COMENIUS UNIVERSITY IN BRATISLAVA
FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS

BIFURCATION AND ASYMPTOTIC PROPERTIES OF PERIODIC SOLUTIONS IN DISCONTINUOUS SYSTEMS

Dissertation thesis

2012 **RNDr. Michal Pospíšil**

COMENIUS UNIVERSITY IN BRATISLAVA
FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS

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Abstrakt

Táto práca je venovaná štúdiu bifurkácií periodických riešení všeobecných n-rozmerných nespojitých autonómnych systémov. Za predpokladu transverzálneho prechodu cez hranicu nespojitosti sú stanovené postačujúce podmienky prežitia jediného periodického riešenia pri neautonómnej perturbácii z jediného riešenia alebo autonómnej perturbácii z nedegenerovaného systému riešení alebo izolovaného riešenia. Ďalej sú študované bifurkácie periodických kĺzavých riešení z kĺzavých periodických riešení neperturbovaných nespojitých rovníc a nútené periodické riešenia impaktných systémov z jediného periodického riešenia neperturbovaných impaktných rovníc. Naviac sú vyšetrované lokálne asymptotické vlastnosti odvodených perturbovaných periodických riešení. Priložené 2-, 3- a 4-rozmerné príklady nespojitých obyčajných diferenciálnych rovníc a impaktných systémov znázorňujú teoretické výsledky.

Kľúčové slová: periodická orbita, kĺzavá periodická orbita, nespojité systémy, impaktné systémy, bifurkácia.

Abstract

This work is devoted to the study of bifurcations of periodic solutions for general n-dimensional discontinuous autonomous systems. By the assumption of a transversal intersection with the discontinuity boundary the sufficient conditions for the persistence of a single periodic solution under nonautonomous perturbation from a single solution or autonomous perturbation from a nondegenerate family of solutions or isolated solution are stated. Furthermore, bifurcations of periodic sliding solutions from sliding periodic solutions of unperturbed discontinuous equations and forced periodic solutions for impact systems from a single periodic solution of unperturbed impact equations are studied. In addition, local asymptotic properties of derived perturbed periodic solutions are also investigated. Examples of 2-, 3- and 4-dimensional discontinuous ordinary differential equations and impact systems are given to illustrate the theoretical results.

Keywords: periodic orbit, sliding periodic orbit, discontinuous systems, impact systems, bifurcation.

Foreword

Discontinuous systems describe many real processes which are characterized by instantaneous change such as electrical switch or bouncing ball. These can be only approximated by classical – smooth-systems theory. This is the reason why there occurred many papers and books on this topic in the last few years. However, the scientists avoid the use of so-called discontinuous Poincaré mapping which maps a point to its position after one period, since the work with the mapping is rather technical. On the other side, by this method we can get results for general dimensions of spatial variables and parameters as well as the asymptotical results such as stability, instability and hyperbolicity. This is the aim of this thesis. After my diploma thesis on periodic orbits in planar discontinuous systems, it was a natural consequence to immerse myself in discontinuous systems and investigate their problems. No one else has ever before studied the discontinuous systems in such general settings. Therefore, our results are original and successfully published. I have to admit that this would not be possible without the aid and ideas of my supervisor.

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Introduction

The persistence/bifurcation/continuation of periodic oscillations is one of the fundamental problems in evolutionary differential systems. The corresponding mathematical theory is well-developed for smooth nonlinear dynamical systems (NDS) and can be found e.g. in books [12, 14, 19]. These books include a lot of references on the classical theory of differential equations. Recently, many interesting results appeared extending the known theory to systems with discontinuous right-hand side. In various cases (for references see e.g. [9, 27, 38]) such systems are much more appropriate for describing the real problems concerning

- a composition of smooth systems – perturbed piecewise-smooth NDS (PPSNDS) [1–9, 26–29, 34–44],
- a smooth system with impact condition – impacting hybrid systems [11,13,18,50].

Discontinuous systems are used for modelling systems with instantaneous change of external forces or parameters of the system, e.g. electrical circuits with switches, diodes or transistors, mechanical devices in which components impact with each other (such as gear assemblies), problems with friction, sliding or squealing and models in the social and financial sciences where continuous change can trigger discrete actions.

In this work we apply the Melnikov method for continuous systems [32] on PPSNDS and hybrid systems. It means that assuming the existence of an isolated periodic solution or a parametrized system of periodic solutions for unperturbed system we construct a discontinuous Poincaré mapping [9, 35, 36] and the corresponding distance function/bifurcation map which zeros imply the periodic solutions for the perturbed system. Using Lyapunov-Schmidt reduction method [12, 45] we derive sufficient conditions in terms of Poincaré-Andronov-Melnikov function on the perturbation for the persistence of a single periodic solution. For the simplicity, we always assume that the original solution hits the boundary transversally. Note that one of the analogical results for the smooth case is Poincaré-Andronov theorem [46].

In Chapter I on PPSNDS we consider two autonomous differential equations connected by a discontinuity boundary/level/set which is usually a smooth submanifold of codimension 1 in $\mathbb{R}^n$. First we investigate the persistence of a T-periodic solution under a small nonautonomous T-periodic perturbation in Section 1, then the bifurcation of a periodic solution from a family of periodic orbits or a single periodic solution under an autonomous perturbation in Section 2 and Section 3, respectively. The final section of the first chapter is devoted to the continuation of a sliding periodic solution of a discontinuous autonomous equation under a nonautonomous perturbation, i.e. we consider a periodic solution which remains on the discontinuity level for some time. In

Chapter II, when a solution of a differential equation hits the boundary, it is immediately mapped by a difference equation to a point of the boundary. So we study the persistence of a periodic solution of a periodically forced autonomous impacting hybrid system.

Moreover, we study the local asymptotic properties of a persisting solution such as hyperbolicity, stability and instability. This is probably the biggest advantage of the method of Poincaré mapping used in this work. In addition, each section contains applications of derived theory to a nontrivial problems, often endowed with numerically computed illustrations which confirm the theoretical results. Furthermore, in comparison to papers and books published by other authors, we have no restriction on the dimensions of a spatial variable and parameters. Finally, we note that this work is based on our papers [21–25] created by the even contribution of both their authors.

Notation:

$[v_1, \dots, v_k]$	linear span of vectors $v_1, \dots, v_k$,
$\langle \cdot, \cdot \rangle$	inner product in $\mathbb{R}^n$,
$\Vert \cdot \Vert$	norm in $\mathbb{R}^n$ generated by $\langle \cdot, \cdot \rangle$,
$B(x, r)$	ball with the center at x and radius r,
$C_b^r(M)$	the set of all functions that are uniformly continuous and bounded on M up to the r-th order,
$\Im\lambda$	imaginary part of $\lambda \in \mathbb{C}$,
$\mathbb{I}$	$n \times n$ identity matrix,
IFT	implicit function theorem [16],
$\mathcal{N}A$	null space of the operator A,
$\Re\lambda$	real part of $\lambda \in \mathbb{C}$,
$\mathcal{R}A$	range of the operator A.

Chapter I

Piecewise-smooth systems

This chapter is devoted to perturbed piecewise-smooth nonlinear dynamical systems under which we understand a differential equation

$$\dot{x} = F(t, x, \chi),$$

where $F(t, x, \chi)$ is a smooth function on $\mathbb{R} \times (\mathbb{R}^n \backslash S) \times \mathbb{R}^m$ periodic in $t \in \mathbb{R}$. Here S denotes the discontinuity set – in this work it is a sufficiently smooth hypersurface of $\mathbb{R}^n$. Moreover, we suppose that $F(t, x, 0) = F(x)$, i.e. function $F(t, x, \chi)$ is independent of t at $\chi_0 = 0$ and the associated autonomous system

$$\dot{x} = F(x)$$

possesses a periodic solution $\gamma(t)$ that transversally hits S. For the case of transverse crossing we prove the persistence of periodically forced isolated solution and the bifurcation from a nondegenerate family or a single periodic solution under autonomous perturbation (here $F(t, x, \chi) = F(x, \chi)$ for any χ). If $\gamma(t)$ does not cross the boundary, we investigate a periodically forced sliding periodic solution. We also investigate hyperbolicity, stability and instability of free solutions.

1 Periodically forced discontinuous systems

In this section, we investigate the persistence of periodic orbit in an autonomous discontinuous system under a small nonautonomous perturbation. More precisely, we assume that the unperturbed equation possesses a periodic solution that transversally crosses the discontinuity boundary and we look for sufficient conditions on the perturbation such that the perturbed equation has a periodic solution which is close to the original one and has the same period.

Let $\Omega \subset \mathbb{R}^n$ be an open set in $\mathbb{R}^n$ and $h(x)$ be a C^r-function on $\overline{\Omega}$, with $r \geq 2$. We set $\Omega_\pm := \{x \in \Omega \mid \pm h(x) > 0\}$, $\Omega_0 := \{x \in \Omega \mid h(x) = 0\}$. Let $f_\pm \in C_b^r(\overline{\Omega})$, $g \in C_b^r(\overline{\Omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^p)$ and $h \in C_b^r(\overline{\Omega}, \mathbb{R})$. Furthermore, we suppose that g is T-periodic in $t \in \mathbb{R}$ and 0 is a regular value of h. Let $\varepsilon, \alpha \in \mathbb{R}$ and $\mu \in \mathbb{R}^p, p \geq 1$ be parameters.

Definition 1.1. We say that a function $x(t)$ is a solution of the equation

$$\dot{x} = f_\pm(x) + \varepsilon g(x, t + \alpha, \varepsilon, \mu), \quad x \in \overline{\Omega}_\pm, \tag{1.1}$$

if it is continuous, piecewise C^1, satisfies equation (1.1) on $\Omega_\pm$ and, moreover, the following holds: if for some t_0 we have $x(t_0) \in \Omega_0$, then there exists $\rho > 0$ such that for any $t \in (t_0 - \rho, t_0)$ we have $x(t) \in \Omega_\pm$, and for any $t \in (t_0, t_0 + \rho)$ we have $x(t) \in \Omega_\mp$.

