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Introduction

The fundamental physical phenomenon concerning sound is a vibration. Every heard sound is a vibration of an instrument. The vibration is transmitted into a human ear through a medium, mostly air. Then the sound is processed in the human ear. The result is an electric impulse going into a brain, and the brain evaluates the impulses and creates such a sensation of the heard sound.

This sensation may not always be related to physical process. The sensation is a subjective phenomenon and is also dependent on a particular person and even on his momentary mental status, while physical process is an objective phenomenon dependent only on physical laws.

One of the psycho-acoustical phenomenon is a deformation of audible spectrum. Due to the Weber-Fechner law, one can consider that audible range is linear in a logarithmic scale. The deformation means that the pitch perception of a pure tone is approximately linear in the middle of hearing range and is deformed near by boundary (threshold of hearing, threshold of pain). We hear nothing outside this range. Of course, these limits are different for different people. The Section 1 is devoted to this effect. In Section 1, we extend Einstein operation to more dimensions such that the operation remains associative and commutative. Hence we obtain an operation of addition of two (and thus an arbitrary finite number) tones.

If one considers a complex tone composed of two or more pure tones, the situation is more complicated. We investigate linear (one-dimensional) musical instruments, i.e. strings and pipes. Their frequency spectrum is linear, which means that their harmonics are f, $2f$, $3f$, $4f$, $\ldots$, where f is a fundamental frequency. This does not apply in the case of two-dimensional instruments like drums or bells. Their spectrum can consist of irrational multiples of the fundamental frequency.

The process of sound analysis in human ear is very similar to Fourier transformation. Linear input, which is a mixture of simple tones, coming into ear is decomposed on individual frequencies. A complex tone produced by a string or pipe is a sum of sine and cosine functions. Each such function corresponds to a pure tone, which is specified by frequency and amplitude. A significant property of complex tones is a timbre. It is a psycho-acoustical impression of heard sound. Roughly speaking, timbre enables to distinguish two different tones of the same loudness and of the same pitch. For example, if we hear middle C playing on piano and violin, we can perceive a difference, but it is unable to say which of those tones is “better” or “nicer”. Therefore the timbre is considered as a multi-dimensional attribute. The timbre is also referred as color of sound.

1 Frontiers of sound perception: relativistic effects

Let us consider Einstein operation $\oplus$ defined as follows

$$u \oplus v = \frac{u+v}{1+\frac{uv}{c^2}}, \tag{1.1}$$

where $u \in (-c, c)$, $v \in (-c, c)$.

The generalization of Einstein addition that is used in relativity theory is defined in the three-dimensional Euclidean space via addition, scalar multiplication, both scalar and vector products. The resulting operation is non-associative and non-commutative. The generalization to other dimensions is based strongly on the Lagrange vector identity in the three-dimensional Euclidean space. There is an extension from three-dimensional Euclidean spaces to every Hilbert space (including finite-dimensional spaces) [24]. Obviously, this generalization of Einstein numbers is only a possibility (not discussing about physics) since the Lagrange identity does not hold for other dimension that three. We suggest a generalization theory which saves commutativity and associativity of the additivity of the original Einstein operation passing to all finite space dimensions. It is applicable for every space dimension and this space dimension can be enlarged (added) or reduced. As an application of our theory, we bring also as an example of a semi-field of dimension 2 equipped with the generalized hyperbolic addition.

The following lemma is an useful helpful tool in various situations. We use it to transform bijectively the problem on the real line: (1) to prove associativity of an unknown operation, (2) to introduce a multiplication when the addition is given.

Lemma 1.1 *Let A, B be sets and the binary operation $*$ on A be associative (commutative, have identity element, etc.). If $F : A \to B$ is a bijective function, then the binary operation $\circledast$ on B defined by $x \circledast y = F(F^{-1}(x) * F^{-1}(y))$, $x, y \in B$, is associative (commutative, have identity element, etc.)*

The system $\mathbb{E}_{(-c,c)} = ((-c, c), \oplus)$ is a group isomorphic to real numbers with usual addition $(\mathbb{R}, +)$. If $c \to \infty$, then the c-ball expands to infinity and Einstein addition reduces to the "normal" addition, i.e. Einstein addition is reduced to the Newtonian addition.

1.1 Case of mutually independent generalized dimensions

Let $\mathbb{V}$ be a vector space over the field of real numbers equipped with a norm $\|\cdot\|$ and an operation of addition $+$. Let us denote the c-ball in $\mathbb{V}$

$$\mathbb{V}_c = \{\mathbf{v} \in \mathbb{V} \mid \|\mathbf{v}\| < c\}.$$

Let us define the function $\phi : \mathbb{V} \to \mathbb{V}_c$ as follows

$$\phi(\mathbf{u}) = \begin{cases} \dfrac{c \tanh \|\mathbf{u}\|}{\|\mathbf{u}\|}\mathbf{u} & \text{if } \mathbf{u} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{u} = \mathbf{0}. \end{cases} \tag{1.2}$$

The function ϕ defined above is a bijection between $\mathbb{V}$ and $\mathbb{V}_c$ and its inverse is

$$\phi^{-1}(\mathbf{u}) = \begin{cases} \operatorname{atanh}\left(\dfrac{\|\mathbf{u}\|}{c}\right) \dfrac{\mathbf{u}}{\|\mathbf{u}\|} & \text{if } \mathbf{u} \neq \mathbf{0}, \\ 0 & \text{if } \mathbf{u} = \mathbf{0}. \end{cases} \tag{1.3}$$

We use this isomorphism to extend the operation of the hyperbolic tangent addition from the real line to every linear normed vector space over real numbers. Define the binary operation $\oplus_{\mathbb{V}_c}$ in the set $\mathbb{V}_c$ as follows

$$\mathbf{u} \oplus_{\mathbb{V}_c} \mathbf{v} = c \tanh(\|A(\mathbf{u}, \mathbf{v})\|) \frac{A(\mathbf{u}, \mathbf{v})}{\|A(\mathbf{u}, \mathbf{v})\|} \tag{1.4}$$

where

$$A(\mathbf{u}, \mathbf{v}) = \operatorname{atanh}\left(\frac{\|\mathbf{u}\|}{c}\right) \frac{\mathbf{u}}{\|\mathbf{u}\|} + \operatorname{atanh}\left(\frac{\|\mathbf{v}\|}{c}\right) \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

It is clear that $(\mathbb{V}_c, \oplus_{\mathbb{V}_c})$ is a commutative group.

1.2 Spaces with added dimensions and generalized addition dependent on the former dimensions

We claim the following theorem to find another possibilities of extending of isomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ to $\varphi_n : D \subset \mathbb{R}^n \to \mathbb{R}^n$ and to define new operations by this isomorphism.

Theorem 1.2 *Let $D_1, D_2, \ldots, D_n \subset \mathbb{R}$, $H_0, H_1, \ldots, H_n \subset \mathbb{R}$. Let the function $f_0 : D_1 \to H_0$ and for all indexes $i = 1, 2, \ldots, n$ $f_i : D_i \to H_i$ are bijective functions, where $f_i(x) \neq 0$ for all $x \in D_i, i = 1, 2, \ldots, n-1$.*

Define $F : D = D_1 \times D_2 \times \ldots \times D_n \to F(D)$ by

$$F(x_1, x_2, \ldots, x_n) = \left(f_0(x_1), \frac{f_2(x_2)}{f_1(x_1)}, \frac{f_3(x_3)}{f_2(x_2)}, \ldots, \frac{f_n(x_n)}{f_{n-1}(x_{n-1})} \right)$$

and put $a = f_1\big(f_0^{-1}(x_1)\big)$.

