of positive integers which satisfies $\lim\limits_{k\to\infty}\sum_{i=1}^{k-1}M_i/M_k=0$. For a sequence $\mathbf{y}_k$, $k=1,2,\dots$, in $[0,1)^s$, let $\mathbf{H}\subset[0,1]^s\times[0,1]^s$ denote the set of all limit points of the sequence $(\mathbf{y}_{k-1},\mathbf{y}_k)$, $k=2,3,\dots$. If the sequence $\mathbf{x}_n$, $n=1,2,\dots$, from $[0,1)^s$ is given by the rule

$$\mathbf{x}_n=\mathbf{y}_k\quad\text{for }\sum_{i=1}^{k-1}M_i\le n<\sum_{i=1}^{k}M_i,$$

then

$$G(\mathbf{x}_n)=\{tc_{\boldsymbol{\alpha}}(\mathbf{x})+(1-t)c_{\boldsymbol{\beta}}(\mathbf{x})\,;\,t\in[0,1],(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathbf{H}\},$$

where the d.f. $c_{\boldsymbol{\alpha}}:[0,1]^s\to[0,1]$ is defined by

$$c_{\boldsymbol{\alpha}}(\mathbf{x})=\begin{cases}1, & \text{if }\mathbf{x}\in[\boldsymbol{\alpha},\mathbf{1}],\\ 0, & \text{otherwise.}\end{cases}$$

Notes: P.J. Grabner, O. Strauch and R.F. Tichy (1997). Consequently: Suppose that for a given a set $\mathbf{H}\subset[0,1]^s$ there exists a sequence $\mathbf{y}_k$, $k=1,2,\dots$, in $[0,1]^s$ such that
(i) $\mathbf{H}$ coincides with the set of all limit points of $\mathbf{y}_k$,
(ii) $\lim\limits_{k\to\infty}(\mathbf{y}_k-\mathbf{y}_{k-1})=\mathbf{0}$.

Then there exists a sequence $\mathbf{x}_n$, $n = 1, 2, \dots$, in $[0, 1)^s$ for which

$$G(\mathbf{x}_n) = \{c_{\boldsymbol{\alpha}}(\mathbf{x})\ ;\ \boldsymbol{\alpha} \in \mathbf{H}\}.$$

RELATED SEQUENCES: 2.12.4

P.J. GRABNER – O. STRAUCH – R.F. TICHY: *Maldistribution in higher dimension*, Math. Panon. **8** (1997), no. 2, 215–223 (MR1476099 (99a:11094); Zbl. 0923.11110).

3.2.3. Let $D_N(\mathbf{x}_n)$ be the extremal discrepancy of the s–dimensional sequence $\mathbf{x}_n = (x_{n,1}, \dots, x_{n,s})$ in $[0, 1)^s$ and $D_N(x_{n,i})$ be the extremal discrepancy of its ith coordinate sequence $x_{n,i}$. Then

$$D_N(\mathbf{x}_n) \geq D_N(x_{n,i}) \quad \text{for } i = 1, 2, \dots, s.$$

NOTES: A similar result holds for the star discrepancy D_N^*, cf. [KN, p. 100, Ex. 1.14].

3.2.4. Let $f(n)$ mod 1 be completely u.d., $q_1, \dots, q_s$ be positive integers, and $\alpha_1, \dots, \alpha_s$ be defined by (with $[\cdot]$ and $\{\cdot\}$ denoting the integral and fractional parts, resp.)

$$\alpha_\nu = \sum_{k=1}^{\infty} \frac{[\{f(sk+\nu)\}q_\nu]}{q_\nu^k}, \quad \nu = 1, \dots, s.$$

Then the sequence

$$\mathbf{x}_n = (\alpha_1 q_1^n, \dots, \alpha_s q_s^n) \bmod 1$$

is

u.d.

N.M. KOROBOV: *On completely uniform distribution and conjunctly normal numbers*, (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. **20** (1956), 649–660 (MR0083522 (18,720d); Zbl. 0072.03801).

3.2.5. If $\mathbf{x}_n = (x_{n1}, \dots, x_{ns}) \in (0, 1]^s$ is u.d. with the discrepancy $D_N(\mathbf{x}_n)$, then the sequence

$$\mathbf{y}_n = \left(\frac{1}{x_{n1}}, \dots, \frac{1}{x_{ns}}\right) \bmod 1$$

has the a.d.f.

$$g(\mathbf{x}) = \prod_{i=1}^{s} \sum_{n=1}^{\infty} \tfrac{x_i}{n(n+x_i)} \text{ for } \mathbf{x} = (x_1, \dots, x_s) \in [0, 1]^s,$$

and

$$D_N(\mathbf{y}_n) \leq 2.12^s (D_N(\mathbf{x}_n))^{\frac{1}{s+1}}.$$

E. Hlawka: *Cremonatransformation von Folgen modulo 1*, Monatsh. Math. **65** (1961), 227–232 (MR0130242 (**24** #A108); Zbl. 0103.27701).

3.2.6. Let (u_n, v_n, a_n, b_n), $n = 1, 2, \dots$, be a sequence in the interval $[0, \delta] \times [0, 1] \times [0, \delta] \times [0, 1]$ which has the limit distribution with density $\rho(u, v, a, b) = \rho_1(u)\rho_2(v)\rho_3(a)\rho_4(b)$ and the extremal discrepancy $D_N((u_n, v_n, a_n, b_n))$ with respect to ρ. If $\Phi(\mu, t) = \frac{1-e^{-\mu t}}{\mu}$ for $t \in [0, \infty)$ and $\mu > 0$ is a constant then the discrepancy of the two–dimensional sequence

$$\left(\frac{u_n}{\delta}\Phi(\mu, t) + \frac{a_n}{\delta}, v_n\Phi(\mu, t) + b_n\right) \bmod 1$$

satisfies

$$D_N \le c\left(\left(\frac{D_N\big((u_n, v_n, a_n, b_n)\big)}{\mu^2}\right)^{\frac{1}{5}} + \delta^2\mu^2\right).$$

Notes: Note that $x = u_n\Phi(\mu, t) + a_n$ and $\omega = v_n\Phi(\mu, t) + b_n$ solve the system of differential equations $\dot{x} = p$, $\dot{\phi} = \omega$, $\dot{p} = -\mu p$, and $\dot{\omega} = -\mu\omega$ in the variable t with the initial condition $p(0) = u_n$ and $\omega(0) = v_n$, where $\mu > 0$ represents the friction.

E. Hlawka: *Gleichverteilung und die willkürlichen Funktionen von Poincaré*, Math. Slovaca **48** (1998), no. 5, 457–506 (MR1697611 (2000j:11120); Zbl 0956.11016).

3.2.7. Let $h(\mathbf{x})$ denote a density on the s–dimensional unit cube $[0, 1]^s$ with corresponding distribution function $g(\mathbf{x}) = \int_{[\mathbf{0},\mathbf{x}]} h(\mathbf{t})\,\mathrm{d}\mathbf{t}$. Suppose that $h(\mathbf{x})$ factors in the form $h(\mathbf{x}) = h_1(x_1)\dots h_s(x_s)$, where $\mathbf{x} = (x_1, \dots, x_s)$ and let g_j denote the d.f. corresponding to h_j for $j = 1, \dots, s$, i.e. $g(\mathbf{x}) = g_1(x_1)\dots g_s(x_s)$. Furthermore, let $\mathbf{x}_n = (x_{n,1}, \dots, x_{n,s})$, $n = 1, \dots, N$, be a sequence in $[0, 1)^s$ and let $\mathbf{y}_n = (y_{n,1}, \dots, x_{n,s})$, $n = 1, \dots, N$, be defined by

$$y_{n,j} = \frac{1}{N}\sum_{i=1}^{N}\big(1 + x_{n,j} - g_j(x_{i,j})\big).$$

Then the discrepancy $D_N(\mathbf{y}_n, g)$ of $\mathbf{y}_n$ with respect to g can be estimated in terms of the usual extremal discrepancy $D_N(\mathbf{x}_n)$ as follows

$$D_N(\mathbf{y}_n, g) \le \left(2 + 6s \sup_{\mathbf{x}\in[0,1]^s} h(\mathbf{x})\right) D_N(\mathbf{x}_n).$$

NOTES: E. Hlawka (1997) applied this estimate to densities $h_t(\mathbf{x})$ which arose from diffusion equations, the Schrödinger equation, the Klein – Gordon equation from optics and thermodynamics, etc. For the one–dimensional case cf. 2.3.10.

E. HLAWKA: *Gleichverteilung und Simulation*, Österreich. Akad. Wiss. Math.–Natur. Kl. Sitzungsber. II **206** (1997 (1998)), 183–216 (MR1632927 (99h:11084); Zbl. 0908.11031).

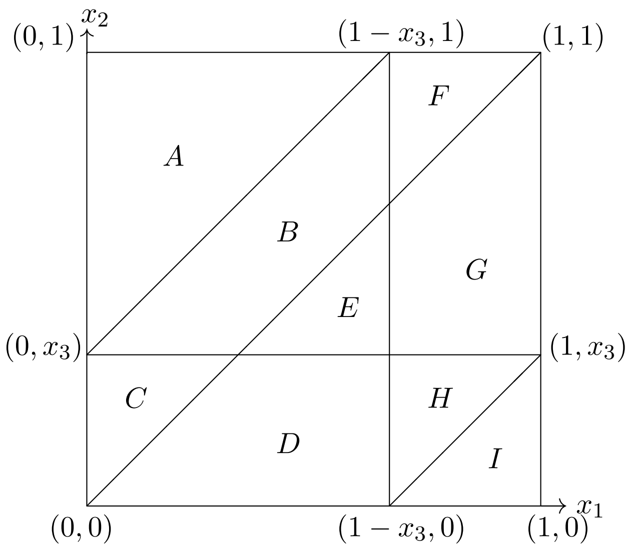
3.2.8. Let u_n and v_n be two u.d. and statistically independent sequences in $[0, 1)$. Then the sequence

$$\mathbf{x}_n = (u_n, v_n, \{u_n - v_n\}), \quad n = 1, 2, \dots,$$

has

the a.d.f. $g(\mathbf{x})$

which can be described as follows:
Divide the unit square $[0, 1]^2$ into regions $A, B, C, D, E, F, G, H, I$ as shown on the following Figure



Then

$$g(x_1,x_2,x_3)=\begin{cases} x_1x_3, & \text{if } (x_1,x_2)\in A,\\ -\frac12(x_1^2+x_2^2+x_3^2)+x_1x_2+x_2x_3, & \text{if } (x_1,x_2)\in B,\\ -\frac12 x_1^2+x_1x_2, & \text{if } (x_1,x_2)\in C,\\ \frac12 x_2^2, & \text{if } (x_1,x_2)\in D,\\ -\frac12 x_3^2+x_2x_3, & \text{if } (x_1,x_2)\in E,\\ -\frac12 x_2^2+x_1x_2+x_1x_3+x_2x_3-x_1-x_3+\frac12, & \text{if } (x_1,x_2)\in F,\\ \frac12 x_1^2+x_1x_3+x_2x_3-x_1-x_3+\frac12, & \text{if } (x_1,x_2)\in G,\\ \frac12(x_1^2+x_2^2+x_3^2)+x_1x_3-x_1-x_3+\frac12, & \text{if } (x_1,x_2)\in H,\\ x_1x_2+x_2x_3-x_2 & \text{if } (x_1,x_2)\in I. \end{cases}$$

NOTES: O. Strauch (2003). The Weyl criterion implies that the two–dimensional sequence

$$(u_n,\{u_n-v_n\})$$

is u.d., thus the face d.f.'s are

$$g(1,x_2,x_3)=x_2x_3,\quad g(x_1,1,x_3)=x_1x_3,\quad g(x_1,x_2,1)=x_1x_2.$$

Another d.f. having these three properties (distinct from the u.d.) is $g(x_1,x_2,x_3)=\min(x_1x_2,x_1x_3,x_2x_3)$.

O. Strauch: *Reconstruction of distribution function by its marginals*, Math. Institute, Slovak Acad. Sci., Bratislava, Slovak Republic, 2003, 10 pp.

3.3 General sequences (Sequences involving continuous functions)

3.3.1. Let $\lambda>3$, $\beta_1>1$ and $0<\beta_2<1$. Suppose that

$$\omega(\nu)\ge\nu^\lambda \quad\text{and}\quad \left(1+\frac{\beta_1}{\nu}\right)\omega(\nu)\le\omega(\nu+1)\le\beta_2\nu\omega(\nu)$$

for every sufficiently large ν. If $f(x)=\sum_{\nu=0}^\infty a_\nu x^\nu$ with $|a_\nu|=e^{-\omega(\nu)}$, then the sequence

$$f(n) \bmod 1$$

is

completely u.d.

NOTES: N.M. Korobov (1950). In (1948) he gave the first example of a completely u.d. sequence of the type $f(n)$ mod 1 where

$$f(x) = \sum_{\nu=0}^{\infty} e^{-e^{\nu}} x^{\nu}.$$

E.D. Knuth (1965) gave the following different construction of a completely u.d. sequence x_n:

- let A_k be the block consisting of 2^{k^2} real numbers $y_{k,i} = \frac{m}{2^k} \mod 1$ for $1 \le m \le 2^k$, where
- $y_{k,1} = \cdots = y_{k,k} = 0$, and for $i > k$
- m is the least integer such that the k–tuple $(y_{k,i-k+1}, \dots, y_{k,i-1}, y_{k,i})$ has not previously occurred in A_k,
- S_k is a $k.2^k$–fold repetition of the segment A_k.

Then the desired x_n is the sequence of blocks S_k, i.e.

$$(x_n)_{n=1}^{\infty} = (S_k)_{k=1}^{\infty}.$$

Knuth proved the completely u.d. of x_n using a result of L.R. Ford, Jr. (1957). M. Vojvoda and M. Šimovcová (2001) replaced A_k in Knuth's construction by a linear recurring sequences associated with a primitive characteristic polynomial.

L.R. FORD, JR.: *A cyclic arrangement of n–tuples*, Rand Corporation, Report P–1070, Santa Monica, Calif., 1957.
E.D. KNUTH: *Construction of a random sequence*, Nordisk. Tidskr. Informations–Behandling **5** (1965), 246–250 (MR0197434 (**33** #5599); Zbl. 0134.35701).
N.M. KOROBOV: *On functions with uniformly distributed fractional parts*, (Russian), Dokl. Akad. Nauk SSSR **62** (1948), 21–22 (MR0027012 (10,235e); Zbl. 0031.11501).
N.M. KOROBOV: *Concerning some questions of uniform distribution*, (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. **14** (1950), 215–238 (MR0037876 (12,321a); Zbl. 0036.31104).
M. VOJVODA – M. ŠIMOVCOVÁ: *On concatenating pseudorandom sequences*, J. Electrical Engineering **52** (2001), no. 10/s, 36–37 (Zbl. 1047.94012).

3.3.2. Let $f(x)$ be an $s+1$ times differentiable function defined for $x \ge 0$ such that
(i) $f^{(s)}(x) \to \infty$ as $x \to \infty$,
(ii) $f^{(s+1)}(x) > 0$ for $x \ge x_0$,
(iii) $f^{(s+1)}(x) \to 0$ as $x \to \infty$.

Then the sequence

$$\left(f(n), f'(n), \dots, f^{(s)}(n)\right) \mod 1$$

is

dense in $[0,1]^s$.

NOTES: John Daily in his Ph.D. dissertation (cf. F.S.Cater, R.B. Crittenden and Ch. Vanden Eynden (1976)). This generalizes the one–dimensional case 2.6.25 of P. Csillag (1929). The two–dimensional case is studied by F.S.Cater, R.B. Crittenden and Ch. Vanden Eynden (1976), cf. 2.6.23.

F.S. CATER – R.B. CRITTENDEN – CH. VANDEN EYNDEN: *The distribution of sequences modulo one*, Acta Arith. **28** (1976), 429–432 (MR0392903 (**52** #13716); Zbl. 0319.10042).
P. CSILLAG: *Über die Verteilung iterierter Summen von positiven Nullfolgen* mod 1, Acta Litt. Sci. Szeged **4** (1929), 151–154 (JFM 55.0129.01).

$\boxed{\textbf{3.3.2.1}}$ **Multidimensional Fejér's theorem.** Let k be a fixed positive integer and $x_n, n = 1, 2, \dots,$ be a sequence of real numbers satisfying (as $n \to \infty$)
(i) $\Delta^k x_n \searrow 0$,
(ii) $\Delta^{k-1} x_n \to \infty$.
Then the condition
(iii) $n\Delta^k x_n \to \infty$
is a necessary and sufficient condition for the k–dimensional sequence

$$(x_n, x_{n+1}, \dots, x_{n+k-1}), \quad n = 1, 2, \dots,$$

to be

u.d.

NOTES: Kemperman (1973, p. 144, Th. 5). For one-dimensional Fejér's theorem see 2.2.10, 2.2.11, 2.6.1.

J.H.B. KEMPERMAN: *Distribution modulo* 1 *of slowly changing sequences*, Nieuw Arch. Wisk. (3) **21** (1973), 138–163 (MR0387224 (**52** #8067); Zbl. 0268.10038).

3.3.3. Denote by $T_{q,y}(x)$ the mapping $T_y : [0,1] \to [0,1]$ defined in 2.7.3 for an integer $q \geq 2$ and every $y \in [0,1]$. Let $y_1 = \frac{1}{q_1}$, $y_2 = \frac{1}{q_2}$, where q_1, q_2 are two integers such that $\gcd(q_1, q_2) = 1$. Then for every $x_1, x_2 \in [0,1]$ the two–dimensional sequence of iterates

$$\left(T^{(n)}_{q_1,y_1}(x_1), T^{(n)}_{q_2,y_2}(x_2)\right), \quad n = 0, 1, 2, \dots,$$

is

u.d.

and has the discrepancy

$$D_N^* \leq \frac{1 + (q_1 - 1)(q_2 - 1)[\log_{q_1}(Nq_1)][\log_{q_2}(Nq_2)]}{N}.$$

NOTES: B. Lapeyre and G. Pagès (1989). For the one–dimensional case see 2.7.3.

B. LAPEYRE – G. PAGÈS: *Familles de suites à discrépance faible obtenues par itération de transformations de* $[0,1]$, C. R. Acad. Sci. Paris, Série I **308** (1989), no. 17, 507–509 (MR0998641 (90b:11076); Zbl. 0676.10038).

3.3.4. Let D_N be the extremal discrepancy of a two–dimensional sequence of the type

$$(x_n, y_n) \bmod 1$$

and $D_N(p,q)$ the one–dimensional extremal discrepancy of the sequence

$$px_n + qy_n.$$

Then there exists an absolute constant c such that for every $\varepsilon > 0$

$$D_N \leq \varepsilon + c\left(D_N(0,1) + D_N(1,0) + \sum_{\substack{(p,q)=1 \\ p>0, q>0}} f(p,q,\varepsilon) D_N(p,q)\right),$$

where $f(p,q,\varepsilon) = \min(|pq|^{-1}, |\varepsilon pq|^{-2})$.
NOTES: This is a quantitative version of the two–dimensional Weyl theorem 1.11.1.3.

W.J. COLES: *On a theorem of van der Corput on uniform distribution*, Proc. Cambridge Philos. Soc. **53** (1957), 781–789 (MR0094329 (**20** #848); Zbl. 0079.07202).

3.4 Sequences of the form $a(n)\boldsymbol{\theta}$

3.4.1. Kronecker sequences.

(I) The s–dimensional Kronecker sequence

$$n\boldsymbol{\theta} = (n\theta_1, \dots, n\theta_s) \bmod 1$$

is

u.d.

if and only if $1, \theta_1, \dots, \theta_s$ are linearly independent over $\mathbb{Z}$ (or equivalently over $\mathbb{Q}$).
(II) If $1, \theta_1, \dots, \theta_s$ are linearly independent (over $\mathbb{Z}$) and if there exists a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(t)/t$ is monotonically increasing and

$$\|\mathbf{h} \cdot \boldsymbol{\theta}\| = \|h_1\theta_1 + \cdots + h_s\theta_s\| \geq \frac{1}{\phi(\max(|h_1|, \dots, |h_s|))} = \frac{1}{\phi(\|\mathbf{h}\|_\infty)}$$

for all $\mathbf{0} \neq \mathbf{h} = (h_1, \dots, h_s) \in \mathbb{Z}^s$, then

$$D_N(n\boldsymbol{\theta}) = \mathcal{O}\left(\frac{\log N \log \phi^{-1}(N)}{\phi^{-1}(N)}\right),$$

where $\phi^{-1}(N)$ denotes the inverse function of $\phi(x)$.
(III) Let

$$\delta_q(\boldsymbol{\theta}) = \max_{1 \leq j \leq s} \|q\theta_j\|$$

for $\boldsymbol{\theta} = (\theta_1, \dots, \theta_s) \in \mathbb{R}^s$ and a positive integer q. If there exists a constant $C > 0$ such that $\delta_q(\boldsymbol{\theta}) \geq C/q^{1/s}$ for every positive integer q then $\boldsymbol{\theta}$ is called **badly approximable**. If $1, \theta_1, \dots, \theta_s$, $s \geq 2$, are linearly independent over $\mathbb{Z}$ then for the isotropic discrepancy (see p. 1 – 87) we have

$$I_N(n\boldsymbol{\theta}) = \mathcal{O}\left(N^{-1/s}\right)$$

if and only if $\boldsymbol{\theta}$ is badly approximable. In other words, if and only if the linear form $L = (\sum_{j=1}^s m_j\theta_j) - m$ is extremal or badly approximable (i.e. there exists a $c > 0$ such that $N^s|L| \geq c$ if $|m_j| \leq N$ for $j = 1, 2, \dots, s$ and all integral $N > 0$).
On the other hand, for every positive integer s there is a positive constant c_s such that for every $\boldsymbol{\theta}$ and every $N = 1, 2, \dots,$ we have

$$I_N(n\boldsymbol{\theta}) > c_s N^{-1/s}.$$

If $s = 2$ then for all $\boldsymbol{\theta} = (\theta_1, \theta_1) \in \mathbb{R}^2$ and for infinitely many N we have

$$N^{1/2} I_N(n\boldsymbol{\theta}) \geq 0.0433\dots.$$

NOTES: (I) u.d. of the Kronecker sequence was proved by H. Weyl (1916).
(II) [DT, p. 70,Th. 1.80].
(IIIa) For the definition of badly approximable numbers cf. [DT, p. 67]. The estimation of the isotropic discrepancy was proved by G. Larcher (1988, 1989), cf.

[DT, p. 71, Th. 1.81]. With the lower bound he extended previous result $I_N(n\boldsymbol{\theta}) > c_s N^{-2/(s+1)}$ proved by W.M. Schmidt (1977). For some related metric theorems, cf. [DT, p. 66–90].

(IIIb) For the cube-discrepancy $D_N^{\mathbf{C}}(n\boldsymbol{\theta})$ (for the def. see 1.11.7) and for dimension $s \geq 2$, G. Larcher (1991) proved that

- $D_N^{\mathbf{C}}(n\boldsymbol{\theta}) > c(s) \max\left(\frac{1}{N}, \frac{r^{s-1}}{N^{1/s}}\right)$ for all $\boldsymbol{\theta} \in \mathbb{R}^s$ and all $r \in (0,1)$,
- $D_N^{\mathbf{C}}(n\boldsymbol{\theta}) < c(s,\boldsymbol{\theta}) \max\left(\frac{1}{N}, \frac{r^{s-1}}{N^{1/s}}\right)$ for badly approximable $\boldsymbol{\theta} \in \mathbb{R}^s$ and all $r \in (0,1)$,
- if $\boldsymbol{\theta} \in \mathbb{R}^s$ is not badly approximable, then for all $r \in (0,1)$ and for all c there is an N such that $D_N^{\mathbf{C}}(n\boldsymbol{\theta}) > c \cdot \frac{r^{s-1}}{N^{1/s}}$.

(IV) A $\boldsymbol{\theta}$ is called a **good point** (cf. L.–K. Hua and Y. Wang (1981, p. 82)) if

$$D_N^*(n\boldsymbol{\theta}) \leq \frac{c(\boldsymbol{\theta},\varepsilon)}{N^{1-\varepsilon}}.$$

They proved (1981, p. 61, Th. 3.3) that if

$$\|\mathbf{h}\cdot\boldsymbol{\theta}\| \geq \frac{c(\boldsymbol{\theta},\varepsilon)}{\|\mathbf{h}\|_\infty^{1+\varepsilon}}$$

holds for any integral vector $\mathbf{h} \neq \mathbf{0}$, then $\boldsymbol{\theta}$ is a good point. For instance:

- If $\theta_1,\dots,\theta_s$ are real algebraic numbers such that $1,\theta_1,\dots,\theta_s$ are linearly independent over $\mathbb{Z}$, then $\boldsymbol{\theta} = (\theta_1,\dots,\theta_s)$ is good point (this follows from W.M. Schmidt (1970)).
- If $\theta_i = e^{r_i}$ where r_i, $i = 1,2,\dots,s$, are different non–zero rational numbers, then $\boldsymbol{\theta}$ is good point (cf. A. Baker (1965)).

(V) Given $\boldsymbol{\theta}$ with $1,\theta_1,\dots,\theta_s$ linearly independent over $\mathbb{Z}$, we say (Niederreiter (1975, Definition 3) that $\boldsymbol{\theta}$ is of finite type $\gamma, \gamma \in \mathbb{R}$, if γ is the infimum of those numbers σ for which there exists a positive constant $c = c(\sigma,\boldsymbol{\theta})$ such that

$$r^\sigma(\mathbf{h})\|\mathbf{h}\cdot\boldsymbol{\theta}\| \geq c$$

holds for all lattice points $\mathbf{h} \in \mathbb{Z}^s$ with $\mathbf{h} \neq \mathbf{0}$ (cf. 2.8.1(V)). H. Niederreiter (1975, Th. 7) proved that if $\boldsymbol{\theta}$ is of finite type $\gamma = 1$, then the Abel discrepancy $D_r(n\boldsymbol{\theta})$ of the sequence $n\boldsymbol{\theta}$, $n = 0,1,2,\dots$, satisfies

$$D_r(n\boldsymbol{\theta}) = \mathcal{O}\left((1-r)^{1-\varepsilon}\right)$$

for every $\varepsilon > 0$. A. Baker (1965) proved that $\boldsymbol{\theta} = (e^{r_1},\dots,e^{r_s})$ with distinct non–zero rationals $r_1,\dots,r_s$ is of type $\gamma = 1$.

(VI) If $1,\theta_1,\dots,\theta_s$ are algebraic numbers linearly independent over $\mathbb{Z}$ then $D_N = \mathcal{O}(N^{-1+\varepsilon})$ for every $\varepsilon > 0$ (cf. H. Niederreiter (1972)).

(VII) Linear independence of $1,\theta_1,\dots,\theta_s$ over $\mathbb{Q}$ for positive real roots

$$\theta_i = \left(\frac{p_i}{q_i}\right)^{1/m_i},$$

where p_i, q_i, m_i are positive integers, $i = 1, 2, \dots, s$, follows from a theorem proved by L.J. Mordell (1953) provided that there is no relation of the form $\theta_1^{n_1} \dots \theta_s^{n_s} \in \mathbb{Q}$ with integers $n_1, \dots, n_s$ unless $n_1 \equiv 0 \pmod{m_1}, \dots, n_s \equiv 0 \pmod{m_s}$. Actually Mordell proved a stronger result generalizing a previous result proved by A.S. Besicovitch (1940), cf. 3.6.5.

(VIII) E.I. Kovalevskaja (2000) proved that if $\boldsymbol{\theta} = (\theta_1, \dots, \theta_s)$ satisfies (II) with $\phi(t) = t^\sigma$, $s < \sigma < s + 1$, then the sequence

$$\mathbf{x}_n = (n\theta_1, \dots, n\theta_s) \bmod 1/2$$

gives a "good approximation of zero".

(IX) I.I. Pjateckiĭ–Šapiro proved (cf. N.N. Korobov (1963, p. 85, Th. 10)) that for every $f(\mathbf{x}) = \sum_{\mathbf{h}\in\mathbb{Z}^s} c_{\mathbf{h}} e^{2\pi i \mathbf{h}\cdot\mathbf{x}}$ with $\sum_{\mathbf{h}\in\mathbb{Z}^s} |c_{\mathbf{h}}| < \infty$, there exits a $\boldsymbol{\theta}$ depending on f such that

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(n\boldsymbol{\theta}) - \int_{[0,1]^s} f(\mathbf{x})\, \mathrm{d}\mathbf{x} \right| = \mathcal{O}\left(\frac{\log N}{N} \right).$$

(X) N.N. Korobov (1963, p. 89, Th. 11) proved: If for $\boldsymbol{\theta}$ and every $\mathbf{0} \neq \mathbf{h} \in \mathbb{Z}^s$ the distance $||\mathbf{h}\cdot\boldsymbol{\theta}||$ of $\mathbf{h}\cdot\boldsymbol{\theta} = h_1\theta_1 + \dots + h_s\theta_s$ to the nearest integer satisfies

$$||\mathbf{h}\cdot\boldsymbol{\theta}|| \geq \frac{c_0}{r(\mathbf{h})(\prod_{i=1}^{s}(\log(r(h_i)+1)))^\gamma},$$

where $\gamma \geq 0$, $c_0 > 0$ are constants independent on $\mathbf{h}$ $\big($here $r(\mathbf{h}) = \prod_{i=1}^{s} r(h_i)$, $r(h_i) = \max(1, |h_i|)\big)$, then for every $f \in E_s^\alpha(c)$ (i.e. $f(\mathbf{x}) = \sum_{\mathbf{h}\in\mathbb{Z}^s} c_{\mathbf{h}} e^{2\pi i \mathbf{h}\cdot\mathbf{x}}$, where $|c_{\mathbf{h}}| \leq c r^{-\alpha}(\mathbf{h})$ for $\mathbf{0} \neq \mathbf{h} \in \mathbb{Z}^s$) we have

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(n\boldsymbol{\theta}) - \int_{[0,1]^s} f(\mathbf{x})\, \mathrm{d}\mathbf{x} \right| = \mathcal{O}\left(\frac{1}{N} \right).$$

(XI)(i) For Kronecker s-dimensional sequence $n\boldsymbol{\theta} = (n\theta_1, \dots, n\theta_s) \bmod 1$ and an interval $\mathbf{I} = I_1 \times \dots \times I_s$ define the local discrepancy function by $D(N, \mathbf{I}) = \big|A(\mathbf{I}; N; n\boldsymbol{\theta}) - N|\mathbf{I}|\big|$. Assume that $1, \theta_1, \dots, \theta_s$ are linearly independent over $\mathbb{Z}$. P. Liardet (1987) proved that $D(N, \mathbf{I})$ is bounded as $N \to \infty$ if and only if there exists an index i such that $|I_i| = k\theta_i \bmod 1$ for some integer k and $|I_j| = 1$ for all $j \neq i$. G. Rauzy (1984) proved a criterion for $D(N, \mathbf{A})$ being bounded for general sets $\mathbf{A} \subset [0, 1]^s$ and S. Ferenczi (1992) for measurable sets $\mathbf{A} \subset [0, 1]^s$.

(ii) Let $s = 2$ and $1, \alpha, \beta$ be linearly independent. Let $\mathbf{I}$ be an interval in $[0, 1)^2$ with sides of length $\{q\alpha\}, \{q\beta\}$ for an integer q. S. Hartman (1948) conjectured that in this case the local discrepancy function $D(N, \mathbf{I})$ is bounded. P. Szüsz (1955) showed that this not true. Namely, using a continued fraction construction he showed that this not true for any irrational β and any one of an uncountable set of α's corresponding to the given β even for $q = 1$. The negative answer also follows from P. Liardet's (1987) result in (XI) (i).

(iii) Let $\mathbf{I}$ by the parallelogram determined by the vectors
$(\min(\{q\alpha\},\{q\beta\})/\max(\{q\alpha\},\{q\beta\}),0)$ and
$(\min(\{q\alpha\},\{q\beta\}),\max(\{q\alpha\},\{q\beta\}))$
for an integer q. P. Szüsz (1954) proved that in this case the local discrepancy function $D(N,\mathbf{I})$ is bounded.
(XII) V.V. Kozlov (1978) investigated the case of dimension $s=1$ and then E.V. Kolomeikina and N.G. Moshchevitin (2003) proved for general s: Let

$$\mathbf{x}_n=(n\theta_1+\psi_1,\dots,n\theta_s+\psi_s) \bmod 1, \quad n=1,2,\dots$$

be the Kronecker sequence, where $\theta_1,\dots,\theta_s,1$ are linearly independent over $\mathbb{Q}$. Let further $f(\mathbf{x})$ be a $\mathbf{1}$-periodic function satisfying $\int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x}=0$ and

(1) $f(\mathbf{x})=p(\mathbf{x})+h(\mathbf{t}\cdot\mathbf{x})$

where $p(\mathbf{x})$ is a trigonometric polynomial, $h(x)$ is a 1-periodic function of a single variable x and $\mathbf{t}\cdot\mathbf{x}$ is the inner product. Then

$$\liminf_{N\to\infty}\sup_{\psi_1,\dots,\psi_s} N\left|\frac{1}{N}\sum_{n=1}^{N} f(\mathbf{x})-\int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x}\right|=0.$$

Note that the vanishing of lim inf characterize the form of $f(x)$ given by (1).

Related sequences: 3.6.5, 3.6.9, 3.6.6, 3.6.7

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P. Szüsz: *Lösung eines Problems von Herrn Hartman* (German), Stud. Math. **15** (1955), 43–55 (MR0074463 (17,589d); Zbl. 0067.02401).
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3.4.1.1 Let $a_{i,n}$, $n = 1, 2, \dots$, $i = 1, 2$, be sequences of positive integers such that $a_{1,1} = a_{2,1} = 1$, $a_{1,2} = a_{2,2} = 2$ and

$$a_{1,n+2} = \begin{cases} 2 + 3^{a_{1,1}a_{1,2}\cdots a_{2,n+1}}, & \text{if } n = 2k, k = 2^m, \\ a_{1,n} + [5\log^3(a_{1,n}+1)], & \text{if } n = 2k, k \neq 2^m, \\ 1 + \left[4^{2^{(a_{1,1}a_{1,2}\cdots a_{2,n})\log a_{1,n}}}\right], & \text{if } n = 2k+1, k = 2^{2m}, \\ a_{1,n} + [\log^2\log^3(a_{1,n} + \log a_{1,n})], & \text{if } n = 2k+1, k \neq 2^{2m}, \end{cases}$$

$$a_{2,n+2} = \begin{cases} 2 + 7^{5^{3^{a_{1,1}a_{1,2}\cdots a_{2,n+1}}}}, & \text{if } n = 2k, k = 2^m, \\ 2 + a_{2,n} + \left[\frac{3n}{2n+2}\log^4\log(a_{2,n}^2 + \sqrt{a_{2,n}})\right], & \text{if } n = 2k, k \neq 2^m, \\ 1 + 3^{4^{5^{3^{a_{1,1}a_{1,2}\cdots a_{2,n+1}}}}}, & \text{if } n = 2k+1, k = 2^{2m}, \\ a_{2,n} + \left[\frac{1}{\sqrt{n^6+4}}\log^3\log^{2n}\log(a_{2,n} + 5\log\log a_{2,n})\right], & \text{if } n = 2k+1, k \neq 2^{2m}, \end{cases}$$

for all $n = 1, 2, \dots$. Then the numbers

$$1, \quad \sum_{n=1}^{\infty}\frac{1}{a_{1,n}}, \quad \sum_{n=1}^{\infty}\frac{1}{a_{2,n}}$$

are linearly independent over the rational numbers.

J. Hančl – P. Rucki – J. Šustek: *A generalization of Sándor's theorem using iterated logarithms*, Kumamoto J. Math. **19** (2006), 25–36 (MR2211630 (2007d:11080); Zbl. 1220.11087).

3.4.1.2 Let $a_{i,n}$, $n = 1, 2, \dots$, $i = 1, 2, 3$, be sequences of positive integers such that $a_{1,1} = 1$, $a_{2,1} = 2$, $a_{3,1} = 3$, and

$$a_{1,n+1} = \begin{cases} a_{1,n} + [\log a_{1,n} \log^{3/2} \log a_{1,n} + \log^3 \log a_{1,n}], & \text{if } n \neq 7^m, \\ 1 + 3^{4^{n(a_{1,1}a_{1,2}\cdots a_{3,n})^2}}, & \text{otherwise}, \end{cases}$$

$$a_{2,n+1} = \begin{cases} a_{2,n} + [\log a_{2,n} \log\log a_{2,n} \log\log^2 \log a_{2,n} \log^2 \log\log\log a_{2,n}], & \text{if } n \neq 7^m, \\ 3 + 2^{2^{2^{2^{3^{4^{(a_{1,1}a_{1,2}\cdots a_{3,n})^3}}}}}}, & \text{otherwise}, \end{cases}$$

$$a_{3,n+1} = \begin{cases} a_{3,n} + [3 \log^{5/7} a_{3,n}], & \text{if } n \neq 7^m, \\ 11 + 2^{2^{2^{2^{2^{3^{4^{n(a_{1,1}a_{1,2}\cdots a_{3,n})^3}}}}}}}, & \text{otherwise}. \end{cases}$$

for all $n = 1, 2, \dots$. Then the numbers

$$1, \quad \sum_{n=1}^{\infty} \frac{1}{a_{1,n}}, \quad \sum_{n=1}^{\infty} \frac{1}{a_{2,n}}, \quad \sum_{n=1}^{\infty} \frac{1}{a_{3,n}}$$

are linearly independent over the rational numbers.

J. Hančl – P. Rucki – J. Šustek: *A generalization of Sándor's theorem using iterated logarithms*, Kumamoto J. Math. **19** (2006), 25–36 (MR2211630 (2007d:11080); Zbl. 1220.11087).

3.4.1.3 Let $K \geq 2$ be an integer. Then for every sequence c_n, $n = 1, 2, \dots$, of positive integers the numbers

$$1, \quad \sum_{n=1}^{\infty} \frac{1}{2^{(K^n - n)} c_n}, \quad \sum_{n=1}^{\infty} \frac{1}{2^{(K^n - 2n)} c_n}, \dots \quad \sum_{n=1}^{\infty} \frac{1}{2^{(K^n - (K-1)n)} c_n}$$

are linearly independent over the rational numbers.

J. Hančl – J. Štěpnička – J. Šustek: *Linearly unrelated sequences and problem of Erdős*, Ramanujan J. **17** (2008), no. 3, 331–342 (MR2456837 (2009i:11089); Zbl. 1242.11049)

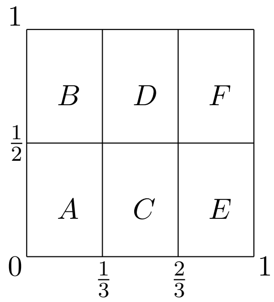
3.4.1.4 For every u.d. sequence $x_n \in [0, 1)$ the two-dimensional sequence

$$(\{2x_n\}, \{3x_n\}), \quad n = 1, 2, \dots,$$

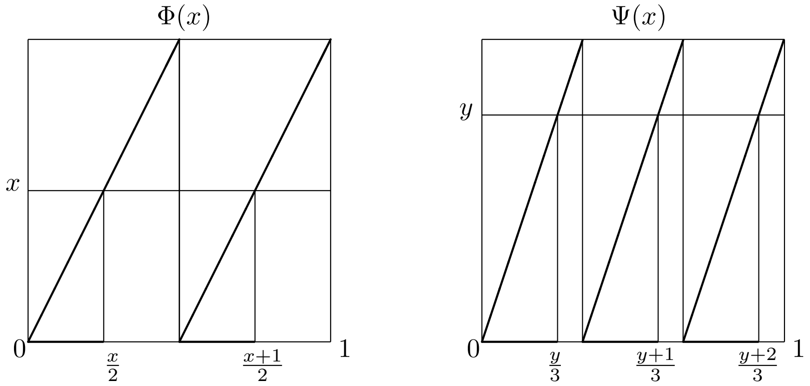
has a.d.f.

$$g(x, y) = \begin{cases} \min\left(\frac{x}{2}, \frac{y}{3}\right), & \text{if } x \in A, \\ \frac{x-1}{2} + \min\left(\frac{x+1}{2}, \frac{y+1}{3}\right), & \text{if } x \in B, \\ \frac{y-2}{3} + \min\left(\frac{x+1}{2}, \frac{y+2}{3}\right), & \text{if } x \in C, \\ \frac{x}{2} + \frac{y-1}{3} + \min\left(\frac{x}{2}, \frac{y}{3}\right), & \text{if } x \in D, \\ \frac{2y-1}{3} + \min\left(\frac{x}{2}, \frac{y+1}{3}\right), & \text{if } x \in E, \\ \frac{x-1}{2} + \frac{2y-2}{3} + \min\left(\frac{x+1}{2}, \frac{y+2}{3}\right), & \text{if } x \in F, \end{cases}$$

where



NOTES:
(I) J. Fialová personal communication.
(II) $g(x,y) = |\Phi^{-1}([0,x)) \cap \Psi^{-1}([0,y))|$, where $\Phi(x) = 2x \bmod 1$ and $\Psi(x) = 3x \bmod 1$, see the following picture



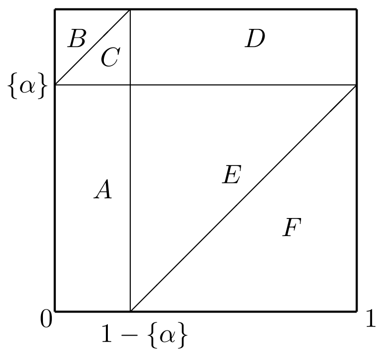
$\boxed{\textbf{3.4.1.5}}$ For every irrational α, $\{\alpha\} > \frac{1}{2}$ the two-dimensional sequence

$$(\{n\alpha\}, \{(n+1)\alpha\}), \quad n = 1, 2, \dots,$$

has the a.d.f.

$$g(x,y) = \begin{cases} 0, & \text{if } (x,y) \in A, \\ x, & \text{if } (x,y) \in B, \\ \{\alpha\} - y, & \text{if } (x,y) \in C, \\ \{\alpha\} - y + x - (1 - \{\alpha\}), & \text{if } (x,y) \in D, \\ x - (1 - \{\alpha\}), & \text{if } (x,y) \in E, \\ y, & \text{if } (x,y) \in F, \end{cases}$$

which is a copula, where



NOTES:
(I) J. Fialová personal communication.
(II) F. Pillichshammer and S. Steinerberger (2009) proved that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\big|\{n\alpha\}-\{(n+1)\alpha\}\big| = 2\{\alpha\}(1-\{\alpha\}).$$

F. PILLICHSHAMMER – S. STEINERBERGER: *Average distance between consecutive points of uniformly distributed sequences*, Unif. Distrib. Theory **4** (2009), no. 1, 51–67 (MR2501478 (2009m:11116); Zbl. 1208.11088).

3.4.2. Let $(f_1(n),\dots,f_s(n))$, $n=1,2,\dots$, be an s–dimensional sequence of positive integers which satisfies
(i) $f_i(n)|f_i(n+1)$ for $i=1,2,\dots,s$, and $n=1,2,\dots$,
(ii) $\frac{f_i(n+1)}{f_i(n)}\to\infty$ as $n\to\infty$ for $i=1,2,\dots,s$.
Let the numbers α_i, $i=1,2,\dots,s$, be defined by
(iii) $\alpha_i=[\alpha_i]+\sum_{n=1}^{\infty}\frac{c_{n,i}}{f_i(n)}$, $c_{n,i}$ are integers and $0\le c_{n,i}<f_i(n)/f_i(n-1)$.
Then the sequence

$$(f_1(n)\alpha_1,\dots,f_s(n)\alpha_s) \bmod 1$$

is

$$\text{u.d.}$$

if and only if

$$c_{n,i}=\left[\{x_i(n)\}\frac{f_i(n)}{f_i(n-1)}\right]$$

and

$$(x_1(n),\dots,x_s(n)) \bmod 1$$

is

$$\text{u.d.}$$

Here we suppose that $f_i(0) = 1$ and as usual, $[x]$ is the integer part and $\{x\}$ is the fractional part of x, resp.

NOTES: P. Gerl (1965) generalized in this way the result from 2.8.16 proved by N.M. Korobov (1950).

P. GERL: *Konstruktion gleichverteilter Punktfolgen*, Monatsh. Math. **69** (1965), 306–317 (MR0184922 (**32** #2393); Zbl. 0144.28801).
N.M. KOROBOV: *Concerning some questions of uniform distribution*, (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. **14** (1950), 215–238 (MR0037876 (12,321a); Zbl. 0036.31104).

3.4.2.1 Let $\alpha_1, \dots, \alpha_s$ be positive real numbers such that $1, \alpha_1, \dots, \alpha_s$ are linearly independent over $\mathbb{Q}$ and let $b_1, \dots, b_s$ be arbitray integers ≥ 2. Then for any $g \in \mathbb{N}$, the sequence

$$\left(\{n\alpha_1/b_1^g\}, \dots, \{n\alpha_s/b_s^g\}\right), \quad n = 0, 1, 2, \dots,$$

is

u.d.

P. GRABNER – P. HELLEKALEK – P. LIARDET: *The dynamical point of view of low-discrepancy sequences*, Unif. Distrib. Theory **7** (2012), no. 1, 11–70 (MR2943160; Zbl. 1313.11093)

3.4.2.2 Let $d_1, \dots, d_s$ be distinct positive integers and let $\alpha_1, \dots, \alpha_s$ be positive irrational numbers. Then for any integers $b_1, \dots, b_s \geq 2$ and any $g \in \mathbb{N}$ the sequence

$$\left(\{n^{d_1}\alpha_1/b_1^g\}, \dots, \{n^{d_s}\alpha_s/b_s^g\}\right), \quad n = 0, 1, 2, \dots,$$

is

u.d.

P. GRABNER – P. HELLEKALEK – P. LIARDET: *The dynamical point of view of low-discrepancy sequences*, Unif. Distrib. Theory **7** (2012), no. 1, 11–70 (MR2943160; Zbl. 1313.11093)

3.4.3. Open problem. Let q_n, $n = 1, 2, \dots,$ be a sequence of positive integers and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_s)$ be an s–dimensional real vector. Describe the distribution of the sequence

$$q_n\boldsymbol{\theta} = (q_n\theta_1, \dots, q_n\theta_s) \bmod 1.$$

NOTES: Generalizing 2.8.5(VIII), H. Albrecher (2002) proved, for the mean value of the weighted L^2 discrepancy $D_N^{(2)}(q_n\boldsymbol{\theta})$ (cf. 1.10.6), that

$$\int_{[0,1]^s} D_N^{(2)}(q_n\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{\theta} = \sum_{m,n=1}^{N} w_m w_n \left(\frac{1}{3} + \right.$$
$$\left. + \frac{1}{12}\frac{(q_m,q_n)^2}{q_m q_n}\right)^s + \sum_{\substack{m,n=1\\ q_m=q_n}}^{N} w_m w_n \left(\frac{1}{2^s} - \left(\frac{5}{12}\right)^s\right) - \frac{1}{3^s},$$

where w_n are weights, $\sum_{n=1}^{N} w_n = 1$ and (q_m,q_n) denotes the g.c.d. of q_m and q_n.

H. ALBRECHER: *Metric distribution results for sequences* $(\{q_n\vec{\alpha}\})$, Math. Slovaca **52** (2002), no. 2, 195–206 (MR 2003h:11083; Zbl. 1005.11036).

3.4.4. The set

$$\left(\sum_{i=1}^{s+1} k_i\alpha_{1,i}, \dots, \sum_{i=1}^{s+1} k_i\alpha_{s,i}\right), \quad k_1, k_2, \dots, k_{s+1} \in \mathbb{N},$$

is

$$\text{dense in } (-\infty,\infty)^s$$

if and only if $\sum_{i=0}^{s} m_i\Delta_i \neq 0$ for all integers m_i, where the determinants Δ_i depend on the coefficients $\alpha_{i,j}$.

M.G. HUDAĬ–VERENOV: *On an everywhere dense set*, Izv. Akad. Nauk Turkmen. SSR Ser. Fiz.–Tech. Him. Geol. Nauk (Russian), **1962** (1962), no. 3, 3–11 (MR0173663 (**30** #3873)).