We assume (see Fig. 1.1)

H1) For $\varepsilon = 0$ equation (1.1) has a T-periodic solution $\gamma(t)$ which has a starting point $x_0 \in \Omega_+$ and consists of three branches

$$\gamma(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [0, t_1], \\ \gamma_2(t) & \text{if } t \in [t_1, t_2], \\ \gamma_3(t) & \text{if } t \in [t_2, T], \end{cases} \tag{1.2}$$

where $0 < t_1 < t_2 < T$, $\gamma_1(t) \in \Omega_+$ for $t \in [0, t_1)$, $\gamma_2(t) \in \Omega_-$ for $t \in (t_1, t_2)$ and $\gamma_3(t) \in \Omega_+$ for $t \in (t_2, T]$ and

$$\begin{aligned} x_1 &:= \gamma_1(t_1) = \gamma_2(t_1) \in \Omega_0, \\ x_2 &:= \gamma_2(t_2) = \gamma_3(t_2) \in \Omega_0, \\ x_0 &:= \gamma_3(T) = \gamma_1(0) \in \Omega_+. \end{aligned} \tag{1.3}$$

H2) Moreover, we also assume that

$$\mathrm{D}h(x_1) f_\pm(x_1) < 0 \quad \text{and} \quad \mathrm{D}h(x_2) f_\pm(x_2) > 0.$$

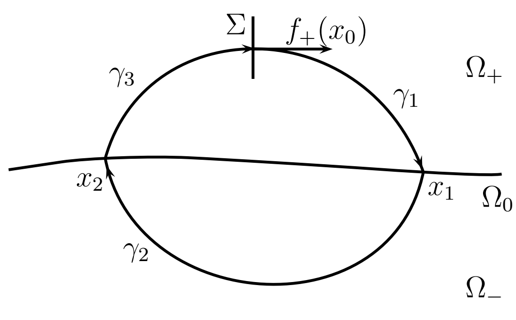

Fig. 1.1: Used notation

Let $x_\pm(\tau, \xi)(t, \varepsilon, \mu, \alpha)$ denote a solution of initial value problem

$$\begin{aligned} \dot{x} &= f_\pm(x) + \varepsilon g(x, t + \alpha, \varepsilon, \mu) \\ x(\tau) &= \xi \end{aligned} \tag*{$(1.4)_\pm$}$$

with corresponding sign.

Using implicit function theorem (IFT) [16] we show that there are some trajectories in the neighbourhood of $\gamma(t)$ and then we select periodic ones from these.

Lemma 1.2. *Assume* H1) *and* H2). *Then there exist* $\varepsilon_3, r_3 > 0$ *and a Poincaré mapping*

$$P(\cdot, \varepsilon, \mu, \alpha) : B(x_0, r_3) \to \Sigma$$

for all fixed $\varepsilon \in (-\varepsilon_3, \varepsilon_3)$, $\mu \in \mathbb{R}^p$, $\alpha \in \mathbb{R}$ *where* $\Sigma = \{x \in \mathbb{R}^n \mid \langle x - x_0, f_+(x_0)\rangle = 0\}$ *and* $B(x, r)$ *is the ball of radius* r *and center at* x. *Moreover,* P *is* C^r*-smooth in all arguments.*

Proof. We denote $\mathcal{A}(\tau, \xi, t, \varepsilon, \mu, \alpha) = h(x_+(\tau, \xi)(t, \varepsilon, \mu, \alpha))$. Since

$$\mathcal{A}(0, x_0, t_1, 0, \mu, \alpha) = 0, \quad \mathrm{D}_t\mathcal{A}(0, x_0, t_1, 0, \mu, \alpha) = \mathrm{D}h(x_1)f_+(x_1) < 0,$$

IFT yields the existence of $\tau_1, r_1, \delta_1, \varepsilon_1 > 0$ and C^r-function

$$t_1(\cdot, \cdot, \cdot, \cdot, \cdot) : (-\tau_1, \tau_1) \times B(x_0, r_1) \times (-\varepsilon_1, \varepsilon_1) \times \mathbb{R}^p \times \mathbb{R} \to (t_1 - \delta_1, t_1 + \delta_1)$$

such that $\mathcal{A}(\tau, \xi, t, \varepsilon, \mu, \alpha) = 0$ for $\tau \in (-\tau_1, \tau_1)$, $\xi \in B(x_0, r_1) \subset \Omega_+$, $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$, $\mu \in \mathbb{R}^p$, $\alpha \in \mathbb{R}$ and $t \in (t_1 - \delta_1, t_1 + \delta_1)$ if and only if $t = t_1(\tau, \xi, \varepsilon, \mu, \alpha)$.

Next we set

$$\mathcal{B}(\tau, \xi, t, \varepsilon, \mu, \alpha) = h(x_-(t_1(\tau, \xi, \varepsilon, \mu, \alpha), x_+(\tau, \xi)(t_1(\tau, \xi, \varepsilon, \mu, \alpha), \varepsilon, \mu, \alpha))(t, \varepsilon, \mu, \alpha)).$$

Then

$$\mathcal{B}(0, x_0, t_2, 0, \mu, \alpha) = 0, \quad \mathrm{D}_t\mathcal{B}(0, x_0, t_2, 0, \mu, \alpha) = \mathrm{D}h(x_2)f_-(x_2) > 0,$$

hence IFT implies that there exist $\tau_2, r_2, \delta_2, \varepsilon_2 > 0$ and C^r-function

$$t_2(\cdot, \cdot, \cdot, \cdot, \cdot) : (-\tau_2, \tau_2) \times B(x_0, r_2) \times (-\varepsilon_2, \varepsilon_2) \times \mathbb{R}^p \times \mathbb{R} \to (t_2 - \delta_2, t_2 + \delta_2)$$

such that $\mathcal{B}(\tau, \xi, t, \varepsilon, \mu, \alpha) = 0$ for $\tau \in (-\tau_2, \tau_2)$, $\xi \in B(x_0, r_2) \subset \Omega_+, \varepsilon \in (-\varepsilon_2, \varepsilon_2)$, $\mu \in \mathbb{R}^p$, $\alpha \in \mathbb{R}$ and $t \in (t_2 - \delta_2, t_2 + \delta_2)$ if and only if $t = t_2(\tau, \xi, \varepsilon, \mu, \alpha)$.

Once more time we use IFT on function $\mathcal{C}$ defined as

$$\begin{aligned}\mathcal{C}(\tau, \xi, t, \varepsilon, \mu, \alpha) = \langle x_+(t_2(\tau, \xi, \varepsilon, \mu, \alpha), x_-(t_1(\tau, \xi, \varepsilon, \mu, \alpha), x_+(\tau, \xi)(t_1(\tau, \xi, \varepsilon, \mu, \alpha), \varepsilon, \mu, \alpha))\\ (t_2(\tau, \xi, \varepsilon, \mu, \alpha), \varepsilon, \mu, \alpha))(t, \varepsilon, \mu, \alpha) - x_0, f_+(x_0)\rangle\,.\end{aligned}$$

Since

$$\mathcal{C}(0, x_0, T, 0, \mu, \alpha) = 0, \quad \mathrm{D}_t\mathcal{C}(0, x_0, T, 0, \mu, \alpha) = \|f_+(x_0)\|^2 > 0,$$

there exist $\tau_3, r_3, \delta_3, \varepsilon_3 > 0$ and C^r-function

$$t_3(\cdot, \cdot, \cdot, \cdot, \cdot) : (-\tau_3, \tau_3) \times B(x_0, r_3) \times (-\varepsilon_3, \varepsilon_3) \times \mathbb{R}^p \times \mathbb{R} \to (T - \delta_3, T + \delta_3)$$

such that $\mathcal{C}(\tau, \xi, t, \varepsilon, \mu, \alpha) = 0$ for $\tau \in (-\tau_3, \tau_3)$, $\xi \in B(x_0, r_3) \subset \Omega_+, \varepsilon \in (-\varepsilon_3, \varepsilon_3)$, $\mu \in \mathbb{R}^p$, $\alpha \in \mathbb{R}$ and $t \in (T - \delta_3, T + \delta_3)$ if and only if $t = t_3(\tau, \xi, \varepsilon, \mu, \alpha)$. Moreover $t_1(0, x_0, 0, \mu, \alpha) = t_1$, $t_2(0, x_0, 0, \mu, \alpha) = t_2$ and $t_3(0, x_0, 0, \mu, \alpha) = T$. Now we can define the Poincaré mapping from the statement

$$\begin{aligned}P(\xi, \varepsilon, \mu, \alpha) = x_+(t_2(0, \xi, \varepsilon, \mu, \alpha), x_-(t_1(0, \xi, \varepsilon, \mu, \alpha), x_+(0, \xi)(t_1(0, \xi, \varepsilon, \mu, \alpha), \varepsilon, \mu, \alpha))\\ (t_2(0, \xi, \varepsilon, \mu, \alpha), \varepsilon, \mu, \alpha))(t_3(0, \xi, \varepsilon, \mu, \alpha), \varepsilon, \mu, \alpha).\end{aligned}$$

Obviously, P maps $B(x_0, r_3)$ to Σ. $\square$

Our aim is to find T-periodic orbits, which is the reason for solving the following system

$$\begin{aligned} P(\xi,\varepsilon,\mu,\alpha) &= \xi \\ t_3(0,\xi,\varepsilon,\mu,\alpha) &= T \end{aligned}$$

for ξ and ε sufficiently close to x_0 and 0, respectively. This problem can be reduced to one equation

$$F(\xi,\varepsilon,\mu,\alpha) := \xi - \widetilde{P}(\xi,\varepsilon,\mu,\alpha) = 0 \tag{1.5}$$

where

$$\begin{aligned} \widetilde{P}(\xi,\varepsilon,\mu,\alpha) = x_+(t_2(0,\xi,\varepsilon,\mu,\alpha), x_-(t_1(0,\xi,\varepsilon,\mu,\alpha), x_+(0,\xi)(t_1(0,\xi,\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha)) \\ (t_2(0,\xi,\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha))(T,\varepsilon,\mu,\alpha) \end{aligned} \tag{1.6}$$

is so-called stroboscopic Poincaré mapping (cf. [9]). It is easy to see that $(\xi,\varepsilon) = (x_0, 0)$ solves equation (1.5) for any $\mu \in \mathbb{R}^p, \alpha \in \mathbb{R}$. However, IFT can not be used here, what is proved in the next lemma (see [35, 36]).