If $ax_2x_3 \cdot \ldots \cdot x_i \in H_i, i = 2, 3, \ldots, n$,

$$F^{-1}(x_1, x_2, \ldots, x_n) = \big(f_0^{-1}(x_1), f_2^{-1}(ax_2), f_3^{-1}(ax_2x_3), \ldots, f_n^{-1}(ax_2x_3 \ldots x_n)\big)$$

for $x_1, x_2, \ldots, x_n \in F(D)$,

Example 1.3 *Let us consider bijective function $F_n : D_n \to H_n$ given by*

$$F_n(\mathbf{x}) = \begin{cases} \left(\tanh(x_1), \dfrac{\tanh(x_2)}{\tanh(x_1)}, \dfrac{\tanh(x_3)}{\tanh(x_2)}, \ldots, \dfrac{\tanh(x_n)}{\tanh(x_{n-1})} \right) & \text{if } x_n > 0, \\ \Big(\tanh(x_1), 0, \ldots, 0 \Big) & \text{if } x_n = 0, \end{cases}$$

for $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in D_n$, where

$$D_n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid 0 < x_n \leq x_{n-1} \leq \ldots \leq x_1\} \cup \big([0, \infty) \times \{0\}^{n-1}\big)$$

and

$$H_n = (0, 1) \times (0, 1]^{n-1} \cup \big([0, 1) \times \{0\}^{n-1}\big).$$

By Theorem 1.2,

$$F_n^{-1}(\mathbf{x}) = \begin{cases} \Big(\operatorname{atanh}(x_1), \operatorname{atanh}(x_1x_2), \ldots, \operatorname{atanh}(x_1x_2 \ldots x_n)\Big), & \text{if } x_n > 0, \\ \Big(\operatorname{atanh}(x_1), 0, \ldots, 0\Big), & \text{if } x_n = 0, \end{cases}$$

for $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in H_n$. Let us define the operation $\bigoplus$ on H_n by Lemma 1.1, thus

$$\mathbf{x} \bigoplus \mathbf{y} = \left(x_1 \oplus y_1, \frac{x_1x_2 \oplus y_1y_2}{x_1 \oplus y_1}, \frac{x_1x_2x_3 \oplus y_1y_2y_3}{x_1x_2 \oplus y_1y_2}, \ldots, \frac{x_1x_2 \ldots x_n \oplus y_1y_2 \ldots y_n}{x_1x_2 \ldots x_{n-1} \oplus y_1y_2 \ldots y_{n-1}} \right), \quad (1.5)$$

if $x_n > 0, y_n > 0$, otherwise

$$\mathbf{x} \bigoplus \mathbf{y} = (x_1 \oplus y_1, 0, \ldots, 0),$$

where $\oplus$ is Einstein addition defined by (1.1) with $c = 1$. By Lemma 1.1, the operation $\bigoplus$ is associative, commutative, identity element is $(0, 0, \ldots, 0)$, since the set D_n with usual coordinate-wise addition has these properties. Thus $(H_n, \bigoplus)$ is a commutative monoid.

1.3 Mean-like Einstein generalized numbers

Another and different generalizations of Einstein numbers can be constructed due to the following assertion, for more details cf. [11]. This construction can be easily extended to arbitrary finite number of coordinates.

Theorem 1.4 *Let $\zeta : B \to B$ be a bijective function, $(A, \oplus, \odot)$ be a field, and $\boxplus : B \times B \to B$, $\boxdot : A \times B \to B$ be such operations that there holds $p \boxdot (x \boxplus y) = (p \boxdot x) \boxplus (p \boxdot y)$ for all $p \in A, x, y \in B$ and $p \boxdot (q \boxdot x) = (p \odot q) \boxdot x$ for all $p, q \in A, x \in B$. Define an operation $\bigoplus : (A \times B) \times (A \times B) \to A \times B$ as follows*

$$(a_1, b_1) \bigoplus (a_2, b_2) = \Big(a_1 \oplus a_2,$$
$$\zeta^{-1}\big[a_1 \odot \{a_1 \oplus a_2\}_A^{-1} \boxdot \zeta(b_1) \boxplus a_2 \odot \{a_1 \oplus a_2\}_A^{-1} \boxdot \zeta(b_2)\big]\Big),$$

where c_A^{-1} means an inverse element to the element c in the set A with respect to $\odot$. We put $(a_1, b_1) \bigoplus (a_2, b_2) = (0, b)$ for some $b \in B$, if $a_1 \oplus a_2 = 0$. If $a_1 \oplus a_2 \neq 0$, $a_2 \oplus a_3 \neq 0$ and $a_1 \oplus a_2 \oplus a_3 \neq 0$, then

$$\Big[(a_1, b_1) \bigoplus (a_2, b_2)\Big] \bigoplus (a_3, b_3) = (a_1, b_1) \bigoplus \Big[(a_2, b_2) \bigoplus (a_3, b_3)\Big].$$

If moreover $\boxplus$ is a commutative operation, then $(a_1, b_1), (a_2, b_2)$ commute with respect to the operation $\bigoplus$.

Example 1.5 *Now let us take operations $\oplus, \boxplus$ as Einstein one-dimensional addition and $\odot, \boxdot$ as multiplication defined by hyperbolic tangent isomorphism, with $A = [0, c)$, $B = (-c, c)$. There are satisfied assumptions of Lemma 1.4. If $\zeta : (-c, c) \to (-c, c)$ is a bijective function, then there exists only function $\eta : \mathbb{R} \to \mathbb{R}$ such that*

$$\zeta(x) = c \tanh\Big[\eta\Big(\operatorname{atanh}\frac{x}{c}\Big)\Big]$$

for all $x \in (-c, c)$. After some manipulations

$$(a_1, b_1) \bigoplus (a_2, b_2) = \left(\frac{a_1 + a_2}{1 + \frac{a_1 a_2}{c^2}},\right.$$
$$\left. c \tanh\left(\eta^{-1}\left[\frac{\eta\big(\operatorname{atanh}\frac{b_1}{c}\big)\big(\operatorname{atanh}\frac{a_1}{c}\big) + \eta\big(\operatorname{atanh}\frac{b_2}{c}\big)\big(\operatorname{atanh}\frac{a_2}{c}\big)}{\operatorname{atanh}\frac{a_1}{c} + \operatorname{atanh}\frac{a_2}{c}}\right]\right)\right).$$

If $a_1 = a_2 = 0$ we put $(a_1, b_1) \bigoplus (a_2, b_2) = (0, 0)$.

(a) If $\eta(x) = x$ for all $x \in \mathbb{R}$ we have

$$(a_1, b_1) \bigoplus\nolimits_A (a_2, b_2) = \left(\frac{a_1 + a_2}{1 + \frac{a_1 a_2}{c^2}},\right.$$
$$\left. c \tanh\left(\frac{\big(\operatorname{atanh}\frac{b_1}{c}\big)\big(\operatorname{atanh}\frac{a_1}{c}\big) + \big(\operatorname{atanh}\frac{b_2}{c}\big)\big(\operatorname{atanh}\frac{a_2}{c}\big)}{\operatorname{atanh}\frac{a_1}{c} + \operatorname{atanh}\frac{a_2}{c}}\right)\right).$$

(b) If $\eta(x) = 1/x$ for all $x \in \mathbb{R} \setminus \{0\}$, $\eta(0) = 0$, then

$$(a_1, b_1) \bigoplus\nolimits_H (a_2, b_2) = \left(\frac{a_1 + a_2}{1 + \frac{a_1 a_2}{c^2}}, c \tanh \frac{\operatorname{atanh}\frac{a_1}{c} + \operatorname{atanh}\frac{a_2}{c}}{\frac{\operatorname{atanh}\frac{a_1}{c}}{\operatorname{atanh}\frac{b_1}{c}} + \frac{\operatorname{atanh}\frac{a_2}{c}}{\operatorname{atanh}\frac{b_2}{c}}} \right).$$

The second coordinate resembles the weighted arithmetic mean in the first case and the weighted harmonic mean in the second case for b_1, b_2. Hence the name "mean-like" Einstein numbers. Both operations form a commutative monoid on the set $[0, c) \times (-c, c)$.