3.4.5.

(I) Let K be an upper bound on the partial quotients of the finite or infinite continued fraction expansion of a number α, and let $N \geq 1$ do not exceed the denominator of α when α is rational. Then for the extreme discrepancy of the finite two–dimensional sequence

$$(0,0), \left(\frac{1}{N}, \alpha\right), \left(\frac{2}{N}, 2\alpha\right), \dots, \left(\frac{N-1}{N}, (N-1)\alpha\right) \bmod 1$$

we have

$$D_N^* \leq c\frac{\log N}{N},$$

where c is a constant which depends only on K.

(II) If all the partial quotients of the continued fraction expansion of α are equal, say, to a positive integer a, and if N is an arbitrary positive integer if α is irrational or does not exceed its denominator if α is rational, then for the L^2 discrepancy of our sequence we have

$$D_N^{(2)} = \mathcal{O}\left(\frac{\log N}{N^2}\right),$$

where the $\mathcal{O}$–constant depends only on a.

NOTES: (I) is proved in S.K. Zaremba (1966) and (II) can be found in V.T. Sós and S.K. Zaremba (1979).

V.T. SÓS – S.K. ZAREMBA: *The mean–square discrepancies of some two–dimensional lattices*, Studia Sci. Math. Hungar. **14** (1979), no. 1–3, 255–271 (1982) (MR0645534 (84a:10054); Zbl. 0481.10048).

S.K. ZAREMBA: *Good lattice points, discrepancy, and numerical integration*, Ann. Mat. Pura Appl. (4) **73** (1966), 293–317 (MR0218018 (**36** #1107); Zbl. 0148.02602).

3.4.6. Let a, N, and k be positive integers with $N \geq 36$, $1 \leq k \leq \sqrt{N}$ and $\gcd(a, N) = 1$. If $\lambda = 2 + \sqrt{2}$, then the extreme discrepancy of the finite two–dimensional sequence

$$(x_n, x_{n+k}) = \left(\frac{a}{N}n^2, \frac{a}{N}(n+k)^2\right) \bmod 1, \quad n = 0, 1, \dots, N-1,$$

satisfies

$$D_N^* < \frac{(3.24)\lambda^{\omega(N)}(\log N)^2 + 392(2\lambda)^{\omega(N)}\log N}{\sqrt{N}},$$

where $\omega(N)$ denotes the number of distinct prime divisors of N.

NOTES: D.L. Jagerman (1964). For the autocorrelation (cf. 2.15.1) he proved that

$$\psi(k) = \frac{1}{N}\sum_{n=0}^{N-1}((1/2)-x_n)((1/2)-x_{n+k}) < \frac{(0.81)\lambda^{\omega(N)}(\log N)^2 + 33(2\lambda)^{\omega(N)}\log N}{\sqrt{N}}.$$

D.L. JAGERMAN: *The autocorrelation and joint distribution functions of the sequences* $\{\frac{a}{m}j^2\}$, $\{\frac{a}{m}(j+\tau)^2\}$, Math. Comp. **18** (1964), 211–232 (MR0177499 (**31** #1762); Zbl. 0134.14801).

3.4.7. The two–dimensional finite sequence (called Roth sequence, see 3.18.2)

$$\left(\frac{t_1}{2}+\frac{t_2}{2^2}+\cdots+\frac{t_n}{2^n},\frac{t_n}{2}+\frac{t_{n-1}}{2^2}+\cdots+\frac{t_1}{2^n}\right),\ \text{with } t_i=0 \text{ or } 1,$$

has $N=2^n$ terms, and for its extreme discrepancy we have

$$\frac{n}{3}<ND_N^*<\frac{n}{3}+3$$

for all $n=1,2,\dots$.

H. Gabai: *On the discrepancy of certain sequences* mod 1, Illinois J. Math. **11** (1967), 1–12 (MR0209252 (**35** #154); Zbl. 0129.03102).

3.4.8. If $\alpha_1,\alpha_2,\alpha_3$ are real numbers such that α_1, $\alpha_2+\alpha_1\alpha_3$, and 1 are linearly independent over the rationals, then the two–dimensional sequence

$$\big(\alpha_1 n,\alpha_2 n+\alpha_3[\alpha_1 n]\big) \bmod 1$$

is

u.d.

Furthermore, if for any integers k_1,k_2,k_3 with $k_1^2+k_2^2\neq 0$, we have

$$\big|\alpha_1k_1+(\alpha_2+\alpha_1\alpha_3)k_2-k_3\big|\geq\frac{c}{r^u},$$

where $r=\max(|k_1|,|k_2|)$, and $c>0$, then

$$D_N=\mathcal{O}\left(N^{-1/(8u+12)}\right).$$

The $\mathcal{O}$–constant depends on c, s, and α_3.

B. Karimov: *On the distribution of the fractional parts of certain linear forms in a unit square*, (Russian), Izv. Akad. Nauk UzSSR Ser. Fiz.–Mat. Nauk **10** (1966), no. 1, 19–22 (MR0204399 (**34** #4241); Zbl. 0135.10704).

[a] B. Karimov: *On the question of the number of fractional parts of certain linear forms in a rectangle*, (Russian), Izv. Akad. Nauk UzSSR Ser. Fiz.–Mat. Nauk **10** (1966), no. 2, 21–28 (MR0205928 (**34** #5753); Zbl. 0144.28602).

3.5 Sequences involving sum–of–digits functions

For the def. of the sum–of–digits function $s_q(n)$ see 2.9.

3.5.1. Let $q_1, \dots, q_s$ be pairwise coprime integers ≥ 1. Then the s–dimensional sequence

$$\mathbf{x}_n = (s_{q_1}(n)\theta_1, \dots, s_{q_s}(n)\theta_s) \bmod 1$$

is

u.d.

if and only if $\theta_1, \dots, \theta_s$ are irrational. Moreover, if $\theta_1, \dots, \theta_s$ are irrational and there exists $\eta \geq 1$ and a constant $c_1 > 0$ such that for all integers $h > 0$ and for every $j = 1, 2, \dots, s$, we have $\|h\theta_j\| \geq c_1 h^{-\eta}$, then the discrepancy D_N of $\mathbf{x}_0, \dots, \mathbf{x}_{N-1}$ is bounded by

$$D_N \leq c(c_1, \eta, q) \left(\frac{\log \log N}{\log N} \right)^{1/(2\eta)}$$

for every N. Conversely, if for some $\eta \geq 1$ and some constant $c_2 > 0$ there exists j such that $\|h\theta_j\| \leq c_2 h^{-\eta}$ for infinitely many integers $h > 0$ then

$$D_N \geq c'(c_2, \eta, q) \frac{1}{(\log N)^{1/(2\eta)}}$$

for infinitely many N.

M. Drmota – G. Larcher: *The sum–of–digits–function and uniform distribution modulo 1*, J. Number Theory **89** (2001), 65–96 (MR1838704 (2002e:11094); Zbl. 0990.11053).

$\boxed{\textbf{3.5.1.1}}$ Let $\gamma = (\gamma_0, \gamma_1, \dots)$ be a sequence in $\mathbb{R}$ and let $q \in \mathbb{N}$, $q \geq 2$. Given an $n \in \mathbb{N}_0$ with base q representation $n = n_0 + n_1 q + n_2 q^2 + \cdots$, define the **weighted q-ary sum-of-digits function** by

$$s_{q,\gamma}(n) := \gamma_0 n_0 + \gamma_1 n_1 + \gamma_2 n_2 + \cdots .$$

Then for $d \in \mathbb{N}$, weight-sequences $\gamma^{(j)} = (\gamma_0^{(j)}, \gamma_1^{(j)}, \dots)$ in $\mathbb{R}$ and $q_j \in \mathbb{N}$, $q_j \geq 2$, $j \in \{1, \dots, d\}$, define

$$s_{q_1,\dots,q_d,\boldsymbol{\gamma}}(n) := (s_{q_1,\gamma^{(1)}}(n), \dots, s_{q_d,\gamma^{(d)}}(n)),$$

where $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1, \dots)$ with $\boldsymbol{\gamma}_k = (\gamma_k^{(1)}, \dots, \gamma_k^{(d)})$ for $k \in \mathbb{N}_0$.

Open question: Let $q_1, \ldots, q_d \geq 2$ be pairwise coprime integers. Which conditions imposed on the weight-sequences $\gamma^{(j)} = (\gamma_0^{(j)}, \gamma_1^{(j)}, \ldots)$ in $\mathbb{R}$, $j \in \{1, \ldots, d\}$, implies that the sequence

$$s_{q_1,\ldots,q_d,\boldsymbol{\gamma}}(n) \bmod 1, \quad n = 0, 1, 2, \ldots, \tag{1}$$

is

$$\text{u.d.} \mod 1?$$

NOTES:
(I) Proposed by F. Pillichshammer (2007).
(II) If $\gamma_k^{(j)} = q_j^{-k-1}$ for all $j \in \{1, \ldots, d\}$ and all $k \in \mathbb{N}_0$, then we obtain the d-dimensional van der Corput-Halton sequence which is u.d. modulo one.
(III) If $\gamma_k^{(j)} = q_j^k \alpha_j$ for all $j \in \{1, \ldots, d\}$ and all $k \in \mathbb{N}_0$, then the sequence (1) has the form $(\{n(\alpha_1, \ldots, \alpha_d)\})_{n \geq 0}$ which is u.d. modulo one if and only if $1, \alpha_1, \ldots, \alpha_d$ are linearly independent over $\mathbb{Q}$.
(IV) If $\gamma_k^{(j)} = \alpha_j \in \mathbb{R}$ for all $j \in \{1, \ldots, d\}$ and all $k \in \mathbb{N}_0$, then M. Drmota and G. Larcher (2001) showed that the sequence (1) is u.d. mod 1 if and only if $\alpha_1, \ldots, \alpha_d \in \mathbb{R} \setminus \mathbb{Q}$.
(V) If $q_1 = \cdots = q_d = q$ then F. Pillichshammer (2007) showed that the sequence (1) is u.d. mod 1 if and only if for every $\boldsymbol{h} \in \mathbb{Z}^d \setminus \{\boldsymbol{0}\}$ one of the following properties holds: Either

$$\sum_{\substack{k=0 \\ \langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle q \notin \mathbb{Z}}}^{\infty} \|\langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle\|^2 = \infty$$

or there exists a $k \in \mathbb{N}_0$ such that $\langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle \notin \mathbb{Z}$ and $\langle \boldsymbol{h}, \boldsymbol{\gamma}_k \rangle q \in \mathbb{Z}$. Here $\| \cdot \|$ denotes the distance to the nearest integer, i.e., for $x \in \mathbb{R}$, $\|x\| = \min_{k \in \mathbb{Z}} |x - k|$ and $\langle \cdot, \cdot \rangle$ is the standard inner product.
(VI) R. Hofer, G. Larcher and F. Pillichshammer (2007) found a generalization where the weighted sum-of-digits function is replaced by a generalized weighted digit-block-counting function.
(VII) R. Hofer (2007): Let $q_1, \ldots, q_d \geq 2$ be pairwise coprime integers and $\gamma^{(1)}, \ldots, \gamma^{(d)}$ be given weight sequences in $\mathbb{R}$. If the sum

$$\sum_{i=0}^{\infty} \left\| h\left(\gamma_{2i+1}^{(j)} - q_j \gamma_{2i}^{(j)}\right) \right\|^2$$

is divergent for every dimension $j \in \{1, \ldots, d\}$ and every nonzero integer h, then the sequence (1) is u.d. in $[0, 1)^d$.

M. Drmota – G. Larcher: *The sum–of–digits–function and uniform distribution modulo 1*, J. Number Theory **89** (2001), 65–96 (MR1838704 (2002e:11094); Zbl. 0990.11053).
R. Hofer: *Note on the joint distribution of the weighted sum-of-digits function modulo one in case of pairwise coprime bases*, Unif. Distrib. Theory **2** (2007), no. 1, 35–47 (MR2357507 (2008i:11102); Zbl. 1153.11036)

R. Hofer – G. Larcher – F. Pillichshammer: *Average growth-behavior and distribution properties of generalized weighted digit-block-counting functions*, Monatsh. Math **154** no. 3, (2008), 199–230.(MR2413302 (2009d:11118); Zbl. 1169.11006).
F. Pillichshammer: *Uniform distribution of sequences connected with the weighted sum-of-digits function*, Unif. Distrib. Theory **2** (2007), no. 1, 1–10 (MR2318528 (2008f:11082); Zbl. 1201.11081).

3.5.2. Let α be an irrational number with a continued fraction expansion $[a_0; a_1, a_2, \dots]$ and let q_i, $i = 0, 1, \dots,$ be the sequence of the denominators of its convergents. Then the α–adic expansion of a positive integer n is defined by $n = \sum_{k=0}^{L(n)} \varepsilon_k(n) q_k$ (also called Ostrowski expansion, cf. 2.8.1 (IV)). Put $\sigma_\alpha(n) = \sum_{k=0}^{L(n)} \varepsilon_k(n)$ and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_s)$. If $1, \theta_1, \dots, \theta_s$ are algebraic and linearly independent over the rationals, then for every $\varepsilon > 0$ there exists a constant $c = c(\boldsymbol{\theta}, \varepsilon, \alpha)$ such that for the sequence

$$\mathbf{x}_n = \sigma_\alpha(n)\boldsymbol{\theta}$$

we have

$$D_N \le cL(N)^{-\frac{1}{2s}+\varepsilon}.$$

Related sequences: 2.9.13

N. Kopecek – G. Larcher – R.F. Tichy – G. Turnwald: *On the discrepancy of sequences associated with the sum–of–digits functions*, Ann. Inst, Fourier (Grenoble) **37** (1987), no. 3, 1–17 (MR0916271 (89c:11119); Zbl. 0601.10038).

3.5.3.

Notes: A number system with base q of an order $\mathcal{O}$ of a number field is called **canonical** if every element $z \in \mathcal{O}$ has the unique representation of the form $z = \sum_{j=0}^{k(z)} a_j q^j$ where $a_j \in \{0, 1, 2, \dots, |N(q)| - 1\}$. Then the **sum-of-digits function** is defined by

$$s_q(z) = \sum_{j=0}^{k(z)} a_j.$$

In the ring of Gaussian integers $\mathbb{Z}[i]$ the all bases of the canonical number systems are given by $q = -b \pm i$ where b is a positive integer.

. . . .

Order the Gaussian integer $\mathbb{Z}[i]$ to a sequence z_n, $n = 1, 2, \dots,$ according to their norm $|z|$ and let q be a canonical base in $\mathbb{Z}[i]$. If θ is irrational then the two–dimensional sequence

$$(\arg z_n, \{s_q(z_n)\theta\})$$

is (cf. 1.5)

$$\text{almost u.d. in } (-\pi,\pi]\times[0,1)$$

with respect to the sequence of indices $[\pi N]$, $N=1,2,\dots$, i.e.

$$\lim_{N\to\infty}\frac{\#\big\{z\in\mathbb{Z}[i]\,;\ |z|<\sqrt{N},\arg z\in I,\{s_q(z)\theta\}\in J\big\}}{\pi N}=\frac{|I|}{2\pi}\cdot|J|$$

for all intervals $I\subset(-\pi,\pi]$ and $J\subset[0,1)$. If θ is of finite type γ $\big($cf. 2.8.1, par. (V), (ii)$\big)$ then for every $\varepsilon>0$ we have

$$D_{[\pi N]}\le\frac{C(b,\theta,\varepsilon)}{(\log N)^{\frac{1}{2\gamma}-\varepsilon}}.$$

Notes: P.J. Grabner and P. Liardet (1999). For the characterization of canonical bases consult I. Kátai and J. Szabó (1975).

Related sequences: 2.9.14

P.J. Grabner – P. Liardet: *Harmonic properties of the sum–of–digits function for complex base*, Acta Arith. **91** (1999), no. 4, 329–349 (MR1736016 (2001f:11126); Zbl. 0949.11004).

I. Kátai – J. Szabó: *Canonical number systems for complex integers*, Acta Sci. Math. (Szeged) **37** (1975), no. 3–4, 255–260 (MR0389759 (**52** #10590); Zbl. 0309.12001).

3.6 Sequences involving primes

See also: 3.10.7, 3.15.1, 3.15.5

3.6.1. If $0<\alpha_1<\dots<\alpha_s<1$ and p_n stands for the nth prime then the sequence

$$\mathbf{x}_n=(p_n^{\alpha_1},\dots,p_n^{\alpha_s}),$$

is

$$\text{u.d.}$$

with

$$D^*_{\pi(N)}=\mathcal{O}\left(\frac{\log^{s+9}N}{N^\delta}\right),$$

where

$$\delta=\frac{1}{3}\min_{i,j=1,\dots,s}\left(\frac{1}{4},1-\alpha_j,\alpha_i,|\alpha_i-\alpha_j|\right).$$

Notes: I.D. Tolev (1991). On the other hand, S.Srinivasan and R.F. Tichy (1993) proved that $D^*_{\pi(N)}=\mathcal{O}((\log N)/N^\delta)$ for any s–tuple of distinct positive non–integral exponents $\alpha_1,\dots,\alpha_s$. They conjecture that $D^*_{\pi(N)}=\mathcal{O}((\log N)/N^\delta)$ for the s–dimensional sequence $(p_{n+1}^{\alpha_1},\dots,p_{n+s}^{\alpha_s})$.

S. Srinivasan – R.F. Tichy: *Uniform distribution of prime power sequences*, Anz. Österreich. Akad. Wiss. Math.–Natur. Kl. **130** (1993), 33–36 (MR1294872 (95h:11071); Zbl. 0807.11037).
D.I. Tolev: *On the simultaneous distribution of the fractional parts of different powers of prime numbers*, J. Number Theory **37** (1991), 298–306 (MR1096446 (92d:11085); Zbl. 0724.11043).

3.6.2. Let
- $p_1 < p_2 < \dots$ be an arbitrary increasing sequence of prime numbers for which $p_{\nu+1} < e^{\frac{1}{3}p_\nu^2}$, and
- $\psi(\nu)$ be an arbitrary integer–valued arithmetical function such that $\nu\left(\frac{p_{\nu+1}}{p_\nu}\right)^3 < \psi(\nu) < e^{p_\nu^2}$.

Moreover define
- two sequences τ_ν, n_ν, $\nu = 1, 2, \dots$, by the relations $\tau_\nu = p_\nu^2(p_\nu - 1)$, $n_{\nu+1} = n_\nu + \psi(\nu)\tau_\nu$ with $n_1 = 0$.

If
$$f(x) = \sum_{\nu=1}^{\infty}\left(\frac{1}{q^{n_\nu}} - \frac{1}{q^{n_{\nu+1}}}\right)\frac{x^\nu}{p_\nu^2}, \quad \text{where } q > 1 \text{ is an integer,}$$
then, for every $s \geq 1$, the sequence
$$\mathbf{x}_n = \bigl(f(n+1)q^{n+1}, \dots, f(n+s)q^{n+s}\bigr) \bmod 1$$
is
$$\text{u.d.}$$
i.e. $f(n)q^n \bmod 1$ is completely u.d.

N.M. Korobov: *On completely uniform distribution and conjunctly normal numbers*, (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. **20** (1956), 649–660 (MR0083522 (18,720d); Zbl. 0072.03801).

3.6.3. Let p_n denote the nth prime and let $N_n = \left[1 + e^{n^3}\right]$ for every $n = 1, 2, \dots$. Define the block
$$A_n = \bigl(\log p_1, \dots, \log p_n, 2\log p_1, \dots, 2\log p_n, \dots, N_n \log p_1, \dots, N_n \log p_n\bigr) \bmod 1$$
Then the block sequence
$$\omega = (A_n)_{n=1}^{\infty}$$
is
$$\text{completely u.d.}$$

NOTES: This example was given by L.P. Starčenko (1959), and another proof can be found in A.G. Postnikov (1960).

A.G. POSTNIKOV: *Arithmetic modeling of random processes*, Trudy Math. Inst. Steklov. (Russian), **57** (1960), 1–84 (MR0148639 (**26** #6146); Zbl. 0106.12101).
L.P. STARČENKO: *The contribution of a completely uniformly distributed sequence* (Russian), Dokl. Akad. Nauk SSSR **129** (1959), 519–521 (MR0108474 (**21** #7190); Zbl. 0087.04401).

3.6.4. Let $\sigma(f)$ be the abscissa of absolute convergence of the general Dirichlet series $f(z) = \sum_{n=1}^{\infty} a_n n^{-z}$ (not necessarily possessing the Euler product decomposition). Let p_n be the sequence of primes in the ascending order. For any real sequence x_n the following statements are equivalent:
(i) If f is a Dirichlet series f and $\sigma > \sigma(f)$ then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |f(\sigma + ix_n)|^2 = \lim_{T\to\infty} \frac{1}{T} \int_0^T |f(\sigma + it)|^2 \, \mathrm{d}t,$$

(ii) For every integer $s > 0$, the s–dimensional sequence

$$\frac{x_n}{2\pi}(\log p_1, \log p_2, \dots, \log p_s) \bmod 1, \quad n = 1, 2, \dots,$$

is u.d. in $[0, 1]^s$,
(iii) The ∞–dimensional sequence

$$\frac{x_n}{2\pi}(\log p_1, \log p_2, \dots, \log p_s, \dots) \bmod 1, \quad n = 1, 2, \dots,$$

is u.d. in $[0, 1]^\infty$, i.e. with respect to the compact abelian group $(\mathbb{R}/\mathbb{Z})^\infty$.
For example conditions (i) – (iii) are satisfied for

$$x_n = p_n; \quad x_n = n; \quad \text{or} \quad \text{a} \;\; x_n \text{ u.d. } \bmod 1.$$

NOTES: This was proved by A. Reich (1981) who also gives a countable set $A \subset \mathbb{R}$ so that (i) holds for all $x_n = \theta n$ with $\theta \notin A$ (for instance $1 \notin A$ and $\alpha\pi \notin A$ with $\alpha \neq 0$ algebraic).

A. REICH: *Dirichletreihen und gleichverteilte Folgen*, Analysis **1** (1981), 303–312 (MR0727881 (85g:11061); Zbl. 0496.10026).

3.6.5. If $p_1, \dots, p_s$ is a finite sequence of different primes then $1, \sqrt{p_1}, \dots, \sqrt{p_s}$ are linearly independent over $\mathbb{Z}$, and consequently Kronecker sequence 3.4.1 of the type

$$\mathbf{x}_n = (n\sqrt{p_1}, \dots, n\sqrt{p_s}) \bmod 1$$

is

u.d.

NOTES: (I) This sequence $\mathbf{x}_n$ was firstly employed by R.D. Richtnyer (1951). F. James, J. Hoogland and R. Kleiss (1997) reportes that it behaved better in a dimension of about 15 than the other types of sequences they tested. For a dimension $s > 2$ it is not known whether it is a low discrepancy sequence.
(II) The linear independence of $1, \sqrt{p_1}, \dots, \sqrt{p_s}$ over $\mathbb{Q}$ follows from the following theorem of A.S. Besicovitch (1940): Let $a_i = b_i p_i$, $i = 1, 2, \dots, s$, where p_i are different primes, b_i are positive integers not divisible by any of these primes and m_i are positive integers. If $\theta_i = a_i^{1/m_i}$ are positive real roots and $P(\theta_1, \dots, \theta_s)$ is a polynomial with rational coefficients of a degree less than or equal to $m_i - 1$ with respect to θ_i, then $P(\theta_1, \dots, \theta_s)$ can vanish only if all its coefficients vanish. For a generalization cf. L.J. Mordell (1953) or 3.4.1.
(III) Since the roots $\sqrt{p_1}, \dots, \sqrt{p_s}$ generate an algebraic number field of degree 2^s over $\mathbb{Q}$, H. Niederreiter (1978, p. 994) notes that the choices from 3.6.6 and 3.6.7 are to be preferred because in these cases the coordinates belong to an algebraic number field of degree $s + 1$ over $\mathbb{Q}$.

A.S. BESICOVITCH: *On the linear independence of fractional powers of integers*, J. London Math. Soc. **15** (1940), 3–6 (MR0002327 (2,33f); Zbl. 0026.20301).
F. JAMES – J. HOOGLAND – R. KLEISS: *Multidimensional sampling for simulation and integration:measures, discrepancies, and quasi–random numbers*, Comp. Phys. Comm **99** (1997), 180–220 (Zbl. 0927.65041).
L.J. MORDELL: *On the linear independence of algebraic numbers*, Pacific J. Math. **3** (1953), 625–630 (MR0058649 (15,404e); Zbl. 0051.26801).
R.D. RICHTNYER: *The evaluation of definite integrals, and quasi–Monte Carlo method based on the properties of algebraic numbers*, Report LA–1342, Los Almos Scientific Laboratory, Los Almos, NM, 1951.

3.6.6. If p is a prime of the form $p = 2s + 3$ and

$$\boldsymbol{\theta} = \left(2\cos\left(\frac{2\pi}{p}\right), 2\cos\left(\frac{4\pi}{p}\right), \dots, 2\cos\left(\frac{2\pi s}{p}\right)\right)$$

then the Kronecker sequence

$$\mathbf{x}_n = n\boldsymbol{\theta} \bmod 1$$

is

u.d.

NOTES: In Niederreiter (1972) an error in the quasi–Monte Carlo integration similar to that in 3.4.1(X) was proved using algebraic irrational points satisfying certain conditions. The above algebraic irrational points were proposed in H. Niederreiter (1978, p. 994) as an example (cf. also 3.6.7) of algebraic irrational points fulfilling these consitions.

H. NIEDERREITER: *On a number–theoretic integration method*, Aequationes Math. **8** (1972), 304–311 (MR0319910 (**47** #8451); Zbl. 0252.65023).
H. NIEDERREITER: *Quasi–Monte Carlo methods and pseudo–random numbers*, Bull. Amer. Math. Soc. **84** (1978), no. 6, 957–1041 (MR0508447 (80d:65016); Zbl. 0404.65003).

3.6.7. If p is a prime, $\xi = p^{\frac{1}{s+1}}$ and

$$\boldsymbol{\theta} = (\xi, \xi^2, \dots, \xi^s)$$

then the Kronecker sequence

$$\mathbf{x}_n = n\boldsymbol{\theta} \bmod 1$$

is

u.d.

NOTES: In Niederreiter (1972) an error in the quasi–Monte Carlo integration similar to that in 3.4.1(X) was proved using algebraic irrational points satisfying certain conditions. The above algebraic irrational points were proposed in H. Niederreiter (1978, p. 994) as an example (cf. also 3.6.6) of algebraic irrational points fulfilling these consitions.

H. NIEDERREITER: *On a number–theoretic integration method*, Aequationes Math. **8** (1972), 304–311 (MR0319910 (**47** #8451); Zbl. 0252.65023).
H. NIEDERREITER: *Quasi–Monte Carlo methods and pseudo–random numbers*, Bull. Amer. Math. Soc. **84** (1978), no. 6, 957–1041 (MR0508447 (80d:65016); Zbl. 0404.65003).

3.6.8. If $p \geq 5$ is a prime and $a_1, \dots, a_s$, $b_1, \dots, b_s$ are integers such that $b_1 a_1^{-1}, \dots, b_s a_s^{-1}$ are distinct $\pmod p$ and

$$\mathbf{z}_n = \left((a_1 n + b_1)^{-1}, \dots, (a_s n + b_s)^{-1}\right) \pmod p,$$

then the sequence

$$\frac{\mathbf{z}_n}{p} \bmod 1, \quad n = 0, 1, 2, \dots, p-1,$$

has discrepancy

$$D_N < 1 - \left(1 - \frac{1}{p}\right)^s + \\ + \left(\frac{2s-2}{\sqrt{p}} + \frac{s+1}{p} + \frac{s}{N}(2\sqrt{p}+1)\left(\frac{4}{\pi^2}\log p + 0.38 + \frac{0.64}{p}\right)\right) \times \\ \times \left(\frac{4}{\pi^2}\log p + 1.38 + \frac{0.64}{p}\right)^s$$

for $s \geq 1$ and $N < p$.

H. Niederreiter: *On a new class of pseudorandom numbers for simulation methods*, (In: Stochastic programming: stability, numerical methods and applications (Gosen, 1992)), J. Comput. Appl. Math. **56** (1994), 159–167 (MR1338642 (96e:11101); Zbl. 0823.65010).

3.6.9. Let $p_1, \dots, p_s$ be a finite sequence of different primes of the form $4k+1$. In the ring of Gaussian integers $\mathbb{Z}(\sqrt{-1})$ they can be decomposed as $p_j = \pi_j.\overline{\pi}_j$, where π_j, $j = 1, \dots, j$, are Gaussian primes. Then the s–dimensional sequence

$$\mathbf{x}_n = \left(\frac{n.\arg(\pi_1/|\overline{\pi}_1|)}{2\pi}, \dots, \frac{n.\arg(\pi_s/\overline{\pi}_s)}{2\pi}\right) \bmod 1$$

is

u.d.

and for its discrepancy we have

$$D_N \leq c\frac{(\log N)^s}{N^{1/s}}$$

with an absolute constant c.

E. Hlawka: *Gleichverteilung und die willkürlichen Funktionen von Poincaré, Teil II*, Österreich. Akad. Wiss. Math.–Natur. Kl. Sitzungsber. II **208** (1999), 31–78 (MR1908803 (2003h:11087); Zbl. 1004.11045).

3.6.10. Let $h_1(\mathbf{x}), \dots, h_k(\mathbf{x})$ be polynomials where $\mathbf{x} = (x_1, \dots, x_s)$, and p be a prime. Let A_p be the block of s–dimensional points of the form

$$\frac{\mathbf{x}}{p} = \left(\frac{x_1}{p}, \dots, \frac{x_s}{p}\right) \bmod 1,$$

where $\mathbf{x}$ runs through the all different values $\mathbf{x} \pmod p$ for which $h_1(\mathbf{x}) \equiv \dots \equiv h_k(\mathbf{x}) \equiv 0 \pmod p$. If

(i) $h_1(\mathbf{x}),\dots,h_k(\mathbf{x})$ are all of degree of at least 2 and at most d,
(ii) the system $h_1(\mathbf{x})=\cdots=h_k(\mathbf{x})=0$ defines an absolute variety of the dimension $s-k$ over rationals,
(iii) for all sufficiently large primes p and all k–tuples $(a_1,\dots,a_k)$ of integers which are not all divisible by p, the hypersurface $a_1h_1(\mathbf{x})+\cdots+a_kh_k(\mathbf{x})$ over $\mathbb{Z}/p\mathbb{Z}$ is non–singular at infinity, and
(iv) $s>2k$,

then the sequence A_p of blocks with p running over the primes is

u.d.

G. Myerson: *The distribution of rational points on varieties defined over a finite field*, Mathematika **28** (1981), 153–159 (MR0645095 (83h:10041); Zbl. 0469.10002).

3.6.11. Let $f\ge 2$ be fixed and p be a prime such that $p\equiv 1 \pmod f$. Let A_p be the block of s–dimensional points

$$\left(\frac{a}{p},\frac{a\zeta}{p},\dots,\frac{a\zeta^{s-1}}{p}\right),\quad a=0,\dots,p-1,$$

where ζ is a primitive fth root of unity $(\operatorname{mod} p)$ and $s=\varphi(f)$. Then the sequence of individual blocks A_p, as p goes to infinity, is

u.d.

Notes: G. Myerson (1993, p. 186, Th. 57), also cf. 3.15.1(IV).

G. Myerson: *A sampler of recent developments in the distribution of sequences*, in: *Number theory with an emphasis on the Markoff spectrum (Provo, UT 1991)*, (A.D. Pollington and W. Moran eds.), Lecture Notes in Pure and App.Math., Vol. 147, Marcel Dekker, New York, Basel, Hong Kong, 1993, pp. 163–190 (MR1219333 (94a:11112); Zbl. 0789.11043).

3.7 Sequences involving number–theoretical functions

3.7.1. Given a rational number r, let $s(r)$ be the Dedekind sum as defined in 2.20.30, and let $v(n)$ be the Farey sequence of the reduced rational numbers in $[0,1)$ ordered by increasing denominators. Then for any non–zero real number α, the 2–dimensional sequence

$$\mathbf{u}(n)=\big(v(n),\alpha s(v(n))\big)$$

is

u.d.

G. Myerson: *Dedekind sums and uniform distribution*, J. Number Theory **28** (1988), 233–239 (MR0932372 (89e:11026); Zbl. 0635.10033).

3.7.2. Let $a_1 = 1 < a_2 < \cdots < a_{\varphi(n)}$, $0 < a_i < n$, be the sequence of all integers coprime to n and define a_i^* by the congruence $a_i a_i^* \equiv 1 \pmod n$. Then the sequence of blocks

$$A_n = \left(\left(\frac{a_1}{n}, \frac{a_1^*}{n}\right), \left(\frac{a_2}{n}, \frac{a_2^*}{n}\right), \dots, \left(\frac{a_{\varphi(n)}}{n}, \frac{a_{\varphi(n)}^*}{n}\right)\right), \quad n = 1, 2, \dots,$$

is

u.d.

and for its discrepancy we have

$$D_{\varphi(n)} \le 17 \left(\frac{3}{2}\right)^2 \frac{d(n)\sqrt{n}}{\varphi(n)} (\log \varphi(n))^2$$

for $n \ge 8$.

Notes: (I) This follows from the classical bound for the Kloosterman sums (cf. O. Strauch, M. Paštéka and G. Grekos (2003))

$$\left| \sum_{j=1}^{\varphi(n)} e^{2\pi i \left(a \frac{a_j}{n} + b \frac{a_j^*}{n}\right)} \right| \le \sqrt{(a, b, n)}\, d(n)\sqrt{n}$$

where a, b, $b \ne 0$, are integers, $d(n)$ is the divisor function and (a, b, n) denotes the greatest common divisor of a, b, and n.

(II) If $n = p^\alpha$ is a power of an odd prime p with $\alpha > 2$, and $p \nmid a, b$ then the result mentioned in (I) was proved by H. Salié (1931) in the form $\le cp^{\frac{\alpha}{2}}$ with an absolute constant c, see also A.L. Whiteman (1945).

(III) If $n = p$ is an odd prime and $p \nmid a, b$ then (I) was proved by A. Weil (1948) in the form $\le 2\sqrt{p}$.

(IV) The estimate (I) was proved by T. Esterman (1961) for general n, see also H.M. Andruhaev (1964).

(V) Multiple Kloosterman sums were introduced by A.V. Malyšev (1960), L. Carlitz (1965) and others. For a history cf. the book by R. Lidl and H. Niederreiter (1983, Chap. 5, Comments).

(VI) It is immediate that the continued fraction expansion $\frac{a_i}{n} = [0; b_1, b_2, \dots, b_k]$ implies

$$[0; b_k, b_{k-1}, \dots, b_1] = \begin{cases} \dfrac{a_i^*}{n}, & \text{if } k \text{ is odd,} \\ \dfrac{n - a_i^*}{n}, & \text{if } k \text{ is even.} \end{cases}$$

(VII) Let p_1, p_2 be two primes, $N = p_1p_2$, $n = \varphi(N) = (p_1-1)(p_2-1)$. In the RSA public key cryptosystem with modulus N, the public exponent a_i and the secret one a_i^* are related by $a_ia_i^* \equiv 1 \pmod{n}$. The encryption and decryption algorithms are $C \equiv M^{a_i} \pmod{N}$, $M \equiv C^{a_i^*} \pmod{N}$, where M is a message to be encrypted. M.J. Wiener (1990) proved that if $p_1 < p_2 < 2p_1$ and $a_i < \frac{1}{3}n^{1/4}$, then a_i^* is the denominator of a convergent of the continued fraction expansion of $\frac{a_i}{N}$ and he described a polynomial time algorithm for computing a_i^* and recovering p_1, p_2 (for some improvements see A. Dujella (2003)).
(VIII) An open problem is to characterize n's for which the interval $I = [0,1] \times [0, 1/(3n^{3/4})]$ contains some elements of A_n $\big($e.g. to characterize n's with discrepancy $D_n < |I| = \frac{1}{3n^{3/4}}\big)$.

H.M. Andruhaev: *A sum of Kloosterman type*, in: *Certain Problems in the Theory of Fields*, Izd. Saratov. Univ., Saratov, 1964, pp. 60–66 (MR0205939 (**34** #5764); Zbl. 0305.10032).
A. Dujella: *Continued fractions and RSA with small secret exponent*, Tatra. Mt. Math. Publ. **29** (2004), 101–112 (MR2201657 (2006j:94062); Zbl. 1114.11008).
T. Estermann: *On Kloosterman's sum*, Mathematika **8** (1961), 83–86 (MR0126420 (**23** #A3716); Zbl. 0114.26302).
R. Lidl – H. Niederreiter: *Finite Fields*, Encyclopeadia of Mathematics and its Applications, Vol. 20, Addison – Wesley Publishing Company, Reading, MA, 1983 (MR0746963 (86c:11106); Zbl. 0554.12010).
A.V. Malyšev: *Gauss and Kloosterman sums*, (Russian), Dokl. Akad. Nauk SSSR **133** (1960), 1017–1020 (English translation: Soviet. Math. Dokl. **1** (1961), 928–932 (MR0126419 (**23** #A3715); Zbl. 0104.04204)).
O. Strauch – M. Paštéka – G. Grekos: *Kloosterman's uniformly distributed sequence*, J. Number Theory **103** (2003), no. 1, 1–15 (MR2008062 (2004j:11081); Zbl. 1049.11083).
A. Weil: *On some exponential sums*, Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 204–207 (MR0027006 (10,234e); Zbl. 0032.26102).
A.L. Whiteman: *A note on Kloosterman sums*, Bull. Amer. Math. Soc. **51** (1945), 373–377 (MR0012105 (6,259f); Zbl. 0060.10903).
M.J. Wiener: *Cryptanalysis of short RSA secret exponents*, IEEE Trans. Inform. Theory **36** (1990), no. 3, 553–558 (MR1053848 (91f:94018); Zbl. 0703.94004).

3.7.2.1 Given a prime number $p > 2$ and an integer n, $0 < n < p$, define n^* by the congruence $nn^* \equiv 1 \pmod{p}$, $0 < n^* < p$. Then the s-dimensional sequence

$$\left(\frac{n^*}{p}, \frac{(n+1)^*}{p}, \dots, \frac{(n+s-1)^*}{p}\right), \qquad n = 1, 2, \dots, p,$$

$$\text{is u.d. as } p \to \infty,$$

and the discrepancy bound is

$$D_p^* = \mathcal{O}\left(\frac{(\log p)^s}{\sqrt{p}}\right)$$

for all $s \geq 2$, and that this estimate is essentially best possible up to the logarithmic factor.

NOTES:
(I) H. Niederreiter (1994). A generalization is given in H. Niederreiter and A. Winterhof (2000).
(II) Tsz Ho Chan (2004) proved that

$$\frac{1}{p}\sum_{n=1}^{p-2}\left|\frac{n^*}{p}-\frac{(n+1)^*}{p}\right| = \frac{1}{3}+\mathcal{O}\left(\frac{(\log p)^3}{\sqrt{p}}\right)$$

for every prime $p > 2$.

H. NIEDERREITER: *Pseudorandom vector generation by the inverse method*, ACM Trans. Model. Comput. Simul. **4** (1994), no. 2, 191–212 (Zbl. 0847.11039).
H. NIEDERREITER – A. WINTERHOF: *Incomplete exponential sums over finite fields and their applications to new inverse pseudorandom number generators*, Acta Arith. **XCIII** (2000), no. 4, 387–399 (MR1759483 (2001d:11120); Zbl. 0969.11040).
TSZ HO CHAN: *Distribution of difference between inverses of consecutive integers modulo P*, J. Number Theory (to appear).

3.7.3. Let $1 = a_1 < a_2 < \cdots < a_{\varphi(n)} = n - 1$ be the integers coprime to n. Denote by A_n the block

$$A_n = \left(\frac{a_2-a_1}{n/\varphi(n)}, \frac{a_3-a_2}{n/\varphi(n)}, \dots, \frac{a_{\varphi(n)}-a_{\varphi(n)-1}}{n/\varphi(n)}\right)$$

and its s–fold cartesian product denote by

$$\mathbf{A}_n = A_n \times \cdots \times A_n$$

The sequence $\mathbf{A}_n$ of individual blocks has relative to the sequence of indices n for which $n/\varphi(n) \to \infty$ in $[0,\infty)^s$ the d.f. $g(\mathbf{x})$ of the form

$$g(\mathbf{x}) = \prod_{i=1}^{s}(1-e^{-x_i}), \quad \text{where } \mathbf{x} = (x_1,\dots,x_s).$$

RELATED SEQUENCES: For the one–dimensional version cf. 2.23.3.

CH. HOOLEY: *On the difference between consecutive numbers prime to n. III*, Math. Z. **90** (1965), 355–364 (MR0183702 (**32** #1182); Zbl. 0142.29202).

3.7.4. For $n = 1, 2, \dots,$ define the block

$$A_n = \left(\left(\frac{1}{n},\frac{1^2}{n}\right),\left(\frac{2}{n},\frac{2^2}{n}\right),\dots,\left(\frac{n}{n},\frac{n^2}{n}\right)\right) \bmod 1.$$

Then the sequence A_n, $n = 1, 2, \dots$, of individual blocks is

u.d.

NOTES: This follows directly from the Weyl criterion and from the well–known estimate for the quadratic Gauss sum

$$\sum_{x=1}^{n} e^{2\pi i\left(\frac{ax^2+bx}{n}\right)} = \mathcal{O}(\sqrt{n}).$$

3.7.5. Let $0 \le \alpha < \beta \le 1$ be fixed. For large N consider the set H_N of pairs (p, q) with coprime coordinates p, and q which satisfy the conditions $0 \le p < q$ and $\alpha N < q < \beta N$. Given a couple (p, q), let x be the integer solution of the diophantine equation $px - qy = 1$ which satisfies $|x| \le q/2$ (i.e. x is the least absolute solution). For our H_N define the block

$$A_N = \left\{ \left(\frac{p}{q}, \frac{|x|}{q} \right) \; ; \; (p, q) \in H_N \right\}.$$

Then the sequence A_N, $N = 1, 2, \dots$, of individual blocks is

u.d.

which respect to the interval $[0, 1] \times [0, 1/2]$ and for its extremal discrepancy we have

$$D_{\#H_N} = \mathcal{O}\left(\frac{1}{\sqrt{N}} \right).$$

NOTES: D.I. Dolgopyat (1994) extended in this way 2.20.37 of E.I. Dinaburg and Ya.G. Sinaĭ. Here the u.d. is defined by

$$F_N(x, y) = \frac{\#\{(p, q) \in H_N \; ; \; p/q \in [0, x), |x|/q \in [0, y)\}}{\#H_N} \to 2xy \quad \text{as } N \to \infty$$

and the extremal discrepancy by

$$D_{\#H_N} = \sup_{I \subset [0,1], I' \subset [0,1/2]} \left| \frac{\#\{(p, q) \in H_N \; ; \; p/q \in I, |x|/q \in I'\}}{\#H_N} - 2|I||I'| \right|.$$

RELATED SEQUENCES: For the one–dimensional case cf. 2.20.37.

E.I. DINABURG – YA.G. SINAĬ: *The statistics of the solutions of the integer equation $ax - by = \pm 1$*, (Russian), Funkts. Anal. Prilozh. **24** (1990), no. 3, 1–8,96 (English translation: Funct. Anal. Appl. **24** (1990), no. 3, 165–171). (MR1082025 (91m:11056); Zbl. 0712.11018).

D.I. DOLGOPYAT: *On the distribution of the minimal solution of a linear diophantine equation with random coefficients*, (Russian), Funkts. Anal. Prilozh. **28** (1994), no. 3, 22–34, 95 (English translation: Funct. Anal. Appl. **28** (1994), no. 3, 168–177 (MR1308389 (96b:11111); Zbl. 0824.11046)).

3.7.6. Let $\varphi(n)$ denote the Euler function and $\sigma(n)$ the sum of divisors of n. Put

$$x_n = \frac{\varphi(n)}{\varphi(n-1)}, \qquad \text{for } n = 2, 3, \dots,$$

or

$$x_n = \frac{\sigma(n)}{\sigma(n-1)}, \qquad \text{for } n = 2, 3, \dots .$$

Then for every $s \geq 1$ the s–dimensional sequence

$$(x_{n+1}, \dots, x_{n+s})$$

is

$$\text{dense in } [0, \infty)^s,$$

i.e. x_n is completely dense in $[0, \infty)$. Quantitatively, for every $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_s) \in [0, \infty)^s$ and every $\varepsilon > 0$ there exist positive constants $c = c(\boldsymbol{\alpha}, \varepsilon)$ and $x_0 = x_0(\boldsymbol{\alpha}, \varepsilon)$ such that the number of positive integers $n \leq x$ which satisfy $|x_{n+i} - \alpha_i| < \varepsilon$ for $i = 1, 2, \dots, s$, is greater than $cx/\log^{s+1} x$, whenever $x > x_0$.

Notes:
(I) The complete density of x_n was proved A. Schinzel (1954, 1955).
(II) This quantitative result was proved by A. Schinzel and Y. Wang (1958). They also proved the complete density of $d(n)/d(n-1)$ where $d(n)$ is the divisor function.
(III) P.–T. Shao (1956) extended this result to all multiplicative positive functions $f_k(n)$, which satisfy the following conditions:
(i) $\lim_{p\to\infty}(f_k(p^l)/p^{lk}) = 1$ for any positive integer l, where p runs over the all primes,
(ii) there exists an interval $[a, b]$ with $a = 0$ or $b = \infty$ such that for any integer $M > 0$ the set of numbers $f_k(N)/N^k$ with $(N, M) = 1$ is dense in $[a, b]$.
(IV) P. Erdős and A. Schinzel (1961) generalized the results of A. Schinzel and Y. Wang (1958) and P.–T. Shao (1956) to all positive multiplicative functions $f_k(x)$ satisfying $\sum_p (f_k(p) - p^k)^2 p^{-2k-1} < \infty$ (the sum is over primes) and satisfying (ii) and with the lover estimate $cx/\log^{s+1} x$ replaced by cx.

P. Erdős – A. Schinzel: *Distributions of the values of some arithmetical functions*, Acta Arith. **6** (1960/1961), 437–485 (MR0126410 (**23** #A3706); Zbl 0104.27202).
A. Schinzel: *Quelques théoremes sur les fonctions $\varphi(n)$ et $\sigma(n)$*, Bull. Acad. Polon. Sci. Cl. III **2** (1954), 467–469 (MR0067141 (16,675g); Zbl. 0056.27003).

A. Schinzel: *On functions $\varphi(n)$ and $\sigma(n)$*, Bull. Acad. Polon. Sci. Cl. III **3** (1955), 415–419 (MR0073625 (17,461c); Zbl. 0065.27103).
A. Schinzel – Y. Wang: *A note on some properties of the functions $\phi(n)$, $\sigma(n)$ and $\theta(n)$*, Bull. Acad. Polon. Sci. Cl. III **4** (1956), 207–209 (MR0079024 (18,17c); Zbl. 0070.04201).
A. Schinzel – Y. Wang: *A note on some properties of the functions $\phi(n)$, $\sigma(n)$ and $\theta(n)$*, Ann. Polon. Math. **4** (1958), 201–213 (MR0095149 (**20** #1655); Zbl. 0081.04203).
P.–T. Shao: *On the distribution of the values of a class of arithmetical functions*, Bull. Acad. Polon. Sci. Cl. III **4** (1956), 569–572 (MR0083514 (18,719d); Zbl. 0072.03304).

$\boxed{\textbf{3.7.6.1}}$ Let $\varphi(n)$ be the Euler function and F_n be the nth Fibonacci number. The sequence

$$\left(\frac{\varphi(F_{n+1})}{\varphi(F_n)}, \frac{\varphi(F_{n+2})}{\varphi(F_n)}, \dots \frac{\varphi(F_{n+k})}{\varphi(F_n)}\right), \quad n = 1, 2, \dots$$

is dense in $[0, \infty)^k$ for every $k = 1, 2, \dots$.