Lemma 1.3. *Let $\widetilde{P}(\xi,\varepsilon,\mu,\alpha)$ be defined by (1.6). Then $\widetilde{P}_\xi(x_0,0,\mu,\alpha)$ has eigenvalue 1 with corresponding eigenvector $f_+(x_0)$, i.e. $\widetilde{P}_\xi(x_0,0,\mu,\alpha)f_+(x_0) = f_+(x_0)$, where $\widetilde{P}_\xi$ denotes the partial derivative of $\widetilde{P}$ with respect to ξ.*

Proof. Let V be a sufficiently small neighbourhood of 0. Then

$$\begin{aligned} x_+(0, x_+(0,x_0)(t,0,\mu,\alpha))(t_1(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha),0,\mu,\alpha) \\ = x_+(0,x_0)(t + t_1(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha),0,\mu,\alpha) \end{aligned} \tag{1.7}$$

for any $t \in V$, where the left-hand side of (1.7) is from Ω_0 and the right-hand side is a point of $\gamma(t)$. Thereafter $t + t_1(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha) = t_1$, i.e. it is constant for all $t \in V$. Similarly

$$\begin{aligned} & x_-(t_1(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha), x_+(0,x_+(0,x_0)(t,0,\mu,\alpha))(t_1(0,x_+(0,x_0) \\ & \quad (t,0,\mu,\alpha),0,\mu,\alpha),0,\mu,\alpha))(t_2(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha),0,\mu,\alpha) \\ & = x_-(t_1(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha), x_+(0,x_0)(t_1,0,\mu,\alpha)) \\ & \quad (t_2(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha),0,\mu,\alpha) \\ & = x_-(t_1 - t, x_+(0,x_0)(t_1,0,\mu,\alpha))(t_2(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha),0,\mu,\alpha) \\ & = x_-(t_1,x_1)(t_2(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha) + t,0,\mu,\alpha) \end{aligned}$$

and we obtain $t + t_2(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha) = t_2$ for all $t \in V$.
With these results we can derive

$$\begin{aligned} & \widetilde{P}(x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha) \\ & = x_+(t_2(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha), x_-(t_1(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha), \\ & \quad x_+(0,x_+(0,x_0)(t,0,\mu,\alpha))(t_1(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha),0,\mu,\alpha)) \\ & \quad (t_2(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha),0,\mu,\alpha))(T,0,\mu,\alpha) \\ & = x_+(t_2(0,x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha),x_2)(T,0,\mu,\alpha) \\ & = x_+(t_2 - t, x_2)(T,0,\mu,\alpha) = x_+(t_2,x_2)(T + t,0,\mu,\alpha) \end{aligned}$$

and finally

$$\begin{aligned}
\widetilde{P}_\xi(x_0,0,\mu,\alpha)f(x_0) &= \mathrm{D}_t\left[\widetilde{P}(x_+(0,x_0)(t,0,\mu,\alpha),0,\mu,\alpha)\right]_{t=0}\\
&= \mathrm{D}_t\left[x_+(t_2,x_2)(T+t,0,\mu,\alpha)\right]_{t=0}\\
&= f(x_+(t_2,x_2)(T+t,0,\mu,\alpha))|_{t=0} = f(x_0).
\end{aligned}$$

□

In the next step we construct the linearization $\widetilde{P}_\xi(x_0,0,\mu,\alpha)$ which will be important in further work.

Differentiating $(1.4)_+$ with respect to ξ at the point $(\tau,\xi,\varepsilon)=(0,x_0,0)$ we get

$$\begin{aligned}
\dot{x}_{+\xi}(0,x_0)(t,0,\mu,\alpha) &= \mathrm{D}f_+(\gamma(t))x_{+\xi}(0,x_0)(t,0,\mu,\alpha)\\
x_{+\xi}(0,x_0)(0,0,\mu,\alpha) &= \mathbb{I}
\end{aligned}$$

where $\mathbb{I}$ denotes $n\times n$ identity matrix. Denote by $X_1(t)$ the matrix solution satisfying this linearized equation on $[0,t_1]$, i.e.

$$\begin{aligned}
\dot{X}_1(t) &= \mathrm{D}f_+(\gamma(t))X_1(t)\\
X_1(0) &= \mathbb{I}.
\end{aligned} \tag{1.8}$$

So $x_{+\xi}(0,x_0)(t,0,\mu,\alpha)=X_1(t)$. By differentiation $(1.4)_+$ with respect to τ at the same point we get

$$\begin{aligned}
\dot{x}_{+\tau}(0,x_0)(t,0,\mu,\alpha) &= \mathrm{D}f_+(\gamma(t))x_{+\tau}(0,x_0)(t,0,\mu,\alpha)\\
x_{+\tau}(0,x_0)(0,0,\mu,\alpha) &= -f_+(x_+(0,x_0)(0,0,\mu,\alpha)).
\end{aligned}$$

Hence

$$x_{+\tau}(0,x_0)(t,0,\mu,\alpha) = -X_1(t)f_+(x_0)$$

for $t\in[0,t_1]$. Also the derivative of $(1.4)_+$ with respect to ε at $(0,x_0,0)$ will be needed. We obtain the initial value problem

$$\begin{aligned}
\dot{x}_{+\varepsilon}(0,x_0)(t,0,\mu,\alpha) &= \mathrm{D}f_+(\gamma(t))x_{+\varepsilon}(0,x_0)(t,0,\mu,\alpha)+g(\gamma(t),t+\alpha,0,\mu)\\
x_{+\varepsilon}(0,x_0)(0,0,\mu,\alpha) &= 0
\end{aligned}$$

which solved by variation of constants gives equality

$$x_{+\varepsilon}(0,x_0)(t,0,\mu,\alpha) = \int_0^t X_1(t)X_1^{-1}(s)g(\gamma(s),s+\alpha,0,\mu)ds$$

holding on $[0,t_1]$.

First intersection point on Ω_0 fulfils

$$h(x_+(\tau,\xi)(t_1(\tau,\xi,\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha))=0$$

for all (τ,ξ,ε) sufficiently close to $(0,x_0,0)$ and $\mu\in\mathbb{R}^p$, $\alpha\in\mathbb{R}$. Thus differentiating the latter identity with respect to ξ, τ and ε at $(\tau,\xi,\varepsilon)=(0,x_0,0)$ yields

$$\begin{aligned}
\mathrm{D}h(x_1)(X_1(t_1)+f_+(x_1)t_{1\xi}(0,x_0,0,\mu,\alpha)) &= 0\\
t_{1\xi}(0,x_0,0,\mu,\alpha) &= -\frac{\mathrm{D}h(x_1)X_1(t_1)}{\mathrm{D}h(x_1)f_+(x_1)},
\end{aligned}$$

$$
\begin{aligned}
\mathrm{D}h(x_1)(-X_1(t_1)f_+(x_0) + f_+(x_1)t_{1\tau}(0,x_0,0,\mu,\alpha)) &= 0\\
t_{1\tau}(0,x_0,0,\mu,\alpha) &= \frac{\mathrm{D}h(x_1)X_1(t_1)f_+(x_0)}{\mathrm{D}h(x_1)f_+(x_1)}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{D}h(x_1)\left(f_+(x_1)t_{1\varepsilon}(0,x_0,0,\mu,\alpha) + \int_0^{t_1} X_1(t_1)X_1^{-1}(s)g(\gamma(s),s+\alpha,0,\mu)ds\right) &= 0\\
t_{1\varepsilon}(0,x_0,0,\mu,\alpha) = -\frac{\mathrm{D}h(x_1)\int_0^{t_1} X_1(t_1)X_1^{-1}(s)g(\gamma(s),s+\alpha,0,\mu)ds}{\mathrm{D}h(x_1)f_+(x_1)}&,
\end{aligned}
$$

respectively.

Next, differentiating $(1.4)_-$ with respect to ξ, τ and ε at the point $(\tau,\xi,\varepsilon) = (t_1,x_1,0)$ we obtain

$$
\begin{aligned}
\dot{x}_{-\xi}(t_1,x_1)(t,0,\mu,\alpha) &= \mathrm{D}f_-(\gamma(t))x_{-\xi}(t_1,x_1)(t,0,\mu,\alpha)\\
x_{-\xi}(t_1,x_1)(t_1,0,\mu,\alpha) &= \mathbb{I},
\end{aligned}
$$

$$
\begin{aligned}
\dot{x}_{-\tau}(t_1,x_1)(t,0,\mu,\alpha) &= \mathrm{D}f_-(\gamma(t)))x_{-\tau}(t_1,x_1)(t,0,\mu,\alpha)\\
x_{-\tau}(t_1,x_1)(t_1,0,\mu,\alpha) &= -f_-(x_-(t_1,x_1)(t_1,0,\mu,\alpha))
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{x}_{-\varepsilon}(t_1,x_1)(t,0,\mu,\alpha) &= \mathrm{D}f_-(\gamma(t))x_{-\varepsilon}(t_1,x_1)(t,0,\mu,\alpha) + g(\gamma(t),t+\alpha,0,\mu)\\
x_{-\varepsilon}(t_1,x_1)(t_1,0,\mu,\alpha) &= 0,
\end{aligned}
$$

respectively, for $t \in [t_1,t_2]$. Using matrix solution $X_2(t)$ of the first equation satisfying

$$
\begin{aligned}
\dot{X}_2(t) &= \mathrm{D}f_-(\gamma(t))X_2(t)\\
X_2(t_1) &= \mathbb{I},
\end{aligned}
\tag{1.9}
$$

i.e. $x_{-\xi}(t_1,x_1)(t,0,\mu,\alpha) = X_2(t)$, we can rewrite the other two solutions as

$$
\begin{aligned}
x_{-\tau}(t_1,x_1)(t,0,\mu,\alpha) &= -X_2(t)f_-(x_1),\\
x_{-\varepsilon}(t_1,x_1)(t,0,\mu,\alpha) &= \int_{t_1}^{t} X_2(t)X_2^{-1}(s)g(\gamma(s),s+\alpha,0,\mu)ds
\end{aligned}
$$

for $t \in [t_1,t_2]$.