2 Multi-polar structure of sound

One of the first who studied timbre was Helmholtz in the second half of the 19th century, cf. [13]. He referred the timbre as *quality of a tone*. He proposed that the timbre is dependent only on ratios of amplitudes of partial tones. In 1982, Pollard and Janssen [19] introduced the timbre as a tri-stimulus (three-valued quantity). They studied a truncated Fourier series

$$\sum_{k=1}^{N} a_k \sin(2\pi f_k t - \varphi_k),$$

where a_k are amplitudes related to frequencies f_k, φ_k is a phase difference, and $N > 4$. The harmonics are aggregated into three groups. There are defined three quantities

$$S_1 = \frac{a_1}{S}, \quad S_2 = \frac{a_2 + a_3 + a_4}{S}, \quad S_3 = \frac{\sum\limits_{k=5}^{N} a_k}{S}, \qquad S = \sum_{k=1}^{N} a_k.$$

Obviously, there holds

$$S_1 + S_2 + S_3 = 1.$$

Hence, the timbre is determined by only two quantities. The space of timbres is very similar to the gamut in the RGB model of colors. This correspondence is remarkable in [18, 23]. In the figure 1, there is a projection of tri-stimulus to two dimensions with axes S_1 and S_2. There is also an approximately location of timbre for trumpet, viola, piano and flute, which is given by experimental measurements, cf. [14]. In the figure 2, there is a space of human color vision. The gamut is a section of the space, it usually is a triangle.

The theory of multi-polarity is a useful tool for investigating multi-dimensional quantities. Multi-polarity provides a uniform frame to deal with usual colors and also sound colors. The idea of multi-polarity (sets equipped with arithmetic operations with arbitrary number of "poles") is relatively new and comes from physics, cf. [17]. However, there was no explanation of the multi-polarity idea in the mathematical literature, i.e., a description of a vector-space-like structure with a chosen number $K \in \mathbb{N}$ of poles. There are many applications of K-polar vector spaces to electricity and magnetism, e.g. we know the 3-phase, K-phase electric streams, K-polar oscillators, receivers, magnets, microscopes, telescopes, etc., [16]. The principle of multi-polarity can be easily observed in the case of K-phase electric current, $K \in \mathbb{N}$. Particularly,

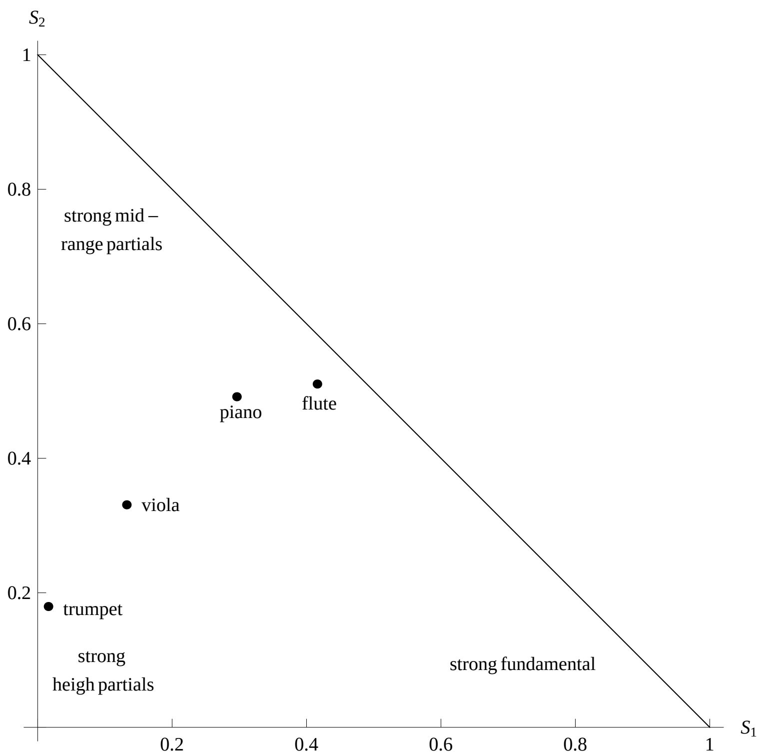

Figure 1: Timbre space, [14]

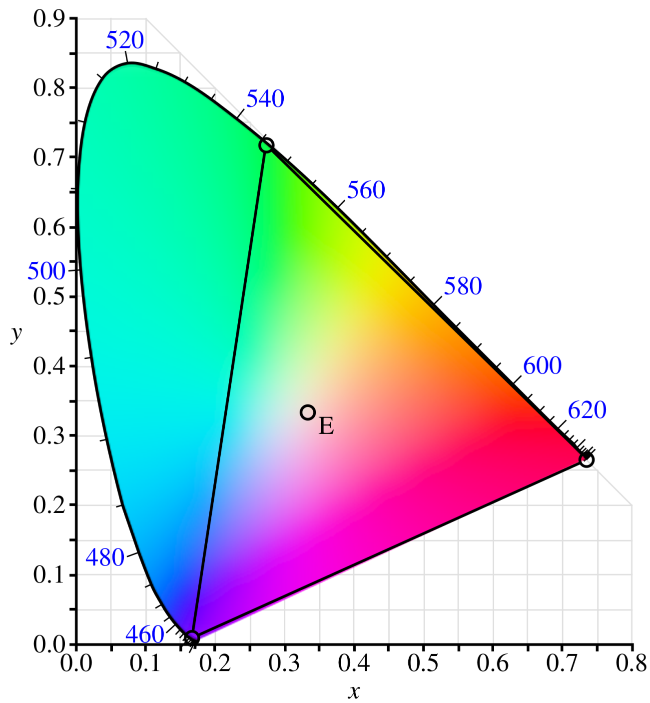

Figure 2: Color space with a triangular gamut, [4]

if $K = 3$ we say about three-phase current. The sum of all three phases of equal amplitude in three-phase current is zero at each moment. We call this fact to be *the cancellation law*. We introduce a mathematical description of multi-polarity, cf. [9]. There is presented the complex plane as a three-polar space.

The notion of semi-field is closely related to multi-polarity. In the last part 2.5, there is investigated four-dimensional semi-field of hyperbolic commutative quaternions. We apply this semi-field to construct a three-polar space based on the first six harmonics in Fourier series. Those six harmonics are usual enough to model musical tones. The seventh harmonics is often regarded as inharmonious. For example, the seventh harmonics on the piano is suppressed by striking the seventh of the string.

2.1 Three-polar space

Semi-fields play a crucial role in the theory of multi-polarity. The set X with two binary operations $+$ and $\cdot$ is called to be a semi-field, $\mathcal{X} = (X, +, \cdot)$, if $(X, +)$ is a commutative semi-group (i.e. $+$ is a commutative associative operation), $(X, \cdot)$ is a commutative group, and multiplication $\cdot$ distributes over addition $+$. If the semi-group $(X, +)$ has a neutral element 0, then we request to be $(X \setminus \{0\}, \cdot)$ a commutative group, and the system $(X, +, \cdot)$ is called to be a semi-field with zero 0. We denote 1 the neutral element with respect to multiplication $\cdot$.

Let us consider three poles A, B, C from a semi-field X to a set Y and an operation $\otimes$ given by the Latin square

$$\begin{array}{c|ccc} \otimes & A & B & C \\ \hline A & A & B & C \\ B & B & C & A \\ C & C & A & B \end{array}. \tag{2.1}$$

The mappings A, B, C are called *polar operators* or *poles*. There is a neutral element A with respect to the operation $\otimes$. Next, we assume that $A(X)$, $B(X)$, $C(X)$ are isomorphic copies of X and

$$A(0) = B(0) = C(0), \tag{2.2}$$

where 0 is a neutral element in the semi-group $(X, \oplus)$. The isomorphic images $A(X), B(X)$ and $C(X)$ are called *polar axes*. The condition (2.2) shows that all polar axes intersect in a single point which can be viewed as an "origin" of the multi-polar space.