Notes: F. Luca, V.J. Mejía Huguet and F. Nicolae (2009). They have the following comments:

- for any positive integer k and every permutation $(i_1, \dots, i_k)$ there exist infinitely many integers n such that $\varphi(F_{n+i_1}) < \varphi(F_{n+i_2}) < \dots < \varphi(F_{n+i_k})$.
- P. Erdős, K. Győry and Z. Papp (1980) call two arithmetic functions $f(n)$ and $g(n)$ *independent* if for every couple of permutations $(i_1, \dots, i_k)$ and $(j_1, \dots, j_k)$ of $(1, \dots, k)$, there exist infinitely many integers n such that both
$f(n + i_1) < f(n + i_2) < \dots < f(n + i_k)$,
$g(n + j_1) < g(n + j_2) < \dots < g(n + j_k)$.
- $\varphi(n)$ and Carmichael $\lambda(n)$ are independent (N. Doyon and F. Luca (2006)).
- $\sigma(\varphi(n))$ and $\varphi(\sigma(n))$ are indepedent (M.O. Hername and F. Luca (2009)).

Open problems (F. Luca, V.J. Mejía Huguet and F. Nicolae (2009)):
- Are the functions $\varphi(F_n)$ and $F_{\varphi(n)}$ independent?
- Are the functions $\varphi(F_n)$ and $\varphi(M_n)$ independent?

N Doyon – F. Luca: *On the local behavior of the Carmichael λ-function*, Michigen Math. J. **54** (2006), 283–300 (MR2253631 (2007i:11133); Zbl. 1112.11047).
P. Erdős – K. Győry – Z. Papp: *On some new properties of functions $\sigma(n)$, $\varphi(n)$, $d(n)$* and *$\nu(n)$*, Mat. Lapok **28** (1980), 125–131 (MR0593425 (82a:10004); Zbl. 0453.10004).
M.O. Hernane – F. Luca: *On the independence of phi* and *σ*, Acta Arith. **138** (2009), 337–346 (MR2534139 (2010f:11147); Zbl. 1261.11061).
F. Luca – V.J. Mejía Huguet – F. Nicolae: *On the Euler function of Fibonacci numbers*, J. Integer Sequences **9** (2009), A09.6.6 (MR2544925 (2010h:11005); Zbl. 1201.11006).

3.7.7. Let $f(n) \geq 1$ be a multiplicative arithmetical function which fulfils the conditions

(i) $\lim_{n\to\infty}(f(p_n))^n = 1$, where p_n is nth prime,
(ii) there exists a positive $c > 0$ such that $f(p^\alpha) \le (f(p))^\alpha + (c/p^2)$ holds for every prime p and positive α,
(iii) $\prod_{n=1}^{\infty} f(p_n) = \infty$.

Then for every $s \ge 1$ the s–dimensional sequence

$$(f(n+1), \dots, f(n+s)), \quad n = 1, 2, \dots,$$

is

$$\text{dense in } [1, \infty)^s,$$

i.e. $f(n)$ is completely dense in $[1, \infty)$. As a consequence we have that any of the following sequences

$$\frac{\sigma(n)}{n}, \quad \frac{n}{\varphi(n)}, \quad \frac{\sigma(n)}{\varphi(n)}$$

are completely dense in $[1, \infty)$.

Notes: This result was proved by J.T. Tóth (1997, Th. 20). J. Bukor and J.T. Tóth (1998) gave the following related result: Let $f(n) > 0$ be a multiplicative arithmetical function which satisfies
(i) $\lim_{n\to\infty}(f(p_n))^n = 1$, where p_n is nth prime,
(ii) $\prod_{\substack{n=1\\ f(p_n)<1}}^{\infty} f(p_n) = 0$,
(iii) $\prod_{\substack{n=1\\ f(p_n)>1}}^{\infty} f(p_n) = \infty$.

Then the sequence

$$f(n), \quad n = 1, 2, \dots,$$

is completely dense in $[0, \infty)$.

J. Bukor – J.T. Tóth: *On completely dense sequences*, Acta Math. Inform. Univ. Ostraviensis **6** (1998), no. 1, 37–40 (MR1822513 (2001k:11147); Zbl. 1024.11052).
J.T. Tóth: *Everywhere dense ratio sequences* (Slovak), Ph.D. Thesis, Comenius' University, Bratislava, Slovakia, 1997.

3.7.8. Let $f(n)$ be an additive arithmetical function which satisfies
(i) $\sum_p \frac{|f(p)|^*}{p} < \infty$,
(ii) $\sum_p \frac{(f(p)^*)^2}{p} < \infty$,

(iii) $\sum_{f(p)\neq 0} \frac{1}{p} = \infty$,

where $x^* = x$ for $|x| \le 1$ and $x^* = 1$ for $|x| > 1$. Then for every $s \ge 1$ the s–dimensional sequence

$$\big(f(n+1), \dots, f(n+s)\big)$$

has in $(-\infty, \infty)^s$ the a.d.f.

$$g(\mathbf{x}) \text{ which is continuous.}$$

NOTES: This result was proved by P. Erdős and A. Schinzel (1961, Th. 3). A similar result was proved by I. Kátai (1969) (cf. A.G. Postnikov (1971, p. 366)).

P. ERDŐS – A. SCHINZEL: *Distributions of the values of some arithmetical functions*, Acta Arith. **6** (1960/1961), 437–485 (MR0126410 (**23** #A3706); Zbl 0104.27202).
I. KÁTAI: *On the distribution of arithmetical functions*, Acta Math. Acad. Scient. Hungar. **20** (1969), no. 1–2, 69–87 (MR0237446 (**38** #5728); Zbl. 0175.04103).
A.G. POSTNIKOV: *Introduction to Analytic Number Theory*, (Russian), Izd. Nauka, Moscow, 1971 (MR0434932 (**55** #7895); Zbl. 0231.10001). (for the English translation see (MR0932727 (89a:11001); Zbl. 0641.10001)).

3.7.9. The two–dimensional sequence

$$\left(\log\left(\frac{\phi(n)}{n}\right), \log\left(\frac{\sigma(n)}{n}\right)\right)$$

has in $(-\infty, \infty)^2$ the a.d.f.

$$g(x, y)$$

which characteristic function is

$$\prod_p \left(1 - \frac{1}{p}\right)\left(1 + \sum_{r=1}^{\infty} \frac{1}{p^r}\left(1 - \frac{1}{p}\right)^{i(s-t)}\left(1 - \frac{1}{p^{irt}}\right)\right)$$

and for its star discrepancy we have

$$D_N^* = \mathcal{O}\left(\frac{\log_2^2 N}{\log N \log_3 N}\right).$$

NOTES: This was proved by A.S. Badarëv (1972), cf. D.S. Mitrinović, J. Sándor and J. Crstici (1996, p. 95).

A.S. BADARËV: *A two–dimensional generalized Esseen inequality and the distribution of the values of arithmetic functions* (Russian), Taškent Gos. Univ. Naučn. Trudy (1972), no. 418, Voprosi Math., 99–110, 379 (MR0344214 (**49** #8954)).

D.S. Mitrinović – J. Sándor – J. Crstici: *Handbook of Number Theory*, Mathematics and its Applications, Vol. 351, Kluwer Academic Publishers Group, Dordrecht, Boston, London, 1996 (MR1374329 (97f:11001); Zbl. 0862.11001).

3.7.10. Let $\rho(n) = \beta(n) + i\gamma(n)$ be the sequence of the non–trivial zeros of the Riemann zeta function ζ in the upper half of the critical strip and ordered by $0 < \gamma(1) \leq \gamma(2) \leq \dots$. Then, for every s–tuple $\alpha_1, \dots, \alpha_s$ linearly independent over the rationals, the sequence

$$\mathbf{x}_n = (\alpha_1\gamma(n), \dots, \alpha_s\gamma(n)) \bmod 1$$

is

u.d.

Notes: (I) In 1956 H.A. Rademacher (1974, p. 455) proved that under the Riemann hypothesis the sequence $t\gamma(n)$ is u.d. for every non–zero real t. P.D.T.A. Elliot (1972, p. 105–106) established this result unconditionally by using a result of A. Selberg. This implies (cf. for instance E. Hlawka (1984, p. 122–123)) the u.d. of our sequence if α's are linearly independent over the rationals.
(II) Assume that α and β are linearly independent over the rationals and β/α is an irrational number of the type $< \psi$ (i.e., $n.\|n(\alpha/\beta)\| \geq 1/\psi(n)$). A. Fujii (1995) proved that for the discrepancy of the two–dimensional sequence

$$\left(\frac{\alpha\gamma(n)}{2\pi}, \frac{\beta\gamma(n)}{2\pi}\right) \bmod 1$$

we have

$$D_N^* = \mathcal{O}\left(\frac{\log\log T}{\log T}\right) + \mathcal{O}\left(\sqrt{\frac{\log T}{T}}(\psi(CT) + \log T)\right),$$

where $N = N(T) = \sum_{0<\gamma(n)\leq T} \sim (T/2\pi)\log T$ and C is a positive constant.

Related sequences: See 2.20.25 for the one–dimensional case.

P.D.T.A. Elliott: *The Riemann zeta function and coin tossing*, J. Reine Angew. Math. **254** (1972), 100–109 (MR0313206 (**47** #1761); Zbl. 0241.10025).
A. Fujii: *On a problem and a conjecture of Rademacher's*, Commen. Math. Univ. St. Paul. **44** (1995), no. 1, 69–92 (MR1336419 (96m:11072); Zbl. 0837.11046).
E. Hlawka: *The Theory of Uniform Distribution*, A B Academic Publishers, Berkhamsted, 1984 (translation of the original German edition Hlawka (1979)) (MR0750652 (85f:11056); Zbl. 0563.10001).
H.A. Rademacher: *Collected Papers of Hans Rademacher*, Vol. II, Mathematicians of our times 4, The MIT Press, Cambridge (Mass.), London (England), 1974 (MR0505096 (**58** #21343b); Zbl. 0311.01023).

3.7.11. Let $z_1, \dots, z_s$ be distinct fixed complex numbers such that $\frac{1}{2} < \Re(z_i) < 1$ for $i = 1, 2, \dots, s$. Let $\Delta > 0$ be arbitrary but fixed, and $\zeta(z)$ be the standard Riemann zeta function. Then the sequence

$$\mathbf{x}_n = \left(\log|\zeta(z_1 + in\Delta)|, \dots, \log|\zeta(z_s + in\Delta)|\right), \quad n = 1, 2, \dots,$$

is

dense in $\mathbb{R}^s$.

NOTES: Follows from Voronin's theorem (1975) on the universality of the Riemann zeta function, cf. S.M. Voronin and A.A. Karacuba (A.A. Karatsuba) (1994, p. 240), and K. Bitar, N.N. Khuri and H.C. Ren (1991).

K. BITAR – N.N. KHURI – H.C. REN: *Path integrals as discrete sums*, Physical Review Letters **67** (1991), no. 7, 781–784 (MR1128186 (92g:81101); Zbl. 0990.81533).
S.M. VORONIN: *A theorem of "universality" of the Riemann zeta function*, (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. **39** (1975), no. 3, 475–486 (MR0472727 (**57** #12419); Zbl. 0315.10037).
S.M. VORONIN – A.A. KARACUBA (A.A. KARATSUBA): *The Riemann Zeta Function*, (Russian), Fiziko–Matematicheskaya Literatura, Moscow, 1994 (MR1918212 (2003b:11088); Zbl. 0836.11029).

3.8 Polynomial sequences

3.8.1. Let $p_i(x) = \sum_{j=1}^{m} a_{ij}x^j$, $i = 1, 2$, be real polynomials of degree m. (i.e. $a_{1m} \neq 0, a_{2m} \neq 0$). Let real numbers λ_1, λ_2 satisfy the diophantine conditions that the inequalities $\|\lambda_1 q\| < q^{-1-\gamma_1}$, $\|\lambda_2 q\| < q^{-1-\gamma_2}$, $\|\lambda_1 q + \lambda_2 p\| < \max(|q|, |p|)^{2-\tau}$ have only finitely many solutions, where τ, γ_1, γ_2 be given in such a way that $0 < \tau$, $0 < \gamma_1 \leq 1 + \tau$, and $0 < \gamma_2 \leq 1 + \tau$. Then

for any real numbers η_1 and η_2 the two–dimensional sequence

$$(\lambda_1 p_1(n) - \eta_1, \lambda_2 p_2(n) - \eta_2) \bmod 1$$

is

u.d.

and for its star discrepancy we have

$$D_N^* = \mathcal{O}\left(N^{1-\min(\beta_1,\beta_2)}\right),$$

where β_1 and β_2 can be given explicitly.

NOTES: This result was proved by È.I. Kovalevskaja (1971) and the proof uses the method of trigonometric sums.

È.I. Kovalevskaja (Kovalevskaya): *The simultaneous distribution of the fractional parts of polynomials*, (Russian), Vescī Akad. Navuk BSSR, Ser. Fīz.-Mat. Navuk, (1971), no. 5 13–23 (MR0389818 (**52** #10648); Zbl 0226.10051).

3.8.2. Define the s–dimensional r–multiple sequence

$$\mathbf{x}_{\mathbf{n}} = (f_1(\mathbf{n}), \dots, f_s(\mathbf{n})) \bmod 1 \quad \text{with } \mathbf{n} = (n_1, \dots, n_r),$$

where $1 \le n_1 \le N_1, \dots, 1 \le n_r \le N_r$, and

$$f_j(x_1, \dots, x_r) = \sum_{t_1=0}^{k_1} \cdots \sum_{t_r=0}^{k_r} \alpha_j(t_1, \dots, t_r) x_1^{t_1} \dots x_r^{t_r}$$

are polynomials with real coefficients which satisfy

$$0 \le \alpha_j(t_1, \dots, t_r) < 1, \quad \text{for } j = 1, 2, \dots, s,\ 0 \le t_1 \le k_1, \dots, 0 \le t_r \le k_r.$$

The set E of all s–tuples $(f_1, \dots, f_s)$ of such polynomials can be decomposed into two classes E_1 and E_2 such that for the star discrepancy (with respect to the u.d.) of E_2 we have

$$D^*_{N_1 \dots N_r} = \mathcal{O}\left(\frac{e^{32\varkappa}}{N^{\rho_1}}\right), \quad \text{where } \rho_1 = \frac{1}{33k\varkappa \log(8k\varkappa)},$$

and for E_1 we have

$$D^*_{N_1 \dots N_r} = \mathcal{O}\left(\frac{1}{(Q_0\delta_0)^{\nu-\varepsilon}}\right).$$

Here we used the following notation
- $k = (k_1 + 1) \dots (k_r + 1)$, $1 < N_1 = \min(N_1, \dots, N_r)$,
- $\nu = \big(\max(k_1, \dots, k_r)\big)^{-1}$,
- $\varkappa = k_1\nu_1 + \dots + k_r\nu_r$, where $\nu_1, \dots, \nu_r$ are positive integers such that

$$-1 < \frac{\log N_i}{\log N_1} - \nu_i \le 0,$$

- $N = (N_1^{k_1} \dots N_r^{k_r})^{1/\varkappa}$, $\Delta = N^{-2\rho}$, $\rho = (32k\varkappa \log 8k\varkappa)^{-1}$,
- for $d_i \in \mathbb{Z}$ with $|d_i| \le \Delta^{-1}$, $i = 1, \dots, s$, and $t_j \in \mathbb{N}$, $j = 1, \dots, r$, define

$$B = d_1\alpha_1(t_1, \dots, t_r) + \dots + d_s\alpha_s(t_1, \dots, t_r)$$

- and find $a, q \in \mathbb{Z}$ and $z \in \mathbb{R}$ such that

$$B = \frac{a}{q} + z, q \geq 1, \quad \gcd(a,q) = 1, \quad |z| \leq \frac{1}{q\tau}, \quad \tau = N_1^{t_1} \dots N_r^{t_r} N^{-1/3}.$$

- for fixed $d_1, \dots, d_s$ and variables $0 \leq t_1 \leq N_1, \dots, 0 \leq t_r \leq N_r$ with $t_1 + \dots + t_r \geq 1$ denote by
- $Q = Q(d_1, \dots, d_s)$ the least common multiple of $q = q(t_1, \dots, t_r)$, and put
- $\delta = \delta(d_1, \dots, d_s) = \max_{t_1,\dots,t_r} N_1^{t_1} \dots N_r^{t_r} |z(t_1, \dots, t_r)|$,
- $Q_0 = \max_{d_1,\dots,d_s} Q(d_1, \dots, d_s)$,
- $\delta_0 = \max_{d_1,\dots,d_s} \delta(d_1, \dots, d_s)$,
- $(f_1, \dots, f_s)$ belongs to E_1 if $Q_0 \leq N^{0.1}$ and to E_2 in the opposite case.

Notes: G.I. Archipov, A.A. Karacuba and V.N. Čubarikov (1987, p. 216, Th. 5). In the case $s = 2$ they illustrate the result by the following example (1987, p. 219): Let $f_1(x, y)$ and $f_2(x, y)$ be two polynomials with real coefficients which degrees in every variable does not exceed $k \geq 4$. Let$N_1 = N_2 = N$ and $\alpha_1(i,j) = \sqrt{2}$, $\alpha_2(i,j) = \sqrt{3}$ for some $0 \leq i, j \leq k$ and $i + j \geq 1$. Then $D^*_{N^2} = \mathcal{O}(N^{-\rho})$, where $\rho = c(k^3 \log k)^{-1}$ for some absolute constant $c > 0$.

G.I. Archipov – A.A. Karacuba – V.N. Čubarikov: *Theory of Multiplies Trigonometric Sums*, (Russian), Nauka, Moscow, 1987 (MR 89h:11050; Zbl. 0638.10037).

3.8.3. Let $p(x) = a_0 + a_1 x + \dots + a_k x^k$ be a polynomial with at least one irrational coefficient $a_s, \dots, a_k$ where $s < k$. Then the s–dimensional sequence

$$(p(n+1), \dots, p(n+s)) \bmod 1$$

is

u.d.

Notes: This generalization of Weyl's result was proved by L.D. Pustyľnikov (1993). The case $s = k$ can be found in [KN, p. 52, Ex. 6.10] and in G. Rauzy (1976, p. 47, 2.3. Application) and in this case the result also follows from a strictly ergodic dynamical system on the k–dimensional torus introduced by H. Furstenberg (1967). Note that polynomials are not completely u.d., cf. N.M. Korobov (1950).

Related sequences: 2.14.1

H. Furstenberg: *Disjointness in ergodic theory, minimal sets and a problem in diophantine approximation*, Math. Systems Theory **1** (1967), no. 1, 1–49 (MR0213508 (**35** #4369); Zbl. 0146.28502).
N.M. Korobov: *Concerning some questions of uniform distribution*, (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. **14** (1950), 215–238 (MR0037876 (12,321a); Zbl. 0036.31104).
L.D. Pustyľnikov: *Distribution of the fractional parts of values of a polynomial Weyl sums and ergodic theory*, (Russian), Uspekhi Mat. Nauk **48** (1993), no. 4, 131–166 (MR1257885 (94k:11094); Zbl. 0821.11039).

G. Rauzy: *Propriétés statistiques de suites arithmétiques*, Le Mathématicien, Vol. 15, Collection SUP, Presses Universitaires de France, Paris 1976, 133 pp. (MR0409397 (**53** #13152); Zbl. 0337.10036).

3.9 Power sequences

3.9.1. Let a_i, b_i, c_i, $i = 1, \dots, s$, be non–zero real numbers, and $0 < u_i < 1$, $i = 1, \dots, s$. If v_i, $i = 1, \dots, s$, are such that $0 < u_1v_1 < u_2v_2 < \cdots < u_sv_s$ and $u_iv_i \notin \mathbb{Z}$, $i = 1, \dots, s$, then the s–dimensional sequence

$$\mathbf{x}_n = ((a_1[b_1n^{v_1}] + c_1)^{u_1}, \dots, (a_s[b_sn^{v_s}] + c_s)^{u_s}) \mod 1$$

is

u.d.

P.J. Grabner – O. Strauch – R.F. Tichy: *Maldistribution in higher dimension*, Math. Panon. **8** (1997), no. 2, 215–223 (MR1476099 (99a:11094); Zbl. 0923.11110).

3.9.2. Let $\theta_1, \dots, \theta_s$ be irrational numbers. Then the s–dimensional sequence

$$\big(\theta_1n^s, \theta_2n^{s-1}, \dots, \theta_sn\big) \mod 1$$

is

u.d.

Notes: Cf. [KN, p. 52, Ex. 6.8]. For instance, the two–dimensional sequence

$$\Big(2\sqrt{2}\,n^2, \sqrt{2}\,n\Big) \mod 1$$

is u.d. It is used in I.J. Håland (1993, p. 328) and for the sequence $[\sqrt{2}n]^2\sqrt{2} \mod 1$ cf. 2.16.4.

I.J. Håland: *Uniform distribution of generalized polynomial*, J. Number Theory **45** (1993), 327–366 (MR1247389 (94i:11053); Zbl. 0797.11064).

3.9.3. Let $\alpha_1, \dots, \alpha_s$ be non–zero real numbers, and let $\tau_1, \dots, \tau_s$ be distinct positive numbers not in $\mathbb{Z}$. Then the sequence

$$(\alpha_1n^{\tau_1}, \dots, \alpha_sn^{\tau_s}) \mod 1$$

is

u.d.

Notes: [KN, p. 52, Ex. 6.9].

$\boxed{\textbf{3.9.3.1}}$ Let p be a prime and $r(X)$ be a rational function from $\mathbb{F}_p(X)$. If $r(X)$ is not of the form $r(X) = AX + B$, then for every m and $N < p$, the m-dimensional sequence

$$\left(\frac{r(i)}{p}, \dots, \frac{r(i^m)}{p}\right) \bmod 1, \quad i = 0, 1, \dots, N-1,$$

where i are not the poles of $r(X)$ and are represented by $\{0, 1, \dots, p-1\}$, has discrepancy

$$D_{N-s} = \mathcal{O}(N^{-1}p^{1/2}(\log p)^{m+1}),$$

where s is the number of poles of $r(X)$.

J. Gutierrez – I.E. Shparlinski: *On the distribution of rational functions on consecutive powers*, Unif. Distrib. Theory **3** (2008), no. 1, 85–91 (MR2453512 (2009k:11122); Zbl. 1174.11059).

3.9.4. If α is an arbitrary positive real and β a real irrational number then the sequence

$$\left(\alpha\sqrt{n}, \beta n\right) \bmod 1$$

is

$$\text{dense in } [0,1]^2.$$

Notes: Cf. A.M. Ostrowski (1980; Lem. 3).

A.M. Ostrowski: *On the distribution function of certain sequences* (mod 1), Acta Arith. **37** (1980), 85–104 (MR0598867 (82d:10073); Zbl. 0372.10036).

3.10 Exponential sequences

Notes: H. Niederreiter and R.F. Tichy (1985) proved that if a_n is a sequence of distinct positive integers, then the sequence λ^{a_n} mod 1 is completely u.d. for almost all $\lambda > 1$.

H. Niederreiter – R.F. Tichy: *Solution of a problem of Knuth on complete uniform distribution of sequences*, Mathematika **32** (1985), no. 1, 26–32 (MR0817103 (87h:11070); Zbl. 0582.10036).

3.10.1. Given a real transcendental number $\lambda > 1$ and an integer $p > \lambda$, let

- $n_1 = 0$, and $n_r = \sum_{i=1}^{r-1} p^i$ for $r = 2, 3, \dots$,
- $q_r = 4p^r p^{p^r}$, for $r = 1, 2, \dots$,
- $\alpha_0 = 0$, and $\alpha_k = \sum_{r=1}^{k} \frac{a_r}{q_r \lambda^{n_r}}$ for $k = 1, 2, \dots$, where a_r are integers in the interval $[0, q_r)$ for $r = 1, 2, \dots$.
- Further, if c is an integer, let

$$S_k(c, m_1, \dots, m_{s+1}) = \sum_{x=0}^{p^k-1} \exp\left(2\pi i \left(\alpha_{k-1} + \frac{c}{q_k \lambda^{n_k}}\right) \lambda^{n_k+x} \sum_{\nu=1}^{s} m_\nu \lambda^{\nu-1} + \frac{2\pi i m_{s+1} x}{p^k}\right)$$

and

$$D_{r,k,s}(c) = \sum_{m_1,\dots,m_{s+1}=-r}^{r}{}' \frac{|S_k(c, m_1, \dots, m_{s+1})|}{r(m_1)\dots r(m_{s+1})},$$

where $r(m) = \max(1, |m|)$, and in the sum $\sum'$ the summand with $m_1 = \dots = m_{s+1} = 0$ is excluded.
- The **transcendence measure** of λ is defined by

$$\Phi(\lambda, s, H) = \min \left| \sum_{\nu=1}^{s+1} m_\nu \lambda^{\nu-1} \right|,$$

where the minimum is taken over all m_ν, $\nu = 1, \dots, s+1$, such that $0 < \max_{1 \le \nu \le s+1} |m_\nu| \le H$.
- Finally let, $h_r = 4(r^2 - 1)\left(\log_p(p - \log_\lambda \Phi(\lambda, r-1, r))\right)$ for $r = 1, 2, \dots$.
- Let λ be a transcendental number λ and $p > \lambda$ an integer. Then for any positive integer r there exist integers a_k ($h_{r+1} \ge k > h_r$) such that

$$D_{r,k,s}(a_k) \le r(3 + 2\ln r)^{s+1} \left(6p^k \left(\frac{\lambda}{\lambda - 1}\right)^3\right)^{1/2} = \mathcal{O}(p^k r^{-1}).$$

holds for every integer $s \in [1, r]$.

If the integers a_k fulfil the above conditions and

$$\alpha = \sum_{k=1}^{\infty} \frac{a_k}{q_k \lambda^{n_k}},$$

then for every $s \ge 1$ the sequence

$$\left(\alpha\lambda^{n+1}, \dots, \alpha\lambda^{n+s}\right) \bmod 1, \quad n = 1, 2, \dots,$$

is

u.d.

i.e. $\alpha\lambda^n$ mod 1 is completely u.d.

NOTES: M.B. Levin (1975, Th. 1). He proved the existence of such a_k in Lem. 2. M.F. Kulikova (1962, [a]1962) constructed number α_1 such that $\alpha_1\lambda^n$ mod 1 is u.d., where $\lambda > 1$ is real.

RELATED SEQUENCES: 2.17.9, 3.10.2, 3.10.3

M.F. KULIKOVA: *A construction problem connected with the distribution of fractional parts of the exponential function* (Russian), Dokl. Akad. Nauk SSSR **143** (1962), 522–524 (MR0132737 (**24** #A2574); Zbl. 0116.27105).
[a] M.F. KULIKOVA: *Construction of a number α whose fractional parts $\{\alpha g^\nu\}$ are rapidly and uniformly distributed* (Russian), Dokl. Akad. Nauk SSSR **143** (1962), 782–784 (MR0137694 (**25** #1144); Zbl. 0131.29302).
M.B. LEVIN: *The uniform distribution of the sequences* $\{\alpha\lambda^x\}$, (Russian), Mat. Sb. (N.S.) **98(140)** (1975), no. 2(10), 207–222,333 (MR0406947 (**53** #10732); Zbl. 0313.10035).

3.10.2. Let $\lambda > 1$ be a real transcendental number such that for its transcendence measure (cf. 3.10.1) we have

$$\Phi(\lambda, s, H) \geq H^{-c(s)}, \quad H > H(\lambda),$$

where $c(s)$ is a monotonically increasing function. Let $p > \lambda$ be an integer and

- $n_{1,1} = 0$, $n_{r,i} = \sum_{k=1}^{r-1} kp^k + (i-1)p^r$ for $r = 2, 3, \dots$, and $i = 1, \dots, r+1$,
- $A_r = [p^{r/2}]$,
- $B > c(1)$,
- $\psi(x) \geq 1$ be an integral valued monotonically increasing function such that $\lim_{x\to\infty} \psi(x) = \infty$ and $c(\psi(p^r)) \leq 2Br$,
- $q_r = 4p^r\psi(p^r)p^{p^r} + \psi(p^r)$,
- $a_{r,i} \in [0, q_r)$ be integers for $r = 1, 2, \dots$, and $i = 1, \dots, r$,
- define the numbers $\alpha_{r,j}$ through $\alpha_{r,j} = \sum_{k=1}^{r-1}\sum_{i=1}^{k} \frac{a_{k,i}}{q_k\lambda^{n_{k,i}}} + \sum_{i=1}^{j} \frac{a_{r,i}}{q_r\lambda_{n_{r,i}}}$, for $r = 1, 2, \dots$, and $j = 1, \dots, r$, where $\alpha_{0,0} = 0$ and $\alpha_{r,0} = \alpha_{r-1,r-1}$ for $r = 1, 2, \dots$.
- if c is an integer, put

$$S_{r,j}(c, m_1, \dots, m_{s+1}) = \sum_{x=0}^{p^r-1} \exp\left(2\pi i\left(\alpha_{r,j-1} + \frac{c}{q_r\lambda^{n_{r,j}}}\right)\lambda^{n_{r,j}+x}\sum_{\nu=1}^{s} m_\nu\lambda^{\nu-1} + \frac{2\pi i m_{s+1}x}{p^r}\right)$$

and

$$D'_{r,j,s}(c) = \sideset{}{'}\sum_{m_1,\dots,m_{s+1}=-A_r}^{A_r} \frac{|S_{j,r}(c,m_1,\dots,m_{s+1})|}{r(m_1)\dots r(m_{s+1})},$$

$$D''_{r,j,s}(c) = \sideset{}{'}\sum_{m_1,\dots,m_s=-A_r}^{A_r} \frac{|S_{j,r}(c,m_1,\dots,m_s,0)|}{r(m_1)\dots r(m_s)},$$

where $r(m) = \max(1,|m|)$, and $\sum'$ denotes the sum with $m_1 = \cdots = m_{s+1} = 0$ excluded,

- if λ is a given transcendental and p an integer then there exist integers $a_{m,i}$ with $m = 1,2,\dots$, $i = 1,\dots,m$, such that

$$D'_{m,i,s}(a_{m,i}) = \mathcal{O}\left(p^{m/2}m^{s+1}\psi(p^m)\right), \quad D''_{m,i,s}(a_{m,i}) = \mathcal{O}\left(p^{m/2}m^{s}\psi(p^m)\right)$$

holds for every integer $s \in [1,\psi(p^m)]$.

If the integers $a_{m,i}$, $m = 1,2,\dots$, $i = 1,\dots,m$, fulfil the above conditions and we define

$$\alpha = \sum_{m=1}^{\infty}\sum_{i=1}^{m} \frac{a_{m,i}}{q_m\lambda^{n_{m,i}}}$$

then for every $s \geq 1$ the discrepancy of the sequence

$$(\alpha\lambda^{n+1},\dots,\alpha\lambda^{n+s}) \bmod 1, \quad n = 1,2,\dots,$$

satisfies

$$ND_N = \mathcal{O}\left(N^{1/2}(\log N)^{s+(1/2)}\psi(N)\right).$$

Notes: M.B.Levin (1975, Th. 2). He proved the existence of such integers $a_{r,j}$ in Lem. 4. Note that the numbers belonging to the Mahler's S and T class, especially the numbers e and π, possess the required transcendence measure. Levin (1975) also claims that based on analogical ideas it is possible to prove that for every real algebraic number $\lambda > 1$ of degree s there exists a number α such that

$$ND_N = \mathcal{O}\left(N^{1/2}(\log N)^{s+(1/2)}\right).$$

Related sequences: 3.10.1, 3.10.3

M.B. Levin: *The uniform distribution of the sequences* $\{\alpha\lambda^x\}$, (Russian), Mat. Sb. (N.S.) **98(140)** (1975), no. 2(10), 207–222,333 (MR0406947 (**53** #10732); Zbl. 0313.10035).

K. Mahler: *Zur Approximation der Exponentialfunction und des Logarithmus, I, II*, J. Reine Angew. Math. **166** (1932), 118–150 (MR1581302; Zbl. 0003.38805; JFM 58.0207.01).

3.10.3. Let $\lambda_\nu > 1$, $\nu = 1, \dots, s$, be real numbers. Let $p > \max_{1\le\nu\le s} \lambda_\nu$ be an integer and

- $n_{1,1} = 0$, $n_{r,j} = \sum_{k=1}^{r-1} kp^k + (j-1)p^r$ for $r = 2, 3, \dots$ and $j = 1, \dots, r+1$,
- $q_r = 4p^r p^{p^r}$,
- $A_r = \left[p^{r/2}\right]$,
- $a_{r,j,\nu} \in [0, q_r)$ be integers for $r = 1, 2 \dots$, $j = 1, \dots, r$ and $\nu = 1, \dots, s$.
- Define the numbers $\alpha_{r,j,\nu}$ by

$$\alpha_{r,j,\nu} = \sum_{k=1}^{r-1}\sum_{i=1}^{k} \frac{a_{k,i,\nu}}{q_k \lambda^{n_{k,i}}} + \sum_{i=1}^{j} \frac{a_{r,i,\nu}}{q_r \lambda^{n_{r,i}}},$$

  for $r = 1, 2, \dots$, $j = 1, \dots, r$, and $\nu = 1, \dots, s$, where $\alpha_{0,0,\nu} = 0$, $\alpha_{r,0,\nu} = \alpha_{r-1,r-1,\nu}$ for $r = 1, 2, \dots$, and $\nu = 1, \dots, s$.
- For a vector $\mathbf{c} = (c_1, \dots, c_s) =$ with integral coordinates put

$$D_{r,j}(c_1, \dots, c_s) = \sum_{m_1,\dots,m_{s+1}=-A_r}^{A_r}{}' \frac{|S_{j,r}(\mathbf{c}, m_1, \dots, m_{s+1})|}{r(m_1)\dots r(m_{s+1})},$$
$$D'_{r,j}(c_1, \dots, c_s) = \sum_{m_1,\dots,m_s=-A_r}^{A_r}{}' \frac{|S_{j,r}(\mathbf{c}, m_1, \dots, m_s, 0)|}{r(m_1)\dots r(m_s)},$$

  where $r(m) = \max(1, |m|)$, and $\sum'$ denotes the sum with $m_1 = \dots = m_{s+1} = 0$ excluded, and finally let

$$S_{r,j}(\mathbf{c}, m_1, \dots, m_{s+1}) = \sum_{x=0}^{p^r-1} \exp\left(2\pi i \sum_{\nu=1}^{s} m_\nu \lambda^{n_{r,j}+x}\left(\alpha_{r,j-1,\nu} + \frac{c_\nu}{q_r \lambda_\nu^{n_{r,j}}}\right) + \frac{2\pi i m_{s+1} x}{p^r}\right).$$

- If $\lambda_\nu > 1$, $\nu = 1, \dots, s$, are given real numbers and p an integer such that $p > \max\{\lambda_s\,;\, 1 \le \nu \le s\}$, then there exist integers $a_{m,j,\nu}$ for $m = 1, 2, \dots$, $j = 1, \dots, m$, and $\nu = 1, \dots, s$, such that

$$D_{m,j}(a_{m,j,1}, \dots, a_{m,j,s}) = \mathcal{O}\left(p^{m/2} m^{s+1}\right),$$
$$D'_{m,j}(a_{m,j,1}, \dots, a_{m,j,s}) = \mathcal{O}\left(p^{m/2} m^{s}\right).$$

If the integers $a_{m,j,\nu}$ for $m = 1, 2, \dots,$ $i = 1, \dots, m$, and $\nu = 1, \dots, s$ fulfil the above conditions and

$$\alpha_\nu = \sum_{m=1}^{\infty} \sum_{j=1}^{m} \frac{a_{m,j,\nu}}{q_m \lambda_\nu^{n_{m,j}}}, \quad \nu = 1, \dots, s,$$

then the sequence

$$(\alpha_1 \lambda_1^n, \dots, \alpha_s \lambda_s^n) \bmod 1, \quad n = 1, 2, \dots,$$

is

u.d.

and its discerpancy satisfies

$$ND_N = \mathcal{O}(N^{1/2} (\log N)^{s+(1/2)}).$$

NOTES: cf. M.B. Levin (1975, Th. 2). The existence of $a_{m,j,\nu}$ is proved in Levin (1975, Lem. 6). He also claims that using analogical ideas it is possible to prove that for given real numbers $\lambda > 1$, $c > 0$ there exists a function $f(x)$ such that

$$f(n) - f(m) \geq s \left(\sqrt{n} - \sqrt{m}\right), \quad n \geq m, \quad n, m = 1, 2, \dots,$$

a number α such that the discrepancy of the sequence

$$\alpha \lambda^{f(n)} \bmod 1, \quad n = 1, 2, \dots,$$

satisfies

$$ND_N = \mathcal{O}\left(N^{1/2} (\log N)^{3/2}\right).$$

RELATED SEQUENCES: 3.10.1, 3.10.2

M.B. LEVIN: *The uniform distribution of the sequences* $\{\alpha\lambda^x\}$, (Russian), Mat. Sb. (N.S.) **98(140)** (1975), no. 2(10), 207–222,333 (MR0406947 (**53** #10732); Zbl. 0313.10035).

3.10.4. Let $\nu(\lambda)$ denote the degree of λ if λ is an algebraic number, and $\nu(\lambda) = 1$ if λ is transcendental. Given an arbitrary real sequence λ_n, $n = 1, 2, \dots,$ there exists a real sequence α_n, $n = 1, 2, \dots,$ such that for all integers $j, k_1, \dots, k_j \geq 1$ the sequence

$$\mathbf{x}_n = \left(\alpha_1 \lambda_{k_1}^n, \dots, \alpha_1 \lambda_{k_1}^{n+\nu(\lambda_{k_1})-1}, \dots, \alpha_s \lambda_{k_s}^n, \dots, \alpha_s \lambda_{k_s}^{n+\nu(\lambda_{k_s})-1}\right) \bmod 1$$

is

u.d. in $[0,1]^s$,

where $s = \nu(\lambda_{k_1}) + \cdots + \nu(\lambda_{k_s})$, and its discrepancy satisfies

$$D_N = \mathcal{O}\left(\frac{(\log N)^{s+\frac{3}{2}}}{\sqrt{N}}\right).$$

M.B. Levin: *Simultaneously absolutely normal numbers*, (Russian), Mat. Zametki **48** (1990), no. 6, 61–71 (English translation: Math. Notes **48** (1990), no. 5–6, (1991), 1213–1220). (MR1102622 (92g:11077); Zbl. 0717.11029).

3.10.5. If α is a real number, and $q \geq 2$ an integer, let

$$\mathbf{x}_n = (\alpha q^n, \alpha n q^n) \bmod 1.$$

If there exist constants $c > 0$ and $0 \leq \varepsilon < 1$ such that,

$$\limsup_{N\to\infty} \frac{A_N(I; N; \mathbf{x}_n)}{N} < c|I|\left(1 + \log\frac{1}{|I|}\right)^{1-\varepsilon}$$

for every subinterval $I \subset [0,1]^2$, then the sequence $\mathbf{x}_n$ is

u.d.

A.G. Postnikov: *On distribution of the fractional parts of the exponential function*, Dokl. Akad. Nauk. SSSR (N.S.) (Russian), **86** (1952), 473–476 (MR0050637 (14,359d); Zbl. 0047.05202).

3.10.6. Given an integer $q \geq 2$, a real number θ and a real polynomial $p(x)$, let

- $x_n = \theta q^n \bmod 1$,
- $y_n = p(n) \bmod 1$,
- $\mathbf{x}_n = (x_{n+1}, \dots, x_{n+s})$ and $\mathbf{y}_n = (y_{n+1}, \dots, y_{n+s})$.

If x_n is u.d. (i.e. θ is normal in the base q, cf. 2.18), then for every $s = 1, 2, \dots$, the sequence

$$(\mathbf{x}_n, \mathbf{y}_n), \quad n = 1, 2, \dots,$$

has d.f.'s

$$g(\mathbf{x}, \mathbf{y}) \in G((\mathbf{x}_n, \mathbf{y}_n))$$

only of the form $g(\mathbf{x}, \mathbf{y}) = g_1(\mathbf{x})g_2(\mathbf{y})$ for some $g_1(\mathbf{x}) \in G(\mathbf{x}_n)$ and $g_2(\mathbf{y}) \in G(\mathbf{y}_n)$, i.e. the sequences x_n and y_n are

completely statistically independent,

cf. 1.8.9.

J. Coquet – P. Liardet: *A metric study involving independent sequences*, J. Analyse Math. **49** (1987), 15–53 (MR0928506 (89e:11043); Zbl. 0645.10044).

3.10.7. Let $s \geq 1$, $c > 1$ and $s - 1 < p < q - 1$. If $p_1, \dots, p_s$ be distinct prime numbers, then the discrepancies D_N of the following s–dimensional sequences

$$\mathbf{x}_n = \left(p_j(cN^p - n^p)^{1/q}, j = 1, \dots, s\right) \bmod 1, \quad n = 1, 2, \dots, N,$$
$$\mathbf{y}_n = \left((p_j)^{p/q}(cN^p - n^p)^{1/q}, j = 1, \dots, s\right) \bmod 1, \quad n = 1, 2, \dots, N,$$

satisfy

$$0 < \limsup_{N\to\infty} N^{1/q} D_N < \infty.$$

Related sequences: 2.15.6.

W.–G. Nowak: *Die Diskrepanz der Doppelfolgen* $(cN^p - n^p)^{1/q}$ *und einige Verallgemeinerungen*, Österreich. Akad. Wiss. Math.–Natur. Kl. Sitzungsber. II **187** (1978), no. 8–10, 383–409 (MR0548968 (80m:10029); Zbl. 0411.10025).

3.11 Circle sequences

Notes: Let $f(n)$, $n = 0, 1, 2, \dots$, be a sequence of complex numbers on the unit circle. If the limit

$$\gamma(k) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(n+k)\overline{f(n)}$$

exists for every $k = 0, 1, 2, \dots$, then the sequence $\gamma(k)$, $k = 0, 1, 2, \dots$, is called the **correlation of the sequence** $f(n)$, $n = 0, 1, 2, \dots$. Here $\overline{z}$ is the complex conjugate of z. Since $\gamma(k)$ is a positive definite sequence, the Herglotz – Bochner representation (cf. e.g. P.J. Brockwell and R.A. Davis (1987, p. 115–116))

$$\gamma(k) = \int z^k \,\mathrm{d}\lambda(z),$$

where the integration is taken over the unit circle, determines the **spectral measure** λ on the unit circle corresponding to the sequence $f(n)$.[1] Moreover

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} |\gamma(k)|^2 = \sum_{|z|=1} \left(\lambda(z+) - \lambda(z-)\right)^2.$$

[1] E.g. the spectral measure of the sequence $e^{2\pi i\alpha n^2}$, $n = 0, 1, 2, \dots$, with α is irrational is the Lebesgue measure, cf. 3.11.3.

The **Fourier − Bohr** spectrum $\mathrm{Bsp}(f)$ of a $f : \mathbb{N} \to$ unit circle (cf. 2.4.4) is the set of all numbers $\alpha \in [0,1)$ such that

$$\limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} f(n) e^{2\pi i n\alpha} \right| > 0.$$

The sequence $f(n)$ on the unit circle is said to be **pseudorandom in the sense of Bertrandias** if
(i) $f(n)$ has the correlation sequence $\gamma(k)$, $k = 0,1,2,\dots$, and
(ii) $\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} |\gamma(k)|^2 = 0$,
and it is said to be **pseudorandom in the sense of Bass** if instead of (ii) the following stronger condition
(ii′) $\lim_{k\to\infty} \gamma(k) = 0$
my be applied. J. Bass (1957) defined the notion of the (auto)correlation which was already introduced by N. Wiener (1927, 1930). J. Bass (1959) defined the notion of a pseudorandom function noting that it was N. Wiener (1930) who first call the attention to such functions. This type of functions was studied then by J.P. Bertrandias (1962) who also used them (1964) in the generalization of the van der Corput criterion for u.d.

The above definitions can be used not only for circle sequences $f(n)$ but also for arbitrary complex sequences. For real sequences x_n, $n = 1,2,\dots$, however, the following slight modifications are necessary (cf. J.–P. Allouche (2000), M. Hörnquist (1999, Chap. 2)):

Assume that the limit $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n e^{-2\pi i n\alpha}$ exists for each $\alpha \in [0,1)$. Then
- $c(\alpha) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n e^{-2\pi i n\alpha}$ is called the **Fourier − Bohr coefficient** of the sequence x_n,
- $\mathrm{Bsp}(x_n) = \{\alpha \in [0,1)\,;\, c(\alpha) \neq 0\}$ is called the **Fourier − Bohr spectrum** of the sequence x_n.

Examples:
(i) If $x_n = (-1)^{[n\beta]}$ and β is irrational, then $\mathrm{Bsp}(x_n) = \left\{ \frac{\beta}{2} + k\beta \bmod 1\,;\, k \in \mathbb{Z} \right\}$.
(ii) If x_n is the Thue − Morse sequence 2.26.2, then $\mathrm{Bsp}(x_n) = \emptyset$.
(iii) If x_n is the Rudin − Shapiro sequence 2.26.3, then $\mathrm{Bsp}(x_n) = \emptyset$.

Assume that the limit $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n x_{n+k}$ exists for each integer k. Then
- $\lambda(k) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n x_{n+k}$, $k = 0,1,2,\dots$ are called the **correlation coefficients** of the sequence x_n,
- $\lambda(k) = \int_0^1 e^{2\pi i k t}\,\mathrm{d}g(t)$ for $k = 0,1,2,\dots$, define the unique d.f. $g(x)$ on $[0,1]$ called the **spectral d.f.** (or **spectral measure**) of the sequence x_n,
- $\mathrm{Wsp}(x_n) = \{\alpha \in [0,1)\,;\, g(\alpha+0) - g(\alpha-0) \neq 0\}$ is called the **Wiener spectrum** of the sequence x_n.

Examples:

(i) The spectral measure of the Fibonacci sequence is discrete.
(ii) The spectral measure of the Thue – Morse sequence is singular.
(iii) the spectral measure of the Rudin – Shapiro sequence is the Lebesgue measure.

Properties:

- $\mathrm{Bsp}(x_n)$ of every sequence x_n is countable.
- $c(\alpha) \le \sqrt{g(\alpha+0)-g(\alpha-0)}$ for every $\alpha \in [0,1)$ and every sequence x_n for which $c(\alpha)$ and $\lambda(k)$ exist for $\alpha \in [0,1)$ and $k = 0,1,2,\dots$, cf. J.–P. Bertrandias (1966). An immediate corollary of this theorem is that
- $\mathrm{Bsp}(x_n) \subset \mathrm{Wsp}(x_n)$.

J.–P. Allouche: *Algebraic and analytic randomness*, in: *Noise, oscillators and algebraic randomness. From noise communication system to number theory. Lectures of a school, Chapelle des Bois, France, April 5–10, 1999*, Lect. Notes Phys. 550, 345–356 Springer, Berlin, 2000, (MR1861985 (2002i:68099); Zbl. 1035.68089).
J. Bass: *Sur certaines classes de fonctions admettant une fonction d'autocorrélation continue*, C. R. Acad. Sci. Paris **245** (1957), 1217–1219 (MR0096344 (**20** #2828)); Zbl. 0077.33302).
J. Bass: *Suites uniformément denses, moyennes trigonométrique, fonctions pseudo–aléatores*, Bull. Soc. Math. France **87** (1959), 1–64 (MR0123147 (**23** #A476); Zbl. 0092.33404).
J.-P. Bertrandias: *Fonctions pseudo–aléatiores et fonctions presque périodic*, C.R. Acad. Sc. Paris **255** (1962), 2226–2228 (MR0145279 (**26** #2812); Zbl. 0106.11801).
J.–P. Bertrandias: *Suites pseudo–aléatoires et critères d'équirépartition modulo un*, Compositio Math. **16** (1964), 23–28 (MR0170880 (**30** #1115); Zbl. 0207.05801).
J.–P. Bertrandias: *Espace de fonctions bornées et continues en moyenne asymptotique d'ordere p*, Bull. Soc. Math. France, Mémoire **5** (1966), 1–106 (MR0196411 (**33** #4598); Zbl. 0148.11701).
P.J. Brockwell – R.A. Davis: *Time Series: Theory and Methods*, Springer Series in Statistics, Springer Verlag, New York, 1987 (MR0868859 (88k:62001); Zbl. 0604.62083).
M. Hörnquist: *Aperiodically Ordered Structures in One Dimension*, Department of Physics and Measurement Technology, Linköping University, Ph.D. thesis in theoretical physics, Linköping, Sweden, 1999 (www.ifm.liu.se/~micho/phd).
N. Wiener: *The spectrum of an array and its application to the study of the translation properties of a simple class of arithmetical functions*, J. Math. and Phys. **6** (1927), 145–157 (JFM 53.0265.02).
N. Wiener: *Generalized harmonic analysis*, Acta Math. **55** (1930), 117–258 (MR1555316; JFM 56.0954.02).