Second intersection point is characterized by

$$
h(x_-(t_1(\tau,\xi,\varepsilon,\mu,\alpha),x_+(\tau,\xi)(t_1(\tau,\xi,\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha))(t_2(\tau,\xi,\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha)) = 0.
$$

From that we derive

$$
\begin{aligned}
\mathrm{D}h(x_2)(x_{-\tau}(t_1,x_1)(t_2,0,\mu,\alpha)t_{1\xi}(0,x_0,0,\mu,\alpha) + x_{-\xi}(t_1,x_1)(t_2,0,\mu,\alpha)\\
\times[x_{+\xi}(0,x_0)(t_1,0,\mu,\alpha) + x_{+t}(0,x_0)(t_1,0,\mu,\alpha)t_{1\xi}(0,x_0,0,\mu,\alpha)]\\
+x_{-t}(t_1,x_1)(t_2,0,\mu,\alpha)t_{2\xi}(0,x_0,0,\mu,\alpha)) = 0
\end{aligned}
$$

$$t_{2\xi}(0,x_0,0,\mu,\alpha) = -\frac{\mathrm{D}h(x_2)X_2(t_2)S_1X_1(t_1)}{\mathrm{D}h(x_2)f_-(x_2)},$$

$$\begin{aligned}\mathrm{D}h(x_2)(x_{-\tau}(t_1,x_1)(t_2,0,\mu,\alpha)t_{1\tau}(0,x_0,0,\mu,\alpha) + x_{-\xi}(t_1,x_1)(t_2,0,\mu,\alpha)\\ \times[x_{+\tau}(0,x_0)(t_1,0,\mu,\alpha) + x_{+t}(0,x_0)(t_1,0,\mu,\alpha)t_{1\tau}(0,x_0,0,\mu,\alpha)]\\ +x_{-t}(t_1,x_1)(t_2,0,\mu,\alpha)t_{2\tau}(0,x_0,0,\mu,\alpha)) = 0\end{aligned}$$

$$t_{2\tau}(0,x_0,0,\mu,\alpha) = \frac{\mathrm{D}h(x_2)X_2(t_2)S_1X_1(t_1)f_+(x_0)}{\mathrm{D}h(x_2)f_-(x_2)}$$

and

$$\begin{aligned}\mathrm{D}h(x_2)(x_{-\tau}(t_1,x_1)(t_2,0,\mu,\alpha)t_{1\varepsilon}(0,x_0,0,\mu,\alpha) + x_{-\xi}(t_1,x_1)(t_2,0,\mu,\alpha)\\ \times[x_{+\varepsilon}(0,x_0)(t_1,0,\mu,\alpha) + x_{+t}(0,x_0)(t_1,0,\mu,\alpha)t_{1\varepsilon}(0,x_0,0,\mu,\alpha)]\\ +x_{-t}(t_1,x_1)(t_2,0,\mu,\alpha)t_{2\varepsilon}(0,x_0,0,\mu,\alpha) + x_{-\varepsilon}(t_1,x_1)(t_2,0,\mu,\alpha)) = 0\end{aligned}$$

$$\begin{aligned}t_{2\varepsilon}(0,x_0,0,\mu,\alpha) = -\frac{\mathrm{D}h(x_2)}{\mathrm{D}h(x_2)f_-(x_2)}\Bigg(X_2(t_2)S_1\int_0^{t_1} X_1(t_1)X_1^{-1}(s)\\ \times g(\gamma(s),s+\alpha,0,\mu)ds + \int_{t_1}^{t_2} X_2(t_2)X_2^{-1}(s)g(\gamma(s),s+\alpha,0,\mu)ds\Bigg),\end{aligned}$$

where

$$S_1 = \mathbb{I} + \frac{(f_-(x_1) - f_+(x_1))\mathrm{D}h(x_1)}{\mathrm{D}h(x_1)f_+(x_1)} \tag{1.10}$$

is so-called saltation matrix [35, 42].

Finally, we count derivatives of $(1.4)_+$ with respect to ξ, τ and ε at $(\tau,\xi,\varepsilon) = (t_2,x_2,0)$ to obtain

$$\begin{aligned}\dot{x}_{+\xi}(t_2,x_2)(t,0,\mu,\alpha) &= \mathrm{D}f_+(\gamma(t))x_{+\xi}(t_2,x_2)(t,0,\mu,\alpha)\\ x_{+\xi}(t_2,x_2)(t_2,0,\mu,\alpha) &= \mathbb{I},\end{aligned}$$

$$\begin{aligned}\dot{x}_{+\tau}(t_2,x_2)(t,0,\mu,\alpha) &= \mathrm{D}f_+(\gamma(t))x_{+\tau}(t_2,x_2)(t,0,\mu,\alpha)\\ x_{+\tau}(t_2,x_2)(t_2,0,\mu,\alpha) &= -f_+(x_+(t_2,x_2)(t_2,0,\mu,\alpha))\end{aligned}$$

and

$$\begin{aligned}\dot{x}_{+\varepsilon}(t_2,x_2)(t,0,\mu,\alpha) &= \mathrm{D}f_+(\gamma(t))x_{+\varepsilon}(t_2,x_2)(t,0,\mu,\alpha) + g(\gamma(t),t+\alpha,0,\mu)\\ x_{+\varepsilon}(t_2,x_2)(t_2,0,\mu,\alpha) &= 0,\end{aligned}$$

respectively, on $[t_2,T]$. Matrix solution $X_3(t)$ for first equation that for $t \in [t_2,T]$ satisfies

$$\begin{aligned}\dot{X}_3(t) &= \mathrm{D}f_+(\gamma(t))X_3(t)\\ X_3(t_2) &= \mathbb{I},\end{aligned} \tag{1.11}$$

i.e. $x_{+\xi}(t_2,x_2)(t,0,\mu,\alpha) = X_3(t)$, simplifies expressions for the other two solutions:

$$\begin{aligned}x_{+\tau}(t_2,x_2)(t,0,\mu,\alpha) &= -X_3(t)f_+(x_2),\\ x_{+\varepsilon}(t_2,x_2)(t,0,\mu,\alpha) &= \int_{t_2}^{t} X_3(t)X_3^{-1}(s)g(\gamma(s),s+\alpha,0,\mu)ds\end{aligned}$$

for $t \in [t_2,T]$. Now we can state the following lemma.

Lemma 1.4. *Let $\widetilde{P}(\xi,\varepsilon,\mu,\alpha)$ be defined by* (1.6). *Then*

$$\widetilde{P}_\xi(x_0,0,\mu,\alpha) = X_3(T)S_2X_2(t_2)S_1X_1(t_1), \tag{1.12}$$

$$\widetilde{P}_\varepsilon(x_0,0,\mu,\alpha) = \int_0^T A(s)g(\gamma(s),s+\alpha,0,\mu)ds, \tag{1.13}$$

where $\widetilde{P}_\xi$ and $\widetilde{P}_\varepsilon$ denote the partial derivatives of $\widetilde{P}$ with respect to ξ and ε, respectively, $X_1(t)$, $X_2(t)$ and $X_3(t)$ are matrix solutions of corresponding linearized equations (1.8), (1.9) *and* (1.11), *respectively, S_1 is the saltation matrix given by* (1.10), *S_2 is a second saltation matrix given by*

$$S_2 = \mathbb{I} + \frac{(f_+(x_2)-f_-(x_2))\mathrm{D}h(x_2)}{\mathrm{D}h(x_2)f_-(x_2)} \tag{1.14}$$

and

$$A(t) = \begin{cases} X_3(T)S_2X_2(t_2)S_1X_1(t_1)X_1^{-1}(t) & \text{if } t\in[0,t_1), \\ X_3(T)S_2X_2(t_2)X_2^{-1}(t) & \text{if } t\in[t_1,t_2), \\ X_3(T)X_3^{-1}(t) & \text{if } t\in[t_2,T]. \end{cases} \tag{1.15}$$

Proof. Direct differentiation of (1.6) and the use of previous results give statement of the lemma:

$$\begin{aligned} \widetilde{P}_\xi(x_0,0,\mu,\alpha) = x_{+\tau}(t_2,x_2)(T,0,\mu,\alpha)t_{2\xi}(0,x_0,0,\mu,\alpha) + x_{+\xi}(t_2,x_2)(T,0,\mu,\alpha) \\ \times[x_{-\tau}(t_1,x_1)(t_2,0,\mu,\alpha)t_{1\xi}(0,x_0,0,\mu,\alpha) + x_{-\xi}(t_1,x_1)(t_2,0,\mu,\alpha) \\ \times[x_{+\xi}(0,x_0)(t_1,0,\mu,\alpha) + x_{+t}(0,x_0)(t_1,0,\mu,\alpha)t_{1\xi}(0,x_0,0,\mu,\alpha)] \\ +x_{-t}(t_1,x_1)(t_2,0,\mu,\alpha)t_{2\xi}(0,x_0,0,\mu,\alpha)] \end{aligned}$$

$$\begin{aligned} = X_3(T)f_+(x_2)\frac{\mathrm{D}h(x_2)X_2(t_2)S_1X_1(t_1)}{\mathrm{D}h(x_2)f_-(x_2)} + X_3(T)\left[X_2(t_2)f_-(x_1)\frac{\mathrm{D}h(x_1)X_1(t_1)}{\mathrm{D}h(x_1)f_+(x_1)}\right. \\ \left. + X_2(t_2)\left[X_1(t_1) - f_+(x_1)\frac{\mathrm{D}h(x_1)X_1(t_1)}{\mathrm{D}h(x_1)f_+(x_1)}\right] - f_-(x_2)\frac{\mathrm{D}h(x_2)X_2(t_2)S_1X_1(t_1)}{\mathrm{D}h(x_2)f_-(x_2)}\right] \\ = X_3(T)S_2X_2(t_2)S_1X_1(t_1). \end{aligned}$$

Equality (1.13) can be shown by the same way. □

For the further work, we recall the following well-known result (cf. [33]).