Now let us consider the element $(a, b, c) \in X^3$. It can be very formally written as a sum

$$(a, b, c) = Aa + Bb + Cc.$$

The important function has the *cancellation law*. We will say that a triple $(u, v, w) \in X^3$ is equal to the triple $(x, y, z) \in X^3$ in the sense of cancellation law ("null of addition"), we write $(u, v, w) \cong (x, y, z)$, if there exists $d \in X$ such that $(u + d, v + d, w + d) = (x, y, z)$ or $(u, v, w) = (x + d, y + d, z + d)$. The relation $\cong$ is an equivalence relation and $(a, b, c) \cong (0, 0, 0) \iff a = b = c$. Formally, for all $a \in X$

$$Aa + Ba + Ca \cong (0, 0, 0).$$

2.2 Operations in the three-polar space

We introduce operations analogical to the classical operations of addition, subtraction, conjugation, multiplication and division. All these operations are defined on X^3.

Addition For (u, v, w), $(x, y, z) \in X^3$,

$$(u, v, w) \boxplus (x, y, z) := (u + x, v + y, w + z). \tag{2.3}$$

Clearly, the operation $\boxplus$ is associative, commutative.

Subtraction For all $x, y, z \in X$ and due to the cancellation law, we define the operation of subtraction as follows

$$(u, v, w) \boxminus (x, y, z) = (u, v, w) \boxplus (y + z, x + z, x + y) = (u + y + z, v + x + z, w + x + y). \tag{2.4}$$

It is obvious that $\boxminus(x, y, z) = (0, 0, 0) \boxminus (x, y, z)$ is just an inverse element to (x, y, z) with respect to $\boxplus$ because

$$(x, y, z) \boxminus (x, y, z) = (x + y + z, x + y + z, x + y + z) \cong (0, 0, 0).$$

Multiplication We define the operation of multiplication polynomial-like ("each one with each one") using the table (2.1) for operation $\otimes$,

$$(u, v, w) \boxdot (x, y, z) = (u \cdot x + v \cdot z + w \cdot y, u \cdot y + v \cdot x + w \cdot z, u \cdot z + v \cdot y + w \cdot x). \tag{2.5}$$

It is easy to see that this operation is associative and commutative, its neutral element is $(1, 0, 0))$, and distributes over addition $\boxplus$.

The special case of this operation is *multiplication by scalar*,

$$k(u, v, w) = (k, 0, 0) \boxdot (u, v, w) = (k \cdot u, k \cdot v, k \cdot w),$$

where $k \in X$.

Conjugation This operation is expressed in terms of three-polar space in the following way

$$(u, v, w)^* = (u, w, v). \tag{2.6}$$

The operation of conjugation has the following natural properties.

Lemma 2.1 *Let be $(u, v, w) \in X^3, (x, y, z) \in X^3$. Then there holds*

1. $((u, v, w)^*)^* = (u, v, w)$,
2. $((u, v, w) \boxplus (x, y, z))^* = (u, v, w)^* \boxplus (x, y, z)^*$,
3. $((u, v, w) \boxdot (x, y, z))^* = (u, v, w)^* \boxdot (x, y, z)^*$.

Division The element $(u,v,w) \in X^3$ is called to be *invertible* if $(u,v,w) \boxdot (u,v,w)^*$ belongs to $X \setminus \{0\}$. We denote the set of all invertible element by X_i^3. For an invertible element $(u,v,w) \in X_i^3$, an inverse element with respect to multiplication $\boxdot$ is given as follows:

$$(u,v,w)^{-1} = ((u,v,w) \boxdot (u,v,w)^*)^{-1} (u,v,w)^* \cong (u \cdot d, w \cdot d, v \cdot d), \tag{2.7}$$

where

$$d = 2/\left[(u-v)^2 + (v-w)^2 + (w-u)^2\right].$$

Now we define an operation of division in the following way:

$$(u,v,w) \boxslash (x,y,z) = (u,v,w) \boxdot (x,y,z)^{-1} \tag{2.8}$$

for $(u,v,w) \in X^3$ and $(x,y,z) \in X_i^3$.

Lemma 2.2 *All operations of addition, subtraction, multiplication and division are independent on the choice of representatives of the classes equivalences given by the cancellation law.*

2.3 The case of the semi-field of non-negative real numbers

If one considers a semi-field of non-negative real numbers $\mathbb{R}_0^+$, there holds $(u,v,w) \boxdot (u,v,w)^* \in \mathbb{R}_0^+$ for all $(u,v,w) \in \mathbb{R}_0^+$, $(u,v,w) \ncong (0,0,0)$. Due to this fact, we obtain a field which is isomorphic to usual complex numbers. From this point of view, the three-polar space over the semi-field of non-negative real numbers is only another expression of complex numbers.

Theorem 2.3 *Let be $\mathbb{C}_3$ a three-polar space over the non-negative real numbers with usual addition and multiplication. The operations of the standard complex field $\mathbb{C}$ and operations in $\mathbb{C}_3$ are isomorphic.*

2.4 The case of the semi-field of hyperbolic complex numbers

In this section, we deal with the system of hyperbolic complex numbers. It is an extension of the system of non-negative real numbers with usual operations. It has some better properties than the field of complex numbers, e.g. there is a non-trivial sub-semi-field.

If we go from one-dimensional real numbers to two-dimensional generalized complex numbers, the situation is more complicated. We will represent generalized complex numbers as follows

$$z = (a,b) = a + \imath b,$$

where $a, b \in \mathbb{R}$ and $\imath \notin \mathbb{R}$. If we consider usual coordinate-wise addition, there are three possible multiplication operations which reflect how we define the element $\imath$ [1, 12, 15]. If we put $\imath^2 = -1$ we speak about elliptic complex numbers $\mathbb{C}_e$ or only complex numbers, if $\imath^2 = +1$ then we have hyperbolic complex numbers $\mathbb{C}_h$, and if $\imath^2 = 0$ then the system is called parabolic complex numbers $\mathbb{C}_p$. We can general write

$$(a,b) \oplus (c,d) = (a+c, b+d),$$

$$(a,b) \odot (c,d) = (ac + \imath^2 bd, ad + bc).$$

The system $\mathbb{C}_e$ is a field, while $\mathbb{C}_h$, $\mathbb{C}_p$ are neither fields nor semi-fields, they are only commutative rings with unit. Clearly, non-negative real numbers are sub-semi-fields in all three system. But one can find subsets of hyperbolic and parabolic complex numbers which are semi-fields not contained in real numbers. We have semi-field $\{(a,b) \in \mathbb{R}^2; a > |b| \text{ or } a = b = 0\}$ for hyperbolic complex numbers, and semi-field $\{(a,b) \in \mathbb{R}^2; a > 0 \text{ or } a = b = 0\}$ for parabolic complex numbers. These sets are the only connected (in usual Euclidean topology) non-real semi-fields in $\mathbb{C}_h$, and $\mathbb{C}_p$ respectively. In usual complex numbers, there does not exist a connected sub-semi-field besides real numbers and whole complex numbers.

Now, we will consider the system of hyperbolic complex numbers, which we will denote by $\mathbb{D}$ due to the other names *double numbers* (there is no standard terminology for hyperbolic complex numbers; we can find the following terms in literature: double numbers, hyperbolic numbers, semi-complex numbers, split-complex numbers, perplex numbers). We will use the shorter name “double numbers” instead of the more concise name “hyperbolic complex numbers”.

Arithmetic operations of two double (hyperbolic complex) numbers $(a_1, a_2), (b_1, b_2) \in \mathbb{D}$ are as follows

$$\begin{aligned}(a_1, a_2) \oplus (b_1, b_2) &= (a_1 + b_1, a_2 + b_2)\,,\\ (a_1, a_2) \ominus (b_1, b_2) &= (a_1 - b_1, a_2 - b_2)\,,\\ (a_1, a_2) \odot (b_1, b_2) &= (a_1 b_1 + a_2 b_2, a_1 b_2 + a_2 b_1)\,,\end{aligned}$$

and if $b_1^2 \neq b_2^2$ then

$$(a_1, a_2) \oslash (b_1, b_2) = \left(\frac{a_1 b_1 - a_2 b_2}{b_1^2 - b_2^2}, \frac{a_2 b_1 - a_1 b_2}{b_1^2 - b_2^2}\right).$$

All operations on the right sides are usual operations of real numbers. In the sequel, for convenience, we will denote these operations $\oplus$, $\ominus$, $\odot$, $\oslash$ in double numbers in the same way as those in real numbers $+$, $-$, $\cdot$, $/$, respectively. If appropriate we return to the “o-notation”.