3.11.1. Let a_k, $k = 0,1,2,\dots$, be a sequence of real numbers and s_k, $k = 0,1,2,\dots$, be a strictly increasing sequence of positive numbers tending to infinity. Let

$$f(n) = e^{2\pi i \sum_{k=0}^{\infty} a_k \left[\frac{n}{s_k}\right]}.$$

(I) If $s_k = q^k$ with $k = 0,1,2,\dots$, and $q \ge 2$ is a positive integer, then the following three statements are equivalent

- $f(n)$ is pseudorandom in the sense of Bertrandias (cf. 3.11),
- $\sum_{k=0}^{\infty} |a_k|^2 = \infty$,
- the Fourier – Bohr spectrum of $f(n)$ is empty.

(II) The same holds under the assumption that $s_k | s_{k+1}$ for $k = 0,1,2,\dots$.

(III) If $(s_k, s_{k+1}) = 1$ for $k = 0, 1, 2, \dots$, and $\sum_{k=0}^{\infty} \frac{1}{s_k} < \infty$, then $f(n)$ is pseudorandom in the sense of Bertrandis if and only if $\sum_{k=0}^{\infty} |a_k|^2 = \infty$. On the other hand, $f(n)$ cannot be pseudorandom in the sense of Bass.
(IV) If $s_k = \tau^k$ for $k = 0, 1, 2, \dots$, where τ is a real transcendental number, then $f(n)$ is pseudorandom in the sense of Bertrandias if and only if $\sum_{k=0}^{\infty} |a_k|^2 = \infty$.
Notes: (I) was proved by J. Coquet and M. Mendès France (1977), (II) by J. Coquet (1977), and (III) with (IV) by J. Coquet (1978).

J. Coquet: *Fonctions q–multiplicatives. Applications aux nombres de Pisot – Vijayaraghavan*, Séminaire de Théorie des Nombres (1976–1977), 17, Lab. Théorie des Nombres, Centre Nat. Recherche Sci., Talence, 1977, 15 pp. (MR0509630 (80g:10051); Zbl. 0383.10032).
J. Coquet: *Sur certain suites pseudo–alétoires*, Acta Sci. Math. (Szeged) **40** (1978), no. 3–4, 229–235 (MR0515203 (80g:10052); Zbl. 0349.10043).
J. Coquet – M. Mendès France: *Suites à spectre vide et suites pseudo–aléatoires*, Acta Arith. **32** (1977), no. 1, 99–106 (MR0435019 (**55** #7981); Zbl. 0303.10047).

3.11.2. Let $g : \mathbb{N} \to [0, 1]$ be a q–additive function ($q > 1$), i.e. if $n = \sum_{k=0}^{\infty} a_k(n)q^k$ is the q–adic digit expansion of an $n = 1, 2, \dots$, then (cf. 2.10)

$$g(n) = \sum_{k=0}^{\infty} g(a_k(n)q^k) \quad \text{and } g(0) = 0.$$

Then for the sequence

$$f(n) = e^{2\pi i g(n)}, \quad n = 0, 1, 2, \dots,$$

the following assertions are equivalent:
- is pseudorandom in the sense of Bertrandias,
- the Fourier – Bohr spectrum $\mathrm{Bsp}(f)$ of $f(n)$ is empty,
- $\sum_{k=0}^{\infty} \sum_{a=0}^{q-1} \|g(aq^k) + \alpha a q^k\|^2 = +\infty$ for all $\alpha \in [0, 1)$, where $\|x\| = \min(\{x\}, 1 - \{x\})$,
- $\sum_{k=0}^{\infty} \sum_{a=0}^{q-1} \|g(aq^k) - aq^k\|^2 = +\infty$,
- $g_y(n) - \alpha n_y \mod 1$ is essentially divergent for all α, where $n_y = \sum_{0 \le k \le y} a_k(n)q^k$ and $g_y(n) = \sum_{0 \le k \le y} g(a_k(n)q^k)$.

Notes: J. Coquet (1979). The last item was found by J.–L. Mauclaire (1993), and later in (1997) he proved a generalization of it. J. Coquet, T. Kamae and M. Mendés France (1977) proved that the circle sequence

$$f(n) = e^{2\pi i \alpha s_q(n)}, \quad n = 0, 1, 2, \dots,$$

where $s_q(n)$ is a sum–of–digits function in the base q (see the def. in 2.9), is pseudorandom if and only if α is not of the form $\frac{k}{q-1}$ with k integer.

J. Coquet: *Sur la mesure spectrale des suites q–multiplicatives*, Ann. Inst. Fourier (Grenoble) **29** (1979), no. 3, 163–170 (MR0552963 (82a:10064); Zbl. 0386.10031).
J. Coquet – T. Kamae – M. Mendès France: *Sur la measure spectrale de certaines suites arithmétiques*, Bull. Soc. Math. France **105** (1977), no. 4, 369–384 (MR0472749 (**57** #12439); Zbl. 0383.10035).
J.–L. Mauclaire: *Sur la réparation des fonctions q–additives*, J. Théor. Nombres Bordeaux **5** (1993), no. 1, 79–91 (MR1251228 (94k:11089); Zbl. 0788.11032).
J.–L. Mauclaire: *Some consequences a result of J. Coquet*, J. Number Theory **62** (1997), no. 1, 1–18 (MR1429999 (98i:11062); Zbl. 0871.11050).

3.11.3. If for a real sequence x_n and for every $h = 1, 2, \dots$, the sequence of differences

$$x_{n+h} - x_n \bmod 1$$

is

u.d.

then the sequence

$$f(n) = e^{2\pi i x_n}$$

has the correlation sequence $\gamma(k)$, $k = 0, 1, 2, \dots$, which spectral measure λ is

the Lebesgue measure.

A. Bellow: *Some remarks on sequences having a correlation*, in: *Proceedings of the conference commemorating the* 1st *centennial of the Circlo Matematico di Palermo (Italia, Palermo, 1984)*, Rend. Circ. Mat. Palermo (2), Suppl. No. 8, 1985, pp. 315–320 (MR0881409 (88f:11070); Zbl. 0629.28012).

3.11.4. Let $q > 1$ be an integer base and let $k \geq 0$ be the number of different prime factors p_j of q with $p_j \equiv 1 \pmod 4$, $j = 1, \dots, k$. Let Π_q be the set of points on the unit circle with finite q–adic digit expansions of their coordinates. In case of $k > 0$ both coordinates of the points $P \in \Pi_q$ have the same number of digits in the base q after the q–adic point. If the points of Π_q are arranged according to this number of digits in any way, then the arising sequence $P_0, P_1, \dots$ is

u.d. on the unit circle.

Notes: P. Schatte (2000). He also notes the following corollary of his results: Let $q > 1$ be an integer base with a prime factor $p \equiv 1 \pmod 4$. Then every point on the unit circle can be approximated with arbitrary accuracy by points also on the unit circle but with finite q–adic digit expansions.

P. Schatte: *On the points on the unit circle with finite b–adic expansions*, Math. Nachr. **214** (2000), 105–111 (MR1762054 (2001f:11125); Zbl. 0967.11028).

3.11.5.

NOTES: If $\mathbf{z}_n = x_n + iy_n$ is a complex sequence then we define $\mathbf{z}_n \bmod 1$ by the rule $\mathbf{z}_n \bmod 1 = x_n \bmod 1 + y_n \bmod 1$ and the distribution of $\mathbf{z}_n \bmod 1$ we identify with the distribution of two–dimensional sequence $(x_n, y_n) \bmod 1$ in $[0,1]^2$.

. . . .

If $\mathbf{u}$ and $\mathbf{v}$ are complex numbers, then the complex exponential sequence

$$\mathbf{z}_n = \mathbf{u}.\mathbf{v}^n \bmod 1$$

is

u.d.

provided there exists a constant $c > 0$ such that

$$\limsup_{N\to\infty} \frac{A(I; N; \mathbf{z}_n)}{N} \le c|I|$$

for all subintervals $I \subset [0,1]^2$.

RELATED SEQUENCES: 2.18.19

A.G. POSTNIKOV: *A criterion for testing the uniform distribution of an exponential function in the complex domain*, Vestnik Leningrad. Univ. (Russian), **12** (1957), no. 13, 81–88 (MR0101859 (**21** #666); Zbl. 0093.05302).

3.11.6. Curve generated by u.d. sequences. Let $\Gamma = (x(t), y(t))$, $t \in [0,\infty)$ be a continuous and locally rectifiable curve. Let

- Γ_t be the initial segment of Γ having length t,
- $\mathrm{Diam}(\Gamma) = \sup\{d(\mathbf{x},\mathbf{y}) \; ; \; \mathbf{x},\mathbf{y} \in \Gamma\}$, where d is the Euclidean distance of the space $\mathbb{R}^2$,
- $\Gamma^\varepsilon = \{\mathbf{y} \; ; \; \mathbf{x} \in \Gamma, d(\mathbf{x},\mathbf{y}) < \varepsilon\}$,
- if $\mathrm{Diam}(\Gamma) = \infty$ and $\lim_{t\to\infty} t/\mathrm{Diam}(\Gamma_t) = \infty$, then the curve Γ is called **superficial**,
- if $\mathrm{Diam}(\Gamma) < \infty$ and $\lim_{\varepsilon\to 0} |\Gamma^\varepsilon|/\varepsilon = \infty$, then the curve Γ is called **superficial**, too, where $|X|$ is the 2–dimensional Lebesgue measure of the plane set X.

For a one–dimensional real sequence x_n define in the complex plane $\mathbb{C}$

- $\mathbf{z}_0 = 0$, $\mathbf{z}_n = \mathbf{z}_{n-1} + e^{2\pi i x_{n-1}}$, $n = 1, 2, \dots$,
- $\Gamma(x_n)$ is the curve Γ which passes successively through the complex points $\mathbf{z}_n$, $n = 0, 1, \dots$, in such way that the points $\mathbf{z}_n$ and $\mathbf{z}_{n+1}$ are connected by a line segment.

The sequence x_n mod 1 is u.d. if and only if for each positive integer h the curve $\Gamma(hx_n)$ is superficial.

NOTES: F.M. Dekking and M. Mendès France (1981). If

$$\underline{\dim}\Gamma = \liminf_{\varepsilon\to 0}\liminf_{t\to\infty}\frac{\log(\varepsilon^{-1}|\Gamma_t^\varepsilon|)}{\log(\varepsilon^{-1}\mathrm{Diam}(\Gamma_t^\varepsilon))}$$

then they also proved that the relation $\underline{\dim}\Gamma > 1$ implies that Γ is superficial. In the paper figures of some parts of $\Gamma(x_n)$ for $x_n = n\sqrt{17}$, $x_n = n^2\sqrt{2}$, $x_n = n^2 e$, $x_n = n^2\pi$, $x_n = s_2(n)(1/4)$, $x_n = s_2(n)\sqrt{3}$, $x_n = n^{2/5}$, and $x_n = (n+1)\log(n+1)$ can be found. Mendès France (1984) also studied the relation between the u.d. of x_n and the entropy of $\Gamma(x_n)$.

F.M. Dekking – M. Mendès France: *Uniform distribution modulo one: a geometrical viewpoint*, J. Reine Angew. Math. **329** (1981), 143–153 (MR0636449 (83b:10062); Zbl. 0459.10025).
M. Mendès France: *Entropy of curves and uniform distribution*, in: *Topics in classical number theory, Vol. I, II (Budapest, 1981)*, (G.Halász ed.), Colloq. Math. Soc. János Bolyai, Vol. 34, North–Holland Publishing Co., Amsterdam, New York, 1984, pp. 1051–1067 (MR0781175; Zbl. 0547.10047).

3.12 Sequences involving trigonometric functions

3.12.1. Let $1, \omega_1, \omega_2$ be linearly independent over the rational numbers. Then the sequence

$$(\cos 2\pi n\omega_1, \cos 2\pi n\omega_2)$$

has the a.d.f.

$$\begin{aligned} g(x,y) =& 4\left(\frac{1}{4} - g_1(x)\right)\left(\frac{1}{4} - g_1(y)\right) + 2\left(\frac{1}{4} - g_1(x)\right)\big(1 - 2g_2(y)\big) \\ &+ 2\big(1 - 2g_2(x)\big)\left(\frac{1}{4} - g_1(y)\right) + \big(1 - 2g_2(x)\big)\big(1 - 2g_2(y)\big), \end{aligned}$$

where

$$g_1(x) = \frac{1}{2\pi}\arccos x \quad \text{and} \quad g_2(x) = \frac{1}{2\pi}\arccos(x-1).$$

R.F. Tichy: *On the asymptotic distribution of linear recurrence sequences*, in: *Fibonacci numbers and their applications (Patras, 1984)*, Math. Appl., 28, Reidel, Dordrecht, Boston (Mass.), 1986, pp. 273–291 (MR0857831 (87i:11095); Zbl. 0578.10053).

3.13 Sequences involving logarithmic function

3.13.1. Let $p_1,\dots,p_s$ be mutually coprime positive integers and $j>1$. Then the set of all d.f.'s of the s–dimensional sequence

$$\mathbf{x}_n=\left((-1)^{[[\log^{(j)}n]^{1/p_1}]}[\log^{(j)}n]^{1/p_1},\dots,(-1)^{[[\log^{(j)}n]^{1/p_s}]}[\log^{(j)}n]^{1/p_s}\right)\bmod 1$$

is

$$G(\mathbf{x}_n)=\{c_{\boldsymbol{\alpha}}(\mathbf{x})\,;\,\boldsymbol{\alpha}\in[0,1]^s\},$$

where

$$c_{\boldsymbol{\alpha}}(\mathbf{x})=\begin{cases}1, & \text{for } \mathbf{x}\in[\boldsymbol{\alpha},\mathbf{1}],\\ 0, & \text{otherwise.}\end{cases}$$

P.J. Grabner – O. Strauch – R.F. Tichy: *Maldistribution in higher dimension*, Math. Panon. **8** (1997), no. 2, 215–223 (MR1476099 (99a:11094); Zbl. 0923.11110).

3.13.2. Let $p_1,\dots,p_s$ be mutually coprime positive integers and $j>1$. Then the set of all d.f.'s of the s–dimensional sequence

$$\mathbf{x}_n=\left([\log^{(j)}n]^{1/p_1},\dots,[\log^{(j)}n]^{1/p_s}\right)\bmod 1.$$

is

$$G(\mathbf{x}_n)=\left\{tc_{\boldsymbol{\alpha}}(\mathbf{x})+(1-t)c_{\boldsymbol{\beta}}(\mathbf{x})\right\},$$

where $t\in[0,1]$, $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_s),\boldsymbol{\beta}=(\beta_1,\dots,\beta_s)\in[0,1]^s$, and if $\alpha_i\neq\beta_i$ then $\alpha_i=1,\beta_i=0$ for $i=1,\dots,s$.
Related sequences: 3.13.1.

P.J. Grabner – O. Strauch – R.F. Tichy: *Maldistribution in higher dimension*, Math. Panon. **8** (1997), no. 2, 215–223 (MR1476099 (99a:11094); Zbl. 0923.11110).

3.13.3. Let $1,\alpha_1,\dots,\alpha_s$ be linearly independent over the rationals. Then the set $G(\mathbf{x}_n)$ of all its d.f.'s of the sequence

$$\mathbf{x}_n=(\alpha_1\log\log n,\dots,\alpha_s\log\log n)\bmod 1$$

satisfies

$$G(\mathbf{x}_n)\supset\left\{c_{\boldsymbol{\alpha}}(\mathbf{x})\,;\,\boldsymbol{\alpha}\in[0,1]^s\right\}.$$

Notes: In other words, the sequence is uniformly maldistributed. This an example was given by G. Myerson (1993).

G. Myerson: *A sampler of recent developments in the distribution of sequences*, in: *Number theory with an emphasis on the Markoff spectrum (Provo, UT 1991)*, (A.D. Pollington and W. Moran eds.), Lecture Notes in Pure and App.Math., Vol. 147, Marcel Dekker, New York, Basel, Hong Kong, 1993, pp. 163–190 (MR1219333 (94a:11112); Zbl. 0789.11043).

3.13.4. The 2–dimensional sequence

$$(n^2 \log n, n \log n) \bmod 1$$

is

u.d..

Notes: cf. [KN, p. 52, Ex. 6.11].

3.13.5. The 2–dimensional sequence

$$(\log n, \log\log n) \bmod 1$$

is

everywhere dense in $[0,1]^2$ but not u.d.

More precisely, let $c_v(x)$ be the one–jump d.f. which has the jump of height 1 at $x = v$, i.e.

$$c_v(x) = \begin{cases} 0, & \text{for } 0 \le x < v, \\ 1, & \text{for } v \le x \le 1. \end{cases}$$

Then the set of all d.f.'s of our sequence is

$$\begin{aligned} G(\{\log n\}, \{\log\log n\}) =& \{g_{u,v}(x,y)\ ;\ u \in [0,1], v \in [0,1]\} \cup \\ & \cup \{g_{u,0,j,\alpha}(x,y)\ ;\ \alpha \in A, u \in [0,1], j = 1, 2, \dots\} \cup \\ & \cup \{g_{u,0,0,\alpha}(x,y)\ ;\ \alpha \in A, u \in [\alpha, 1]\}, \end{aligned}$$

where A is the set of all limit points of the sequence $e^n \bmod 1$, and

$$\begin{aligned} g_{u,v}(x,y) =& g_u(x) \cdot c_v(y), \\ g_{u,0,j,\alpha}(x,y) =& g_{u,0,j,\alpha}(x) \cdot c_0(y), \\ g_{u,0,0,\alpha}(x,y) =& g_{u,0,0,\alpha}(x) \cdot c_0(y), \end{aligned}$$

for $(x,y) \in [0,1)^2$, and

$$
\begin{aligned}
g_u(x) =& \frac{e^{\min(x,u)}-1}{e^u} + \frac{1}{e^u}\frac{e^x-1}{e-1},\\
g_{u,0,j,\alpha}(x) =& \frac{e^{\max(\alpha,x)}-e^\alpha}{e^{j+u}} + \frac{e^{\min(x,u)}-1}{e^u} + \frac{1}{e^u}\frac{e^x-1}{e-1}\left(1-\frac{1}{e^{j-1}}\right),\\
g_{u,0,0,\alpha}(x) =& \frac{e^{\max(\min(x,u),\alpha)}-e^\alpha}{e^u},
\end{aligned}
$$

while

$$
\begin{gathered}
g_{u,v}(x,1) = g_{u,0,j,\alpha}(x,1) = g_{u,0,0,\alpha}(x,1) = g_u(x)\\
g_{u,v}(1,y) = c_v(y), \quad g_{u,0,j,\alpha}(1,y) = h_\beta(y), \quad \text{with } \beta = 1 - \frac{1}{e^{j+u-\alpha}}.
\end{gathered}
$$

Here, in the definition of $g_{u,0,j,\alpha}(1,y)$, the constant d.f. $h_\beta(y) = \beta$, if $y \in (0,1)$ for $j = 0,1,2,\dots$, but if $j=0$, then $u \ge \alpha$ in the definition of β.

NOTES: O. Strauch and O. Blažeková (2003). In the above notation, the step d.f. $F_{N_k}(x,y)$ (for the def. see 1.11) converges to $g(x,y)$ with $k \to \infty$ as follows:

- $F_{N_k}(x,y) \to g_{u,v}(x,y)$ if $\{\log N_k\} \to u$ and $\{\log\log N_k\} \to v > 0$,
- $F_{N_k}(x,y) \to g_{u,0,j,\alpha}(x,y)$ if $\{\log N_k\} \to u$, $\{\log\log N_k\} \to 0$, $\{e^{[\log\log N_k]}\} \to \alpha$, $[\log N_k] - [e^{[\log\log N_k]}] = j > 0$, and
- $F_{N_k}(x,y) \to g_{u,0,0,\alpha}(x,y)$ if $\{\log N_k\} \to u$, $\{\log\log N_k\} \to 0$, $\{e^{[\log\log N_k]}\} \to \alpha$, $[\log N_k] - [e^{[\log\log N_k]}] = 0$.

Note that $\{g_u(x)\,;\, u \in [0,1]\}$ coincides with $G(\{\log n\})$, see 2.12.1. The description of the set A of all limit points of the sequence e^n mod 1, $n = 1,2,\dots$, is an open problem, cf. 2.17.2. The set $G(\log(n\log n))$ in 2.12.16 can be obtained from the $G(\{\log n\},\{\log\log n\})$ applying 2.3.21.

O. Strauch – O. Blažeková: *Distribution of the sequence p_n/n mod 1*, Math. Institute, Slovak Acad. Sci., Bratislava, Slovak Republic, 2003, 15 pp.

3.13.5.1 The two-dimensional sequence

$$(\{\log n\}, \{\log(n+1)\}), \quad n = 1,2,\dots,$$

has the set of d.f.s

$$g_u(x,y) = \frac{e^{\min(x,y)}-1}{e-1}\cdot\frac{1}{e^u} + \frac{e^{\min(x,y,u)}-1}{e^u},$$

where $u \in [0,1]$ and if $\{\log N\} \to u$ then

$$F_N(x,y) = \frac{\{n \le N; (\{\log n\}, \{\log(n+1)\}) \in [0,x) \times [0,y)\}}{N} \to g_u(x,y).$$

NOTES:
(I) The directly of computation of the integral gives

$$\int_0^1 \int_0^1 |x-y| \,\mathrm{d}_x \,\mathrm{d}_y\, g_u(x,y) = 0 = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\{\log(n+1)\} - \{\log n\}| \,.$$

Note, that this can be also proved without using d.f.'s.
(II) Put $g_u(x,1) = g_u(x) = \frac{e^x-1}{e-1} \cdot \frac{1}{e^u} + \frac{e^{\min(x,u)}-1}{e^u}$. Then by Sklar theorem in 3.19.7.3(IV) we have $g_u(x,y) = c_u(g_u(x), g_u(y))$, were the copula $c_u(x,y) = \min(x,y)$ for every $u \in [0,1]$.

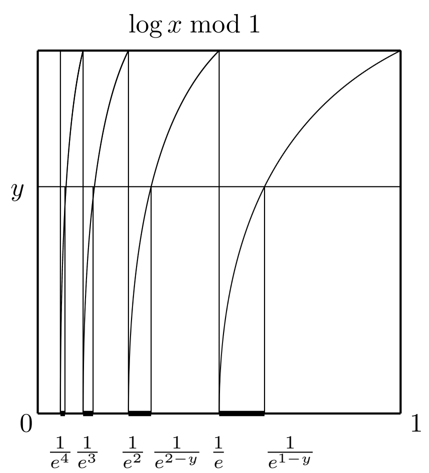
3.13.5.2 For a u.d. sequence $x_n \in (0,1)$ the two-dimensional sequence

$$(x_n, \{\log x_n\}), \quad n = 1, 2, \dots,$$

has a.d.f.

$$g(x,y) = \frac{e^y - e}{(e-1)e^i} + \min\left(x, \frac{1}{e^{i-y}}\right) \text{ if } x \in \left[\frac{1}{e^i}, \frac{1}{e^{i-1}}\right)$$

where $i = 1, 2, \dots$.
NOTES:
(I) J. Fialová personal communication.
(II) The result follows directly from the figure



3.13.5.3 For every u.d. sequence $x_n \in [0,1)$ the two-dimensional sequence

$$(x_n, \{\log n\}), \quad n = 1, 2, \dots,$$

has the set of d.f.s

$$g_u(x,y) = x g_u(y), \text{ where } g_u(y) = \frac{e^y - 1}{e - 1}\frac{1}{e^u} + \frac{e^{\min(y,u)} - 1}{e^u}.$$

NOTES: Every u.d. sequence $x_n \in [0,1)$, $n = 1, 2, \dots$, is statistically independent with the sequence $\{\log n\}$, $n = 1, 2, \dots$.

3.13.6.

NOTES: Let $r(\mathbf{h}) = \prod_{1\le j\le s} \max(1, |h_j|)$ for the integer vector $\mathbf{h} = (h_1, \dots, h_s)$ and let $\|x\|$ be the distance of x to the nearest integer. A real vector $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_s)$ is called of **finite type** η if

$$\eta = \inf\{\tau > 1\,;\, \exists_{c>0}\forall_{\mathbf{h}\neq\mathbf{0}}\ (r(\mathbf{h}))^\tau \|\mathbf{h}\cdot\boldsymbol{\alpha}\| \ge c\}$$

and is called of **constant type** (may not exist for $s > 1$) if

$$r(\mathbf{h})\,\|\mathbf{h}\cdot\boldsymbol{\alpha}\| \ge c$$

for some constant $c > 0$.

. . . .

Let $\boldsymbol{\beta} \neq \mathbf{0}$. The discrepancy of the sequence

$$\boldsymbol{\alpha} n + \boldsymbol{\beta}\log n \bmod 1$$

(I) for $\boldsymbol{\alpha}$ of the finite type η satisfies

$$D_N \ll N^{-\frac{1}{\eta+1/2}+\varepsilon}$$

(II) and if $\boldsymbol{\alpha}$ is of the constant type then

$$D_N \ll \frac{(\log N)^s}{N^{2/3}}.$$

NOTES: K. Goto and Y. Ohkubo (2000) extended the corresponding one–dimensional result of Ohkubo (1999), cf. 2.12.31.

K. GOTO – Y. OHKUBO: *The discrepancy of the sequence* $(n\boldsymbol{\alpha} + (\log n)\boldsymbol{\beta})$, Acta Math. Hungar. **86** (2000), no. 1–2, 39–47 (MR1728588 (2001k:11149); Zbl. 0980.11032).

Y. OHKUBO: *Notes on Erdős – Turán inequality*, J. Austral. Math. Soc. A **67** (1999), no. 1, 51–57 (MR1699155 (2000d:11100); Zbl. 0940.11029).

3.14 Sequences of rational numbers

3.14.1. Let $N \geq 1$ and $M \geq 2$ be integers. For an s–dimensional integer sequence $\mathbf{y}_n = (y_{n,1}, \dots, y_{n,s})$, $n = 1, \dots, N$, define

$$\mathbf{x}_n = \frac{\mathbf{y}_n}{M} \bmod 1.$$

Then the extremal discrepancy of $\mathbf{x}_n$ can be estimated as follows

$$D_N \leq 1 - \left(1 - \frac{1}{M}\right)^s + \sum_{\substack{\mathbf{h}=(h_1,\dots,h_s)\neq\mathbf{0} \\ -\frac{M}{2}<h_i\leq\frac{M}{2}, i=1,\dots,s}} \frac{1}{r(\mathbf{h}, M)} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \mathbf{h}\cdot\mathbf{x}_n} \right|,$$

and

$$D_N^* \geq 1 - \left(1 - \frac{1}{M}\right)^s,$$

where $r(\mathbf{h}, M) = \prod_{j=1}^{s} r(h_j, M)$ with

$$r(h, M) = \begin{cases} M \sin\left(\frac{\pi|h|}{M}\right), & \text{for } h \neq 0 \text{ and } -\frac{M}{2} < h \leq \frac{M}{2}, \\ 1, & \text{for } h = 0. \end{cases}$$

Notes: (I) H. Niederreiter (1992, p. 34, Th. 3.10; p. 41, Th. 3.14).
(II) If $M = 2^m$ with a positive integer m then P. Hellekalek ([a]1994) proved the estimate

$$D_N \leq 1 - \left(1 - \frac{1}{M}\right)^s + \left(m + \frac{1}{2}\right)^s \cdot \max_{\substack{\mathbf{h}\in\mathbb{Z}^s, \mathbf{h}\neq\mathbf{0} \\ 0\leq h_j<M \text{ for } j=1,\dots,s}} \left| \frac{1}{N} \sum_{n=1}^{N} H_{\mathbf{h}}(\mathbf{x}_n) \right|,$$

where $H_{\mathbf{h}} = \prod_{i=1}^{s} H_{h_i}(x_i)$ with $\mathbf{h} = (h_1, \dots, h_s)$ and $\mathbf{x} = (x_1, \dots, x_s)$ denotes the $\mathbf{h}$th normalized Haar function on $[0,1)^s$, i.e. if $h \geq 1$ and $h = 2^a + b$ with $0 \leq b < 2^a$ and $x \in [0,1)$ then

$$H_h(x) = \begin{cases} 1, & \text{for } x \in [b2^{-a}, b2^{-a} + 2^{-a-1}), \\ -1, & \text{for } x \in [b2^{-a} + 2^{-a-1}, (m+1)2^{-a}), \\ 0, & \text{otherwise.} \end{cases}$$

and $H_0(x) = 1$ for all $x \in [0,1]$. (For the theory of Haar functions cf. F. Schipp *et al.* (1990).) He also proved a similar formula using Walsh functions in (1994).

P. Hellekalek: *General discrepancy estimates: The Walsh function system*, Acta Arith. **67** (1994), 209–218 (MR1292735 (95h:65003); Zbl. 0805.11055).

[a] P. Hellekalek: *General discrepancy estimates II: The Haar function system*, Acta Arith. **67** (1994), no. 4, 313–322 (MR1301821 (96c:11088); Zbl. 0813.11046).
H. Niederreiter: *Random Number Generation and Quasi–Monte Carlo Methods*, CBMS–NSF Regional Conference Series in Applied Mathematics, 63, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992, vi+241 pp. (MR1172997 (93h:65008); Zbl. 0761.65002).
F. Schipp – W.R. Wade – P. Simon – J. Pál: *An Introduction to Dyadic Harmonic Analysis*, Adam Hilger, Ltd., Bristol, 1990 (MR1117682 (92g:42001); Zbl. 0727.42017).

3.14.2. The finite sequence of N^2 rational points in $[0,1)^2$

$$\left(\frac{i}{N},\frac{j}{N}\right),\qquad i=0,1,\dots,N-1,\quad j=0,1,\dots,N-1,$$

has discrepancy

$$D_{N^2}=\frac{2}{N}-\frac{1}{N^2}$$

and for every continuously differentiable $f(x,y)$ defined on $[0,1]^2$ we have

$$\lim_{N\to\infty} N\left(\int_0^1\int_0^1 f(x,y)\,\mathrm{d}x\,\mathrm{d}y-\frac{1}{N^2}\sum_{i,j=0}^{N-1} f\left(\frac{i}{N},\frac{j}{N}\right)\right)=$$
$$=\frac{1}{2}\int_0^1\big(f(1,y)-f(0,y)\big)\,\mathrm{d}y+\frac{1}{2}\int_0^1\big(f(x,1)-f(x,0)\big)\,\mathrm{d}x.$$

Notes: The limit was published in Mathematics Today (1986, p. 202).

Mathematics Today. 1986, (Russian), (A.J. Dorogovcev ed.), Golovnoe Izdateľstvo Izdateľskogo Ob"edineniya "Vishcha Shkola", Kiev, 1986 (MR0867889 (87h:00011); Zbl. 0596.00002).

3.14.3. Suppose that $s\ge 2$, $m_1,\dots,m_s$ are s positive integers, $N=m_1\dots m_s$ and $m=\min(m_1,\dots,m_s)$. Then the discrepancy of the s–dimensional finite sequence of N points

$$\left(\frac{a_1}{m_1},\frac{a_2}{m_2}\dots,\frac{a_s}{m_s}\right),\qquad 0\le a_i<m_i,\quad 1\le i\le s,$$

satisfies

$$\frac{1}{2m}\le D_N^*\le\frac{2^s}{m}.$$

Notes: L.–K. Hua and Y. Wang (1981, pp. 70–71, Th. 4.1–2).

L.–K. Hua – Y. Wang: *Applications of Number Theory to Numerical Analysis*, Springer Verlag & Science Press, Berlin, Heidelberg, New York, Beijing, 1981 (MR0617192 (83g:10034); Zbl. 0465.10045). (Chinese edition: Science Press, Beijing, 1978 (MR0617192 (83g:10034); Zbl. 0451.10001)).

3.14.3.1 Let $b \geq 2$, $k \geq 1$, and $t \geq 1$ be integers. Let the points

$$\mathbf{z}_0, \mathbf{z}_1, \dots, \mathbf{z}_{N-1} \in [0,1)^t$$

be such that all their coordinates are rational numbers with the denominator b^k. Then for the discrepancy D_N of these points we have

$$D_N = O_t\left(\frac{1}{b^k} + \frac{1}{N}\sum_{\mathcal{H}\in C(b)^{t\times k}}^{*} W_b(\mathcal{H})\left|\sum_{n=0}^{N-1} e^{2\pi\mathrm{i}\left(\frac{1}{b}\mathcal{H}\otimes\mathbf{z}_n\right)}\right|\right).$$

Here
$C(b) := (-b/2, b/2] \cap \mathbb{Z}$ is the least absolute residue system modulo b;
$C(b)^k$ is the set of k-tuples of elements of $C(b)$;
$C(b)^{t\times k}$ is the set of $t \times k$ matrices with entries from $C(b)$;
$\mathcal{H} = (h_{j,l}) \in C(b)^{t\times k}$;
$W_b(\mathcal{H}) := \prod_{j=1}^{t} Q_b(h_{j,1}, \dots, h_{j,k})$, where
$Q_b(h_1, \dots, h_k) := b^{-d}\csc(\pi|h_d/b)|$ if $(h_1, \dots, h_k) \neq \mathbf{0}$ and
$Q_b(h_1, \dots, h_k) := 1$ if $(h_1, \dots, h_k) = \mathbf{0}$;
$\mathcal{H}\otimes\mathbf{z} := \sum_{j=1}^{t}\sum_{l=1}^{k} h_{j,l} w_l^{(j)}$, where $\mathbf{z} = \left(z^{(1)}, \dots, z^{(t)}\right) \in [0,1)^t$ with
$z^{(j)} = \sum_{l=1}^{k} w_l^{(j)} b^{-l}$ with all $w_l^{(j)} \in \{0, 1, \dots, b-1\}$;
the asterisk $\sum^{*}$ denotes that the zero matrix is omitted from the range of summation.

H. Niederreiter – A. Winterhof: *Discrepancy bounds for hybrid sequences involving digital explicit inversive pseudorandom numbers*, Unif. Distrib. Theory **6** (2011), no. 1, 33–56 (MR2817759 (2012g:11143); Zbl. 1249.11075).

3.14.3.2 Let $b \geq 2$, $k \geq 1$, $s \geq 1$, and $t \geq 1$ be integers. Let the points

$$\mathbf{x}_n = (\mathbf{y}_n, \mathbf{z}_n) \in [0,1)^{s+t}, \quad n = 0, 1, \dots, N-1,$$

be such that $\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_{N-1} \in [0,1)^s$ are arbitrary and the coordinates of all points $\mathbf{z}_0, \mathbf{z}_1, \dots, \mathbf{z}_{N-1} \in [0,1)^t$ are rational numbers with the denominator

b^k. Let D_N be the discrepancy of $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{N-1}$. Then for any integer $H \geq 1$ we have

$$D_N = O_{s,t}\left(\frac{1}{b^k} + \frac{1}{H} + \frac{1}{N} \sum^*_{\substack{\mathbf{h}\in\mathbb{Z}^s,\ \mathcal{H}\in C(b)^{t\times k} \\ M(\mathbf{h})\leq H}} \frac{W_b(\mathcal{H})}{r(\mathbf{h})}\left|\sum_{n=0}^{N-1} e^{2\pi\mathrm{i}\left(\mathbf{h}\cdot\mathbf{y}_n + \frac{1}{b}\mathcal{H}\otimes\mathbf{z}_n\right)}\right|\right),$$

where
$M(\mathbf{h}) := \max_{1\leq i\leq s} |h_i|$;
$r(\mathbf{h}) := \prod_{i=1}^s \max(|h_i|, 1)$;
the symbol $\cdot$ denotes the standard inner product in $\mathbb{R}^s$, and
$W_b(\mathcal{H})$ and $\mathcal{H}\otimes\mathbf{z}_n$ are defined in 3.14.3.1.
The asterisk $\sum^*$ denotes that the pair $(\mathbf{h}, \mathcal{H}) = (\mathbf{0}, 0)$ is omitted from the range of summation.

H. Niederreiter: *Further discrepancy bounds and Erdős-Turán-Koksma inequality for hybrid sequences*, Monatsh. Math. **161** (2010), 193–222 (MR2680007 (2011i:11120); Zbl. 1273.11117).
H. Niederreiter: *A discrepancy bound for hybrid sequences involving digital explicit inversive pseudorandom numbers*, Unif. Distrib. Theory **5** (2010), no. 1, 53–63 (MR2804662 (2012f:11143); Zbl. 1249.11074).

3.15 Good lattice points

See also: 1.8.19

3.15.1. Good lattice points sequences.

(I) If $\mathbf{g} = (g_1, g_2, \dots, g_s) \in \mathbb{Z}^s$ are integral vectors depending on N such that the sequence

$$\mathbf{x}_n = \frac{n}{N}\mathbf{g} = \left(\frac{ng_1}{N}, \frac{ng_2}{N}, \dots, \frac{ng_s}{N}\right) \bmod 1, \quad n = 1, \dots, N,$$

has discrepancy

$$D_N(\mathbf{x}_n) = \mathcal{O}\left(\frac{(\log N)^s}{N}\right),$$

with $\mathcal{O}$–constant not depending on N, then $\mathbf{g}$ is called a sequence of **good lattice points** mod N (abbreviated g.l.p.)[2]

[2] It is also often convenient to call the vector $\mathbf{g}$ itself a **good lattice point**.

(II) For every dimension s there is a constant c_s such that for all $N \in \mathbb{N}$ there is a $\mathbf{g} \in \mathbb{Z}^s$ such that the sequence $\mathbf{x}_n = \frac{n}{N}\mathbf{g} \bmod 1$, $n = 1, 2, \ldots, N$, has discrepancy

$$D_N^* \leq c_s \frac{(\log N)^s}{N}.$$

(III) Especially, for a prime p and $s \geq 2$ there are g.l.p.'s $\mathbf{g}_p \in \mathbb{Z}^s$ and an effectively computable constant c_s which only depends on s such that the sequence

$$\mathbf{x}_n = \frac{n}{p}\mathbf{g}_p = \left(\frac{ng_{1,p}}{p}, \frac{ng_{2,p}}{p}, \ldots, \frac{ng_{s,p}}{p}\right) \bmod 1, \quad n = 1, \ldots, p,$$

has discrepancy

$$D_p \leq c_s \frac{(\log p)^s}{p}\ .$$

Furthermore, given an M with $1 \leq M \leq p$, there exits a $\mathbf{g}_p \in \mathbb{Z}^s$ such that the discrepancy of the sequence $\mathbf{x}_n = \frac{n}{p}\mathbf{g}_p \bmod 1$, $n = 1, 2, \ldots, M$, satisfies the inequality

$$D_M^* \leq c_s \frac{(\log p)^{s+1}}{M}.$$

(IV) If p is a prime then there exists a primitive root $g \pmod p$ such that for $\mathbf{g}_p = (1, g, \ldots, g^{s-1})$ the discrepancy of the sequence

$$\mathbf{x}_n = \frac{n}{p}\mathbf{g}_p = \left(\frac{n}{p}, \frac{ng}{p}, \ldots, \frac{ng^{s-1}}{p}\right) \bmod 1, \quad n = 1, \ldots, p,$$

satisfies

$$D_p^* = \mathcal{O}\left(\frac{(\log p)^s \log\log p}{p}\right).$$

(V) In the two–dimensional case we know that for every positive integer N there exits a lattice point $\mathbf{g} = (1, g)$ with gcd $(g, N) = 1$ such that for the sequence $\mathbf{x}_n = \frac{n}{N}\mathbf{g} \bmod 1$, $n = 1, 2, \ldots, N$, we have

$$D_N^* \leq c_2 \frac{(\log N)(\log\log N)^2}{N}.$$

(VI) Suppose that, for the s–dimensional lattice point $\mathbf{g} = (g_1, \ldots, g_s) \in \mathbb{Z}^s$ the congruence

$$\mathbf{g} \cdot \mathbf{x} = \sum_{i=1}^{s} g_i x_i \equiv 0 \pmod N$$

has no integral solution in the domain $\|\mathbf{x}\|_\infty \le M$, $\mathbf{x} \ne \mathbf{0}$, $\mathbf{x} = (x_1, \dots, x_s) \in \mathbb{Z}^s$ (here $\|\mathbf{x}\|_\infty = \max_{1\le i\le s}(|x_i|)$). Then the star discrepancy of the finite sequence $\mathbf{x}_n = \frac{n}{N}\mathbf{g}$, $n = 1, 2, \dots, N$, satisfies

$$D_N^* \le c_s \frac{(\log 3M)^s}{M}.$$

(VII) If $\mathbf{g} \in \mathbb{Z}^s$, $s \ge 2$, and $N \ge 2$ is an integer then for the discrepancy of the sequence $\mathbf{x}_n = \frac{n}{N}\mathbf{g} \bmod 1$, $n = 1, 2, \dots, N$, we have

$$D_N \le \frac{s}{N} + \frac{1}{2} R_N,$$

where

$$R_N = \sum_{\substack{\mathbf{h}=(h_1,\dots,h_s)\in\mathbb{Z}^s, \mathbf{h}\ne\mathbf{0} \\ -N/2<h_i\le N/2, \text{ for } i=1,\dots,s, \\ \mathbf{h}\cdot\mathbf{g}\equiv\mathbf{0} \,(\mathrm{mod}\, N)}} \frac{1}{r(\mathbf{h})}$$

and $r(\mathbf{h}) = \prod_{i=1}^s \max(1, |h_i|)$. If we denote

$$\rho_N = \min_{\substack{\mathbf{h}\in\mathbb{Z}^s, \mathbf{h}\ne\mathbf{0} \\ \mathbf{h}\cdot\mathbf{g}\equiv\mathbf{0} \,(\mathrm{mod}\, N)}} r(\mathbf{h}),$$

then we have

$$\frac{1}{\rho_N} \le R_N \le c(s) \frac{(\log N)^s}{\rho_N},$$

where $c(s)$ depends only on s. Furthermore,

$$R_N \ge c'(s) \frac{(\log N)^s}{N}.$$

Notes: (I) The sequence of the form

$$\mathbf{x}_n = \left(\frac{ng_1}{p}, \frac{ng_2}{p}, \dots, \frac{ng_s}{p} \right)$$

was first investigated by N.M. Korobov (1959, [a]1959) in connection with the numerical computation of multiple integrals. His results are summarized in the book Korobov (1963).
(II) The existence of g.l.p.'s modulo a prime was proved by E. Hlawka (1962) and

Korobov (1963, p. 96, Lemma 20) (for a proof cf. [KN, p. 154, Th. 5.7] and L.–K. Hua and Y. Wang (1981, p. 92, Th. 4.29)). In Hlawka (1964) further explicit error–estimates for numerical computations of multiple integrals can be found.

• The existence of g.l.p.'s $\mathbf{g}$ for composite integers N was proved by S.K. Zaremba (1973, 1974) in the form

$$\sum_{\substack{0<\|\mathbf{h}\|_\infty<N\\ \mathbf{h}\cdot\mathbf{g}\equiv\mathbf{0}\,(\mathrm{mod}\,N)}} \frac{1}{r(\mathbf{h})} < \frac{1}{N}(c+2\log N)^s,$$

with $c = 2$. H. Niederreiter (1978/79) improved this to $c = 1.4$ and for N a prime or a prime power to $c = 0.81$. If N is a prime then there exits a $\mathbf{g} \in \mathbb{Z}^s$ such that (cf. [KN, p. 156])

$$\sum_{\substack{0<\|\mathbf{h}\|_\infty<p\\ \mathbf{h}\cdot\mathbf{g}\equiv\mathbf{0}\,(\mathrm{mod}\,p)}} \frac{1}{r(\mathbf{h})} < \frac{2}{p}(5\log p)^s.$$

(IIa) Korobov (1963) did not use the name g.l.p. but in (1963, p. 96) he called coordinates of $\mathbf{g} = (g_1, \dots, g_s)$ the **optimal coefficients modulo** N **with index** β provided g_i, $i = 1, 2, \dots, s$, are coprime to N and

$$\sum_{\substack{0<\|\mathbf{h}\|_\infty<N\\ \mathbf{h}\cdot\mathbf{g}\equiv\mathbf{0}\,(\mathrm{mod}\,N)}} \frac{1}{r(\mathbf{h})} \le c_0\frac{(\log N)^\beta}{N}$$

(more precisely, if this inequality is true for infinitely many N and corresponding $\mathbf{g} = \mathbf{g}(N)$, $c_0 = c_0(s)$, and $\beta = \beta(s)$). He proved (1963, p. 141. Th. 22) that $\mathbf{g}$ is optimal (for some index β) if and only if $D_N = \mathcal{O}((\log N)^{\beta_1}/N)$ with a $\beta_1 = \beta_1(s)$. If $\mathbf{g}$ is of the form $\mathbf{g} = (1, g, \dots, g^{s-1})$ with $N = p$ a prime and $1 < g < p$, then Korobov (1963, p. 148, Th. 23) proved that $\mathbf{g}$ is optimal if g minimalizes the function

$$H(g) = \frac{3^s}{p}\sum_{k=1}^{p}\prod_{i=0}^{s-1}\left(1-2\left\{\frac{kg^i}{p}\right\}\right).$$

In the case $s = 2$ the value $H(g)$ can be computed using $\mathcal{O}(\log p)$ arithmetical operations. The method is based on the continued fraction machinery provided we know the continued fraction expansion $g/p = [0; a_1, a_2, \dots]$, see N.M. Dobrovol'skiĭ, A.R. Esayan, S.A. Pikhtil'kov, O.V. Rodionova and A.E. Ustyan (1999) and N.M. Dobrovol'skiĭ and O.V. Rodionova (2000).