Lemma 1.5. *Let $X(t)$ be a fundamental matrix solution of equation $X' = UX$. Then $X(t)^{-1*}$ is a fundamental matrix solution of adjoint equation*

$$\left(X(t)^{-1*}\right)' = -U^*X(t)^{-1*}.$$

We solve equation (1.5) via Lyapunov-Schmidt reduction. As it was already shown in Lemma 1.3, dim $\mathcal{N}(\mathbb{I}-\widetilde{P}_\xi(x_0,0,\mu,\alpha)) \geq 1$. From now on we suppose that

H3) dim $\mathcal{N}(\mathbb{I}-\widetilde{P}_\xi(x_0,0,\mu,\alpha)) = 1$

and therefore codim $\mathcal{R}(\mathbb{I} - \widetilde{P}_\xi(x_0, 0, \mu, \alpha)) = 1$. We denote

$$R_1 = \mathcal{R}(\mathbb{I} - \widetilde{P}_\xi(x_0, 0, \mu, \alpha)), \qquad R_2 = \Big[\mathcal{R}(\mathbb{I} - \widetilde{P}_\xi(x_0, 0, \mu, \alpha))\Big]^\perp \tag{1.16}$$

the image of the corresponding operator and its orthogonal complement in $\mathbb{R}^n$. Then two linear projections are considered $\mathcal{P} : \mathbb{R}^n \to R_2$ and $\mathcal{Q} : \mathbb{R}^n \to R_1$ defined by

$$\mathcal{P}y = \frac{\langle y, \psi\rangle}{\|\psi\|^2}\psi, \qquad \mathcal{Q}y = (\mathbb{I} - \mathcal{P})y = y - \frac{\langle y, \psi\rangle}{\|\psi\|^2}\psi$$

where $\psi \in R_2$ is fixed. We assume that initial point ξ of perturbed periodic trajectory is an element of Σ. Equation (1.5) for $(\xi, \alpha) \in \Sigma \times \mathbb{R}$ is equivalent to a couple of equations

$$\mathcal{Q}F(\xi, \varepsilon, \mu, \alpha) = 0, \qquad \mathcal{P}F(\xi, \varepsilon, \mu, \alpha) = 0$$

for $(\xi, \alpha) \in \Sigma \times \mathbb{R}$ with parameters $(\varepsilon, \mu) \in \mathbb{R} \times \mathbb{R}^p$. The first one can be solved via IFT which implies the existence of $r_0, \varepsilon_0 > 0$ and a C^r-function

$$\xi : (-\varepsilon_0, \varepsilon_0) \times \mathbb{R}^p \times \mathbb{R} \to B(x_0, r_0) \cap \Sigma$$

such that $\mathcal{Q}F(\xi, \varepsilon, \mu, \alpha) = 0$ for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $\mu \in \mathbb{R}^p$, $\alpha \in \mathbb{R}$ and $\xi \in B(x_0, r_0) \cap \Sigma$ if and only if $\xi = \xi(\varepsilon, \mu, \alpha)$. Moreover $\xi(0, \mu, \alpha) = x_0$.

Then the second equation has the form

$$\langle \xi(\varepsilon, \mu, \alpha) - \widetilde{P}(\xi(\varepsilon, \mu, \alpha), \varepsilon, \mu, \alpha), \psi\rangle = 0. \tag{1.17}$$

Again, if $\varepsilon = 0$ this equation is satisfied for any $(\mu, \alpha) \in \mathbb{R}^p \times \mathbb{R}$. Differentiation with respect to ε at 0 gives

$$\begin{aligned}
&\Big\langle \xi_\varepsilon(0, \mu, \alpha) - \widetilde{P}_\xi(x_0, 0, \mu, \alpha)\xi_\varepsilon(0, \mu, \alpha) - \widetilde{P}_\varepsilon(x_0, 0, \mu, \alpha), \psi \Big\rangle \\
&= \Big\langle (\mathbb{I} - \widetilde{P}_\xi(x_0, 0, \mu, \alpha))\xi_\varepsilon(0, \mu, \alpha) - \widetilde{P}_\varepsilon(x_0, 0, \mu, \alpha), \psi \Big\rangle \\
&= \Big\langle (\mathbb{I} - \widetilde{P}_\xi(x_0, 0, \mu, \alpha))\xi_\varepsilon(0, \mu, \alpha), \psi \Big\rangle - \Big\langle \widetilde{P}_\varepsilon(x_0, 0, \mu, \alpha), \psi \Big\rangle \\
&= -\Big\langle \int_0^T A(s)g(\gamma(s), s + \alpha, 0, \mu)ds, \psi \Big\rangle \\
&= -\int_0^T \langle A(s)g(\gamma(s), s + \alpha, 0, \mu), \psi\rangle ds \\
&= -\int_0^T \langle g(\gamma(s), s + \alpha, 0, \mu), A^*(s)\psi\rangle ds
\end{aligned}$$

where

$$A^*(t) = \begin{cases} X_1^{-1*}(t)X_1^*(t_1)S_1^*X_2^*(t_2)S_2^*X_3^*(T) & \text{if } t \in [0, t_1), \\ X_2^{-1*}(t)X_2^*(t_2)S_2^*X_3^*(T) & \text{if } t \in [t_1, t_2), \\ X_3^{-1*}(t)X_3^*(T) & \text{if } t \in [t_2, T]. \end{cases} \tag{1.18}$$

Note that by Lemma 1.5, $A^*(t)$ solves the adjoint variational equation

$$\begin{aligned} X' &= -\mathrm{D}f_+^*(\gamma(t))X \quad \text{if } 0 < t < t_1, \\ X' &= -\mathrm{D}f_-^*(\gamma(t))X \quad \text{if } t_1 < t < t_2, \\ X' &= -\mathrm{D}f_+^*(\gamma(t))X \quad \text{if } t_2 < t < T \end{aligned} \tag{1.19}$$

of (1.1). Differentiation of the left-hand side of (1.17) with respect to ε and α at $\varepsilon = 0$ gives

$$-\int_0^T \langle \mathrm{D}_t g(\gamma(s), s+\alpha, 0, \mu), A^*(s)\psi\rangle ds.$$

In conclusion, we obtain the next result.

Theorem 1.6. *Let conditions* H1), H2), H3) *hold,* $\gamma(t)$*,* R_2 *and* $A^*(t)$ *be defined by* (1.2), (1.16) *and* (1.18)*, respectively, and* $\psi \in R_2$ *be arbitrary and fixed. If* $\alpha_0 \in \mathbb{R}$ *is a simple root of function* $M^{\mu_0}(\alpha)$ *given by*

$$M^{\mu}(\alpha) = \int_0^T \langle g(\gamma(t), t+\alpha, 0, \mu), A^*(t)\psi\rangle dt, \tag{1.20}$$

i.e. $M^{\mu_0}(\alpha_0) = 0$*,* $\mathrm{D}M^{\mu_0}(\alpha_0) \neq 0$ *then there exists a neighbourhood* U *of the point* $(0, \mu_0)$ *in* $\mathbb{R} \times \mathbb{R}^p$ *and a* C^{r-1}*-function* $\alpha(\varepsilon, \mu)$*, with* $\alpha(0, \mu_0) = \alpha_0$*, such that equation* (1.1) *with* $\alpha = \alpha(\varepsilon, \mu)$ *possesses a unique* T*-periodic piecewise* C^1*-smooth solution for each* $(\varepsilon, \mu) \in U$*.*

Proof. Let us denote

$$\mathcal{D}(\varepsilon, \mu, \alpha) = \begin{cases} \frac{1}{\varepsilon}\langle \xi(\varepsilon, \mu, \alpha) - \widetilde{P}(\xi(\varepsilon, \mu, \alpha), \varepsilon, \mu, \alpha), \psi\rangle & \text{for } \varepsilon \neq 0, \\ \mathrm{D}_\varepsilon\langle \xi(\varepsilon, \mu, \alpha) - \widetilde{P}(\xi(\varepsilon, \mu, \alpha), \varepsilon, \mu, \alpha), \psi\rangle & \text{for } \varepsilon = 0. \end{cases}$$

Then $\mathcal{D}$ is C^{r-1}-smooth and the assumptions on M^{μ_0} are fulfilled if and only if

$$\mathcal{D}(0, \mu_0, \alpha_0) = 0, \qquad \mathrm{D}_\alpha \mathcal{D}(0, \mu_0, \alpha_0) \neq 0.$$

IFT implies the existence of the function $\alpha(\varepsilon, \mu)$ from the statement of the theorem. ☐

Function $M^{\mu}(\alpha)$ is a Poincaré-Andronov-Melnikov function for system (1.1).

Remark 1.7.

1. If g is discontinuous in x, i.e.

$$g(x, t, \varepsilon, \mu) = \begin{cases} g_+(x, t, \varepsilon, \mu) & \text{if } x \in \Omega_+, \\ g_-(x, t, \varepsilon, \mu) & \text{if } x \in \Omega_-, \end{cases}$$

   it is possible to show that Theorem 1.6 still holds. Of course, g has to be T-periodic in t.

2. It can be shown that in Theorem 1.6 we can take any other solution of the adjoint variational system consisting of the adjoint variational equation (1.19) and corresponding impulsive and boundary conditions (see Lemma 2.4).

1.1 Geometric interpretation of condition H3)

Consider the linearization of unperturbed problem of (1.1) along $\gamma(t)$, given by

$$\dot{x} = \mathrm{D}f_{\pm}(\gamma(t))x. \tag{1.21}$$

Then (1.21) splits into two unperturbed equations

$$\begin{aligned} \dot{x} &= \mathrm{D}f_{+}(\gamma(t))x \qquad \text{if } t \in [0, t_1] \cup [t_2, T], \\ \dot{x} &= \mathrm{D}f_{-}(\gamma(t))x \qquad \text{if } t \in (t_1, t_2) \end{aligned}$$

with impulsive conditions [35, 36, 42]

$$x(t_1+) = S_1 x(t_1-), \qquad x(t_2+) = S_2 x(t_2-)$$

where $x(t\pm) = \lim_{s\to t^{\pm}} x(s)$. We already know (from (1.8), (1.9), (1.11)) that they have the fundamental matrices $X_1(t)$ resp. $X_3(t)$ and $X_2(t)$ satisfying $X_1(0) = X_2(t_1) = X_3(t_2) = \mathbb{I}$. Consequently, the fundamental matrix solution of discontinuous variational equation (1.21) is given by

$$X(t) = \begin{cases} X_1(t) & \text{if } t \in [0, t_1), \\ X_2(t)S_1X_1(t_1) & \text{if } t \in [t_1, t_2), \\ X_3(t)S_2X_2(t_2)S_1X_1(t_1) & \text{if } t \in [t_2, T]. \end{cases}$$

Then a T-periodic solution of (1.21) with an initial point ξ fulfils $\xi = X(T)\xi$ or equivalently $(\mathbb{I} - X(T))\xi = 0$. Now one can easily conclude the following result.