It is easy to check that double numbers form a commutative ring with unit but not a semi-field. If we take the set $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 = 0\}$, then double numbers are reduced on the semi-field of non-negative real numbers. There exist also a non-trivial subset of double numbers which is closed under addition, multiplication and division. It is a set $\mathbb{D}^0_+ = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > |x_2| \text{ or } x_1 = x_2 = 0\}$. Obviously the system $\left(\mathbb{D}^0_+, \oplus, \odot\right)$ is a semi-field.

We introduce the following definition for the lemma of “existence of the square of the difference”.

Definition 2.4 Let be $a = (a_1, a_2) \in \mathbb{D}^0_+$ and $b = (b_1, b_2) \in \mathbb{D}^0_+$. We will say that a, b *are in the relation* $\overset{+}{\sim}$ ($\overset{-}{\sim}$), $a \overset{+}{\sim} b$ ($a \overset{-}{\sim} b$), if $a_1 + a_2 = b_1 + b_2$ ($a_1 - a_2 = b_1 - b_2$).

Lemma 2.5 *Let be* $a = (a_1, a_2) \in \mathbb{D}^0_+$ *and* $b = (b_1, b_2) \in \mathbb{D}^0_+$. *Then*

1. $(a - b) \cdot (a - b) \in \mathbb{D}^*_+ = \mathbb{D}^0_+ \cup \{(x_1, x_2) \in \mathbb{R}^2 | x_1 = |x_2|\}$,
2. $(a - b) \cdot (a - b) \in \mathbb{D}_+ = \mathbb{D}^0_+ \setminus \{(0,0)\}$ *if and only if* $a \overset{+}{\not\sim} b$ *and* $a \overset{-}{\not\sim} b$.

We construct the three-polar space using the semi-field $X = \mathbb{D}^0_+$. The necessary and sufficient condition for divisibility is given by the following lemma.

Lemma 2.6 *Let be $(u,v,w) \in X^3, (x,y,z) \in X^3$. Then there holds*

$$(u,v,w) \boxdot (u,v,w)^* \cong (d,0,0)$$

for some $d \in X_+$ if and only if there does not hold true that $u \overset{+}{\sim} v \overset{+}{\sim} w$ or $u \overset{-}{\sim} v \overset{-}{\sim} w$.

2.4.1 Cancellation law in split-quaternions

Let us suppose the operators A, B, C in the form

$$B_i : \mathbb{R}^4 \to \mathbb{R}^4, i = 1,2,3,$$

where $B_{1,2,3} = A, B, C$ are square matrices 4×4 with real coefficients.

According to (2.1), we request the following equations

$$\begin{array}{ccccccc} B_1B_1 & = & B_2B_3 & = & B_3B_2 & = & B_1, \\ B_1B_2 & = & B_2B_1 & = & B_3B_3 & = & B_2, \\ B_1B_3 & = & B_2B_2 & = & B_3B_1 & = & B_3, \end{array} \tag{2.9}$$

where multiplication of matrices is understood in the usual sense. We assume that $A = B_1$ is an identity matrix. This way, the system (2.9) is reduced to the following four equations

$$B_2B_3 = B_1, \quad B_3B_2 = B_1, \quad B_3B_3 = B_2, \quad B_2B_2 = B_3. \tag{2.10}$$

These equations lead to a system of 48 non-linear equations with 32 unknown. We do not have a complete solution of this equation system. We considered only a special cases of the so-called split-quaternions. The operation of multiplication of two split-quaternions $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ and $(y_1, y_2, y_3, y_4) \in \mathbb{R}^4$ can be expressed in the "matrix-vector" notation as follows

$$(x_1, x_2, x_3, x_4) \cdot (y_1, y_2, y_3, y_4) = \begin{pmatrix} x_1 & -x_2 & x_3 & x_4 \\ x_2 & x_1 & -x_4 & x_3 \\ x_3 & -x_4 & x_1 & x_2 \\ x_4 & x_3 & -x_2 & x_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}.$$

According to this, the operators A_2, A_3 can be represented by quadruples of real numbers (b_1, b_2, b_3, b_4) and (c_1, c_2, c_3, c_4), respectively. The operator A_1 is represented in this way as $(1,0,0,0)$.

In this case, the system (2.10) has only 8 unknown and is easy solvable. It was solved by Mathematica. All solutions are of two kinds. The first one is a trivial solution $A_2 = A_3 = (1,0,0,0)$ which is not interesting. The second type is parametrizable as follows

$$\begin{array}{rcl} A_2(r,s,t) & = & \left(-\dfrac{1}{2}, \dfrac{r}{2}\sqrt{3+4s^2+4t^2}, s, t\right), \\ A_3(r,s,t) & = & \left(-\dfrac{1}{2}, -\dfrac{r}{2}\sqrt{3+4s^2+4t^2}, -s, -t\right), \end{array} \tag{2.11}$$

where $r \in \{1, -1\}$, $s \in \mathbb{R}$, $t \in \mathbb{R}$.

If $s = t = 0$, then operators A_2, A_3 correspond to rotation in the complex plane by angle $2\pi/3$ and $4\pi/3$, respectively.

We observe that for all $x \in \mathbb{R}^4$,

$$A_1x + A_2(r,s,t)x + A_3(r,s,t)x = (0,0,0,0) \tag{2.12}$$

for all admissible r, s, t. Thus Equation (2.12) can play the role of the cancellation law.

2.4.2 Commutative rings on the set $\mathbb{R}^4$

Using operators $A_1, A_2(r,s,t), A_3(r,s,t)$, we construct a class of commutative rings of three-polar spaces over the semi-field of double numbers.

Let be $m : \mathcal{M} = X^3 \to \mathbb{R}^4$, $(a,b,c) \mapsto A_1 a + A_2(r,s,t)b + A_3(r,s,t)c$, where $a = (a_1, a_2)$, $b = (b_1, b_2)$, $c = (c_1, c_2) \in X$. The mapping m is independent on the choice of representatives of equivalence classes $\mathbb{M} = \mathcal{M}\backslash_{\cong}$. Thus it can be considered as a mapping $\mathbb{M} \to \mathbb{R}^4$ that is injective. If $s^2 + t^2 > 0$, then the mapping m is onto. Moreover, m is linear, i.e.

$$\begin{aligned} m((u,v,w) \boxplus (x,y,z)) &= m(u,v,w) + m(x,y,z), \\ m(k \boxdot (u,v,w)) &= km(u,v,w), \end{aligned}$$

for all $(u,v,w), (x,y,z) \in X^3$, $k \in X$.

If $s^2+t^2 > 0$, then m is a bijection, thus there is an inverse mapping, denote it $n : \mathbb{R}^4 \to \mathbb{M}$. Also this mapping is linear. Through these two mapping we can introduce new operations, say addition and multiplication, on $\mathbb{R}^4$ as follows

$$\begin{aligned} p \bigoplus q &= m(n(p) \boxplus n(q)), \\ p \bigodot q &= m(n(p) \boxdot n(q)), \end{aligned} \tag{2.13}$$

$p, q \in \mathbb{R}^4$.

Immediately, by linearity of both mappings,

$$p \bigoplus q = p + q, \tag{2.14}$$

where $+$ is usual coordinate-wise addition in $\mathbb{R}^4$.

More interesting situation arises from the operation of multiplication. Let us denote four base elements in $\mathbb{R}^4$ as $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ and let be

$$p = \sum_{i=1}^{4} p_i \mathbf{e}_i, \quad q = \sum_{j=1}^{4} q_j \mathbf{e}_j.$$

By properties of operations $\boxplus, \boxdot$ and mappings m, n,

$$p \bigodot q = \sum_{i,j=1}^{4} p_i q_j (\mathbf{e}_i \boxdot \mathbf{e}_j),$$

and thus it suffices to find elements $e_i \bigodot e_j = m(n(\mathbf{e}_i) \boxdot n(\mathbf{e}_j))$.