In the case of general $\mathbf{g} = (g_1, \dots, g_s)$ if $N = p > 2$ is a prime number and $g_1 = 1$, then Korobov (1963, p. 120, the proof of Th. 18) proved that $\mathbf{g}$ has optimal coordinates (coprime with p) if for every $i = 1, 2, \dots, s-1$, the coordinate g_{i+1} minimalizes the expression

$$\sum_{k=1}^{p-1}\prod_{j=1}^{i+1}\left(1-2\log\left(2\sin\pi\left\{\frac{kg_j}{p}\right\}\right)\right).$$

(III) Hua and Wang (1981, p. 93, Th. 4.30).
(IV) H. Niederreiter (1977, 1978). Korobov (1959) pointed out that a g.l.p. may take the form $\mathbf{g} = (1, g, \dots, g^{s-1})$ and in (1960, 1963) he noted that to find the integer g it requires $\mathcal{O}(p^2)$ elementary operations (for $s = 2$ see (IIa)).
(V) G. Larcher (1986) improved (II) for $s = 2$. It is conjectured that for an arbitrary dimension s the result (II) can be improved to

$$D_N^* \le c_s \frac{(\log N)^{s-1}(\log\log N)^{k(s)}}{N}$$

with a suitable $k(s)$.
(VI) Hua and Wang (1981; p. 57, Th. 3.2)
(VII) H. Niederreiter (1992, p. 107, Th. 5.6; p. 108, (5.11)). The lower bound was given by G. Larcher (1987).
(VIII) G. Harman (1998) proved that for every prime p there exits a lattice point $\mathbf{g} \in \mathbb{Z}^s$ such that the ball discrepancy (cf. the def. in 1.11.8) of the sequence $\mathbf{x}_n = \frac{n}{p}\mathbf{g}$, $n = 1, 2, \dots, p$, satisfies

$$D_N^{\mathbf{B}(r)} \le c(s)\left(\frac{(pr^s)^{\frac{s-1}{s+1}}}{p} + \frac{1}{p}\right)$$

for all $r \in (0, 1)$.
(IX) A.I. Saltykov (1963) computed g.l.p.'s if the modulus N is a prime number for dimensions $s = 3, 4, 5, 6$ and if N is a product of two primes and $s = 3, 4, 5, 6, 7, 8$, see Korobov (1963, p. 217–222 Appendix). Y. Wang, G.S. Xu and R.X. Zhang (1978) computed tables of g.l.p.'s modulo N in s dimensions for $N \le 5.5 \times 10^7$ and $s \le 18$. H. Sugiura (1995) compiled tables for g.l.p.'s up to the number of sample points $N = 23644$ for $s = 3$, up to $N = 4590$ for $s = 4$ and up to $N = 1230$ for $s = 5$. He deduced efficient formulas in $3, 4, 5$, and 6 dimensions from the tables, see also G. Kedem and S.K. Zaremba (1974).
(X) For the sequence $\mathbf{x}_n = \frac{n}{N}\mathbf{g}(N) \bmod 1$, $n = 1, 2, \dots, N$, P. Zinterhof (1987) uses the so–called **practical lattice points** of the form

$$\mathbf{g}(N) = ([Ne^{r_1}], [Ne^{r_2}], \dots, [Ne^{r_s}]),$$

where $r_i = p_i/p_{s+1}$ with p_i denoting the ith prime.
(XI) Suppose that f is represented by the absolutely convergent Fourier series $f(\mathbf{x}) = \sum_{\mathbf{h}\in\mathbb{Z}^s} c_{\mathbf{h}} e^{2\pi i \mathbf{h}\cdot\mathbf{x}}$ ($\mathbf{x} \in \mathbb{R}^s$) with Fourier coefficients $c_{\mathbf{h}} = \int_{[0,1]^s} f(\mathbf{x}) e^{-2\pi i \mathbf{h}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{x}$. Then

$$\frac{1}{N}\sum_{n=1}^{N} f\left(\frac{n}{N}\mathbf{g}\right) - \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = \sum_{\substack{\mathbf{h}\in\mathbb{Z}^s, \mathbf{h}\neq\mathbf{0} \\ \mathbf{h}\cdot\mathbf{g}\equiv\mathbf{0} \pmod N}} c_{\mathbf{h}}$$

(cf. Korobov (1963, p. 98, Lemma 21) and Niederreiter (1992, p. 103)).
(XII) Denote by $E_s^\alpha(c)$ the set of all functions f on $\mathbb{R}^s$ represented by the multiple

Fourier series $f(\mathbf{x}) = \sum_{\mathbf{h}\in\mathbb{Z}^s} c_{\mathbf{h}} e^{2\pi i \mathbf{h}\cdot\mathbf{x}}$, where $|c_{\mathbf{h}}| \le c r^{-\alpha}(\mathbf{h})$ for every $\mathbf{h} \ne \mathbf{0}$ and for given $c > 0$ and $\alpha > 1$ (this class was first investigated by Korobov (1963, p. 29)). Then for any $\mathbf{g} \in \mathbb{Z}^s$ and any integer $N \ge 1$ we have (cf. Korobov (1963, p. 104, Formula 128) and Niederreiter (1992, p. 104, Th. 5.3))

$$\max_{f\in E_s^\alpha(c)} \left| \frac{1}{N}\sum_{n=1}^{N} f\left(\frac{n}{N}\mathbf{g}\right) - \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \right| = c. \sum_{\substack{\mathbf{h}\in\mathbb{Z}^s,\mathbf{h}\ne\mathbf{0}\\ \mathbf{h}\cdot\mathbf{g}\equiv\mathbf{0}\ (\mathrm{mod}\ N)}} \frac{1}{(r(\mathbf{h}))^\alpha},$$

where $\alpha > 1$ and $c > 0$ are real.

• If $r(\mathbf{h}_0) = \min\{r(\mathbf{h})\,;\, \mathbf{h}\cdot\mathbf{g} \equiv 0 \pmod N, \mathbf{0} \ne \mathbf{h} \in \mathbb{Z}^s\}$ and $f \in E_s^\alpha(c)$ then we have

$$\left| \frac{1}{N}\sum_{n=1}^{N} f\left(\frac{n}{N}\mathbf{g}\right) - \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \right| \le 4c\alpha \left(\frac{3\alpha^2}{\alpha-1}\right)^s \frac{(1+\log r(\mathbf{h}_0))^{s-1}}{(r(\mathbf{h}_0))^\alpha},$$

see N.S. Bachvalov (1959) and Korobov (1963, p. 126, Th. 19).

• If $\mathbf{g}$ is an s–dimensional g.l.p. modulo a prime p (cf. (II)) and $f \in E_s^\alpha(c)$ then we have (cf. [KN, p. 156–157, Ex. 5.4])

$$\left| \frac{1}{p}\sum_{n=1}^{p} f\left(\frac{n}{p}\mathbf{g}\right) - \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \right| \le c(1+2\zeta(\alpha))^s \frac{1+2^\alpha(5\log p)^{s\alpha}}{p^\alpha},$$

where $\zeta(\alpha)$ is the Riemann zeta function.

• If $\mathbf{g} \pmod N$ is a g.l.p. and $f \in E_s^\alpha(c)$ then the error term is again $\mathcal{O}\left(\frac{(\log N)^{s\alpha}}{N^\alpha}\right)$ (see Korobov (1963, p. 101, Th. 12)), and for every $\mathbf{g} \in \mathbb{Z}^s$, $N > 2^s$, $\alpha > 1$, $c > 1$, there exists an $f \in E_s^\alpha(c)$ such that

$$\left| \frac{1}{N}\sum_{n=1}^{N} f\left(\frac{n}{N}\mathbf{g}\right) - \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \right| \ge c.c' \frac{(\log N)^{s-1}}{N^\alpha},$$

where $c' = c'(\alpha, s)$ (see Korobov (1963, p. 104, Th. 13)).

(XIII) In the series of papers (1994-1997) S.M. Voronin described a method how to find an integer vector $\mathbf{g}$ and a prime p such that

$$\int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = \frac{1}{p}\sum_{n=1}^{p} f\left(\frac{n}{p}\mathbf{g}\right)$$

if a Fourier polynomial $f(\mathbf{x}) = \sum_{\mathbf{h}\in A\subset\mathbb{Z}^s} c_{\mathbf{h}} e^{2\pi i\mathbf{h}\cdot\mathbf{x}}$ (A is finite) is given. His technique is based on the theory of divisors in algebraic number fields.

For example, if $s = 2$ let $p \equiv 1 \pmod 4$ be a prime such that $p \nmid (h_1^2 + h_2^2)$ for every $\mathbf{h} = (h_1, h_2) \in A$ and $\mathbf{h} \ne \mathbf{0}$. Then $\mathbf{g} = (b_1, b_2)$, where $p = b_1^2 + b_2^2$. If $s = q - 1$ with q a prime, Voronin and V.I. Skalyga (1996) proved the existence of a prime p

and an integer a which satisfy $p \equiv 1 \pmod q$, $\gcd(a,p)=1$, $a^{\frac{p-1}{q}} \not\equiv 1 \pmod p$ such that for the lattice point $\mathbf{g} = (1, a^{(p-1)/q}, \dots, a^{(q-2)(p-1)/q})$ we have $\int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = \frac{1}{p}\sum_{n=1}^{p} f\big(\frac{n}{p}\mathbf{g}\big)$.

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3.15.2. Zaremba conjecture. Let $N \geq 2$ be an integer. Then the conjecture claims the existence of an integer $a \geq 1$ with $\gcd(a, N) = 1$ such that in the simple continued fraction expansion

$$\frac{a}{N} = [a_0; a_1, a_2, \dots, a_l]$$

we have the inequality $a_i \leq 5$ for all partial quotients a_i, $i = 1, 2, \dots, l$.
There exists an interesting connection between good lattice points in the two–dimensional case and the continued fractions of rational numbers. Let $K = \max_{1 \leq i \leq k} a_i$, then the quantity

$$\rho_N = \min_{\substack{\mathbf{h} \in \mathbb{Z}^2, \mathbf{h} \neq \mathbf{0} \\ \mathbf{h} \cdot \mathbf{g} \equiv \mathbf{0} \pmod N}} r(\mathbf{h})$$

(for the def. of $r(\mathbf{h})$ see p. 1 – 68) computed for the two–dimensional lattice point $\mathbf{g} = (1, a)$ satisfies the inequality

$$\frac{N}{K+2} \leq \rho_N \leq \frac{N}{K}$$

and applying 3.15.1(VII) to the sequence

$$\mathbf{x}_n = \frac{n}{N} \mathbf{g} \bmod 1, \quad n = 1, 2, \dots, N,$$

we obtain the estimate

$$D_N \leq \frac{2}{N} + \frac{1}{2} c(2)(K+2) \frac{(\log N)^2}{N} .$$

This implies that for smaller values of K we get better good lattice points modulo a fixed N.

NOTES: (I) This conjecture was formulated by S.K. Zaremba (1972, pp. 69 and 76) on the basis of unspecified numerical evidence. H. Niederreiter (1986) proved the conjecture for N of the type $N = 2^n$ or $N = 3^n$, $n = 1, 2, \dots$. More precisely his result states that there exists an integer $1 \le a < N$ with gcd $(a, N) = 1$ such that $K \le 3$. Niederreiter (1992, p. 146) recommends for further reading: I. Borosh and H. Niederreiter (1983), T.W. Cusick (1985, 1989), or J.W. Sander (1987).
(II) If $\frac{a}{N} = [a_0; a_1, a_2, \dots, a_l]$ and $\frac{p_j}{q_j} = [a_0; a_1, a_2, \dots, a_j]$, $j = 0, 1, 2, \dots, l$, then

$$\rho_N = \min_{0 \le j \le l} q_j |q_j a - p_j N|,$$

see Niederreiter (1992, p. 122, Th. 5.15).
(III) Niederreiter (1992, p. 123, (5.39))

$$D_N \le \frac{1 + \sum_{j=1}^{l} a_j}{N}.$$

It is conjectured that

$$\min_{\substack{a \in \mathbb{Z} \\ \gcd(a,N)=1}} \sum_{j=1}^{l} a_j = \mathcal{O}(\log N).$$

G. Larcher (1986) proved the bound $\mathcal{O}((\log N)(\log \log N)^2)$.
(IV) If F_m, $m = 1, 2, \dots$, are Fibonacci numbers then $\mathbf{g} = (1, F_{m-1})$ for $N = F_m$, while $K = 1$, as it is well–known. This selection of two–dimensional good lattice points $\mathbf{g}$ was first explicitly used by N.S. Bachvalov (1959). V.N. Temlyakov (1989) proved the optimality of $\mathbf{g}$ in numerical integration of a class of functions with bounded mixed derivates.
(V) If a is a positive integer then G. Larcher ([b]1986) proved that for the sequence

$$\mathbf{x}_n = \left(\tfrac{n}{N}, \tfrac{na}{N}\right) \bmod 1, \quad n = 1, 2, \dots, N,$$

we have (for the def. of the dispersion d_N^∞ cf. 1.11.17)

$$d_N^\infty = \max(A_{Q(N)}, A_{Q(N)-1}),$$

where (using the continued fraction expansion $a/N = [0; a_1, a_2, \dots]$)

- $A_k = \min\left(f_{k-1} - [h_k] \cdot f_k, \frac{q_{k-1} + ([h_k]+1).q_k}{N}\right)$,
- $h_k = \frac{N f_{k-1} - q_{k-1}}{q_k + N.f_k}$,
- $f_k = \left\| q_k \cdot \frac{a}{N} \right\|$ (the distance to the nearest integer),
- $Q(N)$ is defined by $q^2_{Q(N)} \le N < q^2_{Q(N)+1}$.

Applying this to

$$\mathbf{x}_n = \left(\frac{n}{F_m}, \frac{nF_{m-1}}{F_m}\right) \bmod 1, \qquad n = 1, 2, \dots, F_m,$$

he found that

$$d^\infty_{F_m} = \frac{F_{[m/2]+1}}{F_m}$$

which yields

$$\lim_{N \to \infty} \min_{a \in \mathbb{N}} d_N^\infty \sqrt{N} = \frac{1}{\sqrt{2}}.$$

(VI) J. Bourgain and A. Kontorovich (2014) proved Zaremba's conjecture for almost all N and for $K = 50$.
(VII) D. Hensley (2006), p. 38, Th. 3.2, proved that there are at least on the order x^δ integers N between x and $x + \varepsilon x$ for which $a_i \leq K$. Here δ is the Hausdorff dimension of the Cantor set C_K consisting of all numbers x with continued fraction expansion $x = [0; a_1, a_2, \dots, a_i \dots]$ with $a_i \leq K$, $i = 1, 2, \dots$.
(VIII) Also, Hensley (2006), p. 34, conjectured the strong version of Zaremba's conjecture saying that $a_i \leq K = 2$ for all sufficiently large N.
(IX) Also consult with S.Y. Huang (2015), I.D. Kan and D.A. Frolenkov (2014) and I.D. Kan (2015).

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3.15.3. Let $\mathbb{Q}_s$ be a real algebraic number field of degree s and $\omega_1, \dots, \omega_s$ be an integer basis of $\mathbb{Q}_s$, where $\omega_2, \dots, \omega_s$ are irrational numbers, and let $N, h_1, \dots, h_s$ be integers which satisfy

$$\left| \frac{h_j}{N} - \omega_j \right| \le \frac{c(\mathbb{Q}_s, \varepsilon)}{N^{1+\frac{1}{s-1}}}, \quad j = 2, 3, \dots, s.$$

Then the discrepancy of the finite s–dimensional sequence

$$\mathbf{x}_n = \left(\frac{n}{N}, \frac{nh_2}{N}, \dots, \frac{nh_s}{N} \right) \bmod 1, \quad n = 1, 2, \dots, N,$$

satisfies

$$D_N^* \le \frac{c(\mathbb{Q}_s, \varepsilon)}{N^{\frac{1}{2}+\frac{1}{2(s-1)}-\varepsilon}},$$

with ε being an arbitrary pre–assigned positive number. If $1 \le M \le N^{\frac{1}{2}+\frac{1}{2(s-1)}}$ then for the sequence

$$\left(\frac{nh_2}{N}, \dots, \frac{nh_s}{N} \right) \bmod 1, \quad n = 1, 2, \dots, M,$$

we have

$$D_M^* \le \frac{c(\mathbb{Q}_s, \varepsilon)}{M^{1-\varepsilon}}.$$

NOTES: L.–K. Hua and Y. Wang (1981, p. 86, Th. 4.16–17)).

L.–K. HUA – Y. WANG: *Applications of Number Theory to Numerical Analysis*, Springer Verlag & Science Press, Berlin, Heidelberg, New York, Beijing, 1981 (MR0617192 (83g:10034); Zbl. 0465.10045). (Chinese edition: Science Press, Beijing, 1978 (MR0617192 (83g:10034); Zbl. 0451.10001)).

3.15.4. Let $\mathbb{R}_s = \mathbb{Q}\left(\cos \frac{2\pi}{m}\right)$ be the real cyclotomic field (of degree $s = \frac{\varphi(m)}{2}$). Let the integers c_j be such that

$$\left| \frac{c_j}{N} - 2\cos\frac{2\pi(j-1)}{m} \right| \le \frac{c(\mathbb{R}_s, \varepsilon)}{N^{1+\frac{1}{s-1}}}, \quad j = 2, 3, \dots, s,$$

where $c(\mathbb{R}_s, \varepsilon)$ depends on $\mathbb{R}_s$ and ε. Then the star discrepancy of the finite s–dimensional sequence

$$\mathbf{x}_n = \left(\frac{nc_1}{N}, \frac{nc_2}{N}, \dots, \frac{nc_s}{N} \right) \bmod 1, \quad n = 1, 2, \dots, N,$$

satisfies the inequality

$$D_N^* \leq \frac{c(\mathbb{R}_s, \varepsilon)}{N^{\frac{1}{2}+\frac{1}{2(s-1)}-\varepsilon}},$$

where ε is an arbitrary pre–assigned positive number. If $1 \leq M \leq N^{\frac{1}{2}+\frac{1}{2(s-1)}}$ then for the sequence

$$\left(\frac{nc_2}{N}, \dots, \frac{nc_s}{N}\right) \bmod 1, \quad n = 1, 2, \dots, M,$$

we have

$$D_M^* \leq \frac{c(\mathbb{R}_s, \varepsilon)}{M^{1-\varepsilon}}.$$

NOTES: L.–K. Hua and Y. Wang (1981, p. 87, Th. 4.18–19)).

L.–K. HUA – Y. WANG: *Applications of Number Theory to Numerical Analysis*, Springer Verlag & Science Press, Berlin, Heidelberg, New York, Beijing, 1981 (MR0617192 (83g:10034); Zbl. 0465.10045). (Chinese edition: Science Press, Beijing, 1978 (MR0617192 (83g:10034); Zbl. 0451.10001)).

3.15.5. Let p be a prime greater than the integer s. Then the star discrepancy of the finite sequence

$$\mathbf{x}_n = \left(\frac{n}{p^2}, \frac{n^2}{p^2}, \dots, \frac{n^s}{p^2}\right) \bmod 1 \quad n = 1, 2, \dots, p^2,$$

satisfies

$$D_{p^2}^* \leq c_s \frac{(\log p)^s}{p}.$$

If $f \in E_s^\alpha(c)$ (i.e. $f(\mathbf{x}) = \sum_{\mathbf{h}\in\mathbb{Z}^s} c_{\mathbf{h}} e^{2\pi i \mathbf{h}\cdot\mathbf{x}}$, $|c_{\mathbf{h}}| \leq \frac{c}{(r(\mathbf{h}))^\alpha}$, $\alpha > 1$) then for the error term we have

$$\left| \frac{1}{p^2} \sum_{n=1}^{p^2} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \right| \leq \frac{(s-1)\sigma}{p} + \frac{\beta^s c}{p^\alpha},$$

where $\beta < 4 + \frac{2}{\alpha-1}$ and $\sigma = \sum_{\mathbf{h}\in\mathbb{Z}^s} |c_{\mathbf{h}}|$. In the opposite case, for every $\mathbf{g} = (g_1, \dots, g_s) \in \mathbb{Z}^s$ with $\gcd(g_i, p) = 1$ for $i = 1, 2, \dots, s$, there exists an $f \in E_s^\alpha(c)$ such that for

$$\mathbf{x}_n = \left(\frac{g_1 n}{p^2}, \frac{g_2 n^2}{p^2}, \dots, \frac{g_s n^s}{p^2}\right) \bmod 1, \quad n = 1, 2, \dots, p^2,$$

we have

$$\left|\frac{1}{p^2}\sum_{n=1}^{p^2} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x}\right| \ge \frac{c}{p}.$$

NOTES: (I) N.M. Korobov (1957) investigated this sequence in connection with approximation of multiple integrals. The given star discrepancy is from L.–K. Hua and Y. Wang (1981, p. 79, Th. 4.8). The error terms in the quadrature formulas are from Korobov (1963, p. 70, Th. 5; p. 74, Th. 7). It is also true that if $f(\mathbf{x})$ can be expressed as an absolutely convergent Fourier series and the partial derivative $\frac{\partial^{2s} f(\mathbf{x})}{\partial \mathbf{x}^2}$ is continuous and for any integers $j_1, ..., j_r$ the partial derivatives $\frac{\partial^{2r} f(\mathbf{x})}{\partial x_{j_1}^2 \dots \partial x_{j_r}^2}$ are bounded in magnitude by a constant C, then $\left|\frac{1}{p^2}\sum_{n=1}^{p^2} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x}\right| \le \frac{(s-1)\sigma}{p} + \frac{sC}{10p^2}$.
(II) With the aim to approximate the multiple integral equation of the form

$$\phi(\mathbf{x}) = \lambda \textstyle\int_{[0,1]^s} K(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,\mathrm{d}\mathbf{y} + f(\mathbf{x})$$

Korobov (1959) investigated the sequence

$$\mathbf{x}_n = \left(\frac{n}{p}, \frac{n^2}{p}, \dots, \frac{n^s}{p}\right) \bmod 1, \quad n = 1, 2, \dots, p,$$

where $p > s$ is a prime. For this sequence we have (cf. L.–K. Hua and Y. Wang (1981, p. 79, Th. 4.9))

$$D_p^* \le c_s \frac{(\log p)^s}{\sqrt{p}}\,.$$

With the same $\mathbf{x}_n$ and $f \in E_s^\alpha(c)$ we again have $\left|\frac{1}{p}\sum_{n=1}^{p} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x}\right| \le \frac{(s-1)\sigma}{\sqrt{p}} + \frac{\beta^s c}{p^\alpha}$, where $\beta < 4 + \frac{2}{\alpha-1}$, see Korobov (1963, p. 72, Th. 6) and also L.–K. Hua and Y. Wang (1981, p. 134, Th. 7.3).
(III) The star discrepancy of the double sequence

$$\mathbf{x}_{n,k} = \left(\frac{k}{p}, \frac{nk}{p}, \dots, \frac{n^{s-1}k}{p}\right) \bmod 1, \quad n, k = 1, 2, \dots, p,$$

satisfies by L.–K. Hua and Y. Wang (1981, p. 79, Th. 4.7) the inequality

$$D_{p^2}^* \le c_s \frac{(\log p)^s}{p}.$$

L.–K. HUA – Y. WANG: *Applications of Number Theory to Numerical Analysis*, Springer Verlag & Science Press, Berlin, Heidelberg, New York, Beijing, 1981 (MR0617192 (83g:10034); Zbl. 0465.10045). (Chinese edition: Science Press, Beijing, 1978 (MR0617192 (83g:10034); Zbl. 0451.10001)).
N.M. KOROBOV: *Approximate calculation of repeated integrals by number–theoretical methods*, (Russian), Dokl. Akad. Nauk SSSR (N.S.), **115** (1957), 1062–1065 (MR0098714 (**20** #5169); Zbl. 0080.04601).

N.M. Korobov: *Approximate solution of integral equations*, (Russian), Dokl. Akad. Nauk SSSR (N.S.), **128** (1959), 235–238 (MR0112260 (**22** #3114); Zbl. 0089.04202).

3.16 Lattice points involving recurring sequences

3.16.1. Let F_m be the mth Fibonacci number. Then the star discrepancy of the two–dimensional finite sequence

$$\left(\frac{n}{F_m}, \frac{nF_{m-1}}{F_m}\right) \bmod 1, \quad n = 1, 2, \dots, F_m,$$

satisfies

$$D^*_{F_m} \le c\frac{(\log 3F_m)^2}{F_m}.$$

Notes: L.–K. Hua and Y. Wang (1981, p. 92, Th. 4.28). Actually they proved that if Q_n, $n = 1, 2, \dots$, is a linear recurring sequence of positive integers such that

- $Q_1 \le Q_2$ with $\gcd(Q_1, Q_2) = 1$,
- $Q_n = a_nQ_{n-1} + Q_{n-2}$, $n = 3, 4, \dots$, where
- $a_3, a_4, \dots$ is a sequence of positive integers, $a_n \le M$ with M a constant,

then there exists a constant $c(Q_1, Q_2, M)$ such that for the star discrepancy of the sequence

$$\left(\frac{n}{Q_m}, \frac{nQ_{m-1}}{Q_m}\right) \bmod 1, \quad n = 1, 2, \dots, Q_m,$$

we have

$$D^*_{Q_m} \le c(Q_1, Q_2, M)\frac{(\log 3Q_m)^2}{Q_m}.$$

L.–K. Hua – Y. Wang: *Applications of Number Theory to Numerical Analysis*, Springer Verlag & Science Press, Berlin, Heidelberg, New York, Beijing, 1981 (MR0617192 (83g:10034); Zbl. 0465.10045). (Chinese edition: Science Press, Beijing, 1978 (MR0617192 (83g:10034); Zbl. 0451.10001)).

3.16.2. Assume that

- α is a P.V. number of degree s, i.e. $\alpha > 1$ and its conjugates satisfy $|\alpha^{(2)}| \le \dots \le |\alpha^{(s)}| < 1$,
- α is the root of the irreducible polynomial $x^s - a_{s-1}x^{s-1} - \dots - a_1x - a_0 = 0$,
- Q_n, $n = 0, 1, 2, \dots$, is a sequence of integers defined by the recurrence relation
- $Q_n = a_{s-1}Q_{n-1} + \dots + a_1Q_{n-s+1} + a_0Q_{n-s}$, $n = s, s+1, \dots$, where
- $Q_0 = Q_1 \dots = Q_{s-2} = 0$, $Q_{s-1} = 1$, and denote

- $Q_n(j) = Q_{n+j-1} - a_{s-1}Q_{n+j-2} - \cdots - a_{s-j+2}Q_{n+1} - a_{s-j+1}Q_n$, $j = 2, 3, \ldots, s$.

Then the discrepancy of the s–dimensional finite sequence

$$\mathbf{x}_n = \left(\frac{n}{Q_m}, \frac{nQ_m(2))}{Q_m}, \ldots, \frac{nQ_m(s)}{Q_m}\right) \bmod 1, \quad n = 1, 2, \ldots, |Q_m|,$$

satisfies

$$D^*_{|Q_m|} \le \frac{c(\alpha, \varepsilon)}{|Q_m|^{\frac{1}{2}+\frac{\varrho}{2}-\varepsilon}},$$

where $\rho = -\frac{\log|\alpha^{(s)}|}{\log \alpha}$ and ε is an arbitrary pre–assigned positive number. If M fulfils the inequalities $1 \le M \le |Q_m|^{\frac{1}{2}+\frac{\varrho}{2}}$ then for the $(s-1)$–dimensional sequence

$$\left(\frac{nQ_m(2))}{Q_m}, \ldots, \frac{nQ_m(s)}{Q_m}\right) \bmod 1, \quad n = 1, 2, \ldots, M,$$

we have

$$D^*_M \le \frac{c(\alpha, \varepsilon)}{M^{1-\varepsilon}}$$

with ε being an arbitrary pre–assigned positive number.
NOTES: L.–K. Hua and Y. Wang (1981, p. 88, Th. 4.20–1).

L.–K. HUA – Y. WANG: *Applications of Number Theory to Numerical Analysis*, Springer Verlag & Science Press, Berlin, Heidelberg, New York, Beijing, 1981 (MR0617192 (83g:10034); Zbl. 0465.10045). (Chinese edition: Science Press, Beijing, 1978 (MR0617192 (83g:10034); Zbl. 0451.10001)).

3.16.3. Let F_n, $n = 0, 1, 2, \ldots$, be the sequence of integers (the so–called s–dimensional Fibonacci sequence) defined by the recurrence relation

- $F_n = F_{n-1} + \cdots + F_{n-s+1} + F_{n-s}$, $n = s, s+1, \ldots$, where
- $F_0 = F_1 = \cdots = F_{s-2} = 0$, $F_{s-1} = 1$, and denote
- $F_n(j) = F_{n+j-1} - F_{n+j-2} - \cdots - F_{n+1} - F_n$ for $j = 2, 3, \ldots, s$.

Then for the star discrepancy of the s–dimensional finite sequence

$$\mathbf{x}_n = \left(\frac{n}{F_m}, \frac{nF_m(2))}{F_m}, \ldots, \frac{nF_m(s)}{F_m}\right) \bmod 1, \quad n = 1, 2, \ldots, F_m,$$

we have

$$D^*_{F_m} \le \frac{c(s)}{F_m^{\frac{1}{2}+\frac{1}{2^{s+1}\log 2}+\frac{1}{2^{2s+3}}}}$$

and if M satisfies $1 \le M \le F_m^{\frac{1}{2}+\frac{1}{2^{s+1}\log 2}+\frac{1}{2^{2s+2}}}$, then for the $(s-1)$–dimensional sequence

$$\left(\frac{nF_m(2))}{F_m},\dots,\frac{nF_m(s)}{F_m}\right) \bmod 1, \quad n=1,2,\dots,M,$$

we have

$$D_M^* \le \frac{c(s,\varepsilon)}{M^{1-\varepsilon}}$$

with ε being an arbitrary pre–assigned positive number.
NOTES: L.–K. Hua and Y. Wang (1981, p. 89, Th. 4.22–3).

L.–K. HUA – Y. WANG: *Applications of Number Theory to Numerical Analysis*, Springer Verlag & Science Press, Berlin, Heidelberg, New York, Beijing, 1981 (MR0617192 (83g:10034); Zbl. 0465.10045). (Chinese edition: Science Press, Beijing, 1978 (MR0617192 (83g:10034); Zbl. 0451.10001)).

3.17 Lattice rules

C.f. H. Niederreiter (1992, pp. 125–146) and for the def. see 1.8.20.
(I) For every s–dimensional N–point lattice rule with $N \ge 2$, the node set

$$\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{N-1}$$

consists exactly of all the fractional parts

$$\left\{\sum_{i=1}^r \frac{k_i}{n_i}\mathbf{g}_i\right\} \text{ with integers } 0 \le k_i < n_i \text{ and } 1 \le i \le r,$$

where the integer r with $1 \le r \le s$ and the integers $n_1,\dots,n_r \ge 2$ with $n_{i+1}|n_i$ for $1 \le i \le r-1$ and $n_1 \dots n_r = N$ are uniquely determined. Furthermore, the vectors $\mathbf{g}_1,\dots,\mathbf{g}_s \in \mathbb{Z}^s$ are linearly independent, and, for each $1 \le i \le r$ the coordinates of $\mathbf{g}_i$ and n_i are coprime. (I.H. Sloan and J.N. Lyness (1989), cf. Niederreiter (1992, p. 130, Th. 5.28)).

- The integer r is called the **rank** of the lattice rule.
- The integers $n_1,\dots,n_r$ are called the **invariants** of the lattice rule.

(II) If $f(\mathbf{x})$ is a periodic function represented by its absolutely convergent Fourier series $f(\mathbf{x}) = \sum_{\mathbf{h}\in\mathbb{Z}^s} c_{\mathbf{h}} e^{2\pi i\mathbf{h}\cdot\mathbf{x}}$ with Fourier coefficients given by $c_{\mathbf{h}} = \int_{[0,1]^s} f(\mathbf{x})e^{-2\pi i\mathbf{h}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{x}$ then

$$\frac{1}{N}\sum_{n=0}^{N-1} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x} = \sum_{\mathbf{h}\in L^\perp,\mathbf{h}\ne\mathbf{0}} c_{\mathbf{h}}.$$

For the set $E_s^\alpha(c)$ of all $f(\mathbf{x})$ for which $|c_{\mathbf{h}}| \le c(r(\mathbf{h}))^{-\alpha}$ for all non–zero $\mathbf{h} \in \mathbb{Z}^s$ and some constants $c > 0$ and $\alpha > 1$ we have (cf. Niederreiter (1992, p. 127, Th. 5.23))

$$\max_{f\in E_s^\alpha(c)} \left| \frac{1}{N}\sum_{n=0}^{N-1} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \right| = c.P_\alpha(L),$$

where the **discrepancy** $P_\alpha(L)$ **of lattice rule** L is defined by

$$P_\alpha(L) = \sum_{\mathbf{h}\in L^\perp,\mathbf{h}\neq\mathbf{0}} (r(\mathbf{h}))^{-\alpha}$$

and $L^\perp$ is the dual lattice of L, see 1.8.20. The maximum is attained at $f(\mathbf{x}) = c\cdot\sum_{\mathbf{h}\in\mathbb{Z}^s} \frac{e^{2\pi i\mathbf{h}\cdot\mathbf{x}}}{(r(\mathbf{h}))^\alpha}$.

(III) For the extremal discrepancy D_N of the node set $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{N-1}$ of an s–dimensional point lattice rule L with $s \ge 2$ and $N \ge 2$ we have (see Niederreiter (1992, p. 136, Th. 5.35, and p. 138, Th. 5.37) and also Niederreiter (1985))

$$\frac{1}{c_s\rho(L)} \le D_N < \frac{s}{N} + \frac{1}{\rho(L)}\left(\frac{2}{\log 2}\right)^{s-1}\left((\log N)^s + \frac{3}{2}(\log N)^{s-1}\right),$$

where $c_2 = 4$, $c_3 = 27$, and $c_s = \frac{2}{\pi}((\pi+1)^s - 1)$ for $s \ge 4$. Here

• for any s–dimensional lattice L the **figure of merit** $\rho(L)$ is defined by

$$\rho(L) = \min_{\mathbf{h}\in L^\perp,\mathbf{h}\neq\mathbf{0}} r(\mathbf{h}).$$

The lattice discrepancy $P_\alpha(L)$ defined in (II) satisfies the estimates

$$\frac{1}{(\rho(L))^\alpha} \le P_\alpha(L) = \mathcal{O}\left(\frac{(1+\log\rho(L))^{s-1}}{(\rho(L))^\alpha}\right),$$

cf. I.H. Sloan and P.J. Kachoyan (1987).

H. Niederreiter ([a]1992, Coroll. 2) proved that for every $s \ge 2$ and any prescribed invariants n_1 and n_2, there exists an s–dimensional N–point (where $N = n_1n_2$) lattice rule of rank 2 such that the discrepancy of the node set $\mathbf{x}_0, \dots, \mathbf{x}_{N-1}$ satisfies

$$D_N \le c(s)\left(\frac{(\log N)^s}{N} + \frac{\log N}{n_1}\right).$$

The general lower bound of D_N for any s–dimensional N-point lattice rule is (cf. Niederreiter ([a]1992))

$$D_N \geq \frac{1}{n_1}.$$

NOTES: (IV) the g.l.p. sequence (see 3.15.1)

$$\mathbf{x}_n = \frac{n}{N}\mathbf{g} \bmod 1, \quad n = 0, 1, \ldots, N-1,$$

with $\mathbf{g} = (g_1, \ldots, g_s) \in \mathbb{Z}^s$ and $\gcd(g_1, \ldots, g_s, N) = 1$ is the lattice rule of rank 1.
(V) The s–dimensional lattice rule

$$\left(\frac{k_1}{m}, \ldots, \frac{k_s}{m}\right), \quad k_i \in \mathbb{Z}, 0 < k_i \leq m \text{ for } 1 \leq i \leq s,$$

has the rank s, invariants $n_i = m$ for $1 \leq i \leq s$, and $N = m^s$ points. If $f \in E_s^\alpha(c)$ (see the def. in (II)) then N.M. Korobov (1963, p. 49, Th. 3) proved the error term estimate

$$\left| \frac{1}{m^s} \sum_{k_1,\ldots,k_s=1}^{m} f\left(\frac{k_1}{m}, \ldots, \frac{k_s}{m}\right) - \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \right| = \mathcal{O}\left(\frac{1}{m^\alpha}\right)$$

and the order of the error is the best possible because it is attained for some $f \in E_s^\alpha(c)$.
L.–K. Hua and Y. Wang (1981, p. 131, Th. 7.1) estimated the supremum of the left–hand side with f running over $E_s^\alpha(c)$ by $\leq c(2\zeta(\alpha)+1)^s m^{-\alpha}$with a suitable c.
(VI) We have

$$1 \leq \rho(L) \leq n_1,$$

where n_1 is the first invariant of L. This implies that

$$\rho(L) \leq N/2$$

for rank ≥ 1, see Niederreiter (1992, p. 133, Lemma 5.32, Rem. 5.33).
(VII) Niederreiter (1992, p. 144, Th. 5.44) proved: If L is an s–dimensional N–point lattice rule and $k \geq 2$ is an integer, then $k^{-1}L$ is a $k^s N$–point lattice rule of rank s with dual lattice $(k^{-1}L)^\perp = kL^\perp$.
(VIII) Niederreiter (1992, p. 139, Lemma 5.39): For $s \geq 2$, let a rank $1 \leq r \leq s$ and invariants $n_1, \ldots, n_r \geq 2$ with $n_{i+1} | n_i$ for $i = 1, 2, \ldots, r-1$ be given. Let L_1 be an s–dimensional n_1–point lattice rule of rank 1 generated by $\mathbf{g}_1 = (g_{1,1}, \ldots, g_{1,s}) \in \mathbb{Z}^s$, with $\gcd(g_{1,1}, n_1) = 1$. Then an s–dimensional lattice rule L with rank r and invariants $n_1, \ldots, n_r$ exists such that the node set of L contains the node set of L_1.
(IX) F.J. Hickernell (1998) expressed the discrepancy $P_{2k}(L)$ in terms of Bernoulli polynomials $B_{2k}(x)$

$$P_{2k}(L) = -1 + \tfrac{1}{N} \textstyle\sum_{n=0}^{N-1} \prod_{j=1}^{s} \left(1 - \tfrac{(-1)^k (2\pi)^{2k}}{2k!} B_{2k}(x_{n,j})\right),$$

where $\mathbf{x}_n = (x_{n,1}, \dots, x_{n,s})$ and $\mathbf{x}_0, \dots, \mathbf{x}_{N-1}$ is the s–dimensional N–point lattice rule.
(X) I.H. Sloan and L. Walsh (1990) gave several examples of lattice rules of rank 2.

F.J. Hickernell: *Lattice rules: How well do they measure up?* (P. Hellekalek and G. Larcher eds.), in: *Random and quasi-random point sets*, Lecture Notes in Statistics **138**, pp. 109–166, Springer, New York, NY, 1998 (MR1662841 (2000b:65007); Zbl. 0920.65010).
L.–K. Hua – Y. Wang: *Applications of Number Theory to Numerical Analysis*, Springer Verlag & Science Press, Berlin, Heidelberg, New York, Beijing, 1981 (MR0617192 (83g:10034); Zbl. 0465.10045). (Chinese edition: Science Press, Beijing, 1978 (MR0617192 (83g:10034); Zbl. 0451.10001)).
H. Niederreiter: *The serial test for pseudo–random numbers generated by the linear congruential method*, Numer. Math **46** (1985), 51–68 (MR0777824 (86i:65010); Zbl. 0541.65004).
H. Niederreiter: *Random Number Generation and Quasi–Monte Carlo Methods*, CBMS–NSF Regional Conference Series in Applied Mathematics, 63, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992, vi+241 pp. (MR1172997 (93h:65008); Zbl. 0761.65002).
[a]H. Niederreiter: *The existence of efficient lattice rules for multidimensional numerical integration*, Math. Comp. **58** (1992), no. 197, 305–314 (MR1106976 (92e:65023); Zbl. 0743.65018).
I.H. Sloan – P. Kachoyan: *Lattice methods for multiple integration: Theory, error analysis and examples*, SIAM J. Numer. Anal. **24** (1987), 116–128 (MR0874739 (88e:65023); Zbl. 0629.65020).
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I.H. Sloan – L. Walsh: *A computer search of rank–2 lattice rules for multidimensional quadrature*, Math. Comp. **54** (1990), no. 189, 281–302 (MR1001485 (91a:65061); Zbl. 0686.65012).

3.18 Sequences involving radical inverse function

3.18.1. Halton sequence. Let $n = \sum_{j=0}^{\infty} a_j(n) q^j$, $a_j \in \{0, 1, \dots, q-1\}$, be the q–adic digit expansion of the integer n, where $q \geq 2$ is an integer. The **radical inverse function** (cf. 2.11.2) or the **Monna map** in the base q is defined by $\gamma_q(n) = \sum_{j=0}^{\infty} a_j(n) q^{-j-1}$ for $n = 0, 1, 2, \dots$. The Halton sequence in the bases $q_1, \dots, q_s$ is defined by

$$\mathbf{x}_n = (\gamma_{q_1}(n), \dots, \gamma_{q_s}(n)), \quad n = 0, 1, 2, \dots.$$

For the pairwise coprime bases $q_1, \dots, q_s$ the Halton sequence is

u.d.

and for the discrepancy of $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{N-1}$ we have

$$D_N(\mathbf{x}_n) = \mathcal{O}\left(\frac{(\log N)^s}{N}\right),$$

more precisely

$$D_N^* < \frac{s}{N} + \frac{1}{N} \prod_{i=1}^{s} \left(\frac{q_i - 1}{2 \log q_i} \log N + \frac{q_i + 1}{2} \right)$$

for $N = 1, 2, \dots$.

NOTES: (I) J.H. Halton (1960) and for the discrepancy cf. H. Niederreiter (1992, p. 29, Th. 3.6).

(II) I.M. Soboľ (1969, p. 176, Th. 3) gave the estimate

$$D_N^* \le \frac{1}{N} \prod_{i=1}^{s} \left(\frac{q_i - 1}{\log q_i} \log N + 2q_i - 1 \right).$$

(III) L.–K. Hua and Y. Wang (1981, p. 74, Th. 4.3) proved

$$D_N^* \le \frac{1}{N} \prod_{i=1}^{s} \left(\frac{q_i \log(q_i N)}{\log q_i} \right)$$

but for $\mathbf{x}_n$ with $n = 1, 2, \dots, N$ (not with $n = 0, 1, \dots, N-1$).

(IV) If $2 \le q_1 < q_2 < \cdots < q_s$ are pairwise coprime bases then G. Larcher (1986) proved that for the isotropic discrepancy I_N (cf. 1.11.9) of $\mathbf{x}_n$ we have

$$N^{1/s} I_N \le c.q_1^2 q_2 \dots q_{s-1} q_s^s.$$

(V) P. Hellekalek and H. Niederreiter (2011): The s-dimensional Halton sequence $\mathbf{x}_n$, $n = 0, 1, 2, \dots$, is u.d. **if and only if** the bases q_i, $i = 1, 2, \dots, s$ are pairwise coprime.

(VI) P. Grabner, P. Hellekalek and P. Liardet (2012): Moreover, if $q_1, \dots, q_s$ are pairwise coprime, then the Halton sequence is also well-distributed [Coroll.33, p. 28].

RELATED SEQUENCES: 2.11.2, 3.18.3.

J.H. HALTON: *On the efficiency of certain quasi–random sequences of points in evaluating multi–dimensional integrals*, Numer. Math. **2** (1960), 84–90 (MR0121961 (**22** #12688); Zbl. 0090.34505).

P. GRABNER – P. HELLEKALEK – P. LIARDET: *The dynamical point of view of low-discrepancy sequences*, Unif. Distrib. Theory **7** (2012), no. 1, 11–70 (MR2943160; Zbl. 1313.11093)

P. HELLEKALEK – H. NIEDERREITER: *Constructions of uniformly distributed sequences using the b-adic method*, Unif. Distrib. Theory **6** (2011), no. 1 185–200.(MR2817766; Zbl. 1333.11071)

L.–K. HUA – Y. WANG: *Applications of Number Theory to Numerical Analysis*, Springer Verlag & Science Press, Berlin, Heidelberg, New York, Beijing, 1981 (MR0617192 (83g:10034); Zbl. 0465.10045). (Chinese edition: Science Press, Beijing, 1978 (MR0617192 (83g:10034); Zbl. 0451.10001)).

G. LARCHER: *Über die isotrope Discrepanz von Folgen*, Arch. Math. (Basel) **46** (1986), no. 3, 240–249 (MR0834843 (87e:11091); Zbl. 0568.10029).

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J.G. VAN DER CORPUT: *Verteilungsfunktionen I – II*, Proc. Akad. Amsterdam **38** (1935), 813–821, 1058–1066 (JFM 61.0202.08, 61.0203.01; Zbl. 0012.34705, 0013.05703).

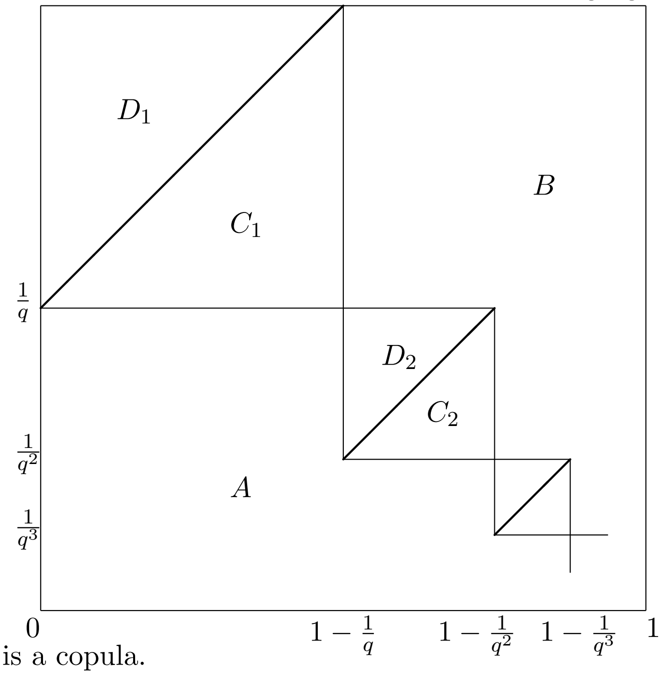
3.18.1.1 For van der Corput sequence $x_n = \gamma_q(n)$, $n = 0, 1, \dots$, in base q the two-dimensional sequence

$$(\gamma_q(n), \gamma_q(n+1)), \quad n = 0, 1, 2, \dots,$$

has the a.d.f.

$$g(x,y) = \begin{cases} 0, & \text{if } (x,y) \in A, \\ 1 - (1-y) - (1-x) = x + y - 1, & \text{if } (x,y) \in B, \\ y - \frac{1}{q^i}, & \text{if } (x,y) \in C_i, \\ x - 1 + \frac{1}{q^{i-1}}, & \text{if } (x,y) \in D_i, \end{cases}$$

where A, B, C_i and D_i, $i = 1, 2, \dots$, are as in the following figure



This a.d.f. is a copula.

Notes:

(I) There follows that every point $(\gamma_q(n), \gamma_q(n+1))$, $n = 0, 1, 2, \dots$, lies on the line segment

$$Y = X - 1 + \frac{1}{q^k} + \frac{1}{q^{k+1}}, \quad X \in \left[1 - \frac{1}{q^k}, 1 - \frac{1}{q^{k+1}}\right]$$

for $k = 0, 1, \dots$.

(II) F. Pillichshammer and S. Steinerberger (2009) proved that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |x_{n+1} - x_n| = \frac{2(q-1)}{q^2}.$$

In J. Fialová and O. Strauch (2011) an alternative proof via d.f.'s is given.

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}|x_{n+1}-x_n| = \int_0^1\int_0^1 |x-y|\,\mathrm{d}_x\,\mathrm{d}_y\, g(x,y) = 1-2\int_0^1 g(x,x)\,\mathrm{d}x.$$

J. Fialová – O. Strauch: *On two-dimensional sequences composed by one-dimensional uniformly distributed sequences*, Unif. Distrib. Theory **6** (2011), no. 2, 101–125 (MR2817763 (2012e:11135); Zbl. 1313.11089)
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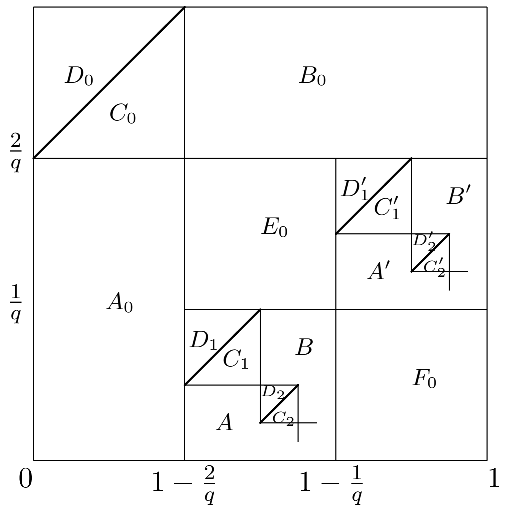
$\boxed{\textbf{3.18.1.2}}$ For van der Corput sequence $x_n = \gamma_q(n)$, $n = 0, 1, \dots$, in base q the two-dimensional sequence

$$(\gamma_q(n), \gamma_q(n+2)), \quad n = 0, 1, 2, \dots,$$

$$g(x,y) = \begin{cases} x, & \text{if } (x,y)\in D_0,\\ y-\frac{2}{q}, & \text{if } (x,y)\in C_0,\\ 0, & \text{if } (x,y)\in A_0,\\ y+x-1, & \text{if } (x,y)\in B_0,\\ x-1+\frac{2}{q}, & \text{if } (x,y)\in E_0,\\ y, & \text{if } (x,y)\in F_0,\\ 0, & \text{if } (x,y)\in A,\\ x+y-1+\frac{1}{q}, & \text{if } (x,y)\in B,\\ x-1+\frac{1}{q}+\frac{1}{q^i}, & \text{if } (x,y)\in D_i,\\ y-\frac{1}{q^{i+1}}, & \text{if } (x,y)\in C_i,\\ \frac{1}{q}, & \text{if } (x,y)\in A',\\ x+y-1, & \text{if } (x,y)\in B',\\ x-1+\frac{1}{q}+\frac{1}{q^i}, & \text{if } (x,y)\in D_i',\\ y-\frac{1}{q^{i+1}}, & \text{if } (x,y)\in C_i', \end{cases}$$

where the regions A's, B's, C's, and D's of $[0,1]^2$ are given as in the following figure



This $g(x,y)$ is a copula.