Proposition 1.8. *Condition* H3) *is equivalent to say that discontinuous variational equation* (1.21) *has a unique T-periodic solution up to a scalar multiple.*

1.2 Nonlinear planar applications

Here we consider the following piecewise nonlinear problem

$$\begin{aligned} &\left.\begin{aligned} \dot{x} &= \omega_1(y-\delta) + \varepsilon g_1(x, y, t+\alpha, \varepsilon, \mu) \\ \dot{y} &= -\omega_1 x + \varepsilon g_2(x, y, t+\alpha, \varepsilon, \mu) \end{aligned}\right\} \quad \text{if } y > 0, \\ &\left.\begin{aligned} \dot{x} &= \eta x + \omega_2(y+\delta) \\ &\quad + \left[x^2 + (y+\delta)^2\right]\left[-ax - b(y+\delta)\right] + \varepsilon g_1(x, y, t+\alpha, \varepsilon, \mu) \\ \dot{y} &= -\omega_2 x + \eta(y+\delta) \\ &\quad + \left[x^2 + (y+\delta)^2\right]\left[bx - a(y+\delta)\right] + \varepsilon g_2(x, y, t+\alpha, \varepsilon, \mu) \end{aligned}\right\} \quad \text{if } y < 0 \end{aligned} \tag{1.22$_\varepsilon$}$$

with assumptions

$$\eta, \delta, \omega_1, \omega_2, \omega, a > 0, \quad b \in \mathbb{R}, \quad \omega_2 - \frac{\eta b}{a} > 0, \quad \frac{\eta}{a} > \delta^2. \tag{1.23}$$

Note the dependence on ε in the notation $(1.22)_\varepsilon$. Hence $(1.22)_0$ refers to the unperturbed system.

Due to linearity, the first part of $(1.22)_0$ can be easily solved, e.g. via matrix exponential. For starting point $(x_0, y_0) = (0, \delta + \sqrt{\frac{\eta}{a}})$ and $t \in [0, t_1]$ the solution is

$$\gamma_1(t) = \left(\sqrt{\frac{\eta}{a}} \sin \omega_1 t, \delta + \sqrt{\frac{\eta}{a}} \cos \omega_1 t \right). \tag{1.24}$$

Time t_1 of the first intersection with discontinuity boundary $\Omega_0 = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}$ and the point (x_1, y_1) of this intersection are obtained from relations $h(\gamma_1(t_1)) = 0$ for $h(x, y) = y$ and $(x_1, y_1) = \gamma_1(t_1)$, respectively:

$$t_1 = \frac{1}{\omega_1} \arccos \left(-\sqrt{\frac{a}{\eta}} \delta \right), \qquad (x_1, y_1) = \left(\sqrt{\frac{\eta}{a} - \delta^2}, 0 \right).$$

After transformation $x = r \cos \theta$, $y + \delta = r \sin \theta$ in the second part of $(1.22)_0$, we get

$$\begin{aligned} \dot{r} &= \eta r - a r^3 \\ \dot{\theta} &= -\omega_2 + b r^2 \end{aligned}$$

from which, one can see that the second part of $(1.22)_0$ possesses a stable limit cycle/circle with the center at $(0, -\delta)$ and radius $\sqrt{\frac{\eta}{a}}$, which intersects boundary Ω_0. Now it is obvious that (x_1, y_1) is a point of this cycle and the direction of rotation remains the same as in $\Omega_+ = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$. Therefore $\gamma_2(t)$ is a part of the circle, given by

$$\begin{aligned} \gamma_2(t) = (&x_1 \cos \omega_3(t - t_1) + \delta \sin \omega_3(t - t_1), \\ &-\delta - x_1 \sin \omega_3(t - t_1) + \delta \cos \omega_3(t - t_1)) \end{aligned} \tag{1.25}$$

for $t \in [t_1, t_2]$, where $\omega_3 = \omega_2 - \frac{\eta b}{a}$. Equation $h(\gamma_2(t_2)) = 0$ together with symmetry of $\gamma_2(t)$ give the couple of equations

$$\begin{aligned} x_1 \cos \omega_3(t_2 - t_1) + \delta \sin \omega_3(t_2 - t_1) &= -x_1, \\ -\delta - x_1 \sin \omega_3(t_2 - t_1) + \delta \cos \omega_3(t_2 - t_1) &= 0. \end{aligned}$$

From these them we obtain

$$t_2 = \frac{1}{\omega_3} \left(\pi + \operatorname{arccot} \frac{-\delta^2 + x_1^2}{2\delta x_1} \right) + t_1.$$

Point (x_2, y_2) is a second intersection point of the limit cycle and Ω_0, i.e.

$$(x_2, y_2) = \gamma_2(t_2) = \left(-\sqrt{\frac{\eta}{a} - \delta^2}, 0 \right).$$

Next, solution $\gamma(t)$ continues in Ω_+ following the solution of the first part of $(1.22)_0$. Thus we have

$$\begin{aligned} \gamma_3(t) = (&x_2 \cos \omega_1(t - t_2) - \delta \sin \omega_1(t - t_2), \\ &\delta - x_2 \sin \omega_1(t - t_2) - \delta \cos \omega_1(t - t_2)) \end{aligned} \tag{1.26}$$

for $t \in [t_2, T]$. Period T obtained from identity $\gamma_3(T) = (x_0, y_0)$ is

$$T = \frac{1}{\omega_1} \arccos \left(-\sqrt{\frac{a}{\eta}} \delta \right) + t_2.$$

The next theorem is due to Diliberto (cf. [12, 46]) and we shall use it to find the fundamental matrix solution of the variational equation.

Theorem 1.9. *Let $\gamma(t)$ be a solution of the differential equation $\dot{x} = f(x)$, $x \in \mathbb{R}^2$. If $\gamma(0) = p$, $f(p) \neq 0$ then the variational equation along $\gamma(t)$*

$$\dot{V} = \mathrm{D}f(\gamma(t))V$$

has the fundamental matrix solution $\Phi(t)$ satisfying $\det \Phi(0) = \|f(p)\|^2$, *given by*

$$\Phi(t) = [f(\gamma(t)), V(t)]$$

where $[\lambda_1, \lambda_2]$ stands for a matrix with columns λ_1 and λ_2 and

$$\begin{aligned} V(t) &= a(t)f(\gamma(t)) + b(t)f^{\perp}(\gamma(t)),\\ a(t) &= \int_0^t \left[2\kappa(\gamma(s))\|f(\gamma(s))\| + \operatorname{div} f^{\perp}(\gamma(s))\right] b(s)ds,\\ b(s) &= \frac{\|f(p)\|^2}{\|f(\gamma(t))\|^2} \mathrm{e}^{\int_0^t \operatorname{div} f(\gamma(s))ds},\\ \operatorname{div} f(x) &= \frac{\partial f_1(x)}{\partial x_1} + \frac{\partial f_2(x)}{\partial x_2}, \qquad \operatorname{div} f^{\perp}(x) = -\frac{\partial f_2(x)}{\partial x_1} + \frac{\partial f_1(x)}{\partial x_2},\\ \kappa(\gamma(t)) &= \frac{1}{\|f(\gamma(t))\|^3}\left[f_1(\gamma(t))\dot{f}_2(\gamma(t)) - f_2(\gamma(t))\dot{f}_1(\gamma(t))\right]. \end{aligned}$$

Lemma 1.10. *Assuming* (1.23), *unperturbed system* $(1.22)_0$ *has fundamental matrices X_1, X_2 and X_3 satisfying* (1.8), (1.9) *and* (1.11), *respectively, given by*

$$\begin{aligned} X_1(t) &= \begin{pmatrix} \cos\omega_1 t & \sin\omega_1 t\\ -\sin\omega_1 t & \cos\omega_1 t \end{pmatrix},\\ X_2(t) &= \frac{a}{\eta}\left[\lambda_1, \lambda_2\right], \qquad X_3(t) = X_1(t - t_2) \end{aligned}$$

where

$$\begin{aligned} \lambda_1 &= \begin{pmatrix} U(-\delta x_1 + \delta x_1 W + x_1^2\widetilde{W}) + V(\delta^2 + x_1^2 W - \delta x_1\widetilde{W})\\ U(-\delta^2 - x_1^2 W + \delta x_1\widetilde{W}) + V(-\delta x_1 + \delta x_1 W + x_1^2\widetilde{W}) \end{pmatrix},\\ \lambda_2 &= \begin{pmatrix} U(x_1^2 + \delta^2 W + \delta x_1\widetilde{W}) + V(-\delta x_1 + \delta x_1 W - \delta^2\widetilde{W})\\ U(\delta x_1 - \delta x_1 W + \delta^2\widetilde{W}) + V(x_1^2 + \delta^2 W + \delta x_1\widetilde{W}) \end{pmatrix},\\ U &= \sin\omega_3(t - t_1), \qquad V = \cos\omega_3(t - t_1),\\ W &= \mathrm{e}^{-2\eta(t-t_1)}, \qquad \widetilde{W} = \frac{b}{a}(1 - W), \end{aligned}$$

and saltation matrices

$$S_1 = \begin{pmatrix} 1 & -\frac{\delta(\omega_1+\omega_3)}{\omega_1 x_1}\\ 0 & \frac{\omega_3}{\omega_1} \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 1 & -\frac{\delta(\omega_1+\omega_3)}{\omega_3 x_1}\\ 0 & \frac{\omega_1}{\omega_3} \end{pmatrix}$$

defined by (1.10), (1.14), *respectively.*

Proof. Matrices $X_1(t)$ and $X_3(t)$ are derived easily because of the linearity of function $f_+(x,y)$. Using