The system $(\mathbb{R}^4, \bigoplus, \bigodot)$ is a commutative ring with unit $(1,0,0,0)$ regardless of the parameters r, s, t, except for the condition $s^2 + t^2 > 0$ because if $s = t = 0$ then the mapping $m : X^3 \to \mathbb{R}^4$ is not surjective and thus the mapping n is not well defined.

2.5 Three-polar space over the semi-field of hyperbolic commutative quaternions

2.5.1 Hyperbolic quaternions

The system of usual real numbers and usual (elliptic) complex numbers are significant, their multiplication have four important properties: 1. associativity; 2. distributivity over addition;

3. commutativity; 4. no zero divisors (i.e. $ab = 0$ implies $a = 0$ or $b = 0$). It can be proved that in more than two dimensions a system with these four properties can not exist, cf. [21]. If such a system satisfies the two first conditions, then it can satisfy just one of the two last conditions. It is clear that associative system can be non-commutative, e.g. classical Hamilton quaternions, but on the other hand commutative systems with unit are associative, cf. [5]. Hamilton quaternions are without zero divisors, what allows to divide arbitrary non-zero numbers, but they are not commutative.

In functional analysis, the greatest attention is paid for commutative systems due to the Scheffers theorem [20]: for a distributive system with the unity the differential and integral calculus exists only if the system is commutative. In the sequel, we will deal with four-dimensional system like classical Hamilton quaternions. Due to the commutativity, we will call such system *commutative (Segre) quaternions*, cf. [22, 2]. Scheffers theorem allows to introduce differential and integral calculus in commutative quaternions and also functions of commutative quaternion variable. It has interesting applications in the relativity theory, cf. [6, 3], or in Maxwell equations, cf. [2]. The multiplication operation of basis elements 1, $\imath_1$, $\imath_2$, $\imath_3$ is given by the table

$$\begin{array}{c|cccc} & 1 & \imath_1 & \imath_2 & \imath_3 \\ \hline 1 & 1 & \imath_1 & \imath_2 & \imath_3 \\ \imath_1 & \imath_1 & \alpha & \imath_3 & \alpha\imath_2 \\ \imath_2 & \imath_2 & \imath_3 & 1 & \imath_1 \\ \imath_3 & \imath_3 & \alpha\imath_2 & \imath_1 & \alpha \end{array}, \tag{2.15}$$

where $\alpha \in \{-1, 0, +1\}$. According to whether α is equal to -1, 0, or $+1$ we say about *elliptic* $\mathbb{Q}_e$, *parabolic* $\mathbb{Q}_p$, and *hyperbolic* $\mathbb{Q}_h$ (commutative) quaternions, respectively. This system can be equivalently viewed as pairs of generalized complex numbers as follows

$$Z = a1 + b\imath_1 + c\imath_2 + d\imath_3 = u + \imath_2 v,$$

where $a, b, c, d \in \mathbb{R}$, u, v are generalized complex numbers related to α (elliptic if $\alpha = -1$, parabolic if $\alpha = 0$, hyperbolic if $\alpha = +1$), and $\imath_2^2 = 1$. Thus, the system is the direct sum of generalized complex numbers

$$\mathbb{Q}_\beta = \mathbb{C}_\beta \oplus \mathbb{C}_\beta,$$

where $\beta \in \{e, p, h\}$.

Due to the fact that these three types types of commutative quaternions can be constructed as a composition of generalized complex numbers, they are called also *decomposable* quaternions or *bi-complex numbers*. Two generalized complex numbers are joined together through a hyperbolic unit $\imath_2^2 = +1$. The construction is very similar to Cayley-Dickson construction of dimensions doubling. Our aim is to preserve properties of multiplication, i.e. associativity, commutativity and existence of inverse elements, with increasing dimensions. If there is an inverse element to every element expect zero, then we have also an operation of division. But, similarly to Cayley-Dickson construction, it fails something. In our case, it will be restriction of domain and only partially defined subtraction operation.

2.5.2 Semi-field of hyperbolic quaternions

In this section, we will concern the system of hyperbolic quaternions, i.e. we assume $\alpha = 1$ in (2.15). We denote the system of hyperbolic quaternions as $\mathbb{H}$.

Let us consider the set of quadruples of real numbers

$$\mathbb{H} = \{a1 + b\jmath_1 + c\jmath_2 + d\jmath_3; a, b, c, d \in \mathbb{R}\},$$

where 1 is a real unit and $\jmath_1$, $\jmath_2$, $\jmath_3$ are all imaginary hyperbolic units. Equivalently, we can write

$$a1 + b\jmath_1 + c\jmath_2 + d\jmath_3 = (a, b, c, d).$$

Define the operation of addition coordinate-wisely, i.e.

$$(a, b, c, d) \oplus (x, y, z, w) = (a + x, b + y, c + z, d + w), \tag{2.16}$$

where a, b, c, d, x, y, z, w are all real numbers.

Similarly, there is also an operation of subtraction given as follows

$$(a, b, c, d) \ominus (x, y, z, w) = (a - x, b - y, c - z, d - w). \tag{2.17}$$

Remark 2.7 The system $(\mathbb{H}, \oplus)$ is a commutative group with zero $\mathbf{0} = (0, 0, 0, 0)$.

According to the multiplication table (2.15) with $\alpha = 1$, the operation of multiplication on $\mathbb{H}$ possesses the form

$$\begin{aligned}(a, b, c, d) \odot (x, y, z, w) = (&ax + by + cz + dw, ay + bx + cw + dz,\\ &az + bw + cx + dy, aw + bz + cy + dx).\end{aligned}$$

Remark 2.8 The system $(\mathbb{H}, \oplus, \odot)$ is a commutative ring with unit $\mathbf{1} = (1, 0, 0, 0)$.

The operation of division is not defined in general. There is a necessary and sufficient condition for existence of an inverse element with respect multiplication and also its expression in the following lemma.

Lemma 2.9 *Let be $(a, b, c, d) \in \mathbb{H}$. If*

$$(a + b + c + d)(a + b - c - d)(a - b + c - d)(a - b - c + d) \neq 0, \tag{2.18}$$

then there exists an inverse element of (a, b, c, d) with respect to multiplication and it is as follows

$$1 \oslash (a, b, c, d) = (k + l + m + n, k + l - m - n, k - l + m - n, k - l - m + n), \tag{2.19}$$

where

$$\begin{aligned} k &= \frac{1}{4(a + b + c + d)}, & l &= \frac{1}{4(a + b - c - d)},\\ m &= \frac{1}{4(a - b + c - d)}, & n &= \frac{1}{4(a - b - c + d)}. \end{aligned}$$

If the condition (2.18) is not satisfied, then there exists $(x, y, z, w) \in \mathbb{H} \setminus \{\mathbf{0}\}$ such that

$$(a, b, c, d) \odot (x, y, z, w) = (0, 0, 0, 0).$$

The elements not satisfying condition (2.18) are zero divisors. If we would like to preserve the division operation, we have to exclude zero divisors. Let us denote

$$\mathbb{H}^+ = \{(a,b,c,d) \in \mathbb{H}; a+b+c+d > 0, a+b > c+d, a+c > b+d, a+d > b+c\}, \quad (2.20)$$

and

$$\mathbb{H}_0^+ = \mathbb{H}^+ \cup \{\mathbf{0}\}. \quad (2.21)$$

Lemma 2.10 *The set $\mathbb{H}^+$ is closed under operations $\oplus$, $\odot$, $\oslash$.*

Corollary 2.11 *The system $\left(\mathbb{H}_0^+, \oplus, \odot\right)$ is a semi-field with zero $\mathbf{0}$.*

The set $\mathbb{H}^+$ can be understood as the set of positive hyperbolic quaternions, the set $\mathbb{H}_0^+$ as the set of non-negative hyperbolic quaternions. In fact, it is a extension of positive real numbers $\mathbb{R}^+$, and non-negative real numbers $\mathbb{R}_0^+$, respectively. Non-negative hyperbolic quaternions are closed under operations of addition, multiplication and division (excluding the point 0). The subtraction operation is defined only partially in the set $\mathbb{H}_0^+$,

$$(a,b,c,d) \ominus (x,y,z,w) = (a-x, b-y, c-z, d-w), \quad (2.22)$$

provided this element belongs to $\mathbb{H}_0^+$, where $(a,b,c,d), (x,y,z,w) \in \mathbb{H}_0^+$.