All terms of the sequence $(\gamma_q(n),\gamma_q(n+2)), n=1,2,...,$ lies in the line segments

$$
\begin{aligned}
&Y = X + \frac{2}{q}, \quad X \in \left[0, 1-\frac{2}{q}\right), \text{ or}\\
&Y = X + \frac{1}{q} + \frac{1}{q^{i+1}} + \frac{1}{q^{i+2}} - 1, \quad X \in \left[1-\frac{1}{q}-\frac{1}{q^{i+1}}, 1-\frac{1}{q}-\frac{1}{q^{i+2}}\right) \text{ or}\\
&Y = X + \frac{1}{q} + \frac{1}{q^{i+1}} + \frac{1}{q^{i+2}} - 1, \quad X \in \left[1-\frac{1}{q^{i+1}}, 1-\frac{1}{q^{i+2}}\right)
\end{aligned}
$$

for $i=0,1,\dots$. Note that for $q=2$, the interval $\left[0,1-\frac{2}{q}\right]\times\left[\frac{2}{q},1\right]$ is empty.

J. Fialová – L. Mišík – O. Strauch: *An asymptotic distribution function of three-dimensional shifted van der Corput sequence*, Applied Mathematics **5** (2014), 2334–2359 (`http://dxdoi.org/10.4236/am.2014515227`).

3.18.1.3 The points

$$(\gamma_q(n),\gamma_q(n+s)), n=0,1,2,\dots$$

lie on the diagonals of intervals

$$
\begin{aligned}
I_0 &= \left[0, 1-\frac{s}{q}\right] \times \left[\frac{s}{q}, 1\right],\\
I_1^{(i)} &= \left[1-\frac{s-1}{q}-\frac{1}{q^i}, 1-\frac{s-1}{q}-\frac{1}{q^{i+1}}\right] \times \left[\frac{1}{q^{i+1}}, \frac{1}{q^i}\right],\ i=1,2,\dots,\\
I_2^{(i)} &= \left[1-\frac{s-2}{q}-\frac{1}{q^i}, 1-\frac{s-2}{q}-\frac{1}{q^{i+1}}\right] \times \left[\frac{1}{q}+\frac{1}{q^{i+1}}, \frac{1}{q}+\frac{1}{q^i}\right],\\
&\quad i=1,2,\dots,\\
I_3^{(i)} &= \left[1-\frac{s-3}{q}-\frac{1}{q^i}, 1-\frac{s-3}{q}-\frac{1}{q^{i+1}}\right] \times \left[\frac{2}{q}+\frac{1}{q^{i+1}}, \frac{2}{q}+\frac{1}{q^i}\right],\\
&\quad i=1,2,\dots,\\
I_4^{(i)} &= \left[1-\frac{s-4}{q}-\frac{1}{q^i}, 1-\frac{s-4}{q}-\frac{1}{q^{i+1}}\right] \times \left[\frac{3}{q}+\frac{1}{q^{i+1}}, \frac{3}{q}+\frac{1}{q^i}\right],\\
&\quad i=1,2,\dots,\\
&\quad \vdots\\
I_{l-1}^{(i)} &= \left[1-\frac{s-l+1}{q}-\frac{1}{q^i}, 1-\frac{s-l+1}{q}-\frac{1}{q^{i+1}}\right] \times \left[\frac{l-2}{q}+\frac{1}{q^{i+1}}, \frac{l-2}{q}+\frac{1}{q^i}\right],\\
&\quad i=1,2,\dots,\\
&\quad \vdots\\
I_s^{(i)} &= \left[1-\frac{1}{q^i}, 1-\frac{1}{q^{i+1}}\right] \times \left[\frac{s-1}{q}+\frac{1}{q^{i+1}}, \frac{s-1}{q}+\frac{1}{q^i}\right],\ i=1,2,\dots.
\end{aligned}
$$

V. Baláž– J. Fialová – M. Hoffer – M.R. Iacó – O. Strauch: *The asymptotic distribution function of the 4-dimensional shifted van der Corput sequence*, Tatra Mt. Math. Publ. **64** (2015), 75–92 (MR3458785; Zbl 06545459).

3.18.1.4

Every point of the sequence

$$(\gamma_q(n), \gamma_q(n+1), \gamma_q(n+2)), n = 1, 2, \dots$$

lies on the diagonals of intervals

$$
\begin{aligned}
I &= \left[0, 1-\frac{2}{q}\right] \times \left[\frac{1}{q}, 1-\frac{1}{q}\right] \times \left[\frac{2}{q}, 1\right],\\
I^{(i)} &= \left[1-\frac{1}{q^i}, 1-\frac{1}{q^{i+1}}\right] \times \left[\frac{1}{q^{i+1}}, \frac{1}{q^i}\right] \times \left[\frac{1}{q}+\frac{1}{q^{i+1}}, \frac{1}{q}+\frac{1}{q^i}\right],\ i=1,2,\dots,\\
J^{(k)} &= \left[1-\frac{1}{q}-\frac{1}{q^k}, 1-\frac{1}{q}-\frac{1}{q^{k+1}}\right] \times \left[1-\frac{1}{q^k}, 1-\frac{1}{q^{k+1}}\right] \times \left[\frac{1}{q^{k+1}}, \frac{1}{q^k}\right],\\
&\quad k=1,2,\dots,
\end{aligned}
$$

where $|I| = 0$ if $q = 2$, and these intervals are maximal with respect to the set inclusion. The a.d.f. of this sequence is given by

$$
\begin{aligned}
g(x,y,z) = &\min\left(|[0,x]\cap I_X|, |[0,y]\cap I_Y|, |[0,z]\cap I_Z|\right)\\
&+\sum_{i=1}^{\infty}\min\left(|[0,x]\cap I_X^{(i)}|, |[0,y]\cap I_Y^{(i)}|, |[0,z]\cap I_Z^{(i)}|\right)\\
&+\sum_{k=1}^{\infty}\min\left(|[0,x]\cap J_X^{(k)}|, |[0,y]\cap J_Y^{(k)}|, |[0,z]\cap J_Z^{(k)}|\right).
\end{aligned}
$$

For example

$$
g(x,x,x) = \begin{cases} 0 & \text{if } x \in \left[0, \frac{2}{q}\right],\\ x-\frac{2}{q} & \text{if } x \in \left[\frac{2}{q}, 1-\frac{1}{q}\right],\\ 3x-2 & \text{if } x \in \left[1-\frac{1}{q}, 1\right]. \end{cases}
$$

J. Fialová – L. Mišík – O. Strauch: *An asymptotic distribution function of three-dimensional shifted van der Corput sequence*, Applied Mathematics **5** (2014), 2334–2359 (`http://dxdoi.org/10.4236/am.2014515227`).

3.18.1.5 The maximal 4-dimensional intervals containing points

$$
(\gamma_q(n), \gamma_q(n+1), \gamma_q(n+2), \gamma_q(n+3)), \quad n = 0,1,2,\dots
$$

on its diagonals are

$$\begin{aligned}
I &= \left[0, 1-\frac{3}{q}\right]\times\left[\frac{1}{q}, 1-\frac{2}{q}\right]\times\left[\frac{2}{q}, 1-\frac{1}{q}\right]\times\left[\frac{3}{q}, 1\right]\\
I^{(i)} &= \left[1-\frac{1}{q^i}, 1-\frac{1}{q^{i+1}}\right]\times\left[\frac{1}{q^{i+1}}, \frac{1}{q^i}\right]\times\left[\frac{1}{q}+\frac{1}{q^{i+1}}, \frac{1}{q}+\frac{1}{q^i}\right]\\
&\quad\times\left[\frac{2}{q}+\frac{1}{q^{i+1}}, \frac{2}{q}+\frac{1}{q^i}\right], \quad i=1,2,\dots,\\
J^{(j)} &= \left[1-\frac{2}{q}-\frac{1}{q^j}, 1-\frac{2}{q}-\frac{1}{q^{j+1}}\right]\times\left[1-\frac{1}{q}-\frac{1}{q^j}, 1-\frac{1}{q}-\frac{1}{q^{j+1}}\right]\\
&\quad\times\left[1-\frac{1}{q^j}, 1-\frac{1}{q^{j+1}}\right]\times\left[\frac{1}{q^{j+1}}, \frac{1}{q^j}\right], \quad j=1,2,\dots,\\
K^{(k)} &= \left[1-\frac{1}{q}-\frac{1}{q^k}, 1-\frac{1}{q}-\frac{1}{q^{k+1}}\right]\times\left[1-\frac{1}{q^k}, 1-\frac{1}{q^{k+1}}\right]\times\left[\frac{1}{q^{k+1}}, \frac{1}{q^k}\right]\\
&\quad\times\left[\frac{1}{q}+\frac{1}{q^{k+1}}, \frac{1}{q}+\frac{1}{q^k}\right], \quad k=1,2,\dots,
\end{aligned}$$

and the a.d.f. of this 4-diemnsional sequence is given by formula

$$\begin{aligned}
&g(x,y,z,u)\\
&=\min\left(|[0,x]\cap I_X|, |[0,y]\cap I_Y|, |[0,z]\cap I_Z|, |[0,u]\cap I_U|\right)\\
&\quad+\sum_{i=1}^{\infty}\min\left(|[0,x]\cap I_X^{(i)}|, |[0,y]\cap I_Y^{(i)}|, |[0,z]\cap I_Z^{(i)}|, |[0,u]\cap I_U^{(i)}|\right)\\
&\quad+\sum_{j=1}^{\infty}\min\left(|[0,x]\cap J_X^{(j)}|, |[0,y]\cap J_Y^{(j)}|, |[0,z]\cap J_Z^{(j)}|, |[0,u]\cap J_U^{(j)}|\right)\\
&\quad+\sum_{k=1}^{\infty}\min\left(|[0,x]\cap K_X^{(k)}|, |[0,y]\cap K_Y^{(k)}|, |[0,z]\cap K_Z^{(k)}|, |[0,u]\cap K_U^{(k)}|\right).
\end{aligned}$$

For example

$$g(x,x,x,x)=\begin{cases}0, & \text{if } x\in\left[0,\frac{3}{q}\right],\\ x-\frac{3}{q}, & \text{if } x\in\left[\frac{3}{q}, 1-\frac{1}{q}\right],\\ 4x-3, & \text{if } x\in\left[1-\frac{1}{q}, 1\right].\end{cases}$$

for $q\ge 4$.

V. Baláž– J. Fialová – M. Hoffer – M.R. Iacó – O. Strauch: *The asymptotic distribution function of the 4-dimensional shifted van der Corput sequence*, Tatra Mt. Math. Publ. **64** (2015), 75–92 (MR3458785; Zbl 06545459).

3.18.1.6

• Let $G_0, G_1, G_2, \dots$ be an enumeration system where G_n is a linear recurrence given by $G_{n+d} = a_0 G_{n+d-1} + \cdots + a_{d-1} G_n$, $n = 0, 1, 2, \dots$.
• Let its characteristic polynomial $x^d = a_0 x^{d-1} + \cdots + a_{d-1}$ has a PV- number β as a root. Then $a_0 \ge a_1 \ge \cdots \ge a_{d-1} \ge 1$ and β-expansion of β is

$$\beta = a_0 + \frac{a_1}{\beta} + \cdots + \frac{d_{d-1}}{\beta^{d-1}}.$$

Let
• $\phi_\beta(n)$ be the Monna map, and $\phi_\beta(n)$, $n = 0, 1, 2, \dots$, is β-van der Corput sequence;
• $T : [0, 1) \to [0, 1)$ be the von Neumann-Kakutani map defined by $\phi_\beta(n)$, i.e. $T^n(0) = \phi_\beta(n)$;
• $n_1, \dots, n_s$ be non-negative integers;
• k_n, $n = 1, 2, \dots$, be Hartman uniformly distributed and L^p-good universal for a $p \in [1, \infty]$ (see 1.8.33 and 1.8.34).
Then the sequence

$$(\phi_\beta(k_n + n_1), \dots, \phi_\beta(k_n + n_s)), n = 1, 2, \dots$$

has the a.d.f in $[0, 1)^s$.

P. Lertchoosakul – A. Jaššová – R. Nair – M. Weber: *Distribution functions for subsequences of generalized van der Corput sequences*, Unif. Distrib. Theory (to appear).
W. Parry: *On the β-expansions of real numbers*, Acta Math. Acad. Sci. Hungar. **11** (1960), 401–416 (MR0142719 (**26** #288); Zbl. 0099.28103)).

3.18.1.7 **Subsequences of Halton sequence.** Let $d_1, \dots, d_s$ be distinct positive integers and let $\alpha_1, \dots, \alpha_s$ be positive irrational numbers. Put $f_i(n) = [\alpha_i n^{d_i}]$ for $1 \le i \le s$ and $n \ge 0$. Then the sequence

$$\big(\gamma_{q_1}(f_1(n)), \dots, \gamma_{q_s}(f_s(n))\big), \quad n = 0, 1, 2, \dots,$$

is

u.d.

for arbitrary (not necessary distinct) integers $q_1, \dots, q_s$, $q_i \geq 2$, $1 \leq i \leq s$.

NOTES: P. Hellekalek and H. Niederreiter (2011) proved generally: The sequence $\big(\gamma_{q_1}(f_1(n)), \dots, \gamma_{q_s}(f_s(n))\big)$ is u.d. if and only if the integer sequence $(f_1(n), \dots, f_s(n))$ is uniformly distributed modulo $(q_1^g, \dots, q_s^g)$ for all $g \in \mathbb{N}$.

P. HELLEKALEK – H. NIEDERREITER: *Constructions of uniformly distributed sequences using the b-adic method*, Unif. Distrib. Theory **6** (2011), no. 1 185–200.(MR2817766; Zbl. 1333.11071)

$\boxed{\textbf{3.18.1.8}}$ Let $\alpha_1, \dots, \alpha_s$ be positive real numbers such that $1, \alpha_1, \dots, \alpha_s$ are linearly independent over $\mathbb{Q}$. Put $f_i(n) = [n\alpha_i]$ for $1 \leq i \leq s$. Then the sequence

$$\big(\gamma_{q_1}(f_1(n)), \dots, \gamma_{q_s}(f_s(n))\big), \quad n = 0, 1, 2, \dots,$$

is

u.d.

for arbitrary integers $q_1, \dots, q_s \geq 2$. Note that of the assumptions in 3.18.1.7 not all d_i's can be equal 1.

P. HELLEKALEK – H. NIEDERREITER: *Constructions of uniformly distributed sequences using the b-adic method*, Unif. Distrib. Theory **6** (2011), no. 1 185–200.(MR2817766; Zbl. 1333.11071)

3.18.2. Hammersley sequence. Let $n = \sum_{j=0}^{\infty} a_j(n) q^j$, $a_j \in \{0, 1, \dots, q-1\}$, be the q–adic digit expansion of the integer n, where $q \geq 2$ is an integer. The van der Corput sequence in the base q is defined by $\gamma_q(n) = \sum_{j=0}^{\infty} a_j(n) q^{-j-1}$, $n = 1, 2, \dots$ (see 2.11.2). If $s \geq 2$, $N \geq 1$ and $q_1, \dots, q_{s-1} \geq 2$ are integers, then the ***N*-terms Hammersley sequence** in the bases $q_1, \dots, q_{s-1}$ is defined by

$$\mathbf{x}_n = \left(\gamma_{q_1}(n), \dots, \gamma_{q_{s-1}}(n), \frac{n}{N}\right), \quad n = 0, 1, 2, \dots, N-1.$$

If the bases $q_1, \dots, q_{s-1}$ are pairwise coprime then for its discrepancy we have

$$D_N(\mathbf{x}_n) = \mathcal{O}\left(\frac{(\log N)^{s-1}}{N}\right),$$

or more precisely that

$$D_N^* < \frac{s}{N} + \frac{1}{N} \prod_{i=1}^{s-1} \left(\frac{q_i - 1}{2 \log q_i} \log N + \frac{q_i + 1}{2}\right).$$

NOTES: (I) J.M. Hammersley (1960) generalized K.F. Roth's (1954) construction of the two–dimensional sequence

$$\left(\frac{n}{N}, \gamma_2(n)\right), \quad n = 0, 1, 2, \dots, N-1.$$

In the case when N is a power of 2 this sequence is known as the **Roth sequence** (cf. H. Niederreiter (1978, p. 977)). P. Peart (1982) proved that in the case $N = 2^k$ the dispersion d_N (cf. 1.11.17) of this sequence satisfies

$$Nd_N = \begin{cases} \sqrt{2N - 2\sqrt{N} + 1}, & \text{if } k \text{ is even,} \\ \sqrt{(5/2)N - \sqrt{8N} + 1}, & \text{if } k \text{ is odd.} \end{cases}$$

He also showed that these types of Hammersley sequences in the unit square attain the best possible order of magnitude.
(II) For the discrepancies, cf. H. Niederreiter (1992, p. 31, Th. 3.8).
(III) I.M. Soboľ (1969, p. 182, Th. 4) proved

$$D_N^* \le \frac{1}{N} \prod_{i=1}^{s-1} \left(\frac{q_i - 1}{\log q_i} \log N + 2q_i - 1 \right), \quad \text{and}$$

(IV) L.–K. Hua and Y. Wang (1981, p. 78, Th. 4.5) proved that

$$D_N^* \le \frac{1}{N} \prod_{i=1}^{s-1} \left(\frac{q_i \log(q_i N)}{\log q_i} \right)$$

but for $\mathbf{x}_n$ with $n = 1, 2, \dots, N$ (instead of for $n = 0, 1, \dots, N-1$) and $N > \max(q_1, \dots, q_{s-1})$.
(V) If $2 \le q_1 < q_2 < \dots < q_{s-1}$ are pairwise coprime bases then G. Larcher (1986) proved that for the isotropic discrepancy I_N (cf. 1.11.9) of $\mathbf{x}_n$ we have

$$N^{1/s} I_N \le c.q_{s-1}.$$

J.M. HAMMERSLEY: *Monte Carlo methods for solving multiple problems*, Ann. New York Acad. Sci. **86** (1960), 844–874 (MR0117870 (**22** #8644); Zbl. 0111.12405).
L.–K. HUA – Y. WANG: *Applications of Number Theory to Numerical Analysis*, Springer Verlag & Science Press, Berlin, Heidelberg, New York, Beijing, 1981 (MR0617192 (83g:10034); Zbl. 0465.10045). (Chinese edition: Science Press, Beijing, 1978 (MR0617192 (83g:10034); Zbl. 0451.10001)).
G. LARCHER: *Über die isotrope Discrepanz von Folgen*, Arch. Math. (Basel) **46** (1986), no. 3, 240–249 (MR0834843 (87e:11091); Zbl. 0568.10029).
H. NIEDERREITER: *Quasi–Monte Carlo methods and pseudo–random numbers*, Bull. Amer. Math. Soc. **84** (1978), no. 6, 957–1041 (MR0508447 (80d:65016); Zbl. 0404.65003).
H. NIEDERREITER: *Random Number Generation and Quasi–Monte Carlo Methods*, CBMS–NSF Regional Conference Series in Applied Mathematics, 63, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992, vi+241 pp. (MR1172997 (93h:65008); Zbl. 0761.65002).

P. Peart: *The dispersion of the Hammersley sequence in the unit square*, Monatsh. Math. **94** (1982), no. 3, 249–261 (MR0683058 (85a:65010); Zbl. 0484.10033).
K.F. Roth: *On irregularities of distribution*, Mathematika **1** (1954), 73–79 (MR0066435 (16,575c); Zbl. 0057.28604).
I.M. Sobol': *Multidimensional Quadrature Formulas and Haar Functions*, (Russian), Library of Applied Analysis and Computational Mathematics, Izd. "Nauka", Moscow, 1969 (MR0422968 (**54** #10952); Zbl. 0195.16903).

3.18.2.1 Digitally shifted Hammersley sequences.

Let $x = 0.x_1x_2\dots x_m$ and $y = 0.y_1y_2\dots y_m$ be two real numbers written in the dyadic expansion. Define $x \oplus y = z = 0.z_1z_2\dots z_m$, where $z_i = x_i + y_i \pmod 2$, $i = 1, 2, \dots, m$. Let $\gamma_2(n)$ be the van der Corput radical inverse function (cf. 2.11.1, 2.11.2(V)) defined by $\gamma_2(n) = 0.a_0a_1\dots a_{m-1}$ for a nonnegative integer $n = a_{m-1}a_{m-2}\dots a_0$ (again in the dyadic expansion). Then for the L^2 discrepancy $D_N^{(2)}$ of the sequence

$$\left(\frac{n}{N}, \gamma_2(n) \oplus x\right), \quad n = 0, 1, \dots, N-1, \quad \text{with } N = 2^m$$

we have

$$N^2 D_N^{(2)} = \frac{m^2}{64} - \frac{19m}{192} - \frac{lm}{16} + \frac{l^2}{16} + \frac{l}{4} + \frac{3}{8} + \frac{m}{16.2^m} - \frac{l}{8.2^m} + \frac{1}{4.2^m} - \frac{1}{72.4^m},$$

where l denotes the number of zeros in the dyadic expansion of x. If m is even and $l = m/2$, then

$$D_N^{(2)} = \mathcal{O}\left(\frac{\log N}{N^2}\right)$$

which is the best possible estimate (cf. 1.11.4.1). A similar situation also holds in the case of odd m and $l = (m-1)/2$.

Notes: (I) P. Kritzer and F. Pillichshammer (2006) and partial results can be found in (2005,Th. 2 and 3).
(II) For the L^2 discrepancy of the 2–dimensional Hammersley sequence (cf. 3.18.2, also called Roth sequence)

$$\left(\frac{n}{N}, \gamma_2(n)\right), \quad n = 0, 1, \dots, N-1, \quad N = 2^m$$

the following exact formula

$$N^2 D_N^{(2)} = \frac{m^2}{64} + \frac{29m}{192} + \frac{3}{8} - \frac{m}{16.2^m} + \frac{1}{4.2^m} - \frac{1}{72.2^{2m}}.$$

was proved by I.V. Vilenkin (1967) and independently by J.H. Halton and S.K. Zaremba (1969).

Related sequences: 2.11.1, 2.11.2(V), 3.18.2, 3.18.4.

J.H. Halton – S.K. Zaremba: *The extreme and L^2 discrepancies of some plane set*, Monatsh. Math. **73** (1969), 316–328 (MR0252329 (**40** #5550); Zbl. 0183.31401).
P. Kritzer – F. Pillichshammer: *Point sets with low L_p–discrepancy*, Math. Slovaca **57** (2007), no. 1, 11-32 (MR2357804 (2009a:11155); Zbl 1153.11037).
P. Kritzer – F. Pillichshammer: *An exact formula for the L_2 of the shifted Hammersley point set*, Unif. Distrib. Theory **1** (2006), no. 1, 1–13 (MR2314263 (2008d:11084); Zbl. 1147.11041).
I.V. Vilenkin: *Plane sets of integration*, (Russian), Zh. Vychisl. Mat. Mat. Fiz. **7** (1967), no. 1, 189–196 (MR0205464 (**34** #5291); Zbl. 0187.10701). (English translation: U.S.S.R. Comput. Math. Math. Phys. **7** (1967), 258–267).

3.18.3. Permuted Halton sequences. Let $q_1,\dots,q_s$ be s pairwise coprime integers and let a permutation π_{q_i} on $\{0,1,2,\dots,q_i-1\}$ and the radical inverse function (cf. 2.11.2) $\gamma_{q_i}(n)=\sum_{j=0}^{k(n)} a_j(n)/q_i^{j+1}$ for the q_i-adic digit expansion of $n=\sum_{j=0}^{k(n)} a_j(n)q_i^j$ $(a_{k(n)}(n)>0)$ be assigned to each q_i, $i=1,\dots,s$. Then the **permuted Halton sequence** (or **scrambled Halton sequence**) over $\pi_{q_1},\dots,\pi_{q_s}$ is defined by

$$\mathbf{x}_n=(x_{n,1},\dots,x_{n,s}),$$

where

$$x_{n,i}=\frac{\pi_{q_i}(a_0(n))}{q_i}+\frac{\pi_{q_i}(a_1(n))}{q_i^2}+\dots+\frac{\pi_{q_i}(a_{k(n)}(n))}{q_i^{k(n)+1}}.$$

The sequence $\mathbf{x}_n$ is

u.d.

and is of low discrepancy because

$$D_N^{(2)}=\mathcal{O}\left(\frac{(\log N)^s}{N^2}\right).$$

Notes: (I) These sequences were introduced by E. Braaten and G. Weller (1979). They used different primes p_i, $i=1,\dots,s$, for the bases and permutations π_{p_i} defined by induction: $\pi_{p_i}(0)=0$ and if we know $\pi_{p_i}(1),\dots,\pi_{p_i}(j)$, we take for $\pi_{p_i}(j+1)$ the element which minimizes the L^2 discrepancy of the sequence

$$\frac{\pi_{p_i}(1)}{p_i},\dots,\frac{\pi_{p_i}(j)}{p_i},\frac{\pi_{p_i}(j+1)}{p_i}.$$

(II) B. Tuffin (1998) used four approaches: Let p_i by the ith prime, permutations $\pi_{p_1},\dots,\pi_{p_j}$ be fixed, and $\pi^{(1)},\dots,\pi^{(K)}$ be given permutations of $0,1,2,\dots,p_{j+1}-1$.

- For permutation $\pi_{p_{j+1}}$ we chose that $\pi^{(k)}$ with k, $1 \le k \le K$, which minimizes the L^2 discrepancy $D^{(2)}_{p_{j+1}-1}$ of the $(j+1)$–dimensional permuted Halton sequence $\mathbf{x}_n$, $n = 1, 2, \dots, p_{j+1} - 1$, over permutation $\pi_{p_1}, \dots, \pi_{p_j}, \pi^{(k)}$.
- The same procedure but with the L^2 discrepancy replaced by the diaphony.
- For fixed s the search goes over a sequence of permutation $\pi_{p_1}, \dots, \pi_{p_s}$ such that $\pi_{p_1}(0) = 0, \dots, \pi_{p_s}(0) = 0$, which minimizes the L^2 discrepancy $D^{(2)}_{p_1 \cdots p_s}$ of the permuted Halton sequence $\mathbf{x}_n$, $n = 1, 2, \dots, p_1 \cdots p_s$.
- The same procedure as the last one but minimizing the diaphony (cf. 1.11.5).

Related sequences: Generalized van der Corput sequence 2.11.3, and Halton sequence 3.18.1.

E. Braaten – G. Weller: *An improved low–discrepancy sequence for multidimensional quasi–Monte Carlo integration*, J. Comput. Phys. **33** (1979), 249–258 (Zbl. 426.65001).
B. Tuffin: *A new permutation choice in Halton sequence*, in: *Monte Carlo and Quasi–Monte Carlo Methods 1996 (Proceedings of a conference at the University of Salzburg, Austria, July 9–12, 1996)*, (H. Niederreiter, P. Hellekalek, G. Larcher, P. Zinterhof eds.), Lecture Notes in Statistics, 127, Springer Verlag, New York, Berlin, 1998, pp. 427–435 (MR1644537 (99d:65018); Zbl. 0885.65025).

3.18.4. Zaremba two–dimensional sequence. Let N be a power of 2, say $N = 2^k$, $k \ge 1$. Define the N terms sequence $\mathbf{x}_n \in [0,1)^2$ by

$$\mathbf{x}_n = \left(\frac{a_{k-1}}{2} + \frac{a_{k-2}}{2^2} + \dots \frac{a_0}{2^k}, \frac{a'_0}{2} + \frac{a'_1}{2^2} + \dots + \frac{a'_{k-1}}{2^k} \right), \quad n = 0, 1, 2, \dots, N-1,$$

where $n = a_{k-1} 2^{k-1} + \dots + a_0$ is the dyadic expansion of n, and $a'_i = a_i$ if i is odd and $a'_i = 1 - a_i$ if i is even for $i = 0, 1, 2, \dots, k-1$. Then the L^2 discrepancy of $\mathbf{x}_n$ satisfies

$$D^{(2)}_N = \mathcal{O}(k 2^{-2k}).$$

J.H. Halton – S.K. Zaremba: *The extreme and L^2 discrepancies of some plane set*, Monatsh. Math. **73** (1969), 316–328 (MR0252329 (**40** #5550); Zbl. 0183.31401).

3.19 (t,m,s)–nets and (t,s)–sequences

For definitions, see 1.8.17 and 1.8.18. In this section $q \ge 2$ will denote a given integer base.

(I) For any (t,m,s)–net $\mathbf{x}_n$, $n = 1, 2, \dots, N$, $N = q^m$, in the base q with $m \ge 1$ we have

$$D^*_N \le B_s(q) q^t \frac{(\log N)^{s-1}}{N} + \mathcal{O}\left(q^t \frac{(\log N)^{s-2}}{N} \right)$$

with

$$B_s(q) = \left(\frac{q-1}{2\log q}\right)^{s-1}$$

if either $s = 2$ or $q = 2$ and $s = 3, 4$, otherwise

$$B_s(q) = \frac{1}{(s-1)!}\left(\frac{[q/2]}{\log q}\right)^{s-1}.$$

The dispersion d_N^∞ of $\mathbf{x}_n$, $n = 1, 2, \dots, N$, $N = q^m$ (for the def. see 1.11.17) satisfies

$$d_N^\infty \le q^{(s-1+t)/s} N^{-1/s}.$$

(II) For any (t, s)–sequence $\mathbf{x}_n$, $n = 1, 2, \dots$, in the base q we have

$$D_N^* \le C_s(q) q^t \frac{(\log N)^s}{N} + \mathcal{O}\left(q^t \frac{(\log N)^{s-1}}{N}\right)$$

for all $N \ge 2$, where the $\mathcal{O}$-constant depends only on q and s, where

$$C_s(q) = \frac{1}{s}\left(\frac{q-1}{2\log q}\right)^s$$

if either $s = 2$ or $q = 2$ and $s = 3, 4$, otherwise we have

$$C_s(q) = \frac{1}{s!}\frac{q-1}{2[q/2]}\left(\frac{[q/2]}{\log q}\right)^s.$$

For the dispersion we have

$$d_N^\infty \le q^{(s+t)/s} N^{-1/s}$$

for all $N \ge 1$. Thus, every (t, s)–sequence is a low discrepancy sequence in $[0, 1)^s$ (for the def. see 1.8.15) and also a low dispersion one (see 1.8.16).

NOTES: (III) Small improvements of the value t lead to considerably better discrepancy bounds for the (t, s)–sequence. The best possible expected case is thus $t = 0$, but for every base $q \ge 2$ and every dimension $s \ge 1$, a necessary condition for the existence of a (t, s)–sequence in the base q is

$$t \ge \frac{s}{q} - \log_q\left(\frac{(q-1)s + q + 1}{2}\right).$$

Let $t_s(q)$ be the least value of t such that there exists a (t, s)–sequence in the base q. If $q = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ with $p_1 < \dots < p_r$, is the canonical factorization of q, then

$$t_s(q) \le c\frac{s}{\log p_1} + 1$$

for all $s \geq 1$ with an absolute constant $c > 0$.
(IV) If q is a prime power then a $(0, s)$–sequence in the base q exists if and only if $s \leq q$.
(V) If there exists a (t, s)–sequence in the base q, then a $(t, m, s+1)$–net in the base q exists for every integer $m \geq t$.
(VI) The following so–called **propagation rules** are true for (t, m, s)–nets in an arbitrary integer base $q \geq 2$:

- Every (t, m, s)–net over $\mathbb{F}_q$ is a (k, m, s)–net over $\mathbb{F}_q$ for $t \leq k \leq m$.
- If $1 \leq r \leq s$ then every (t, m, s)–net over $\mathbb{F}_q$ can be transformed into a (t, m, r)–net over $\mathbb{F}_q$.
- Every (t, m, s)–net over $\mathbb{F}_q$ can be transformed into a (t, k, s)–net over $\mathbb{F}_q$ with $t \leq k \leq m$.
- Every (t, m, s)–net over $\mathbb{F}_q$ can be transformed into a $(t+k, m+k, s)$–net over $\mathbb{F}_q$ for every $k \in \mathbb{N}$.

(VII) A $(0, 2, s)$–net in the base q exists if and only if there exist $s - 2$ mutually orthogonal latin squares of order q (see Niederreiter (1992, p. 60, Th. 4.18)).
(VIII) The general theory of (t, m, s)–nets and (t, s)–sequences was developed by H. Niederreiter (1987). He also gives a detailed information for the star discrepancies (I) and (II) and he also proved (V) (cf. also H. Niederreiter and C.-P. Xing ([a]1996) and Soboľ (1967, Part 5)). Theorem (III) was proved by H. Niederreiter and C.-P. Xing ([a][b]1996). Most of the known constructions of (t, m, s)–nets and (t, s)–sequences are based on the digital method 3.19.1 which was introduced by H. Niederreiter (1987, Sect. 6). Surveys on the subject can be foun in H. Niederreiter (1992, Chap. 4), G. Larcher (1998), H. Niederreiter and C.-P. Xing (1998).
(IX) From the history: Firstly, a formal definition was given by I.M. Soboľ (1966). In (1967) he proved that for every $s \geq 1$ there exist (t, s)–sequences over $\mathbb{F}_2$ with $t = \mathcal{O}(s \log s)$, see 3.19.5. He also for the first time investigated $t_s(2)$ as the least value of t for which there exists a (t, m, s)–net in the base 2 for an infinitely many m and he proved that $t_1(2) = t_2(2) = t_3(2) = 0$, $t_4(2) = 1$, and in general that $t_s(q) = \mathcal{O}(s \log s)$. Soboľ (1967) also mentiones that the number $t_s(2)$ may be used for a geometric characterization of the cube $[0, 1]^s$.
Secondly, H. Faure (1982) proved that for every prime p there exists a $(0, s)$–sequence over $\mathbb{F}_p$ if $s \leq p$, see 3.19.6. Niederreiter (1987) extended this result to every prime power q (see (IV)).
(X) Estimates for the dispersions d_N^∞ of (t, m, s)–nets and (t, s)–sequences was given by Niederreiter (1988).
(XI) For tables of (t, m, s)–nets and (t, s)–sequences cf. G.L. Mullen, A. Mahalanabis and H. Niederreiter (1995).

H. Faure: *Discrépance de suites associées à un système de numération (en dimension s)*, Acta Arith. **41** (1982), 337–351 (MR0677547 (84m:10050); Zbl. 0442.10035).
G. Larcher: *Digital point sets: Analysis and application*, in: *Random and Quasi–Random Point Sets*, (P. Hellekalek, G. Larcher eds.), Lecture Notes in Statistics, 138, Springer Verlag, New York, Berlin, 1998, pp. 167–222 (MR1662842 (99m:11085); Zbl. 0920.11055).
G.L. Mullen – A. Mahalanabis – H. Niederreiter: *Tables of (t, m, s)–net and (t, s)–sequence*

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[a] H. Niederreiter – C.-P. Xing: *Low–discrepancy sequences and global function fields with many rational places*, Finite Fields Appl. **2** (1996), no. 3, 241–273 (MR1398076 (97h:11080); Zbl. 0893.11029).
[b] H. Niederreiter – C.-P. Xing: *Quasi random points and global functional fields*, in: *Finite Fields and Applications (Glasgow, 1995)*, (S. Cohen and H. Niederreiter eds.), London Math. Soc. Lecture Note Ser., 233, Cambridge Univ. Press, Cambridge, 1996, pp. 269–296 (MR1433154 (97j:11037); Zbl. 0932.11050).
H. Niederreiter – C.-P. Xing: *Nets, (t,s)–sequences and algebraic geometry*, in: *Random and Quasi–Random Point Sets*, (P. Hellekalek and G. Larcher eds.), Lecture Notes in Statistics, 138, Springer Verlag, New York, Berlin, 1998, pp. 267–302 (MR1662844 (99k:11121); Zbl. 0923.11113).
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I.M. Sobol': *Distribution of points in a cube and approximate evaluation of integrals*, (Russian), Ž. Vyčisl. Mat. i Mat. Fiz. **7** (1967), 784–802 (MR0219238 (**36** #2321)).
I.M. Sobol': *Multidimensional Quadrature Formulas and Haar Functions*, (Russian), Library of Applied Analysis and Computational Mathematics, Izd. "Nauka", Moscow, 1969 (MR0422968 (**54** #10952); Zbl. 0195.16903).

3.19.1. Digital (t,m,s)–nets.

- Let q be a prime power.
- For $n = 0,1,2,\dots,q^m-1$ let $n = \sum_{r=0}^{m-1} a_r(n)q^r$ be the q–adic digit expansion of n in the base q. Consider the digits $a_0(n),\dots,a_{m-1}(n)$ as elements of the field $\mathbb{F}_q$.
- Let $C^{(1)},\dots,C^{(s)}$ be $m\times m$-matrices over $\mathbb{F}_q$. The matrix $C^{(i)}$ will called the **generator matrix** of the ith coordinate. The jth row of $C^{(i)}$ will be denoted by $\mathbf{C}_j^{(i)}$.
- Let $(y_1^{(i)}(n),\dots,y_m^{(i)}(n))^T = C^{(i)}\cdot(a_0(n),\dots,a_{m-1}(n))^T$ for $i=1,\dots,s$, and
- $\Psi:\mathbb{F}_q\to\{0,1,\dots,q-1\}$.
- Then the finite sequence

$$\mathbf{x}_n = \left(\sum_{j=1}^m \frac{\Psi(y_j^{(1)}(n))}{q^j},\dots,\sum_{j=1}^m \frac{\Psi(y_j^{(s)}(n))}{q^j}\right) \quad \text{for } n=0,1,\dots,q^m-1,$$

is called a **digital net over** $\mathbb{F}_q$ and if it is also a (t, m, s)–net in the base q, then it is called **a digital** (t, m, s)**–net constructed over** $\mathbb{F}_q$.

The sequence

$$\mathbf{x}_0, \ldots, \mathbf{x}_{q^m-1}$$

is a

$$(t, m, s)\text{–net}$$

in the base q if and only if for all integers $0 \le d_1, \ldots, d_s \le m$ with $d_1 + \cdots + d_s = m - t$, the system of vectors $\mathbf{C}_j^{(i)}$, $j = 1, \ldots, d_i$, $i = 1, \ldots, s$, is linearly independent over $\mathbb{F}_q$. For the star discrepancy D_N^*, $N = q^m$, we again have

$$D_N^* = \mathcal{O}\left(q^t \frac{(\log N)^{s-1}}{N}\right),$$

where the $\mathcal{O}$-constant depends only on s and q, see 3.19 (I)

Notes: (I) The concept of digital nets over a ring was introduced by H. Niederreiter (1987, Sect. 6) and he also proved the above criterion. On the other hand, the previous constructions of I.M. Sobol̕ (1967) and H. Faure (1982) also lead to digital nets.

(I′) More precisely, Niederreiter (1992, p. 63, Par. 4.3) understands under the **general construction principles** the following situation:

- R is a commutative ring with identity and the number of its elements is q,
- $\psi_r : \{0, 1, \ldots, q-1\} \to R$ are bijections for $0 \le r \le m-1$,
- $\eta_{i,j} : R \to \{0, 1, \ldots, q-1\}$ are bijections for $1 \le i \le s$ and $1 \le j \le m$,
- $c_{j,r}^{(i)} \in R$ for $1 \le i \le s$, $1 \le j \le m$, and $0 \le r \le m-1$,
- let $n = \sum_{r=0}^{m-1} a_r(n) q^r$ with $a_r(n) \in \{0, 1, \ldots, q-1\}$ for $n = 0, 1, \ldots, q^m - 1$, and put
- $y_{n,j}^{(i)} = \eta_{i,j}\left(\sum_{r=0}^{m-1} c_{jr}^{(i)} \psi_r\big(a_r(n)\big)\right)$,
- $x_n^{(i)} = \sum_{j=1}^{m} y_{n,j}^{(i)} q^{-j}$,

Then the finite s–dimensional sequence $\mathbf{x}_n = (x_n^{(1)}, \ldots, x_n^{(s)}) \in [0,1)^s$, for $n = 0, 1, \ldots, q^m - 1$, is called **the digital net constructed over** R.

(II) H. Niederreiter and C.-P. Xing (1998, Coroll. 4) proved: If there exist n digital (t_k, m_k, s_k)–nets constructed over $\mathbb{F}_{q^{r_k}}$ for $1 \le k \le n$ ($r_1, \ldots, r_n$ are positive integers), then there also exists a digital $(t, \sum_{k=1}^n r_k m_k, \sum_{k=1}^n s_k)$–net constructed over $\mathbb{F}_q$ with $t = \sum_{k=1}^n m_k - \min_{1 \le k \le n}(m_k - t_k)$.

(III) G. Larcher (1998) introduced the concept of a **digital translation net** using the construction

- Let $C^{(i)} \cdot (a_0(n), \ldots, a_{m-1}(n))^T + \mathbf{W}_i^T =: (y_1^{(i)}(n), \ldots, y_m^{(i)}(n))^T$ for $i = 1, \ldots, s$,

where $\mathbf{W}_i$, $i = 1, 2, \ldots, s$, are the so–called **translation vectors**.

(IV) The finite field $\mathbb{F}_q$ can be replaced by the ring $\mathbb{Z}_q$ of all integers $(\bmod\, q)$. G. Larcher, H. Niederreiter and W.Ch. Schmid (1996) proved that: If $q = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$,

$p_1 < \cdots < p_r$, is the canonical factorization of q, then a digital $(0, m, s)$–net constructed over $\mathbb{Z}_q$ exists if and only if $s \le p_1 + 1$. For a general ring of order q the condition is $s \le p_1^{\alpha_1} + 1$.
(V) W.Ch. Schmid (1996) introduced a **shift–nets over** $\mathbb{F}_q$: consider the first $m \times m$ matrix $C^{(1)}$ not as a system $C^{(1)} = (\mathbf{C}_1^{(1)}, \dots, \mathbf{C}_m^{(1)})$ of row vectors (over $\mathbb{F}_q$), but as a system of column vectors $C^{(1)} = (\mathbf{D}_1^{(1)}, \dots, \mathbf{D}_m^{(1)})$. The remaining matrices $C^{(i)}$ are then built using the shift to the left procedure, i.e. $C^{(2)} = (\mathbf{D}_2^{(1)}, \dots, \mathbf{D}_m^{(1)}, \mathbf{D}_1^{(1)})$, $\dots$, $C^{(m)} = (\mathbf{D}_m^{(1)}, \mathbf{D}_1^{(1)} \dots, \mathbf{D}_{m-1}^{(1)})$. In the original construction it was $m = s$ and the corresponding shift–net is a digital (t, s, s)–net. Schmid (1998) gave matrices with provide the binary shift–nets $(0, 3, 3)$, $(1, 4, 4)$, $(1, 5, 5)$, $(2, 6, 6)$, etc.
• Conjecture (Schmid (1998)): Let $t \ge 1$ and $m \ge t$ be integers. If a binary (t, s, s)–shift–net exists, then for each $k \in \mathbb{N}$ also a binary $(t+k, s+k, s+k)$–shift–net exists.
(VI) Larcher ([b]1998): For every s–dimensional digital net over a prime base p and having $N = p^m$ elements and constructed by $m \times m$ matrices $C^{(1)}, \dots, C^{(s)}$ we have

$$D_N^* \le \sum_{w=0}^{s-1} (p-1)^w \sum_{(d_1,\dots,d_w)\in X_w} p^{-(d_1+\cdots+d_w+h(d_1,\dots,d_w))},$$

where X_w is the set of all w–tuples $(d_1, \dots, d_w)$ of positive integers for which the system of vectors $\mathbf{C}_j^{(i)}$ with $j = 1, 2, \dots, d_i$ and $i = 1, 2, \dots, w$ is linearly independent over $\mathbb{F}_p$ $\big(X_0$ contains the "zero–tuple" $()\big)$, and $h(d_1, \dots, d_w) = \max\big\{h \ge 0\,;\ (d_1, \dots, d_w, h) \in X_{w+1}\big\}$.
(VII) If we take a family of suitable test–functions from the class $E_s^{\alpha}(c)$ (cf. p. 3 – 72) then G. Larcher ([c]1998) showed that the integration errors over (t, m, s)–nets in the base $q = 2$ are essentially smaller than for good lattice points sequences (cf. 3.15) and Halton sequences (cf. 3.18.1), see his tables (1998, p. 208–209, Table 4a, 4b).

H. Faure: *Discrépance de suites associées à un système de numération (en dimension s)*, Acta Arith. **41** (1982), 337–351 (MR0677547 (84m:10050); Zbl. 0442.10035).
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3.19.2. Digital (t,s)–sequences.

- Let q be a prime power.
- For $n = 0,1,2,\dots$, let $n = \sum_{r=0}^{\infty} a_r(n)q^r$ be the q–adic digit expansion of n in the base q ($a_r(n) > 0$ only for a finitely many r's). Consider the digits $a_0(n), a_1(n), \dots$ as elements of the field $\mathbb{F}_q$.
- Let $C^{(1)}, \dots, C^{(s)}$ be $\infty \times \infty$-matrices over $\mathbb{F}_q$.
- Let $(y_1^{(i)}(n), y_2^{(i)}(n), \dots)^T = C^{(i)} \cdot (a_0(n), a_1(n), \dots)^T$ for $i = 1,\dots,s$, and
- $\Psi : \mathbb{F}_q \to \{0,1,\dots,q-1\}$ be a bijection.
- Then the infinite sequence

$$\mathbf{x}_n = \left(\sum_{j=1}^{\infty} \frac{\Psi(y_j^{(1)}(n))}{q^j}, \dots, \sum_{j=1}^{\infty} \frac{\Psi(y_j^{(s)}(n))}{q^j} \right), \quad \text{for } n = 0,1,\dots,$$

  is called a **digital sequence constructed over** $\mathbb{F}_q$.
- Let $C^{i,m}$ denote the left upper $m \times m$ submatrix of $C^{(i)}$.

(A) If there exists a function $T : \mathbb{N} \to \mathbb{N}$ such that $C^{(1,m)}, \dots, C^{(s,m)}$ generate a digital $(T(m),m,s)$–net constructed over $\mathbb{F}_q$ for all $m = 1,2,\dots$, then the sequence

$$\mathbf{x}_0, \mathbf{x}_1, \dots,$$

is called a **digital (T,s)–sequence constructed over** $\mathbb{F}_q$. For its first N terms we have the following discrepancy estimate

$$D_N^* \le C_s'(q) \frac{1}{N} \sum_{m=1}^{k} p^{T(m)} m^{s-1},$$

where k is such that $q^k \le N < q^{k+1}$. Here $C_s'(q)$ is a constant which depends only on s and q.

(B) If the digital sequence $\mathbf{x}_n$, $n = 0, 1, 2, \dots$, is a (t, s)–sequence (e.g. if $T(m) \le t$ for $m = 1, 2, \dots$), then it is called **a digital** $(t, s,)$**–sequence constructed over** $\mathbb{F}_q$. For its star discrepancy we may use the estimate 3.19(II), i.e.

$$D_N^* = \mathcal{O}\left(q^t \frac{(\log N)^s}{N}\right)$$

for all $N \ge 2$, where $\mathcal{O}$-constant depends only on s and q.