$$\begin{aligned} f_+(x_1,y_1) &= \begin{pmatrix} -\omega_1\delta \\ -\omega_1\sqrt{\frac{\eta}{a}-\delta^2} \end{pmatrix}, \qquad & f_-(x_1,y_1) &= \begin{pmatrix} \omega_3\delta \\ -\omega_3\sqrt{\frac{\eta}{a}-\delta^2} \end{pmatrix}, \\ f_+(x_2,y_2) &= \begin{pmatrix} -\omega_1\delta \\ \omega_1\sqrt{\frac{\eta}{a}-\delta^2} \end{pmatrix}, \qquad & f_-(x_2,y_2) &= \begin{pmatrix} \omega_3\delta \\ \omega_3\sqrt{\frac{\eta}{a}-\delta^2} \end{pmatrix}, \end{aligned} \tag{1.27}$$

saltation matrices are obtained directly from their definitions. Since $(1.22)_0$ is 2-dimensional and one solution of the second part is already known – the limit cycle, we can apply Theorem 1.9 to derive the fundamental solution of this part. So we get a matrix

$$\widetilde{X}_2(t) = \omega_3 \begin{pmatrix} -x_1U+\delta V & U(\delta W + x_1\widetilde{W}) + V(x_1W - \delta\widetilde{W}) \\ -\delta U - x_1V & U(-x_1W+\delta\widetilde{W}) + V(\delta W + x_1\widetilde{W}) \end{pmatrix}$$

such that

$$\widetilde{X}_2^{-1}(t_1) = \frac{a}{\eta\omega_3}\begin{pmatrix} \delta & -x_1 \\ x_1 & \delta \end{pmatrix}$$

and $\det \widetilde{X}_2(t_1) = \|f_-(x_1,y_1)\|^2 = \frac{\eta}{a}\omega_3^2$. If $X_2(t)$ has to satisfy (1.9) then $X_2(t) = \widetilde{X}_2(t)\widetilde{X}_2^{-1}(t_1)$. □

Now, we can verify the basic assumptions.

Proposition 1.11. *Assuming* (1.23), *unperturbed system* $(1.22)_0$ *has a T-periodic solution with initial point* $(x_0,y_0) = (0, \delta + \sqrt{\frac{\eta}{a}})$ *defined by* (1.2) *with branches* $\gamma_1(t)$, $\gamma_2(t)$ *and* $\gamma_3(t)$ *given by* (1.24), (1.25) *and* (1.26), *respectively. Moreover, conditions* H1), H2) *and* H3) *are satisfied.*

Proof. Condition H1) was already verified. Since $\nabla h(x,y) = (0,1)$ for all $(x,y)\in\mathbb{R}^2$ and (1.27) holds, also condition H2) is fulfilled.

Now suppose that $\dim\ \mathcal{N}(\mathbb{I} - \widetilde{P}_\xi(x_0,0,\mu,\alpha)) > 1$. We recall that $f_+(x_0,y_0) \in \mathcal{N}(\mathbb{I} - \widetilde{P}_\xi(x_0,0,\mu,\alpha))$. Since $\mathcal{N}(\mathbb{I} - \widetilde{P}_\xi(x_0,0,\mu,\alpha))$ is linear, there is a vector

$$\bar{v} \in \mathcal{N}(\mathbb{I} - \widetilde{P}_\xi(x_0,0,\mu,\alpha))$$

such that $\langle \bar{v}, f_+(x_0,y_0)\rangle = 0$. Then we can write $\bar{v} = (0,v)^*$. Using formula (1.12) for $\widetilde{P}_\xi$ we look for the image of $\bar{v}$ by mapping $\widetilde{P}_\xi(x_0,0,\mu,\alpha)$. We subsequently obtain

$$S_1X_1(t_1)\bar{v} = \frac{v}{\omega_1}\sqrt{\frac{a}{\eta}}\begin{pmatrix} \omega_1x_1 + \frac{\delta^2(\omega_1+\omega_3)}{x_1} \\ -\delta\omega_3 \end{pmatrix},$$

$$X_2(t_2)S_1X_1(t_1)\bar{v} = \frac{v}{\omega_1}\sqrt{\frac{a}{\eta}}\begin{pmatrix} \frac{\delta^2}{x_1}(\omega_1+\omega_3) - x_1\omega_1Z - \delta\omega_1\widetilde{Z} \\ \delta(\omega_1+\omega_3) + \delta\omega_1Z - x_1\omega_1\widetilde{Z} \end{pmatrix}$$

where $Z = \mathrm{e}^{-2\eta(t_2-t_1)}$ and $\widetilde{Z} = \frac{b}{a}(1-Z)$ are values of W and $\widetilde{W}$ at $t=t_2$,

$$S_2X_2(t_2)S_1X_1(t_1)\bar{v} = \frac{v}{\omega_3}\sqrt{\frac{a}{\eta}}\begin{pmatrix} -\frac{\delta^2}{x_1}(\omega_1+\omega_3) - (\frac{\delta^2\omega_1}{x_1} + \frac{\eta\omega_3}{ax_1})Z + \delta\omega_1\widetilde{Z} \\ \delta(\omega_1+\omega_3) + \delta\omega_1Z - x_1\omega_1\widetilde{Z} \end{pmatrix}$$

and finally

$$X_3(T)S_2X_2(t_2)S_1X_1(t_1)\bar{v} = v\left(\frac{\delta}{x_1}\frac{\omega_1+\omega_3}{\omega_3} + \frac{\delta}{x_1}\frac{\omega_1+\omega_3}{\omega_3}Z - \frac{\omega_1}{\omega_3}\widetilde{Z} \atop Z\right).$$

Since $Z \le \exp\left\{-\frac{2\eta}{\omega_3}\pi\right\} < 1$, it is obvious that $\bar{v} = \widetilde{P}_\xi(x_0, 0, \mu, \alpha)\bar{v}$ if and only if $\bar{v} = (0,0)^*$. Hence the verification of condition H3) is finished. $\square$

Because, in general, the formula for $A(t)$ is rather awkward, we move to examples with concrete parameters.

Example 1.12. Consider system $(1.22)_\varepsilon$ with

$$\begin{aligned} &a = b = \delta = 1,\ \eta = 2,\ \omega_1 = 1,\ \omega_2 = 5, \\ &g(x,y,t,\varepsilon,\mu) = \begin{cases} (\sin\omega t, 0)^* & \text{if } y > 0, \\ (0,0)^* & \text{if } y < 0. \end{cases} \end{aligned} \tag{1.28}$$

Then we have $\omega_3 = 3$, $T = 2\pi$, initial point $(x_0, y_0) = (0, 1+\sqrt{2})$, saltation matrices

$$S_1 = \begin{pmatrix} 1 & -4 \\ 0 & 3 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 1 & -\frac{4}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$$

and

$$\widetilde{P}_\xi(x_0, y_0, 0, \mu, \alpha) = \begin{pmatrix} 1 & 1 + \frac{5}{3}\mathrm{e}^{-2\pi} \\ 0 & \mathrm{e}^{-2\pi} \end{pmatrix}.$$

Therefore $R_1 = \left[\left(1 + \frac{5}{3}\mathrm{e}^{-2\pi}, \mathrm{e}^{-2\pi} - 1\right)^*\right]$ and $\psi = \left(1 - \mathrm{e}^{-2\pi}, 1 + \frac{5}{3}\mathrm{e}^{-2\pi}\right) \in R_2$. After some algebra we obtain

$$M(\alpha) = \frac{1}{3}\frac{\mathrm{e}^{-2\pi}}{\omega^2 - 1}\left[(\omega A + B)\sin\omega\alpha + (\omega C + D)\cos\omega\alpha\right]$$

for $M(\alpha) = M^\omega(\alpha)$ of (1.20), where

$$\begin{aligned} A &= 4\sqrt{2}\sin\left(\frac{3}{4}\pi\omega\right) + \left(3\mathrm{e}^{2\pi}\sqrt{2} + \sqrt{2}\right)\sin\left(\frac{5}{4}\pi\omega\right) + \left(3\mathrm{e}^{2\pi} - 3\right)\sin(2\pi\omega), \\ B &= -5 - 3\mathrm{e}^{2\pi} - \left(\sqrt{2} + 3\sqrt{2}\mathrm{e}^{2\pi}\right)\cos\left(\frac{3}{4}\pi\omega\right) + 4\sqrt{2}\cos\left(\frac{5}{4}\pi\omega\right) + \left(5 + 3\mathrm{e}^{2\pi}\right)\cos(2\pi\omega), \\ C &= 3\mathrm{e}^{2\pi} - 3 - 4\sqrt{2}\cos\left(\frac{3}{4}\pi\omega\right) - \left(\sqrt{2} + 3\sqrt{2}\mathrm{e}^{2\pi}\right)\cos\left(\frac{5}{4}\pi\omega\right) + \left(3 - 3\mathrm{e}^{2\pi}\right)\cos(2\pi\omega), \\ D &= -\left(\sqrt{2} + 3\sqrt{2}\mathrm{e}^{2\pi}\right)\sin\left(\frac{3}{4}\pi\omega\right) + 4\sqrt{2}\sin\left(\frac{5}{4}\pi\omega\right) + \left(5 + 3\mathrm{e}^{2\pi}\right)\sin(2\pi\omega). \end{aligned} \tag{1.29}$$

For $\omega > 0$, $\omega \ne 1$, $M(\alpha)$ has a simple root if and only if $(\omega A + B)^2 + (\omega C + D)^2 > 0$. Since A and C are 8-periodic functions,

$$\begin{aligned} \sqrt{B^2 + D^2} &\le \left(\left(5 + 3\mathrm{e}^{2\pi} + \sqrt{2} + 3\sqrt{2}\mathrm{e}^{2\pi} + 4\sqrt{2} + 5 + 3\mathrm{e}^{2\pi}\right)^2\right. \\ &\quad \left. + \left(5 + 3\mathrm{e}^{2\pi} + \sqrt{2} + 3\sqrt{2}\mathrm{e}^{2\pi} + 4\sqrt{2}\right)^2\right)^{\frac{1}{2}} \\ &= \sqrt{3}\left(1 + \sqrt{2}\right)\sqrt{9\mathrm{e}^{4\pi} + 30\mathrm{e}^{2\pi} + 25} \le 6739, \end{aligned}$$

and according to Fig. 1.2 we have the estimate

$$\sqrt{(\omega A+B)^2+(\omega C+D)^2} \geq \omega\sqrt{A^2+C^2}-\sqrt{B^2+D^2} \geq 400\omega - 6739$$

and one can see that for $\omega \geq 17$, T-periodic orbit in perturbed system $(1.22)_\varepsilon$ persists for all $\varepsilon \neq 0$ small. It can be proved numerically (see Fig. 1.2) that

$$\frac{1}{|\omega^2-1|}\sqrt{(\omega A+B)^2+(\omega C+D)^2} > 0 \tag{1.30}$$

for $\omega \in (0,17)$. We conclude

Corollary 1.13. *Consider* $(1.22)_\varepsilon$ *with parameters* (1.28). *Then* 2π*-periodic orbit persists for all* $\omega > 0$ *and* $\varepsilon \neq 0$ *small.*