Similarly as $\mathbb{R}_0^+$, also $\mathbb{H}_0^+$ can be equipped by natural partial order, however the order is not total.

Definition 2.12 Let be $(a,b,c,d), (x,y,z,w) \in \mathbb{H}_0^+$. Define a binary relation $\leq$ as follows

$$(a,b,c,d) \leq (x,y,z,w)$$

if and only if there exists $(p,q,r,s) \in \mathbb{H}_0^+$ such that

$$(a,b,c,d) \oplus (p,q,r,s) = (x,y,z,w).$$

The binary relation $\leq$ is a partial order with the least element $\mathbf{0}$ and has similar properties as usual order in non-negative real numbers.

2.5.3 Applications

There does not exist any four-dimensional field. But there is a subset of four-dimensional number system of hyperbolic quaternions which is a semi-field. The semi-field can be used to create a three-polar space which is considered as space of timbres.

Let us consider the first six harmonics in a Fourier series

$$\sum_{k=1}^{6} a_k \sin(2\pi k f t) + b_k \cos(2\pi k f t), \quad (2.23)$$

where a_k, b_k are real coefficients and f is the fundamental frequency. We have twelve coefficients which we merge into three groups

$$\begin{aligned} x &= (a_1, b_1, a_6, b_6), \\ y &= (a_2, b_2, a_5, b_5), \\ z &= (a_3, b_3, a_4, b_4). \end{aligned} \quad (2.24)$$

In each group is a significant harmonic: the unison (a_1, b_1), the fifth (a_3, b_3), and the third (a_5, b_5). The rest three harmonics are less significant. They are octaves from another harmonics: (a_2, b_2) is the octave from the unison, (a_4, b_4) is the octave from the octave, (a_6, b_6) is the octave from the fifth. According to the grouping of harmonics, polar operators correspond to the unison, the fifth, and the third. We define triples

$$(x, y, z) = Ax + By + Cz \in \mathbb{M}_3, \tag{2.25}$$

where x, y, z belong to the semi-field of hyperbolic quaternions $\left(\mathbb{H}_0^+, \oplus, \odot\right)$.

The cancellation law can be understood as a "grey color" in music. It is due to the fact that the cancellation law in the color theory defines various shades of grey, cf. [10]. By factorizing by greys, we obtain "pure colors".

The division operation is a bit complicated, because there are zero divisors. The set of zero divisors describes the following lemma.

Lemma 2.13 *Let be $(a, b, c) \in \mathbb{M}_3$. The element (a, b, c) is a zero divisor if and only if $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4)$, $c = (c_1, c_2, c_3, c_4) \in \mathbb{H}_0^+$ concurrently satisfies at least one of the following conditions*

(a) $\cdot_1 + \cdot_2 + \cdot_3 + \cdot_4 = 0$,

(b) $\cdot_1 + \cdot_2 - \cdot_3 - \cdot_4 = 0$,

(c) $\cdot_1 - \cdot_2 + \cdot_3 - \cdot_4 = 0$,

(d) $\cdot_1 - \cdot_2 - \cdot_3 + \cdot_4 = 0$,

where $\cdot_i$ means a_i, b_i, and c_i, respectively, for all $i = 1, 2, 3, 4$. We will call the elements in $\mathbb{M}_3$ which are not zero divisors invertible elements.

The operations of addition, subtraction, multiplication and division defined operation with truncated Fourier series of the first six harmonics. Especially, the multiplication (and also division) operation is new and requires further investigation.

Conclusion

The aim of the dissertation thesis is to study structure of sound whose source is linear (one-dimensional), i.e. a string or a pipe. The spectrum of it is linear and consists of only natural multiples of the fundamental frequency. The complex tone is a composition of pure tones with only one frequency, i.e. a Fourier series whose components are sine and cosine functions.

This work elaborates two fields concerning sound. The first one is deformation of audible spectrum in the near of its boundaries. We assume that the sound perception is approximately linear in the middle range due to the Weber-Fechner law. We considered a hyperbolic tangent function to model this. It is also defined Einstein operation by this function. We found some extensions of one-dimensional Einstein operation to more dimensions. In the first coordinate, there is Einstein operation and other coordinates depend only on the preceding coordinates. All dimensions are independent on the following ones. The important property of our extensions is associativity and commutativity. In physics, it is used another type of extension of

Einstein operation that works in arbitrary inner product space but it is neither commutative nor associative.

The second field of the study is multi-polarity theory. The theory of multi-polarity is relatively new. The mathematical formalization of multi-polarity is introduced in Section 2.1. We investigated three types of semi-fields to construct three-polar spaces: non-negative real numbers with usual addition and multiplication, cf. [9], hyperbolic complex numbers, cf. [8], and hyperbolic commutative quaternions. In the paper [10], there is studied a three-polar over the semi-field of parabolic complex numbers in the frame of RGB color model. In this case, the author obtains a field.

Semi-fields themselves are interesting from the point of view that they permit to divide by arbitrary non-zero element. But, on the other side, there is not subtraction operation in general. A semi-field in a four-dimensional case is found in Section 2.5, cf. [7]. It is interesting because there is no four-dimensional field. We investigated the system of hyperbolic commutative quaternions. This number system is studied also in physics, e.g. Maxwell equations, relativity theory, because its multiplication is commutative unlike usual Hamiltonian quaternions that are not commutative.

We defined a three-polar space using the semi-field of hyperbolic commutative quaternions. Every element of the semi-field represents a pair of harmonics of a tone (i.e. coefficients of Fourier series). We consider the first six harmonics, thus we model the timbre space as a three-polar space. Polar operators represents the unison, the third and the fifth. The octaves are less significant due to the octave equivalence. There are also operation of addition, subtraction, multiplication and division. These operations define new operations with tones.

Resumé

Cieľom tejto práce je popísať štruktúru zvuku, ktorého zdroj je lineárny (jednorozmerný), t.j. píšťala alebo struna. Spektrum takéhoto zdroja je lineárne a obsahuje iba prirodzené násobky základného tónu. Súčasným znením viacerých tónov vzniká komplexný tón, ktorý sa dá vyjadriť ako Fourierov rad, kde významnú úlohu hrajú amplitúdy a frekvencie jednotlivých bázových funkcií (sínus a kosínus).

V prvej časti, kapitola 1, sa venujeme deformácii vnímania zvuku. Weberov-Fechnerov zákon prevádza exponenciálny nárast nejakého podnetu na lineárny nárast vnemu. Ak sa podnet blíži k hranici vnímania, napr. prah počuteľnosti, prah bolesti, táto linearita sa stráca. Predpokladáme, že v strede oblasti je vnímanie lieárne a pri hraniciach sa deformuje. Na modelovanie tohto javu využívame Einsteinovu operáciu, ktorá je používaná v rôznych oblastiach matematiky a fyziky. Einsteinova operácia je odvodená od funkcie hyperbolický tangens. Einsteinova operácia je jednorozmerná. Jej zovšeobecnenie do viacrozmerných priestorov (dokonca Hilbertových priestov) používané vo fyzike nie je vhodné, pretože nezachováva komutatívnosť ani asociatívnosť, pozri napr. [24]. Našli sme niekoľko typov zovšeobecnení do viacrozmerných priestorov, ktoré zachovávajú obidve tieto vlastnosti. Každá súradnica závisí iba od predchádzajúcich súradníc a nezávisí od nasledujúcich súradníc. Reštrikciou na jednorozmerný prípad dostaneme pôvodnú Einsteinovu operáciu.