Notes: (I) van der Corput sequence in the base q is a $(0, 1)$–sequence in the base q and actually a digital $(0, 1)$–sequence constructed over the ring $\mathbb{Z}_q$.
(II) The discrepancy bound for (T, s)–sequences was given by G. Larcher (1998) as combining the bounds from G. Larcher and H. Niederreiter (1995).
(III) H. Faure (1982) took for the generating matrix $C^{(1)}$ the Pascal's triangle, i.e. $C^{(i)} = (C_{j,k}^{(i)})$, where

$$C_{j,k}^{(i)} = \begin{cases} \binom{j}{k}(i-1)^{j-k}, & \text{for } 0 \le k \le j, \\ 0, & \text{for } k > j. \end{cases}$$

He obtained $T(m) = 0$ identically and the resulting $(0, s)$–sequence had a prime base $p \ge s$. This type of a sequence is also called the **Faure sequence**, see 3.19.6.
(IV) Let $d_s(q)$ be the least value of t for which there exists a digital (t, s)–sequence constructed over $\mathbb{F}_q$. For all $s \ge 1$ we have

- $d_s(2) \ge s \log_2(3/2) - 4 \log_2(s-2) - 23$ for $s \ge 3$ (W.Ch. Schmid (1998)),
- $d_s(5) \le 3s + 1$ (Niederreiter and Xing (1998), p. 281),
- $d_s(27) \le \frac{12}{5}s + 1$ (Niederreiter and Xing (1998), p. 281),
- $d_s(q) \ge \frac{s}{q} - \log_q\left(\frac{(q-1)s+q+1}{2}\right)$ for all prime powers q and all $s \ge 1$ (H. Niederreiter and C.-P. Xing ([b]1996, 1998)),
- $d_s(q) \le \frac{3q-1}{q-1}(s-1) - \frac{(2q+4)(s-1)^{1/2}}{(q^2-1)^{1/2}} + 2$ for every prime power q and every $s \ge 1$ (Xing and Niederreiter (1995)),
- $d_s(q) \le c\frac{s}{\log q} + 1$ for all prime powers q with an absolute constant $c > 0$ (Niederreiter and Xing ([a]1996)),
- $d_s(q) \le c'\frac{s}{q^{1/4}} + 1$ if the (prime power) q is a square, here $c' > 0$ is an absolute constant (Niederreiter and Xing ([a]1996)),
- $d_s(q^2) \le \frac{ps}{q-1}$, for $q = p^r$ and every $s \ge 1$ (Xing and Niederreiter (1995)).

(V) Analogically to 3.19.1(I′) Niederreiter's general schema for construction of digital (t, s)–sequences modulo q is (see H. Niederreiter and Ch. Xing ([a]1998)): We fix the dimension $s \ge 1$ and choose the following
(S1) bijections $\psi_r : \{0, 1, \dots, q-1\} \to \mathbb{F}_q$ for $r = 0, 1, 2, \dots$, with $\psi_r(0) = 0$ for all sufficiently large r,
(S2) maps $\eta_j^{(i)} : \mathbb{F}_q \to \{0, 1, \dots, q-1\}$ for $1 \le i \le s$ and $j \ge 1$,
(S3) elements $c_{j,r}^{(i)} \in \mathbb{F}_q$ for $1 \le i \le s$ and $j \ge 1$ and $r \ge 0$.

Let $n = \sum_{r=0}^{\infty} a_r(n) q^r$ be the q–adic digit expansion of n in the base q with $a_r(n) \in \{0, 1, \dots, q-1\}$ and put

$$\mathbf{x}_n = \left(\sum_{j=1}^{\infty} \frac{\eta_j^{(1)}\left(\sum_{r=0}^{\infty} c_{j,r}^{(1)} \psi_r(a_r(n))\right)}{q^j}, \dots, \sum_{j=1}^{\infty} \frac{\eta_j^{(s)}\left(\sum_{r=0}^{\infty} c_{j,r}^{(s)} \psi_r(a_r(n))\right)}{q^j} \right).$$

If $\mathbf{x}_n$ is a (t, s)–sequence in the base q, then it is called a digital (t, s)–sequence constructed over $\mathbb{F}_q$.

(S4) Let $s = 1$ and let $C = (c_{j,r})$ be a generator matrix from (S3). For a given l, let $a = \rho(C; l)$ be the maximal integer such that the vectors $(c_{j,1} \dots, c_{j,l})$, $j = 1, 2, \dots, a$, are linearly independent. Then the sequence x_n constructed in (S3) for $s = 1$, is a digital $(t, 1)$-sequence, where $t = \sup_{l \geq 1}(l - \rho(C; l))$. Note that the generator matrix $C = (c_{j,r})$ is the unit one in case of van der Corput sequence, thus it is a $(0, 1)$-sequence.

(VI) Let $s = 1$ and let $C = (c_{j,r})$ be a generator matrix as in (S3). For given l, let $a = \rho(C; l)$ be the maximal integer such that the vectors $(c_{j,1} \dots, c_{j,l})$, $j = 1, 2, \dots, a$ are linearly independent. Then the sequence x_n constructed in (S3) for $s = 1$, is a digital $(t, 1)$-sequence, where $t = \sup_{l \geq 1}(l - \rho(C; l))$. Note that for van der Corput sequence the generator matrix $(c_{j,r})$ is the unit matrix, thus it is a $(0, 1)$-sequence.

H. Faure: *Discrépance de suites associées à un système de numération (en dimension s)*, Acta Arith. **41** (1982), 337–351 (MR0677547 (84m:10050); Zbl. 0442.10035).

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[b] H. Niederreiter – C.-P. Xing: *Quasi random points and global functional fields*, in: *Finite Fields and Applications (Glasgow, 1995)*, (S. Cohen and H. Niederreiter eds.), London Math. Soc. Lecture Note Ser., 233, Cambridge Univ. Press, Cambridge, 1996, pp. 269–296 (MR1433154 (97j:11037); Zbl. 0932.11050).

H. Niederreiter – C.-P. Xing: *Nets, (t, s)–sequences and algebraic geometry*, in: *Random and Quasi–Random Point Sets*, (P. Hellekalek and G. Larcher eds.), Lecture Notes in Statistics, 138, Springer Verlag, New York, Berlin, 1998, pp. 267–302 (MR1662844 (99k:11121); Zbl. 0923.11113).

[a] H. Niederreiter – C.-P. Xing: *The algebraic–geometry approach to low–discrepancy sequences*, in: *Monte Carlo and Quasi–Monte Carlo Methods 1996 (Proceedings of a conference at the University of Salzburg, Austria, July 9–12, 1996)*, (H. Niederreiter, P. Hellekalek, G. Larcher, P. Zinterhof eds.), Lecture Notes in Statistics, 127, Springer Verlag, New York, Berlin, 1998, pp. 139–160 (MR1644516 (99d:11081); Zbl. 0884.11031).

W.Ch. Schmid: *Shift-nets: a new class of binary digital (t, m, s)–nets*, in: *Monte Carlo and Quasi–Monte Carlo Methods 1996 (Proceedings of a conference at the University of Salzburg, Austria, July 9–12, 1996)*, (H. Niederreiter, P. Hellekalek, G. Larcher, P. Zinterhof eds.), Lecture Notes in Statistics, 127, Springer Verlag, New York, Berlin, 1998, pp. 369–381 (MR1644533 (99d:65023); Zbl. 0884.11033).

C.-P. Xing – H. Niederreiter: *A construction of low-discrepancy sequences using global function fields*, Acta Arith. **73** (1995), no. 1, 87–102 (MR1358190 (96g:11096); Zbl. 0848.11038).

3.19.3. Niederreiter sequences.

- Let q be a prime power.
- For $n = 0, 1, 2, \dots$, let $n = \sum_{r=0}^{\infty} a_r(n) q^r$ be the q–adic digit expansion of n in the base q.
- Let $p_1(x), \dots, p_s(x) \in \mathbb{F}_q[x]$ be pairwise coprime polynomials over the finite field $\mathbb{F}_q$ and $s \geq 1$ be arbitrary. Let $\deg p_i(x) = e_i \geq 1$ for $1 \leq i \leq s$.
- Let $j \geq 1$ and $g_{ij}(x) \in \mathbb{F}_q[x]$ be such that $\gcd(g_{ij}(x), p_i(x)) = 1$ for $1 \leq i \leq s$ and $\lim_{j \to \infty}(je_i - \deg g_{ij}(x)) = \infty$ for $1 \leq i \leq s$.
- For $0 \leq k < e_i$, $1 \leq i \leq s$, and $j \geq 1$ the elements $a_i(j, k, r) \in \mathbb{F}_q$ are defined by the series expansion

$$\frac{x^k g_{ij}(x)}{p_i(x)^j} = \sum_{r=w}^{\infty} a_i(j, k, r) x^{-r-1}.$$

- Put $c_i(j, r) = a_i(q + 1, u, r) \in \mathbb{F}_q$ for $1 \leq i \leq s$, $j \geq 1$ and $j - 1 = qe_i + u$ where $0 \leq u < e_i$.
- Put $x_i(n, j) = \sum_{r=0}^{\infty} c_i(j, r) a_r(n)$, where $x_i(n, j) \in \mathbb{F}_q$.
- Finally, put $x_{n,i} = \sum_{j=1}^{\infty} x_i(n, j) q^{-j}$ for $1 \leq i \leq s$.

Then the sequence

$$\mathbf{x}_n = (x_{n,1}, \dots, x_{n,s}), \quad n = 1, 2, \dots,$$

is a

$$(t, s)\text{–sequence}$$

in the base q with $t = \sum_{i=1}^{s}(e_i - 1)$.

Notes: This construction was given by H.Niederreiter (1988), see also [DT, p. 383-386]. For the discrepancy bounds 3.19(II) may used.

H. Niederreiter: *Point sets and sequences with small discrepancy*, Monatsh. Math. **104** (1987), 273–337 (MR0918037 (89c:11120); Zbl. 0626.10045).
H. Niederreiter: *Low discrepancy and low–dispersion sequences*, J. Number Theory **30** (1988), 51–70 (MR0960233 (89k:11064); Zbl. 0651.10034).

3.19.4.

- q is a prime power,
- $\mathbb{F}_q$ is the finite field of order q,

- $\mathbb{F}_q(z)$ is the rational function field over $\mathbb{F}_q$,
- $\mathbb{F}_q((z^{-1}))$ is the field of formal Laurent series over $\mathbb{F}_q$,
- if $L = \sum_{k=w}^{\infty} u_k z^{-k} \in \mathbb{F}_q((z^{-1}))$ with $u_w \neq 0$, let $\mathrm{Fr}(L) = \sum_{k=\max(1,w)}^{\infty} u_k z^{-k}$ denote its fractional part and let
- ν be the standard degree valuation on $\mathbb{F}_q((z^{-1}))$ given by $\nu(L) = -w$,
- given $L = \sum_{k=w}^{\infty} u_k z^{-k} \in \mathbb{F}_q((z^{-1}))$, define the associated real number expressed in the base q, by $\Phi(L) = \sum_{k=\max(1,w)}^{\infty} u_k q^{-k}$,
- to a given non–negative integer $n = \sum_{r=0}^{m(n)} a_r(n) q^r$, $a_r(n) \in \mathbb{F}_q$, written in the q–adic digit expansion in the base q associate the polynomial $n(z) \in \mathbb{F}_q[z]$ defined by $n(z) = \sum_{r=0}^{m(n)} a_r(n) z^r$.

If $L_1, L_2, \ldots, L_s \in \mathbb{F}_q((z^{-1}))$ then the associated sequence $\mathbf{x}_n \in [0,1]^s$ is defined by

$$\mathbf{x}_n = (\Phi(n(z)L_1(z)), \ldots, \Phi(n(z)L_s(z))).$$

(A) The sequence $\mathbf{x}_n$ is

u.d.

if and only if $1, L_1, \ldots, L_s$ are linearly independent over the rational function field $\mathbb{F}_q(z)$.

(B) If there is a constant $c \in \mathbb{Z}$ such that for all $Q_1, \ldots, Q_s \in \mathbb{F}_q[z]$ (not all 0) we have

$$\nu(\mathrm{Fr}(\sum_{i=1}^{s} Q_i L_i)) \geq -c - \sum_{i=1}^{s} \deg Q_i,$$

then the sequence $\mathbf{x}_n$ is a

digital $(c-s,s)$–sequence

over $\mathbb{F}_q$ and for its discrepancy we have (applying 3.19(II))

$$D_N^*(\mathbf{x}_n) = \mathcal{O}\left(\frac{(\log N)^s}{N}\right).$$

Notes: (I) G. Larcher and H. Niederreiter (1993,1995), where part(A)=Th.1(1993), part(B)=Th. 2(1995).

(II) This method is digital, since if the used formal Laurent series are $L_i(x) = \sum_{k=w_i}^{\infty} u_{k,i} x^{-k}$, where $w_i \leq 1$, for $1 \leq i \leq s$, then the same $\mathbf{x}_n$ can be constructed by method 3.19.2 using matrices $C^{(i)}$ which have rows

$$\mathbf{C}_j^{(i)} = (u_{j,i}, u_{j+1,i}, u_{j+2,i}, \ldots) \quad \text{for } j = 1, 2, \ldots.$$

(III) If

- $q = 2$,
- $\Phi_m(L) = \sum_{k=\max(1,w)}^{m} u_k q^{-k}$ for $L(x) = \sum_{k=w}^{\infty} u_k x^{-k}$,
- $L_i(x) = g_i(x)/f(x)$, $i = 1, 2, \dots, s$, where
- $f(x) \in \mathbb{F}_2[x]$ with $\deg f = m$,
- $\deg g_i < m$, $\deg 0 = -1$,

then G. Larcher, A. Lauss, H. Niederreiter and W.Ch. Schmid (1996) proved (see also Larcher (1998)): The sequence

$$\mathbf{x}_n = \left(\Phi_m\left(\frac{n(x)g_1(x)}{f(x)}\right), \dots, \Phi_m\left(\frac{n(x)g_s(x)}{f(x)}\right)\right), \quad n = 0, 1, \dots, 2^m - 1,$$

is a (t, m, s)–net over $\mathbb{F}_2$ with

$$t = m - s + 1 - \min \sum_{i=1}^{s} \deg h_i,$$

where the minimum runs over all non–zero s–tuples $(h_1(x), \dots, h_s(x))$ of polynomials from $\mathbb{F}_2[x]$ with $\deg h_i < m$, $i = 1, \dots, s$, and for which f divides $\sum_{i=1}^{s} g_i h_i$.
Polynomial s–tuples $\mathbf{g} = (g_1, \dots, g_s) \pmod f$ which lead to a "small" t is called **good s–tuples $\mathbf{g}$** and if $\mathbf{g} = (1, g, g^2, \dots, g^s) \pmod f$ then g is called **optimal polynomial**. In Larcher, Lauss, Niederreiter and Schmid (1996) various existence results for such $\mathbf{g}$ and g are given.
(IV) Every $L \in \mathbb{F}_q((z^{-1}))$ has the unique continued fraction expansion $L = [A_0; A_1, A_2, \dots]$, where $A_i \in \mathbb{F}_q[z]$ for all $i \geq 0$ and $\deg A_i \geq 1$ for $i \geq 1$. Similar to the case of the simple continued fraction expansion of real numbers, the expansion is finite for rational L and infinite for irrational L. Larcher and Niederreiter (1993) proved: If L is irrational, then for all integers N with $q^{\sum_{i=1}^{H-1} \deg A_i} < N \leq q^{\sum_{i=1}^{H} \deg A_i}$ and $H \geq 1$, the star discrepancy of the first N terms of the one–dimensional sequence

$$x_n = \Phi(n(z)L(z))$$

satisfies

$$ND_N^* \leq \frac{q+1}{q} + \frac{1}{4} \sum_{i=1}^{H} q^{\deg A_i}(1 + q^{-\deg A_i})^2.$$

Consequently, if L has bounded partial quotients (i.e. $\deg A_i \leq K$ for all $i \geq 1$), then $ND_N^* = \mathcal{O}(\log N)$ for all $N \geq 2$. See also Larcher (1998, p. 190–191, Th. 17).
(V) Let $s = 1$. Let $f(z) = \sum_{j=k}^{\infty} x_j z^{-j} \in GF_b\{z\}$ be a formal Laurent series. For $f(z)$ define the Hankel matrix $H(f(z))$ as

$$H(f(z)) = \begin{vmatrix} x_1 & x_2 & x_3 & \cdots & x_n & \cdots \\ x_2 & x_3 & x_4 & \cdots & x_{n+1} & \cdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \cdots \\ x_{n-1} & x_n & x_{n+1} & \cdots & x_{2n-2} & \cdots \\ x_n & x_{n+1} & x_{n+2} & \cdots & x_{2n-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix}.$$

Larcher and Niederreiter (1993) were the first who used a Hankel matrix as a generator matrix in the form $C = H(p(z)/q(z))$, where $\deg q(z) < \deg p(z)$, $\gcd(q(z),p(z)) = 1$. They called the resulting digital $(t,1)$-sequence **the polynomial Weyl sequence**. They determined t as follows: If the continued fraction expansion of $q(z)/p(z)$ over GF_b is $q(z)/p(z) = [0, g_1(z), g_2(z), \dots, g_K(z)]$, then put

$$\rho(H;l) = \begin{cases} d_k, & \text{if } d_k \le l < d_{k+1} \text{ for some } o \le k < K, \\ d_K, & \text{otherwise,} \end{cases}$$

where $d_k = \sum_{i=1}^{k} \deg g_i(z)$ for $k = 1,2,\dots,K$ and $d_0 = 0$. Then $t = \sup_{l\ge1}(l - \rho(H;l))$.

G. Larcher: *Digital point sets: Analysis and application*, in: *Random and Quasi–Random Point Sets*, (P. Hellekalek, G. Larcher eds.), Lecture Notes in Statistics, 138, Springer Verlag, New York, Berlin, 1998, pp. 167–222 (MR1662842 (99m:11085); Zbl. 0920.11055).

G. Larcher – H. Niederreiter: *Kronecker–type sequences and non–Archimedean Diophantine approximations*, Acta Arith. **63** (1993), no. 4, 379–396 (MR1218466 (94c:11063); Zbl. 0774.11039).

G. Larcher – H. Niederreiter: *Generalized (t,s)–sequence, Kronecker–type sequences, and diophantine approximations of formal Laurent series*, Trans. Amer. Math. Soc. **347** (1995), no. 6, 2051–2073 (MR1290724 (95i:11086); Zbl. 0829.11039).

G. Larcher – A. Lauss – H. Niederreiter – W.Ch. Schmid: *Optimal polynomials for (t,m,s)–nets and numerical integration of multivariate Walsh series*, SIAM J. Numer. Anal. **33** (1996), no. 6, 2239–2253 (MR1427461 (97m:65046); Zbl. 0861.65019).

3.19.5. Sobol' sequences. Let $n = \sum_{j=0}^{m(n)} a_j(n)2^j$ be the dyadic expansion of a non–negative integer n. Given any $\gamma,\delta \in [0,1)$ with dyadic expansions $\gamma = 0.c_1c_2\dots$ and $\delta = 0.d_1d_2\dots$ define $\gamma \oplus \delta = 0.e_1e_2\dots$ by $e_i = c_i + d_i \pmod 2$ for all i. In $\mathbb{F}_2$ consider the following recurring formulas for $k = 1,\dots,s$,

- $z^{(k)}_{j+m_k} = b^{(k)}_{m_k-1} z^{(k)}_{j+m_k-1} + \dots + b^{(k)}_1 z^{(k)}_{j+1} + z^{(k)}_j$, $j = 0,1,\dots,$
  with characteristic polynomials
- $p^{(k)}(x) = x^{m_k} + b^{(k)}_{m_k-1} x^{m_k-1} + \dots + b^{(k)}_1 x + 1.$
  Apply this recurrence relation to dyadic rationals with initial values
- $(y^{(k)}_0, \dots, y^{(k)}_{m_k-1}) = (1/2, \dots, 1/2^{m_k})$
  and compute the dyadic sequences
- $y^{(k)}_{j+m_k} = b^{(k)}_{m_k-1} y^{(k)}_{j+m_k-1} \oplus \dots \oplus b^{(k)}_1 y^{(k)}_{j+1} \oplus y^{(k)}_j \oplus \left(y^{(k)}_j/2^{m_k}\right)$, $j = 0,1,2,\dots,$
- $x_{n,k} = a_0(n)y^{(k)}_0 \oplus a_1(n)y^{(k)}_1 \oplus \dots \oplus a_{m(n)}(n)y^{(k)}_{m(n)}$, $n = 0,1,2,\dots,$
  finally also consider the van der Corput sequence 2.11.1
- $x_n = \sum_{j=0}^{m(n)} a_j(n)2^{-(j+1)}$, $n = 0,1,2,\dots$

If, for $k = 1,2,\dots,s$, the recurring formulas $z^{(k)}(j)$ are distinct in $\mathbb{F}_2$ and every non–trivial solution of $z^{(k)}(j)$ has the period $2^{m_k} - 1$ (i.e. equivalently, all characteristic polynomials $p^{(k)}(x)$ in $\mathbb{F}_2[x]$ are distinct, irreducible, and the minimal step i for which $p^{(k)}(x)|x^i+1$ is $i=2^{m_k}-1$; in other words, they are **primitive**), then the sequences

$$\mathbf{x}_n=(x_{n,1},\dots,x_{n,s}),\quad n=0,1,2,\dots,\qquad \text{is a } (t,s)\text{–sequence},$$

and

$$\mathbf{x}_n^*=(x_{n,1},\dots,x_{n,s},x_n),\quad n=0,1,2,\dots,\qquad \text{is a } (t,s+1)\text{–sequence}$$

for

$$t=m_1+\dots+m_s-s.$$

NOTES: (I) I.M. Soboľ (1966, 1967). He also proved that:

- $\varphi_\infty(N)\le 2^{s-1+t}$ for every (t,m,s)–net in the base $q=2$ and $N=2^m$ (here $\varphi_\infty(N)$ is the non–uniformity, cf. 1.11.13). This is also true for every initial N terms ($N=1,2,\dots$) of any (t,s)–sequence in the base $q=2$.
- $ND_N^*\le 2^t\sum_{j=0}^{s-1}\binom{m-t}{j}$, for every (t,m,s)–net in the base $q=2$ with $m\ge s-1+t$ if $N=2^m$.
- $ND_N^*\le 2^t\sum_{j=0}^{s-1}\binom{[\log_2 N]-t+1}{j}$, for every (t,s)–sequences in the base $q=2$ and $N\ge 2^{s-1+t}$.

For details we refer to Soboľ (1969, Chap. 6, Par. 3–5).
(II) Using the lists of all primitive polynomials over $\mathbb{F}_2$ arranged according to their non–decreasing degrees and bearing in mind that the number of primitive polynomials of degree m is $\frac{\varphi(2^m-1)}{m}$, Soboľ (1969, pp. 215–218) found that the minimal t (for fixed s denoted by t_s) for which there exists a (t,s)–sequence satisfies $t_s=\mathcal{O}(s\log s)$.

I.M. SOBOĽ: *Distribution of points in a cube and integration nets*, (Russian), Uspechi Mat. Nauk **21** (1966), no. 5(131), 271–272 (MR0198678 (**33** #6833)).
I.M. SOBOĽ: *Distribution of points in a cube and approximate evaluation of integrals*, (Russian), Ž. Vyčisl. Mat. i Mat. Fiz. **7** (1967), 784–802 (MR0219238 (**36** #2321)).
I.M. SOBOĽ: *Multidimensional Quadrature Formulas and Haar Functions*, (Russian), Library of Applied Analysis and Computational Mathematics, Izd. "Nauka", Moscow, 1969 (MR0422968 (**54** #10952); Zbl. 0195.16903).

3.19.6. Faure sequences. Let $n=\sum_{j=0}^{m(n)}a_j(n)q^j$ be the q–adic digit expansion of a non–negative integer n. Given a $\gamma\in[0,1)$ with q–adic digit expansion $\gamma=0.c_1c_2\dots$ and an infinite integer matrix C, define $C\cdot\gamma$ as $0.d_1d_2\dots$, where $(d_1,d_2,\dots)=C\cdot(c_1,c_2,\dots)^T \pmod q$. Assume that

- q is the smallest prime number with $q\ge s$,
- C is the Pascal triangle matrix and thus for its i-th power $(C)^i=(C_{j,k}^{(i)})$ we have

- $C_{j,k}^{(i)} = \begin{cases} \binom{j}{k} i^{j-k}, & \text{for } 0 \le k \le j, \\ 0, & \text{for } k > j, \end{cases}$
  moreover consider the van der Corput sequence 2.11.1
- $x_{n,1} = \sum_{j=0}^{m(n)} a_j(n) q^{-(j+1)}$, $n = 0, 1, 2, \dots$,
  and define
- $x_{n,i} = (C)^{i-1} \cdot x_{n,1}$, $i = 1, 2, \dots$.

Then

$$\mathbf{x}_n = (x_{n,1}, \dots, x_{n,s}), \quad n = 0, 1, 2, \dots,$$

is a

$$(0, s)\text{–sequence in the base } q.$$

NOTES: (I) H. Faure (1982). He also proved that, for every $N \ge 1$,

$$D_N^*(\mathbf{x}_n) \le F_s(q) \frac{(\log N)^s}{N} + \mathcal{O}\left(\frac{(\log N)^{s-1}}{N} \right)$$

where

$$F_s(q) = \frac{1}{s!} \left(\frac{q-1}{2 \log q} \right)^s.$$

(II) Clearly, $x_{n,i} = \sum_{j=1}^{\infty} \frac{y_j^{(i)}(n)}{q^j}$ where $(y_1^{(i)}(n), y_2^{(i)}, \dots) = (C)^{i-1} \cdot (a_0(n), a_1(n), \dots)$. Thus, $\mathbf{x}_n$ is digital.

H. FAURE: *Discrépance de suites associées à un système de numération (en dimension s)*, Acta Arith. **41** (1982), 337–351 (MR0677547 (84m:10050); Zbl. 0442.10035).

3.19.7. Niederreiter – Xing sequences.

- $q = p^m$, p is a prime,
- $K/\mathbb{F}_q$ is a global function field,
- g is the genus of $K/\mathbb{F}_q$,
- $\nu_P(k)$ is the normalized discrete valuation corresponding to the place P of $K/\mathbb{F}_q$,
- P_∞ is a fixed rational place of $K/\mathbb{F}_q$,
- R is the ring of elements of K that have no pole outside P_∞,
- $n_1 < n_2 < \dots$ are all the so–called pole numbers of P_∞,

Given an integer $s \ge 1$ we choose $k_1, \dots, k_s \in R$ such that

- the zero sets $Z(k_1), \dots, Z(k_s)$ are pairwise disjoint,
- $n_{e_i} - e_i < n_1$ where $e_i = -\nu_{P_\infty}(k_i) \ge 1$ for $1 \le i \le s$,

For every pole number n_r we can find $w_r \in R$ such that $(\omega_r)_\infty = n_r P_\infty$, $r = 1, 2, \dots$. Since $(k_i)_\infty = e_i P_\infty$, each e_i is a pole number of P_∞, consequently $n_{f_i} = e_i$ for uniquely determined positive integer f_i for each $1 \le i \le s$. Define for $1 \le i \le s$

- $\{w_{i,0}, w_{i_1}, \dots, w_{i.e_i-1}\} = \{1, w_1, w_2, \dots, w_{e_i}\} \setminus \{w_{f_i}\}$, and write for $j \le 1$
- $j - 1 = Q(i,j)e_i + u(i,j)$, where $Q(i,j)$ and $u(i,j)$ are integers and $0 \le u(i,j) < e_i$.

Then we have the following expansion at P_∞

$$w_{i,u(i,j)} k^{-Q(i,j)-1} = z^{-g} \sum_{r=0}^{\infty} c_{j,r}^{(i)} z^r$$

where $c_{j,r}^{(i)} \in \mathbb{F}_q$ and z is a local unifromizing parameter at P_∞. The coefficients $c_{j,r}^{(i)} \in \mathbb{F}_q$ can serve as the elements in (S3) in construction 3.19.2(V) of a digital (t, s)–sequence in the base q where

$$t = g + 1 + \sum_{i=1}^{s} (e_i - 1).$$

NOTES: H. Niederreiter and Ch. Xing (1995). In (1996) they call the above procedure the first construction and gave also two others algebraic–geometrical constructions of (t, s)–sequences. Their methods yield (t, s)–sequences in the base 2 with $16 \le s \le 126$ having currently the smallest parameters t.

H. Niederreiter – C.-P. Xing: *Low–discrepancy sequences obtained from algebraic function fields over finite fields*, Acta Arith. **72** (1995), no. 3, 281–298 (MR1347491 (96g:11099); Zbl. 0833.11035).
H. Niederreiter – C.-P. Xing: *The algebraic–geometry approach to low–discrepancy sequences*, in: *Monte Carlo and Quasi–Monte Carlo Methods 1996 (Proceedings of a conference at the University of Salzburg, Austria, July 9–12, 1996)*, (H. Niederreiter, P. Hellekalek, G. Larcher, P. Zinterhof eds.), Lecture Notes in Statistics, 127, Springer Verlag, New York, Berlin, 1998, pp. 139–160 (MR1644516 (99d:11081); Zbl. 0884.11031).

Hybrid sequences

3.19.7.1 Let $z_0, z_1, \dots$ be a digital explicit inversive sequence as defined in 2.25.10.1. Let $q = p^k$ with a prime p and an integer $k \ge 1$. For a given integer t with $1 \le t \le q$, let $0 \le d_1 < d_2 < \cdots < d_t < q$ be integers. Consider the hybrid sequence

$$\mathbf{x}_n = (\{n\alpha\}, z_{n+d_1}, \dots, z_{n+d_t}) \in [0,1)^{s+t}, \quad n = 0, 1, \dots.$$

Let $\alpha \in \mathbb{R}^s$ be of finite type η. Then for $1 \le N \le q$ the discrepancy D_N of the first N terms of the sequence $\mathbf{x}_n$ satisfies

$$D_N = O_{\alpha,t,\varepsilon}\Bigg(\max\big(N^{-1/((\eta-1)s+1)+\varepsilon}, \\ 2^{(k-1)t+k/2}k^{1/2}N^{-1/2}(\log N)^s q^{1/4}(\log q)^t(1+\log p)^{k/2}\big)\Bigg)$$

for all $\varepsilon > 0$, where the implied constant depends only on α, t, and ε.
NOTES:
(I) H. Niederreiter (2010).
(II) If $\eta = 1$ then

$$D_N = O_{\alpha,t}\left(2^{(k-1)t+k/2}k^{1/2}N^{-1/2}(\log N)^s q^{1/4}(\log q)^t(1+\log p)^{k/2}\right).$$

H. NIEDERREITER: *A discrepancy bound for hybrid sequences involving digital explicit inversive pseudorandom numbers*, Unif. Distrib. Theory **5** (2010), no. 1, 53–63 (MR2804662 (2012f:11143); Zbl. 1249.11074).

3.19.7.2 For integers $b \ge 2$ and $n \ge 0$, let $n = \sum_{j=0}^{\infty} a_j(n)b^j$ be the digit expansion of n in the base b, where $a_j(n) \in \{0,1,\dots,b-1\}$ for all $j \ge 0$ and $a_j(n) = 0$ for all sufficiently large j. Then the radical-inverse function γ_b in the base b is $\gamma_b(n) = \sum_{j=0}^{\infty} a_j(n)b^{-j-1}$. For a given dimension $s \ge 1$, let $b_1,\dots,b_s$ be pairwise coprime integers ≥ 2. Then the Halton sequence (in the bases $b_1,\dots,b_s$) is given by $\mathbf{y}_n = \big(\gamma_{b_1}(n),\dots,\gamma_{b_s}(n)\big) \in [0,1)^s$, $\quad n = 0,1,\dots$ It is a classical low-discrepancy sequence.
Let $z_0, z_1, \dots$ be a digital explicit inversive sequence as defined in 2.25.10.1. Let $q = p^k$ with a prime p and an integer $k \ge 1$. For a given integer t with $1 \le t \le q$, let $0 \le d_1 < d_2 < \cdots < d_t < q$ be integers. Consider the hybrid sequence obtained by "mixing" the Halton sequence and a digital explicit inversive sequence

$$\mathbf{x}_n = \big(\gamma_{b_1}(n),\dots,\gamma_{b_s}(n), z_{n+d_1},\dots,z_{n+d_t}\big) \in [0,1)^{s+t}, \quad n = 0,1,\dots$$

The discrepancy D_N of the first N terms of the sequence $\mathbf{x}_n$ satisfies

$$D_N = O_{b_1,\dots,b_s,t}\left(\left(2^k q^{1/2}(1+\log p)^k(\log q)^t N^{-1}\right)^{1/(s(k-1)t+s+1)}\right)$$

where the implied constant depends only on $b_1,\dots,b_s$, and t.

H. Niederreiter – A. Winterhof: *Discrepancy bounds for hybrid sequences involving digital explicit inversive pseudorandom numbers*, Unif. Distrib. Theory **6** (2011), no. 1, 33–56 (MR2817759 (2012g:11143); Zbl. 1249.11075).

Sequences (x_n, y_n) where both x_n and y_n are u.d.

3.19.7.3 All d.f.'s $g(x, y)$ of the sequence (x_n, y_n) has marginals
$g(x, 1) = x$,
$g(1, y) = y$.
These d.f.'s are called copulas and they were introduced by M. Sklar (1959). For basic properties of copulas consult R.B. Nelsen (1999).

Let $G_{2,1}$ be the set of all two-dimensional copulas. Some basic properties of $G_{2,1}$:
(I) $G_{2,1}$ is closed under pointwise limit and convex linear combinations.
(II) For every $g(x, y) \in G_{2,1}$ and every $(x_1, y_1), (x_2, y_2) \in [0, 1]^2$ we have

$|g(x_2, y_2) - g(x_1, y_1)| \le |x_2 - x_1| + |y_2 - y_1|$.
(III) For every $g(x, y) \in G_{2,1}$ we have

$g_3(x, y) = \max(x + y - 1, 0) \le g(x, y) \le \min(x, y) = g_2(x, y)$,
where $g_3(x, y)$ and $g_2(x, y)$ are copulas (the so called Fréchet-Hoeffding bounds, see R.B. Nelsen [1999, p. 9]).
(IV) M. Sklar (1959) proved that for every d.f. $g(x, y)$ on $[0, 1]^2$ there exists a copula $c(x, y) \in G_{2,1}$ such that $g(x, y) = c(g(x, 1), g(1, y))$ for every $(x, y) \in [0, 1]^2$. If $g(x, 1)$ and $g(1, y)$ are continuous, then the copula $c(x, y)$ is uniquely determined (cf. Nelsen [p. 15, Th. 2.3.3]). Furthermore, if $f(x, y)$ is continuous we have
(V) $\int_0^1 \int_0^1 f(x, y)\, \mathrm{d}g(x, y) = \int_0^1 \int_0^1 f(g^{-1}(x, 1), g^{-1}(1, y))\, \mathrm{d}c(x, y)$.
(VI) Examples:

$g_\theta(x, y) = (\min(x, y))^\theta (xy)^{1-\theta}$, where $\theta \in [0, 1]$ (Cuadras-Augé family, cf. Nelsen [1999, p. 12, Ex. 2.5]),

$g_4(x, y) = \frac{xy}{x+y-xy}$ (see Nelsen [1999, p. 19, 2.3.4]),

$\tilde{g}(x, y) = x + y - 1 + g(1 - x, 1 - y)$ for every $g(x, y) \in G_{2,1}$ (Survival copula, see Nelsen [1999, p. 28, 2.6.1]).

Related sequences: 3.18.1.1, 3.18.1.2, 3.4.1.4, 3.4.1.5.

R.B. Nelsen: *An Introduction to Copulas. Properties and Applications*, Lecture Notes in Statistics 139, Springer, New York, NY, 1999 (2nd ed. Springer 2006). (MR1653203 (99i:60028); Zbl. 0909.62052).

M. Sklar: *Fonctions de répartition à n dimensions et leur marges*, Publ. Inst. Stat. Univ. Paris **8** (1960), 229–231 (MR0125600 (**23** #A2899); Zbl. 0100.14202).

3.20 Pseudorandom Numbers Congruential Generators

3.20.1. Matrix generator. The matrix numbers generator produced the s–dimensional vector sequence

$$\mathbf{x}_n = \frac{1}{M}\mathbf{y}_n, \text{ where } \mathbf{y}_{n+1} \equiv \mathbf{A} \cdot \mathbf{y}_n \pmod{M}, \quad n = 0, 1, \dots,$$

where $\mathbf{y}_0$ is an initial s–dimensional integer vector different from $\mathbf{0}$ (mod M), and $\mathbf{A}$ is an $s \times s$ matrix with integer elements and non–singular modulo M. Then we have

- $\mathbf{x}_n$ is purely periodic,
- if $M = p$ is a prime modulus, then $\mathbf{x}_n$ has the maximal period $p^s - 1$ if and only if the characteristic polynomial of $\mathbf{A}$ is primitive over $\mathbb{F}_p$.

Notes: H. Niederreiter (1992, p. 207, Th. 10.2; 1995). For a discrepancy of the sj–dimensional sequence $\mathbf{z}_n = (\mathbf{x}_n, \dots, \mathbf{x}_{n+j-1})$ cf. (1992, p. 209, Th. 10.4).

H. Niederreiter: *Random Number Generation and Quasi–Monte Carlo Methods*, CBMS–NSF Regional Conference Series in Applied Mathematics, 63, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992, vi+241 pp. (MR1172997 (93h:65008); Zbl. 0761.65002).
H. Niederreiter: *New developments in uniform pseudorandom number and vector generation*, in: *Monte Carlo and quasi–Monte Carlo methods in scientific computing (Las Vegas, NV, 1994)*, Lecture Notes in Statist., Vol. 106, Springer Verlag, New York, 1995, pp. 87–120 (MR1445782 (97k:65019); Zbl. 0893.11030; entire collection MR1445777 (97j:65002)).

3.20.2.

- Let p be a prime,
- m a positive integer,
- $q = p^m$,
- $\mathbf{A}$ be a non–singular $m \times m$ matrix over $\mathbb{F}_p$,
- $\mathbf{z}_0 \in (\mathbb{F}_p)^m$ initial vector different from $\mathbf{0}$,
- $\mathbf{z}_{n+1} := \mathbf{z}_n \cdot \mathbf{A}$, $\mathbf{z}_n = (z_{n,1}, \dots, z_{n,m})$,
- $x_n = \sum_{j=1}^m \frac{z_{n,j}}{p^j}$,

The sequence x_n, $n = 0, 1, 2, \dots$, and consequently also $\mathbf{z}_n$, $n = 0, 1, 2, \dots$, is purely periodic and has the maximal possible period $p^m - 1$ if and only if the characteristic polynomial of $\mathbf{A}$ $\big($i.e. $\det(x \cdot \mathbf{E} - \mathbf{A})\big)$ is a primitive polynomial of degree m over $\mathbb{F}_p$. Equivalently, there exists a primitive element σ of $\mathbb{F}_q$ and

a basis $\beta_1, \dots, \beta_m$ of $\mathbb{F}_q$ over $\mathbb{F}_p$ such that $z_{n,j} = \mathrm{Tr}(\beta_j \sigma^n)$ for $j = 1, 2, \dots, m$ and $n = 0, 1, 2, \dots$, where Tr is the trace function $\mathrm{Tr} : \mathbb{F}_q \to \mathbb{F}_p$.
For every s, $2 \le s \le m$, $N = p^m - 1$, and fixed primitive element σ of $\mathbb{F}_q$, the sequence

$$\mathbf{x}_n = (x_n, x_{n+1}, \dots, x_{n+s-1}), \quad n = 0, 1, \dots, N-1,$$

has the discrepancy with the average $D_N^* = \mathcal{O}((\log N)^s/N)$, where the average is taken over all ordered bases of $\mathbf{F}_q$ over itself.

Notes: This method was introduced in full generality in H. Niederreiter (1993) and it was studied in detail in (1995).

H. Niederreiter: *Factorization of polynomials and some linear–algebra problems over finite fields*, in: *Computational linear algebra in algebraic and related problems (Essen, 1992)*, Linear Algebra Appl. **192** (1993), 301–328 (MR1236747 (95b:11114); Zbl. 0845.11042).
H. Niederreiter: *The multiple recursive matrix method for pseudorandom number generation*, Finite Fields Appl. **1** (1995), no. 1, 3–30 (MR1334623 (96k:11103); Zbl. 0823.11041).

3.21 Miscellaneous items

Here we list some sequences which we have find after finishing the work over the manuscript.

3.21.1. Generalized ratio sequences. Let x_n be an increasing sequence of positive integers. If the lover asymptotic density $\underline{d}(x_n) > 0$ (for the def. see p. 1 – 3), then there exists a positive integer k such that the sequence

$$\frac{x_{m_1} x_{m_2} \dots x_{m_k}}{x_{n_1} x_{n_2} \dots x_{n_k}}, \quad m_1, m_2, \dots, m_k, n_1, n_2, \dots, n_k = 1, 2, \dots,$$

is

dense in $[0, \infty)$.

Notes: This complements the result mentioned in 2.22.2. The proof of J. Bukor and J.T. Tóth (2003) is based on the result of O. Strauch and J.T. Tóth (1998) saying that $\overline{d}(x_n) \le 1 - |X|$ for every open set $X \subset [0,1]$ not containing an accumulation point of $\frac{x_m}{x_n}$, $m, n = 1, 2, \dots$, where $|X|$ denotes the Lebesgue measure of X.

J. Bukor – J.T. Tóth: *On accumulation points of generalized ratio sets of positive integers*, Acta Acad. Paedagog. Agriensis, Sect. Mat. (N.S.) **30** (2003), no. 6, 37–43 (MR2054713 (2005h:11020); Zbl. 1050.11012).

O. Strauch – J.T. Tóth: *Asymptotic density of $A \subset \mathbb{N}$ and density of the ratio set $R(A)$*, Acta Arith. **87** (1998), no. 1, 67–78 (correction *ibid.* **103** (2002), no. 2, 191–200). (MR1659159 (99k:11020); Zbl. 0923.11027).

3.21.2. Absolutely abnormal numbers. Let $d_2 = 2^2$ and define recursively $d_j = j^{d_{j-1}/(j-1)}$ for $j = 3, 4, \dots$. Then

$$\theta = \prod_{j=2}^{\infty} \left(1 - \frac{1}{d_j}\right)$$

is a real transcendental number which is

not normal for any base $q \geq 2$.

Notes: This complements the result of 2.18.

G. Martin: *Absolutely abnormal numbers*, Amer. Math. Monthly **108** (2001), no. 8, 746–754 (MR1865662 (2002m:11071); Zbl. 1036.11035).

3.21.3. Generalized two–dimensional Zaremba sequence. Let $q \geq 2$, $m > 0$, a, b be fixed integers. Define

- $n = \sum_{j=0}^{\infty} a_j(n) q^j$ is the expansion of n in the base q,
- $\gamma_q(n) = \sum_{j=0}^{\infty} a_j(n) q^{-j-1}$ is the radical inverse function, see 2.11.2,
- $\gamma'_q(n) = \sum_{j=0}^{\infty} a'_j(n) q^{-j-1}$, where $a'_j(n) \equiv a_j(n) + aj + b \pmod q$ for $j = 0, 1, \dots$,
- $N = q^m$.

Then the finite two–dimensional sequence

$$\mathbf{x}_n = \left(\frac{n}{N}, \gamma'_q(n)\right), \quad n = 0, 1, \dots, N-1,$$

has the q–adic diaphony (see 1.11.5)

$$DF_N(\mathbf{x}_n) = \mathcal{O}\left(\frac{\sqrt{\log N}}{N}\right).$$

Notes: If $a = b = 1$, then we obtain the two–dimensional Zaremba sequence defined in 3.18.4. V.S. Grozdanov and S.S. Stoilova (2003) called the sequence $\mathbf{x}_n$ the **generalized Zaremba net in base** q. They also introduced the notion of the q–adic diaphony.

V.S. Grozdanov – S.S. Stoilova: *On the b–adic diaphony of the Roth net and generalized Zaremba net*, Math. Balkanica (N.S.) **17** (2003), no. 1–2, 103–112 (MR2096244 (2005f:11144); Zbl. 1053.11066).

3.21.4. Sequences on two–dimensional sphere S^2. Let A, B, C be the rotations of the tree–dimensional Euclidean space with respect to the x, y, z–axes, each through an angle of $\arccos\left(-\frac{3}{5}\right)$. Let W_k be the set of nontrivial words in $A, B, C, A^{-1}, B^{-1}, C^{-1}$ of length $\le k$ (all the obvious cancelations such as AA^{-1} have been carried out). Then W_k consists of $N = \frac{3}{2}(5^k - 1)$ elements, say $\psi_1, \psi_2, \dots, \psi_N$. If $P \in S^2$ is a suitable chosen starting point, then the orbital points

$$\mathbf{x}_n = \psi_n(P), \quad n = 1, 2, \dots, N,$$

have spherical–cap discrepancy (for def. see 1.11.10)

$$S_N = \mathcal{O}\left(\frac{(\log N)^{2/3}}{N^{1/3}}\right).$$

Notes: A. Lubotzky, R. Phillips and P. Sarnak (1986). R.F. Tichy (1990) used this sequence for approximate solutions of some initial–valued problems defined on S^2.

A. Lubotzky – R. Phillips – P. Sarnak: *Hecke operators and distributing points on the sphere. I*, Comm. Pure Appl. Math **39** (1986), no. S, suppl., S149–S186 (MR0861487 (88m:11025a); Zbl. 0619.10052).
R.F. Tichy: *Random points in the cube and on the sphere with applications to numerical analysis*, J. Comput. Appl. Math. **31** (1990), no. 1, 191–197 (MR1068159 (91j:65009); Zbl. 0705.65003).

3.21.5. Salem numbers.

Notes: As we have defined in 2.17.7, a Salem numbers is a real algebraic integer, greater than 1, with the property that all its conjugates lie on or within the unit circle, and at least one conjugate lies on the unit circle.

. . . .

Let θ be the Salem numbers of degree greater than or equal to 8. Then the sequence

$$x_n = \theta^n \bmod 1, \quad n = 1, 2, \dots,$$

has

$$\text{a.d.f. } g(x) \neq x$$

which satisfies

$$|(g(y) - g(x)) - (y - x)| \le 2\zeta\left(\frac{\deg(\theta) - 2}{4}\right)(2\pi)^{1 - \frac{\deg(\theta)}{2}}(y - x),$$

where $\zeta(z)$ is the Riemann zeta function, $\deg(\theta)$ is the degree of θ over $\mathbb{Q}$ and $0 \le x < y \le 1$.

NOTES: (I) This was proved by S. Akiyama and Y. Tanigawa (2004). They also proved that if the Salem number θ is of degree 4 or 6 then x_n has a.d.f. $g(x) \ne x$ such that

$$|(g(y) - g(x)) - (y - x)| \le 4\pi^{-\frac{3}{2}}\sqrt{y - x} \quad \text{if } \deg(\theta) = 4,$$

and

$$|(g(y) - g(x)) - (y - x)| \le \frac{y - x}{2\pi^2}\left(\log\frac{1}{y - x} + 1 + (y - x)\right) \text{ if } \deg(\theta) = 6.$$

(II) Salem numbers are the only known concrete numbers whose powers are dense mod 1 in $[0, 1]$, see 2.17.7 and the monograph of M.J. Bertin, A. Decomps-Guilloux, M. Grandet-Hugot, M. Pathiaux-Delefosse, and J.P. Schreiber (1992, pp. 87–89). The survey paper of E. Ghate and E. Hironaka (2001) deals with the following **open problem**: Is the set of Salem numbers bounded away from 1? D.H. Lehmer (1933) found the monic polynomial

$$L(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$$

where its real root $\theta = 1.17628\ldots$ is both the smallest known Salem number.
(III) The result implies that if $\deg(\theta) \to \infty$, then $g(x) \to x$. In 2.4.4.1 Y. Dupain and J. Lesca (1973) proved (see 2.4.4.1): If $\deg(\theta) \to \infty$, then there exists a u.d. subsequence $x_{h(n)}$ such that the asymptotic density of $h(n)$ is arbitrarily close to 1.

A. AKIYAMA – Y. TANIGAWA: *Salem numbers and uniform distribution modulo* 1, Publ. Math. Debrecen **64** (2004), no. 3–4, 329–341 (MR 2058906; Zbl. 1072.11053).
M.–J. BERTIN – A. DECOMPS–GUILLOUX – M. GRANDET–HUGOT – M. PATHIAUX–DELEFOSSE – J.–P. SCHREIBER: *Pisot and Salem numbers*, Birkhäuser Verlag, Basel, 1992 (MR1187044 (93k:11095); Zbl. 0772.11041).
Y. DUPAIN – J. LESCA: *Répartition des sous-suites d'une suite donnée*, Acta Arith. **23** (1973), 307–314 (MR0319884 (**47** #8425); Zbl. 0263.10021).
E. GHATE – E. HIRONAKA: *The arithmetic and geometry of Salem numbers*, Bull. Amer. Math. Soc. (N.S.) **38** (2001), no. 3, 293–314 (MR1824892 (2002c:11137); Zbl. 0999.11064).
D.H. LEHMER: *Factorization of certain cyclotomic functions*, Ann. Math. **34** (1933), 461–469 (MR1503118; Zbl. 0007.19904).

4. Appendix

This Appendix contains some useful technical complementary results to that of Chapter 1 grouped loosely by the subject.

4.1 Technical theorems

We shall list here some important theorems from the mathematical analysis which have applications in the theory of u.d. sequences. We start with the well–known

4.1.1 Basic formulas

1. $|1-e^{2\pi ix}| = 2|\sin\pi x|$ for $x\in\mathbb{R}$,
2. $|1+e^{2\pi ix}| = 2|\cos\pi x|$ for $x\in\mathbb{R}$,
3. $|e^{2\pi ix}-e^{2\pi iy}| \le 2\pi|x-y|$ for $x,y\in\mathbb{R}$,
4. $\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi ihn\theta}\right| \le \frac{1}{N|\sin(\pi h\theta)|}$ for an irrational θ and integer $h\neq 0$, more precisely
5. $\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi ihnx}\right| = \begin{cases}\frac{|\sin\pi Nx|}{N|\sin(\pi hx)|}, & \text{if } x\notin\mathbb{Z},\\ 1, & \text{if } x\in\mathbb{Z},\end{cases}$
6. $\left|\sum_{n=M+1}^{M+N} e^{2\pi in\theta}\right| = \left|\frac{\sin(\pi\theta N)}{\sin(\pi\theta)}\right|$, $\left|\int_x^{x+T} e^{2\pi i\theta t}\,\mathrm{d}t\right| = \left|\frac{\sin(\pi\theta T)}{\pi\theta}\right|$ for irrational θ, where M is an integer and x,T are arbitrary real numbers.
7. $\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi in\mathbf{h}\cdot\boldsymbol{\theta}}\right| \le \frac{1}{2N\|\mathbf{h}\cdot\boldsymbol{\theta}\|}$ for s–dimensional non–zero $\mathbf{h}\in\mathbb{Z}^s$ and $\boldsymbol{\theta}\in\mathbb{R}^s$,
8. $\{x\} = \frac{1}{2} - \sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{\pi k}$ is the Fourier series expansion of the fractional part function $\{x\}$,
9. $\{x\} = \frac{1}{2} - \sum_{k=1}^{K}\frac{\sin(2\pi kx)}{\pi k} + \frac{\theta}{\pi(K+1)\sin(\pi x)}$ for $|\theta|\le 1$,
10. $c_{[0,x)}(t) = x + \sum_{k=1}^{\infty}\frac{\sin(2\pi kt)+\sin(2\pi k(x-t))}{\pi k}$ is the Fourier expansion of the indicator function,

11. $F_N(x) = x + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} c_k \left(\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k x_n} \right)$ where c_k are the coefficients in $c_{[0,x)}(t) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k t}$.
12. $c_{[0,x)}(\{y\}) = x + \sum_{n\neq 0} \frac{1}{2\pi i n}\left(1 - e^{-2\pi i n x}\right) e^{2\pi i n y} + \frac{1}{2} c_{\mathbb{Z}}(y) - \frac{1}{2} c_{\mathbb{Z}+x}(y)$, where $x \in [0,1]$;
13. $c_{[0,\{x-y\}}(\{x\}) = \{x-y\} - \{x\} + \{y\}$ for $x, y \in \mathbb{R}$. ... [J. Schoißengeier (1984), p. 243 and p. 250.]
14. $\left|\int_x^{x+T} e^{2\pi i \theta t}\,\mathrm{d}t\right| = \left|\frac{\sin(\pi\theta T)}{\pi\theta}\right|$, for irrational θ.

J. Schoissengeier: *On the discrepancy of* $(n\alpha)$, Acta Arith. **44** (1984), 241–279 (MR0774103 (86c:11056); Zbl. 0506.10031).

4.1.2 Continued fractions

1. $q_i\alpha - p_i = \frac{(-1)^i}{q_i r_{i+1} + q_{i-1}}$, where $\alpha = [a_0; a_1, a_2, \dots]$ is the continued fraction expansion of α with partial quotients a_i and convergents p_i/q_i;
2. $\frac{p_i}{q_i} = [a_0; a_1, \dots, a_i]$, $p_0 = a_0$, $q_0 = 1$, $p_{-1} = 1$, $q_{-1} = 0$,
3. $r_{i+1} = [a_{i+1}; a_{i+2}, \dots]$, $\frac{q_{i-1}}{q_i} = [0 : a_i, a_{i-1}, \dots, a_1]$,
4. $\{q_i\alpha\} = \begin{cases} |q_i\alpha - p_i| & \text{if } 2 | i, \\ 1 - |q_i\alpha - p_i| & \text{if } 2 \nmid i, \end{cases}$
   see [A. Ya. Khintchine (1963)];
5. $\{(n+q_i)\alpha\} = \{n\alpha\} + q_i\alpha - p_i$ for $n \le q_i$
   [L. Roçadas (2008)];
6. $\{j\alpha\} < \{k\alpha\} \Leftrightarrow \{j(p_{m+1}/q_{m+1})\} < \{k(p_{m+1}/q_{m+1})\}$ if $\max(|j|, |k|, |j-k|) < q_{m+1}$
   [J. Schoißengeier (1984)];
7. $|q_{k-1}\alpha - p_{k-1}| = \frac{1}{q_k} + O\left(\frac{1}{a_{k+1}q_k}\right)$
   [J. Schoißengeier (1987)], more precisely
8. $|q_{k-1}\alpha - p_{k-1}| = \frac{1}{q_k} - \frac{1}{q_k a_k r_{k+1}\left(1 + \frac{1}{a_k r_{k+1}} + \frac{q_{k-2}}{a_k q_{k-1}}\right)}$.

A. Ya. Khintchine (A.J. Chinčin): *Continued Fractions*, P. Noordhoff, Ltd., Groningen, the Netherlands, 1963 (MR0161834 (**28** #5038); Zbl 0117.28503), another translation into English (MR0161833 (**28** #5037); Zbl 0117.28601)). (Russian 2nd. ed.: Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow–Leningrad, 1949 (MR0044586 (13,444e)); German edition B. G. Teubner Verlagsgesellschaft, Leipzig, 1956 (MR0080630 (18,274f); Zbl. 0071.03601)).

L. Roçadas: *Bernoulli polynomials and* $(n\alpha)$*-sequences*, Unif. Distrib. Theory **3** (2008), no. 1, 127–148 (MR2475821 (2009m:11118); Zbl. 1174.11061).
J. Schoissengeier: *On the discrepancy of* $(n\alpha)$, Acta Arith. **44** (1984), 241–279 (MR0774103 (86c:11056); Zbl. 0506.10031).
J. Schoissengeier: *Eine Explizite Formel für* $\sum_{n\le N} B_2(\{n\alpha\})$, in: *Zahlentheoretische Analysis II (Seminar, Wien, 1984–86)*, (E. Hlawka eds.), Lecture Notes in Mathematics, 1262, Springer-Verlag, Berlin-Heidelberg, 1987, pp. 134–138 (MR1012966 (90j:11021); Zbl 0622.10007).

4.1.3 Fractional parts of $n\alpha$

(I) Let α be irrational, I be an interval in $[0,1]$ and assume that $\{n\alpha\} \in I$. Using the notation from Fig. 1 we have

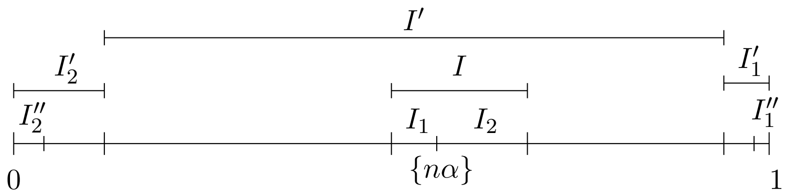

Figure 1:

$$
\begin{aligned}
\{(n+k)\alpha\} \in I_1 &\Leftrightarrow \{k\alpha\} \in I'_1,\\
\{(n+k)\alpha\} \in I_2 &\Leftrightarrow \{k\alpha\} \in I'_2,\\
\{(k+n)\alpha\} \in I &\Leftrightarrow \{k\alpha\} \notin I'.
\end{aligned}
$$

(II) The intervals I_1, I_2, I'_1, I'_2 can be replaced by a parameter $0 < t < 1$ in the form that for every $n, k \in \mathbb{N}$ and every $0 < t < 1$ we have

$$
\begin{aligned}
0 < \{n\alpha\} - \{(n+k)\alpha\} = t &\Leftrightarrow 1 - \{k\alpha\} = t,\\
0 < \{(n+k)\alpha\} - \{n\alpha\} = t &\Leftrightarrow \{k\alpha\} = t.
\end{aligned}
$$

(III) Let $I \subset [0,1]$ be an interval and $k \in \mathbb{N}$. Then both numbers $\{n\alpha\}$ and $\{(n+k)\alpha\}$ lie in I if and only if the sawtooth graph of the function $y = x + \{k\alpha\} \bmod 1$ intersects $I \times I$ and simultaneously $\{n\alpha\}$ lies in the projection of this intersection onto the x-axis, see Fig. 2.
(IV) Given an interval I of the form $I = (0,t)$, $t \le 1/2$, define a and b as the least positive integers such that $\{a\alpha\} \in (0,t)$ and $\{b\alpha\} \in (1-t,1)$. Let $\{n\alpha\} \in (0,t)$ and let k be minimal with $\{(n+k)\alpha\} \in (0,t)$. Then

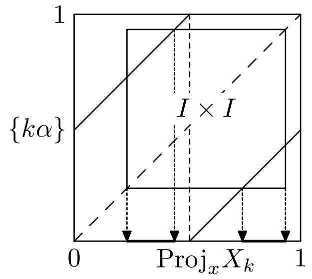

Figure 2:

$t - \{a\alpha\} \le 1 - \{b\alpha\}$ and

$$k = \begin{cases} a, & \text{if } 0 < \{n\alpha\} < t - \{a\alpha\}, \\ a+b, & \text{if } t - \{a\alpha\} < \{n\alpha\} < 1 - \{b\alpha\}, \\ b, & \text{if } 1 - \{b\alpha\} < \{n\alpha\} < t. \end{cases}$$

Moreover a and b are relatively prime.

NOTES: (I), (II) and (III) are due to Š. Porubský and O. Strauch (2010). (IV) is from N.B. Slater (1950, 1967). Š. Porubský and O. Strauch (2010) also give a formula for k similar to the above one for intervals $|I| > 1/2$.

N.B. SLATER: *The distribution of the integers N for which $\{\theta N\} < \phi$*, Proc. Cambridge Philos. Soc. **46** (1950), 525–534 (MR0041891 (13,16e); Zbl. 0038.02802).

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Š. PORUBSKÝ – O. STRAUCH: *Binary sequences generated by sequences $\{n\alpha\}$, $n = 1, 2, \dots$*, Publ. Math. **77** (2010), No. 1-2, 139-170 (MR2675740 (2011f:11092)).

4.1.4 Summation formulas

The following summation formulas are well–known:

Theorem 4.1.4.1 (Euler summation formula). *If $F(t)$ is a complex valued function with a continuous derivative on the interval $[1, N]$, then*

$$\sum_{n=1}^{N} F(n) = \int_1^N F(t)\,\mathrm{d}t + \tfrac{1}{2}(F(1) + F(N)) + \int_1^N (\{t\} - \tfrac{1}{2})F'(t)\,\mathrm{d}t.$$

Cf. [KN, p. 8, formula (2.3)].

Theorem 4.1.4.2 (Sonin summation formula). *Let $F(t)$ be twice continuously differentiable on the interval $(a, b]$, and*

$$\rho(x) = -\{x\} + \frac{1}{2}, \qquad \sigma(x) = \int_0^x \rho(t)\,\mathrm{d}t.$$

Then

$$\sum_{a<n\le b} F(n) = \int_a^b F(x)\,\mathrm{d}x + \rho(b)F(b) - \rho(a)F(a) - \sigma(b)F'(b) + \sigma(a)F'(a) + \\ + \int_a^b \sigma(x)F''(x)\,\mathrm{d}x.$$

Note that $|\rho(x)| \le 1/2$ and $|\sigma(x)| \le 1/8$. Cf. I.M. Vinogradov (1985, p. 37). The above two formulas are special cases of the following one, cf. for instance E. Hlawka, J. Schoissengeier and R. Taschner (1991, pp. 104–5):

Theorem 4.1.4.3 (Euler-McLaurin summation formula). *Let $B_n(x)$ be the nth Bernoulli polynomial. Suppose that $F : [a, b] \to \mathbb{C}$ is q times differentiable with $\int_a^b |F^{(q)}(x)|\,\mathrm{d}x < \infty$. Then for every m, $1 \le m \le q$, we have*

$$\sum_{a<n\le b} F(n) = \\ = \int_a^b F(x)\,\mathrm{d}x + \sum_{k=1}^{m} \frac{(-1)^k}{k!}\Big(B_k(\{b\})F^{(k-1)}(b) - B_k(\{a\})F^{(k-1)}(a)\Big) + \\ + \frac{(-1)^{m+1}}{m!}\int_a^b B_m(\{x\})F^{(m)}(x)\,\mathrm{d}x.$$

If $a, b \in \mathbb{Z}$, then the second sum can also written in the form

$$\sum_{k=1}^{m} \frac{(-1)^k}{k!} B_k\big(F^{(k-1)}(b) - F^{(k-1)}(a)\big).$$

NOTES: Let $B_n = B_n(0)$ be the nth Bernoulli number. The $B_n(x)$ and B_n can be determined using the following recurrence relation:

$$B_n(x) = \sum_{k}^{n} \binom{n}{k} B_k x^{n-k}, \qquad B_n = \sum_{j=1}^{n} \frac{1}{j+1} \sum_{i=1}^{j} (-1)^i \binom{j}{i} i^n, B_0 = 1.$$

The next formula can be found, for instance, in E. Hlawka, J. Schoissengeier and R. Taschner (1991, p. 78):

Theorem 4.1.4.4 (Abel partial summation). *If $f, h : \mathbb{Z}^+ \to \mathbb{C}$ and P, Q are integers with $P \le Q$ then*

$$\sum_{n=P}^{Q} f(n)h(n) = f(Q+1)\sum_{n=P}^{Q} h(n) + \sum_{n=P}^{Q} (f(n) - f(n+1)) \sum_{m=P}^{n} h(m).$$

If moreover, $f : [1,\infty) \to \mathbb{R}$ *has continuous derivative* f' *and* $g(x) = \sum_{n=1}^{[x]} h(n)$*, then the above formula can also be expressed in the form*

$$\sum_{n=1}^{[x]} f(n)h(n) = f(x)g(x) - \int_1^x g(t)f'(t)\,\mathrm{d}t$$

for every $x \in [1,\infty)$.

The exponential sums can be handled using the estimate, cf. (KN, p. 17, Th. 2.7):

Theorem 4.1.4.5 (van der Corput lemma). *Let* a *and* b *be integers with* $a < b$*, and let* f *be twice differentiable on* $[a,b]$ *with* $f''(x) \geq \varrho > 0$ *or* $f''(x) \leq -\varrho < 0$ *for* $x \in [a,b]$*. Then,*

$$\left|\sum_{n=a}^{b} e^{2\pi i f(n)}\right| \leq \left(|f'(b) - f'(a)| + 2\right)\left(\frac{4}{\sqrt{\varrho}} + 3\right).$$

For the summation of $e^{2\pi i f(n)}$ can often be useful also the next result (cf. E.C. Titchmarch (1986, Lemma 4.7)):

Theorem 4.1.4.6. *Let* $f(x)$ *be a real function with continuous and decreasing derivative* $f'(x)$ *on* (a,b) *and put* $f'(b) = A$*,* $f'(a) = B$*. Then*

$$\sum_{a<n\leq b} e^{2\pi i f(n)} = \sum_{A-c<k<B+c} \int_a^b e^{2\pi i(f(x)-kx)}\,\mathrm{d}x + \mathcal{O}(\log(B - A + 2)),$$

where c *is any positive constant less than* 1.

cf. [KN, p. 25, Lemma 3.1] and B. Massé and D. Schneider (2014):

Theorem 4.1.4.7 (van der Corput's Fundamental Inequality). *Let* N *be a positive integer gretaer than* 1*,* $a_1,\ldots,a_N$ *be* N *complex numbers of modulus* 1*. Then there exists an absolute constant* C *such that for all positive integer* $H < N$ *we have*

$$\left|\frac{1}{N}\sum_{n=1}^{N} a_n\right|^2 \leq \frac{C}{H} + \frac{C}{H}\sum_{h=1}^{H}\left|\frac{1}{N-h}\sum_{n=1}^{N-h} a_n\overline{a}_{n+h}\right|.$$

Tsuji's extension:

Theorem 4.1.4.8. *Let* N *be a positive integer greater than* 1*,* $a_1,\ldots,a_N$ *be* N *complex numbers of modulus* 1*, let* w_n *be a sequence of positive weights* *and* $W_n = w_1 + \cdots + w_N$. *Then for all positive integer* $H < N$ *we have*

$$\frac{\left|\sum_{n=1}^{N} w_n a_n\right|^2}{W_{N+H-1}} \le \frac{1}{H^2}\sum_{n=1}^{N} w_n^2 |a_n|^2 \sum_{j=0}^{H-1} \frac{1}{w_{n+j}} + 2\Re\left(\frac{1}{H^2}\sum_{h=1}^{H-1}\sum_{n=1}^{N_h} w_n w_{n+h} a_n \overline{a}_{n+h} \sum_{j=0}^{H-1}\frac{1}{w_{n+j}}\right).$$

The following estimate play a central role in the quantitative theory of u.d. (cf. H. Niederreiter and W. Philipp (1973) and H. Niederreiter (1978)): Let

$$\widehat{F}(h) = \int_0^1 e^{2\pi i h t}\,\mathrm{d}F(t) \quad \text{for every integer } h$$

be the the Fourier – Stieltjes transform of a function $F : [0,1] \to \mathbb{R}$ with bounded variation.

Theorem 4.1.4.9. *Let* F *be a d.f. and let* $G : [0,1] \to \mathbb{R}$ *satisfy the Lipschitz condition* $|G(u) - G(v)| \le L|u - v|$ *for* $u, v \in [0,1]$, *and* $G(0) = 0$ *and* $G(1) = 1$. *Then, for any positive integer* m, *we have*

$$\sup_{u,v\in[0,1]} \left|(F(v) - F(u)) - (G(v) - G(u))\right| \le \le \frac{4L}{m+1} + \frac{4}{\pi}\sum_{h=1}^{m}\left(\frac{1}{h} - \frac{1}{m+1}\right)\left|\widehat{F}(h) - \widehat{G}(h)\right|,$$

or also

$$\sup_{u,v\in[0,1]} \left|(F(v) - F(u)) - (G(v) - G(u))\right| \le \left(\frac{6L}{\pi^2}\sum_{h=1}^{\infty}\frac{1}{h^2}\left|\widehat{F}(h) - \widehat{G}(h)\right|^2\right)^{1/3}.$$

NOTES: Application of this theorem to the step d.f. $F(t) = F_N(t) = \frac{A([0,t);N;x_n)}{N}$ (cf. 1.3) and function $G(t) = t$ yields the Erdős – Turán theorem 1.9.0.8, and thus a discrepancy bound for D_N. The second formula of this Theorem gives immediately the Le Veque theorem 1.9.0.7, and can be found in H. Niederreiter (1975), also cf. H. Niederreiter (1978, pp. 974, 976).

The next formula can be used to express a discrete sum in terms of Riemann – Stieltjes integral.

Theorem 4.1.4.10. *If $x_1, x_2, \ldots, x_N$ is a finite sequence from $[0,1]$ and $f : [0,1] \to \mathbb{R}$ a continuous function then*

$$\frac{1}{N}\sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\,\mathrm{d}F_N(x),$$

where $F_N(x)$ is again the step d.f. given by (cf. 1.3)

$$F_N(x) = \frac{A([0,x);N;x_n)}{N} \quad \text{for } x \in (0,1),$$

and $F_N(0) = 0$ and $F_N(1) = 1$.

NOTES: This theorem is also valid for the Riemann integrable functions f for which none of $x_1, \ldots, x_N$ is a point of its discontinuity.

It is necessary to take into account the possible jumps of $F_N(x)$ at limit points 0 and 1 of the integration, e.g. the integration limits could be $\int_{0-0}^{1+0}$. For the Riemann – Stieltjes integrals the integration method by parts can be used:

Theorem 4.1.4.11. *Let f and g be two functions defined on $[0,1]$. The following Riemann – Stieltjes integrals exist simultaneously and*

$$\int_0^1 f(x)\,\mathrm{d}g(x) = [f(x)g(x)]_0^1 - \int_0^1 g(x)\,\mathrm{d}f(x).$$

Remember that the Riemann – Stieltjes integrals are undefined if f and g have a jump at a common point x.

NOTES: (I) A short account of the theory of Riemann – Stieltjes integration can be found in the book H. Riesel (1985, pp. 358–367, Appendix 9).

(II) The by parts method also yields a proof for the Euler's summation formula: $\sum_{k=m}^{n} f(k) = \int_{m-0}^{n+0} f(x)\,\mathrm{d}[x] = \int_{m-0}^{n+0} f(x)\,\mathrm{d}x - \int_{m-0}^{n+0} f(x)\,\mathrm{d}(x-[x]) = \int_m^n f(x)\,\mathrm{d}x - \int_{m-0}^{n+0} \mathrm{d}(x-[x]-(1/2)) = \int_m^n f(x)\,\mathrm{d}x - [f(x)(x-[x]-(1/2)]_{m-0}^{n+0} + \int_m^n (x-[x]-(1/2))f'(x)\,\mathrm{d}x = \int_m^n f(x)\,\mathrm{d}x + f(m)/2 + f(n)/2 + \int_m^n (x-[x]-(1/2))f'(x)\,\mathrm{d}x$.

(III) S.K. Zaremba (1968) found the following variant of the integration by parts: Let $f(\mathbf{x})$ and $g(\mathbf{x})$ be periodic with the unit period in each of the s coordinates of $\mathbf{x}$. It suffices to assume that one of these functions is continuous and the other is of bounded variation in the sense of Vitali over $[0,1]^s$ (see p. 1 – 73), then

$$\int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}g(\mathbf{x}) = (-1)^s \int_{[0,1]^s} g(\mathbf{x})\,\mathrm{d}f(\mathbf{x}).$$

In the proofs of some integral equations referred to in 4.2 the Helly theorems are systematically used:

Theorem 4.1.4.12 (First Helly theorem). *Any sequence g_n of d.f.'s contains a subsequence g_{k_n} such that the sequence $g_{k_n}(x)$ converges for every $x \in [0,1]$ and its point limit $\lim_{n\to\infty} g_{k_n}(x) = g(x)$ is also a d.f.*
Theorem 4.1.4.13 (Second Helly theorem). *If we have $\lim_{n\to\infty} g_n(x) = g(x)$ a.e. on $[0,1]$, then for the s–dimensional integral of a continuous function $f : [0,1]^s \to \mathbb{R}$ we have*

$$\lim_{n\to\infty} \int_0^1 \dots \int_0^1 f(t_1,\dots,t_s)\,\mathrm{d}g_n(t_1)\dots \mathrm{d}g_n(t_s) = \\ = \int_0^1 \dots \int_0^1 f(t_1,\dots,t_s)\,\mathrm{d}g(t_1)\dots \mathrm{d}g(t_s).$$

The Second Helly theorem is occasionally also called the **Helly – Bray Theorem** (cf. R.G. Laha and V.K. Rohatgi (1979, p. 135, Th. 3.1.3). One of the most important applications of this theorem is the following result:
Theorem 4.1.4.14. *For every sequence x_n in $[0,1]$ and any increasing sequence of indices N_k, $k = 1,2,\dots,$ with $\lim_{k\to\infty} F_{N_k} = g(x)$ a.e. we have*

$$\lim_{k\infty} \frac{1}{N_k^s} \sum_{i_1,\dots,i_s=1}^{N_k} f(x_{i_1},\dots,x_{i_s}) = \int_0^1 \dots \int_0^1 f(t_1,\dots,t_s)\,\mathrm{d}g(t_1)\dots \mathrm{d}g(t_s).$$

If $I = \{i_1,\dots,i_l\} \subset \{1,2,\dots,s\}$ is a non–empty set of indices and $\mathbf{x} = (x_1,\dots,x_s)$ a given vector, then $\mathbf{x}_I$ will denote the vector $(x_{i_1},\dots,x_{i_l})$. Further, if $g(\mathbf{x})$ is an s–dimensional d.f. then the face d.f. $g_I(\mathbf{x})$ is defined by (see 1.11)

$$g_I(\mathbf{x}) = g(1,\dots,1,x_{i_1},1,\dots,1,x_{i_2},1,\dots,1,x_{i_l},1\dots,1)).$$

Theorem 4.1.4.15 (The multi–dimensional second Helly theorem). *Let $f : [0,1]^s \to \mathbb{R}$ be a bounded function, $g_n(\mathbf{x})$, $n = 1,2,\dots,$ and $g(\mathbf{x})$ be s–dimensional d.f.'s (for the def. see 1.11). If $\lim_{n\to\infty}(g_n)_I(\mathbf{x}) = g_I(\mathbf{x})$ at every common continuity point $\mathbf{x}_I \in (0,1)^l$, $l = 1,2,\dots,s$, of $(g_n)_I(\mathbf{x})$, $n = 1,2,\dots,$ and $g_I(\mathbf{x})$, then*

$$\lim_{n\to\infty} \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}g_n(\mathbf{x}) = \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}g(\mathbf{x})$$

provided the all Riemann – Stieltjes integrals exist.

Theorem 4.1.4.16 (Lebesgue theorem on dominant convergence). *If* f_n *for* $n = 1, 2, \ldots,$ *and* g *are Lebesgue integrable on* $[0, 1]$ *then* $\lim_{n\to\infty} f_n(x) = f(x)$ *a.e., and* $|f_n(x)| \le g(x)$ *a.e. over* $[0, 1]$ *for* $n = 1, 2, \ldots,$ *imply that* f *is Lebesgue integrable and*

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x.$$

If f, g are Riemann integrable functions then the following mean value theorems are well–known (cf. for instance Ryshik and Gradstein (1957, pp. 129–130):

Theorem 4.1.4.17 (The first mean value theorem). *Suppose that* $f(x)$ *is continuous throughout the interval* $(0, 1)$ *such that* $m \le f(x) \le M$ *for* $x \in (0, 1)$*, and* $g(x)$ *is integrable over that interval, and that* $g(x)$ *does not change its sign in the interval* $(0, 1)$*. Then there exists at least one* $\xi \in [0, 1]$ *such that*

$$\int_0^1 f(x)g(x)\,\mathrm{d}x = f(\xi)\int_0^1 g(x)\,\mathrm{d}x.$$

Theorem 4.1.4.18 (The second mean value theorem). *Let* $f(x)$ *be a non–negative function on the interval* $(0, 1)$*, and* $g(x)$ *be integrable over the same interval.*

(1) If $f(x)$ *is non–increasing function throughout the interval* $(0, 1)$ *then there exists at least one* $\xi \in [0, 1]$ *such that*

$$\int_0^1 f(x)g(x)\,\mathrm{d}x = f(0)\int_0^\xi g(x)\,\mathrm{d}x.$$

(2) If $f(x)$ *is non–decreasing then for some* $\xi \in [0, 1]$

$$\int_0^1 f(x)g(x)\,\mathrm{d}x = f(1)\int_\xi^1 g(x)\,\mathrm{d}x$$

(3) If the function $f(x)$ *is monotonic then for some* $\xi \in [0, 1]$

$$\int_0^1 f(x)g(x)\,\mathrm{d}x = f(0)\int_0^\xi g(x)\,\mathrm{d}x + f(1)\int_\xi^1 g(x)\,\mathrm{d}x,$$

or generally

$$\int_0^1 f(x)g(x)\,\mathrm{d}x = A\int_0^\xi g(x)\,\mathrm{d}x + B\int_\xi^1 g(x)\,\mathrm{d}x,$$

where $A \ge f(0+0)$ *and* $B \le f(1-0)$ *if* f *is decreasing, and* $A \le f(0+0)$ *and* $B \ge f(1-0)$ *if* f *is increasing.*

Finally, we add the following well–known theorem from the elementary analysis which has many applications in the theory of u.d. (see e.g. 2.22.1, 2.6.18) and which is known under different names, e.g. as Stolz's or Cesàro's theorem, or Cauchy – Stolz theorem [1]

Theorem 4.1.4.19. *If the real–valued sequences x_n and y_n, $n = 1, 2, \dots$, satisfy at least one of the conditions:*
(i) y_n is strictly monotone, $|y_n| \to \infty$,
(ii) y_n is strictly monotone, $x_n \to 0$, $y_n \to 0$,
and if the limit (finite or infinite) $\lim_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n}$ exists, then the limit of the sequence $\frac{x_n}{y_n}$ also exists and

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n}.$$

NOTES: O. Stolz (1888). For application in continued fraction transformations cf. V.L. Danilov *et al.* (1961, p. 272).

Pick's Theorem. $\mathcal{P}$ be a lattice polygon, $B(\mathcal{P})$ denote the number of lattice points on the boundary of $\mathcal{P}$, $I(\mathcal{P})$ denote the number of lattice points inside $\mathcal{P}$ and $A(\mathcal{P})$ denote the area of $\mathcal{P}$. Then every simple lattice polygon $\mathcal{P}$ satisfies

$$\frac{1}{2}B(\mathcal{P}) + I(\mathcal{P}) = A(\mathcal{P}) + 1.$$

(I) G. Pick (1899); J. Beck [p. 27](2014); H. Steinhaus [p. 96](1983); M. Krebs and Th. Wright (2010);
(II) There are some beautiful higher-dimensional extensions of Pick's formula based upon deep work in combinatorial algebraic geometry, in particular around toric varieties. For a readable introduction see R. Morelli (1993).
(III) Another simple result, J. Beck [p. 28](2014): Let $A \subset \mathbb{R}^2$ be a Lebesgue measurable set in the plane with finite measure (that we call the area). Then

$$\int_0^1 \int_0^1 \#\{(A + \mathbf{x}) \cap \mathbb{Z}^2\} \, \mathrm{d}\mathbf{x} = \mathrm{area}(A).$$

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[1] A.L. Cauchy used this theorem for $y_n = n$.

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4.2 Integral identities

The following list of integrals from O. Strauch (1999, pp. 132–135) and his papers (1989, 1990, [a]1994, [b]1994, 1997, 2000) may be instrumental:

(I) For every d.f. g, $\widetilde{g}$, g_1, g_2, g_3, and g_4 we have:

$$\int_0^1\int_0^1 -\frac{|x-y|}{2}\,\mathrm{d}(g_1(x)-g_2(x))\,\mathrm{d}(g_3(y)-g_4(y)) = \\ = \int_0^1 (g_1(x)-g_2(x))(g_3(x)-g_4(x))\,\mathrm{d}x,$$

consequently (cf. O. Strauch (1989, p. 130)[2]

$$\int_0^1\int_0^1 -\frac{|x-y|}{2}\,\mathrm{d}(g(x)-\widetilde{g}(x))\,\mathrm{d}(g(y)-\widetilde{g}(y)) = \int_0^1 (g(x)-\widetilde{g}(x))^2\,\mathrm{d}x$$

and thus

$$\int_0^1 (g(x)-\tilde{g}(x))^2\,\mathrm{d}x = \int_0^1\int_0^1 |x-y|\,\mathrm{d}g(x)\,\mathrm{d}\tilde{g}(x) \\ -\frac{1}{2}\int_0^1\int_0^1 |x-y|\,\mathrm{d}g(x)\,\mathrm{d}g(x) - \frac{1}{2}\int_0^1\int_0^1 |x-y|\,\mathrm{d}\tilde{g}(x)\,\mathrm{d}\tilde{g}(x).$$

Similarly

$$\int_0^1\int_0^1 |x-y|\,\mathrm{d}g(x)\,\mathrm{d}\widetilde{g}(y) = \int_0^1 g(x)\,\mathrm{d}x + \int_0^1 \widetilde{g}(x)\,\mathrm{d}x - 2\int_0^1 g(x)\widetilde{g}(x)\,\mathrm{d}x,$$

or in a special case (cf. O. Strauch ([a]1994, p. 178))

$$\int_0^1\int_0^1 |x-y|\,\mathrm{d}g(x)\,\mathrm{d}g(y) = \\ = 2\left(\int_0^1 g(x)\,\mathrm{d}x - \int_0^1 g^2(x)\,\mathrm{d}x\right) = \\ = 2\int_0^1 \left(\int_0^x g(t)\,\mathrm{d}t\right)\mathrm{d}g(x).$$

In the case of restricted integral range ($0 \le \alpha \le 1$) we have

[2]The multidimensional integrals of the type $\int\int |\mathbf{x}-\mathbf{y}|^\alpha\,\mathrm{d}g(\mathbf{x})\,\mathrm{d}g(\mathbf{y})$ were studied by R. Alexander and K.B. Stolarsky (1974), R. Alexander (1991) and others.

$$\int_0^\alpha\int_0^\alpha |x-y|\,\mathrm{d}g(x)\,\mathrm{d}g(y) = 2\left(g(\alpha)\int_0^\alpha g(x)\,\mathrm{d}x - \int_0^\alpha g^2(x)\,\mathrm{d}x\right).$$

For $0\le\alpha\le\beta\le 1$ we have

$$\begin{aligned}\int_0^1\int_0^1 |x\alpha-y\beta|\,\mathrm{d}g(x)\,\mathrm{d}g(y) &= \\ &= 2\beta\int_0^{\alpha/\beta} g(x)\,\mathrm{d}x + (\beta-\alpha)\left(1-\int_0^1 g(x)\,\mathrm{d}x\right) - \\ &\quad - 2\alpha\int_0^1 g(x)g\left(\frac{x\alpha}{\beta}\right)\mathrm{d}x - \alpha\beta\left(\int_0^1 g(x)\,\mathrm{d}x\right)^2 + \\ &\quad + \alpha\beta\int_0^1 g(x)\,\mathrm{d}x\int_0^{\alpha/\beta} g(x)\,\mathrm{d}x.\end{aligned}$$

O. Strauch (1990, p. 251) proved that

$$\begin{aligned}&\int_0^1\int_0^1 |x-y|^k\,\mathrm{d}(g(x)-x)\,\mathrm{d}(g(y)-y) \\ &= \begin{cases} 0, & \text{if } k=0,\\ -2\int_0^1 (g(x)-x)^2\,\mathrm{d}x, & \text{if } k=1,\\ -k(k-1)\int_0^1\int_0^1 (g(x)-x)(g(y)-y)|x-y|^{k-2}\,\mathrm{d}x\,\mathrm{d}y, & \text{if } k\ge 2,\end{cases}\end{aligned}$$

and that

$$\int_0^1\int_0^1 |x-y|^k\,\mathrm{d}(g(x)-x)\,\mathrm{d}y = -\int_0^1 (g(x)-x)(x^k-(1-x)^k)\,\mathrm{d}x.$$

(II) If $f:[0,1]\to[0,1]$ and $H:[0,1]^2\to\mathbb{R}$ are continuous functions then for

$$g_f(x) = \int_{f^{-1}([0,x))} 1\,\mathrm{d}g(u),$$

we have the following known integral transforms

$$\int_0^1\int_0^1 H(x,y)\,\mathrm{d}g_f(x)\,\mathrm{d}g_f(y) = \int_0^1\int_0^1 H(f(x),f(y))\,\mathrm{d}g(x)\,\mathrm{d}g(y).$$

If $f:[0,1]^2\to[0,1]$ is continuous and $g_f(x)=\int_{f^{-1}([0,x))} 1\,\mathrm{d}g(u)\,\mathrm{d}g(v)$ then

$$\begin{aligned}&\int_0^1\int_0^1 H(x,y)\,\mathrm{d}g_f(x)\,\mathrm{d}g_f(y) = \\ &\quad = \int_0^1\int_0^1\int_0^1\int_0^1 H(f(x,y),f(u,v))\,\mathrm{d}g(x)\,\mathrm{d}g(y)\,\mathrm{d}g(u)\,\mathrm{d}g(v)\end{aligned}$$

and in the special case

$$\int_0^1 h(f(x))\,\mathrm{d}g(x) = \int_0^1 h(x)\,\mathrm{d}g_f(x).$$

(III) If

$$F_{\widetilde{g}}(x,y) = \int_0^1 \widetilde{g}^2(t)\,\mathrm{d}t - \int_x^1 \widetilde{g}(t)\,\mathrm{d}t - \int_y^1 \widetilde{g}(t)\,\mathrm{d}t + 1 - \max(x,y),$$

then (cf. O. Strauch ([b]1994, p. 618))

$$\int_0^1 (g(x) - \widetilde{g}(x))^2\,\mathrm{d}x = \int_0^1 \int_0^1 F_{\widetilde{g}}(x,y)\,\mathrm{d}g(x)\,\mathrm{d}g(y)$$

or more generally

$$\int_0^1 (g_1(x) - \widetilde{g}(x))(g_2(x) - \widetilde{g}(x))\,\mathrm{d}x = \int_0^1 \int_0^1 F_{\widetilde{g}}(x,y)\,\mathrm{d}g_1(x)\,\mathrm{d}g_2(y).$$

(For the proof compute $\int_0^1 \int_0^1 F_{\widetilde{g}}(x,y)\,\mathrm{d}(g_1(x)+g_2(y))\,\mathrm{d}(g_1(y)+g_2(y))$.)

$$\int_0^1 (g_f(x) - \widetilde{g}_f(x))^2\,\mathrm{d}x = \int_0^1 \int_0^1 F_{\widetilde{g}_f}(f(x), f(y))\,\mathrm{d}g(x)\,\mathrm{d}g(y).$$

(IV) If

$$\begin{aligned} &F_{f,h}(x,y) = \\ &= \max(f(x),h(y)) + \max(f(y),h(x)) - \max(f(x),f(y)) - \max(h(x),h(y)) = \\ &\qquad = \frac{1}{2}\left(|f(x)-h(y)| + |f(y)-h(x)| - |f(x)-f(y)| - |h(x)-h(y)|\right), \end{aligned}$$

then (cf. O. Strauch ([b]1994, p. 628)

$$\int_0^1 (g_f(x) - g_h(x))^2\,\mathrm{d}x = \int_0^1 \int_0^1 F_{f,h}(x,y)\,\mathrm{d}g(x)\,\mathrm{d}g(y).$$

There follows from the above that

$$\int_0^1 \int_0^1 F_{f,h}(x,y)\,\mathrm{d}g(x)\,\mathrm{d}\widetilde{g}(y) = \int_0^1 \big(g_f(x) - g_h(x)\big)\big(\widetilde{g}_f(x) - \widetilde{g}_h(x)\big)\,\mathrm{d}x$$

and
$$\int_0^1 g_f^2(x)\,\mathrm{d}x = \int_0^1\int_0^1 \big(1-\max(f(x),f(y))\big)\,\mathrm{d}g(x)\,\mathrm{d}g(y).$$
O. Strauch (2000, p. 427) proved that
$$\iint_{0\le x\le y\le 1}\Big(\big(g_f(y)-g_f(x)\big)-\big(g_h(y)-g_h(x)\big)\Big)^2\,\mathrm{d}x\,\mathrm{d}y =$$
$$= \int_0^1\int_0^1 F_{f,h}^{(1)}(x,y)\,\mathrm{d}g(x)\,\mathrm{d}g(y),$$
where
$$F_{f,h}^{(1)}(x,y) = F_{f,h}(x,y) - \big(f(x)-h(x)\big)\big(f(y)-h(y)\big).$$
This follows from the fact that the integral on the right-hand side is equal to
$$\int_0^1 \big(g_f(x)-g_h(x)\big)^2\,\mathrm{d}x - \left(\int_0^1 \big(g_f(x)-g_h(x)\big)\,\mathrm{d}x\right)^2$$
and $\int_0^1 g_f(x)\,\mathrm{d}x = 1-\int_0^1 f(x)\,\mathrm{d}g(x)$ gives
$$\left(\int_0^1 \big(g_f(x)-g_h(x)\big)\,\mathrm{d}x\right)^2 = \int_0^1\int_0^1 \big(f(x)-h(x)\big)\big(f(y)-h(y)\big)\,\mathrm{d}g(x)\,\mathrm{d}g(y).$$
(V) If g_1 is a strictly increasing solution of $g_f = \widetilde{g}_f$ (with $\widetilde{g}$ fixed) and $f'(x)$ is continuous, then (cf. Strauch (2000, p. 437, Th. 4))
$$\int_0^1 \big(g_f(x)-\widetilde{g}_f(x)\big)^2\,\mathrm{d}x = \int_0^1 \big(g(x)-g_1(x)\big)f'(x)\big(g_f(f(x)-\widetilde{g}_f(f(x))\big)\,\mathrm{d}x.$$
On the other hand, if g_1 is a strictly increasing solution of $g = g_f$, then
$$\int_0^1 \big(g(x)-g_f(x)\big)^2\,\mathrm{d}x =$$
$$= \int_0^1 \big(g(x)-g_1(x)\big)\big(g(x)-g_f(x)+f'(x)\big(g_f(f(x))-g(f(x))\big)\big)\,\mathrm{d}x,$$
and it is also true that
$$\int_0^1 \big(g(x)-g_f(x)\big)^2\,\mathrm{d}x = \int_0^1\int_0^1 F_{g_f}(x,y)\,\mathrm{d}g(x)\,\mathrm{d}g(y).$$
(VI) Let $\psi(y) = a(x)y^2 + b(x)y + c(x)$ be a polynomial in the variable y, where $a(x)$, $b(x)$ and $c(x)$ are integrable functions in $[0,1]$ and put
$$F(x,y) = \int_{\max(x,y)}^1 a(t)\,\mathrm{d}t + \frac{1}{2}\int_x^1 b(t)\,\mathrm{d}t + \frac{1}{2}\int_y^1 b(t)\,\mathrm{d}t + \int_0^1 c(t)\,\mathrm{d}t.$$

Then (cf. O. Strauch (1997, p. 219, Lemma 5))

$$\int_0^1 \psi(g(x))\,\mathrm{d}x = \int_0^1 \int_0^1 F(x,y)\,\mathrm{d}g(x)\,\mathrm{d}g(y)$$

for every d.f. $g(x)$.
(VII) Given a finite sequence $x_1, x_2, \dots, x_N$ in $[0,1)$, a d.f. $g(x)$, and a continuous $f : [0,1] \to \mathbb{R}$, let $F_N(x) = \frac{A([0,x);N;x_n)}{N}$. Then

$$\frac{1}{N}\sum_{n=1}^{N} f(x_n) - \int_0^1 f(x)\,\mathrm{d}g(x) = -\int_0^1 (F_N(x) - g(x)\,\mathrm{d}f(x)$$

which implies

$$\sum_{n=1}^{N} f(x_n) = N\left(\int_0^1 f(x)\,\mathrm{d}g(x) - \int_0^1 (F_N(x) - g(x))\,\mathrm{d}f(x)\right).$$

(VIII) If $F(x,y)$ defined on $[0,1]^2$ is continuous and symmetric, then we have

$$\begin{aligned}\frac{1}{N^2}\sum_{m,n=1}^{N} F(x_m,x_n) - \int_0^1\int_0^1 F(x,y)\,\mathrm{d}g(x)\,\mathrm{d}g(y)&\\ = -2\int_0^1 (F_N(x) - g(x))\,\mathrm{d}_x F(x,1)&\\ + \int_0^1\int_0^1 (F_N(x) - g(x))(F_N(y) + g(y))\,\mathrm{d}_y\,\mathrm{d}_x F(x,y).&\end{aligned}$$

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4.3 Basic statistical notions

Let x_n and y_n be sequences of real numbers.

• The **mean value of** x_n is given by

$$E_N(x_n)=\frac{1}{N}\sum_{n=1}^N x_n.$$

• The **dispersion (variance) of** x_n is defined by

$$\mathbf{D}_N^{(2)}(x_n)=\frac{1}{N}\sum_{n=1}^N(x_n-E_N(x_n))^2=\frac{1}{N}\sum_{n=1}^N x_n^2-(E_N(x_n))^2,$$

while $\sqrt{\mathbf{D}_N^{(2)}(x_n)}$ is the **standard deviation**.

• The **correlation coefficient of** x_n **and** y_n is

$$R_N(x_n,y_n)=\frac{|E_N(x_ny_n)-E_N(x_n)E_N(y_n)|}{\sqrt{\mathbf{D}_N^{(2)}(x_n)\mathbf{D}_N^{(2)}(y_n)}}.$$

(I) Since

$$|E_N(x_ny_n)-E_N(x_n)E_N(y_n)|=\left|\frac{1}{N}\sum_{n=1}^N(x_n-E_N(x_n))(y_n-E_N(y_n))\right|,$$

the Cauchy inequality implies that if

$$R_N(x_n,y_n)=1,\quad \mathbf{D}_N^{(2)}(x_n)>0,\quad \mathbf{D}_N^{(2)}(y_n)>0,$$

then

$$x_n=A_Ny_n+B_N$$

for all $n=1,2,\dots,N$, where

$$A_N=\frac{E_N(x_ny_n)-E_N(x_n)E_N(y_n)}{\mathbf{D}_N^{(2)}(x_n)}\quad\text{and}\quad B_N=E_N(y_n)-A_NE_N(x_n).$$

(II) The Tchebyschev inequality

$$\frac{1}{N}\#\{n \le N\,;\,|x_n - E_N(x_n)| \ge \varepsilon\} \le \frac{\mathbf{D}_N^{(2)}(x_n)}{\varepsilon^2}$$

implies (cf. M. Paštéka and R.F. Tichy (2003))

$$\frac{1}{N}\#\{n \le N\,;\,|x_n - A_N y_n - B_N| \ge \varepsilon\} \le \frac{\big(1-(R_N(x_n,y_n))^2\big)\mathbf{D}_N^{(2)}(x_n)}{\varepsilon^2}.$$

(III) A deterministic model of probability theory and statistics is presented in E. Hlawka (1998). The definition of the probability of an event makes use of u.d. of sequences here. Hlawka also discusses other subjects as the recontre problem, Markov chains, the construction of u.d. sequences with respect to the normal distribution, etc.

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4.3.1 A dynamical system

• Let $(X, \mathcal{B}, \mu)$ be a probability space, i.e. X is a set, $\mathcal{B}$ is a σ-algebra of subsets of X, and μ is a measure on $(X, \mathcal{B})$ such that $\mu(X) = 1$,
• $T : X \to X$ be a measurable map (i.e. $A \in \mathcal{B}$ implies $T^{-1}(A) \in \mathcal{B}$), that is measure-preserving (i.e. $A \in \mathcal{B}$ implies $\mu(T^{-1}A) = \mu(A)$).
• $(X, \mathcal{B}, \mu, T)$ is called a dynamical system.
• The system is ergodic if $T^{-1}A = A$, $A \in \mathcal{B}$ means that either $\mu(A)$ or $\mu(X - A) = 0$.
• The system is uniquely ergodic if there is only one such ergodic T.

Theorem 4.3.1.1. *[(Birkhoff 1931)]. For all L^1 integrable $f : X \to X$ the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \tilde{f}(x)$$

exists for μ-almost all $x \in X$. If the dynamical system is ergodic, then $\tilde{f}(x) = \int_X f(y)\,\mathrm{d}\mu(y)$ holds μ-almost everywhere.

Theorem 4.3.1.2. *$(X, \mathcal{B}, \mu, T)$ is uniquely ergodic if and only if for all continuous $f : X \to X$ the relation*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \int_X f(y)\,\mathrm{d}\mu(y)$$

holds uniformly in x.

Theorem 4.3.1.3. *Suppose that k_n is Hartman uniformly distributed and L^2-good universal and $f : X \to X$ is continuous. Then $(X, \mathcal{B}, \mu, T)$ is uniquely ergodic if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^{k_n} x) = \int_X f(y)\,\mathrm{d}\mu(y)$$

for all $x \in X$.

NOTES:
(I) Theorem 4.3.1.2, see the expository paper P.J. Grabner, P. Hellekalek and P. Liardet (2012).
(II) Theorem 4.3.1.3, see A. Jaššová, P. Lertchoosakul and R. Nair.

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