V druhej časti rozpracúvame teóriu multipolarity, ktorá je pomerne nová. V časti 2.1 zavádzame formálnu definíciu multipolárneho priestoru. Základná myšlienka multipolarity je predstavená na príklade multipolárnych komplexných čísel. Komplexné čísla sú vyjadrené v trojpolárnom tvare, pričom všetky operácie (sčítanie, odčítanie, násobenie, delenie) sú izomorfné, takže trojpolárny priestor je pole. Indukciou sa dá tento postup aplikovať na K-polárny priestor, kde $K > 2$. Všetky tieto K-polárne priestory sú definované nad polopoľom nezáporných reálnych čísel.

V časti 2.4 skúmame trojpolárny priestor, ktorý je definovaný nad polopoľom hyperbolických komplexných čísel, ktoré sú rozšírením polopoľa nezáporných reálnych čísel. V článku [10] sa skúma trojpolárny priestor nad polopoľom parabolických komplexných čísel v rámci RGB modelu farieb. V tomto prípade sa dostáva pole.

Polopolia umožňujú delenie ľubovoľným nenulovým prvkom. No na druhej strane, nie je v nich vo všeobecnosti definované odčítanie. V časti 2.5 hľadáme polopolia definované v štvorrozmernom priestore hyperbolických komutatívnych kvaterniónov. Hyperbolické komutatívne kvaternióny majú aplikácie najmä vo fyzike, napr. Maxwellove rovnice, teória relativity. Štvorrozmerné polopolia sú zaujímavé i z toho pohľadu, že neexistuje štvorrozmerné pole.

Polopole hyperbolických komutatívnych kvaterniónov využijeme na konštrukciu trojpolárneho priestoru. Prvky polopoľa reprezentujú dvojicu harmoník Fourierovho radu. Každá harmonika má dva koeficienty (sínus, kosínus). Uvažujeme prvých šesť harmoník Fourierovho radu. Šesť harmoník zvyčajne stačí na aproximáciu komplexného tónu. Siedma harmonika sa považuje za neharmonickú. Vyššie harmoniky sú zvyčajne buď slabé (majú malú amplitúdu) alebo sú vysoko (majú vysokú frekvenciu mimo rozsahu počutia). Póly reprezentujú tri základné intervaly - unisono, kvinta, tercia. V trojpolárnom priestore sú zavedené aritmetické operácie, ktoré definujú nové operácie s tónmi.

References

[1] F. Antonuccio. “Semicomplex analysis and mathematical physics (corrected version)”. In: *ArXiv General Relativity and Quantum Cosmology e-prints* (1993). arXiv: `gr-qc/9311032`.

[2] F. Catoni. “Commutative (Segre’s) quaternion fields and relation with Maxwell equations”. In: *Advances in Applied Clifford Algebras* 18.1 (2007), pp. 9–28.

[3] F. Catoni et al. *Geometry of Minkowski Space-Time.* SpringerBriefs in Physics. Berlin, Heidelberg: Springer-Verlag, 2011.

[4] *CIE 1931 color space — Wikipedia, The Free Encyclopedia.* [Online; accessed 20.4.2016]. URL: `https://en.wikipedia.org/wiki/CIE_1931_color_space#/media/File:CIE1931xy_CIERGB.svg`.

[5] M. Cipolla. “The system of hypercomplex numbers [in Italian]”. In: *Periodico di Matematica* 20 (1905).

[6] C. M. Davenport. *A Commutative Hypercomplex Calculus with Applications to Special Relativity.* University of Tennessee, 1991.

[7] T. Gregor. *On a semi-field in hyperbolic commutative quaternions.* 2016.

[8] T. Gregor. “Three-polar space over the semi-field of double numbers”. In: *Tatra Mt. Math. Publ.* 61 (2014), pp. 163–173.

[9] T. Gregor and J. Haluška. “Lexicographical ordering and field operations in the complex plane”. In: *Mat. Stud.* 41.2 (2014), pp. 123–133.

[10] J. Haluška. “On fields inspired with the polar HSV – RGB theory of colour”. In: *Modern Real Analysis.* Ed. by J. Hejduk et al. Łódź: Łódź University Press, 2015, pp. 69–88.

[11] J. Haluška and O. Hutník. “On algebras of symmetrical associative aggregation operators related to means”. In: *Tatra Mt. Math. Publ.* 34 (2006), pp. 119–133.

[12] A. A. Harkin and J. B. Harkin. “Geometry of generalized complex numbers”. In: *Math. Mag.* 77.2 (2004), pp. 118–129.

[13] H. L. F. Helmholtz. *On the Sensations of Tone as a Physiological Basis for the Theory of Music.* Third. London: Longmans, Green, and Co., 1895.

[14] K. Jensen and G. Marentakis. *Hybrid perception.* 1st Seminar on Auditory Models. Lyngby, Denmark, 2001.

[15] V. V. Kisil. “Erlangen programme at large: An overview”. In: *ArXiv Complex Variables e-prints* (2011). arXiv: `1106.1686`.

[16] N. Kutka, S. Goceikis, and S. Adomavičius. *Multipolarity radio telescope and radio transmission systems.* Materials of the Multi polarity laboratory of cryptography.

[17] V. Lenski. *Generation of multipolar electromagnetic energy.* US patent 0112111. 2008.

[18] H. F. Pollard. “Tonal portrait of a pipe organ”. In: *The Journal of the Acoustical Society of America* 106.1 (1999), pp. 360–370.

[19] H. F. Pollard and E. V. Jansson. “A tristimulus method for the specification of musical timbre”. In: *Acustica* 51 (1982), pp. 162–171.

[20] G. Scheffers. “Sur la généralisation des fonctions analytiques [in French]”. In: *C. R. Acad. Sci.* 116 (1893), pp. 1114–1117.

[21] G. Scorza. *Corpi numerici e algebre [in Italian].* Messina: Principato, 1921.

[22] C. Segre. “The real representations of complex elements and extension to bicomplex systems [in Italian]”. In: *Math. Ann.* 40 (1892), pp. 413–467.

[23] R. S. Sequera. “Timbrescape: a musical timbre and structure visualization method using tristimulus data”. In: *Proceedings of the 9th International Conference on Music Perception and Cognition (ICMPC), Bologna.* 2006.

[24] A. A. Ungar. “Einstein’s velocity addition law and its hyperbolic geometry”. In: *Comput. Math. Appl.* 53 (2007), pp. 1228–1250.

List of my publications

Journal articles

GREGOR, Tomáš - HALUŠKA, Ján. Lexicographical ordering and field operations in the complex plane. In Matematichni Studii, 2014, vol. 41, s. 123-133.

GREGOR, Tomáš - HALUŠKA, Ján. Generalizations of Einstein Numbers by Adding New Dimensions to Domains Preserving Commutativity and Associativity. In Journal of Advanced Mathematics and Applications, 2015, vol. 4, no. 1, p. 1-8. (2015 - WOS). ISSN 2156-7565.

GREGOR, Tomáš. Three-polar space over the semi-field of double numbers. In Tatra Mountains Mathematical Publications, 2014, vol. 61, p. 163-173. (2014 - SCOPUS). ISSN 1210-3195. <http://tatra.mat.savba.sk/>.

Citation:
HALUŠKA, Ján. On fields inspired with the polar HSV - RGB theory of colour. In Modern Real Analysis. Eds. J. Hejduk et al. Łódź: Wydawnictwo Uniwersytetu Lodzkiego, 2015. ISBN 978-83-7969-663-5, p. 69-88.

Conferences

The 27th International Summer Conference on Real Functions Theory, Niedzica, Poland, September 1 - 6, 2013.

Complex Analysis and Related Topics, Lviv, Ukraine, September 23 - 28, 2013.

International Student Conference on Applied Mathematics and Informatics, Malenovice, Czech Republic, March 27 - 30, 2014.

The 28th International Summer Conference on Real Functions Theory, Stará Lesná, Slovakia, August 31 - September 5, 2014.

16. Konferencia košických matematikov, Herľany, Slovakia, March 25 - 28, 2015.

The 29th International Summer Conference on Real Functions Theory, Niedzica, Poland, September 6 - 11, 2015.