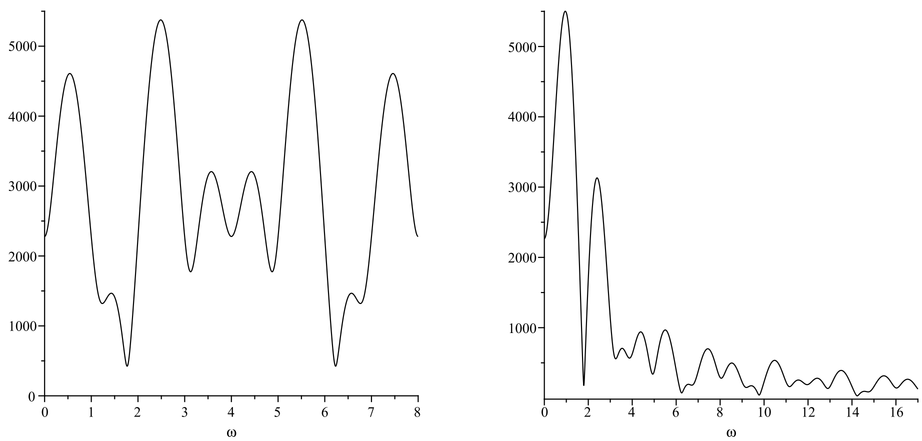

Fig. 1.2: Graphs of the functions $\sqrt{A^2+C^2}$ and the left-hand side of (1.30)

Example 1.14. Consider system $(1.22)_\varepsilon$ with

$$a=b=\delta=1,\ \eta=2,\ \omega_1=1,\ \omega_2=5,$$
$$g(x,y,t,\varepsilon,\mu)=\begin{cases}\mu_1(\sin\omega t,0)^* & \text{if } y>0,\\ \mu_2(x+y,0)^* & \text{if } y<0.\end{cases} \tag{1.31}$$

Consequently, Poincaré-Andronov-Melnikov function of (1.20) is

$$M(\alpha)=\mu_1\frac{1}{3}\frac{\mathrm{e}^{-2\pi}}{\omega^2-1}\left[(\omega A+B)\sin\omega\alpha+(\omega C+D)\cos\omega\alpha\right]+\mu_2 E$$

where A, B, C, D are given by (1.29) and

$$E=\frac{\sqrt{2}}{975}\left(739-223\mathrm{e}^{-2\pi}\right).$$

Function $M(\alpha)$ possesses a simple root if and only if

$$|\mu_2| < \frac{1}{3}\frac{\mathrm{e}^{-2\pi}}{|\omega^2-1|}\frac{\sqrt{(\omega A+B)^2+(\omega C+D)^2}}{E}|\mu_1|. \tag{1.32}$$

Applying Theorem 1.6 we obtain the next result.

Corollary 1.15. *Consider* $(1.22)_\varepsilon$ *with parameters* (1.31). *If* μ_1, μ_2 *and* ω *satisfy* (1.32) *then* 2π*-periodic orbit persists for* $\varepsilon \neq 0$ *small.*

Remark 1.16. Inequality (1.32) means that if the periodic perturbation is sufficiently large (with respect to non-periodic part of perturbation) then the T-periodic trajectory persists. Note that the right-hand side of (1.32) can be estimated from above by

$$\frac{\sqrt{c_1\omega^2 + c_2\omega + c_3}}{|\omega^2 - 1|}$$

for appropriate constants c_1, c_2, c_3, which tends to 0, if ω tends to $+\infty$. Hence the bigger frequency ω, the bigger $|\mu_1|$ is needed for fixed $\mu_2 \neq 0$ for persistence of the T-periodic orbit after Theorem 1.6.

1.3 Piecewise linear planar application

Now we consider the system

$$\begin{aligned}
&\begin{aligned} \dot{x} &= b_1 + \varepsilon\mu_1 \sin\omega t \\ \dot{y} &= -2a_1b_1x + \varepsilon\mu_2\cos\omega t \end{aligned} \qquad \text{if } y > 0, \\
&\begin{aligned} \dot{x} &= -b_2 + \varepsilon\mu_1 \sin\omega t \\ \dot{y} &= -2a_2b_2x + \varepsilon\mu_2\cos\omega t \end{aligned} \qquad \text{if } y < 0
\end{aligned} \tag{1.33$_\varepsilon$}$$

where all constants a_i, b_i for $i = 1, 2$ are assumed to be positive and $(\mu_1, \mu_2) \neq (0, 0)$, $\omega > 0$.

The starting point can be chosen in the form $(x_0, y_0) = (0, y_0)$ with $y_0 > 0$. Then with $h(x, y) = y$ we obtain results similar to those of the nonlinear previous case.

First, one can easily find the periodic trajectory starting at $(0, y_0) \in \Omega_+ = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$, intersecting transversally $\Omega_0 = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}$ to $\Omega_- = \{(x, y) \in \mathbb{R}^2 \mid y < 0\}$ and returning back to Ω_+ transversally through Ω_0.

Lemma 1.17. *For any* $y_0 > 0$, *unperturbed system* $(1.33)_0$ *possesses a unique periodic solution starting at* $(0, y_0)$ *given by*

$$\gamma(t) = \begin{cases} \gamma_1(t) = (b_1t, -a_1b_1^2t^2 + y_0) & \text{if } t \in [0, t_1], \\ \gamma_2(t) = (x_1 - b_2(t - t_1), a_2(x_1 - b_2(t - t_1))^2 - a_2x_1^2) & \text{if } t \in [t_1, t_2], \\ \gamma_3(t) = (x_2 + b_1(t - t_2), -a_1(x_2 + b_1(t - t_2))^2 + a_1x_2^2) & \text{if } t \in [t_2, T] \end{cases}$$

where

$$t_1 = \frac{1}{b_1}\sqrt{\frac{y_0}{a_1}}, \qquad (x_1, y_1) = \left(\sqrt{\frac{y_0}{a_1}}, 0\right), \qquad t_2 = \frac{2}{b_2}\sqrt{\frac{y_0}{a_1}} + t_1,$$

$$(x_2, y_2) = (-x_1, 0), \qquad T = \frac{1}{b_1}\sqrt{\frac{y_0}{a_1}} + t_2.$$

Fundamental and saltation matrices are described in the next lemma.

Lemma 1.18. *Unperturbed system* $(1.33)_0$ *has the corresponding fundamental matrices*

$$X_1(t) = \begin{pmatrix} 1 & 0 \\ -2a_1b_1t & 1 \end{pmatrix}, \qquad X_2(t) = \begin{pmatrix} 1 & 0 \\ -2a_2b_2(t-t_1) & 1 \end{pmatrix},$$
$$X_3(t) = \begin{pmatrix} 1 & 0 \\ -2a_1b_1(t-t_2) & 1 \end{pmatrix}$$

and saltation matrices

$$S_1 = \begin{pmatrix} 1 & \frac{\sqrt{a_1}(b_1+b_2)}{2a_1b_1\sqrt{y_0}} \\ 0 & \frac{a_2b_2}{a_1b_1} \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 1 & \frac{\sqrt{a_1}(b_1+b_2)}{2a_2b_2\sqrt{y_0}} \\ 0 & \frac{a_1b_1}{a_2b_2} \end{pmatrix}.$$

Proof. Because of the linearity of this case, fundamental matrices are obtained from equalities

$$X_1(t) = \mathrm{e}^{At}, \qquad X_2(t) = \mathrm{e}^{B(t-t_1)}, \qquad X_3(t) = \mathrm{e}^{A(t-t_2)}$$

where

$$A = \begin{pmatrix} 0 & 0 \\ -2a_1b_1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ -2a_2b_2 & 0 \end{pmatrix}$$

are Jacobi matrices of the functions $f_+(x,y)$ and $f_-(x,y)$, respectively. Saltation matrices are given by their definitions in (1.10) and (1.14) where $\nabla h(x,y) = (0,1)$ in $\overline{\Omega}$ and

$$\begin{aligned} f_+(x_1,y_1) &= \begin{pmatrix} b_1 \\ -2a_1b_1\sqrt{\frac{y_0}{a_1}} \end{pmatrix}, \quad f_-(x_1,y_1) = \begin{pmatrix} -b_2 \\ -2a_2b_2\sqrt{\frac{y_0}{a_1}} \end{pmatrix}, \\ f_+(x_2,y_2) &= \begin{pmatrix} b_1 \\ 2a_1b_1\sqrt{\frac{y_0}{a_1}} \end{pmatrix}, \quad f_-(x_2,y_2) = \begin{pmatrix} -b_2 \\ 2a_2b_2\sqrt{\frac{y_0}{a_1}} \end{pmatrix}. \end{aligned} \tag{1.34}$$

□

In this case, the corresponding matrices can be easily multiplied to derive the following result.

Lemma 1.19. *Function* $A(t)$ *of* (1.15) *for the system* $(1.33)_\varepsilon$ *possesses the form*

$$A(t) = \begin{cases} \begin{pmatrix} 1 - \frac{2\sqrt{a_1}b_1t(b_1+b_2)}{b_2\sqrt{y_0}} & -\frac{b_1+b_2}{b_2\sqrt{a_1y_0}} \\ 2a_1b_1t & 1 \end{pmatrix} & \text{if } t \in [0,t_1), \\ \begin{pmatrix} -1 - \frac{2b_1}{b_2} + \frac{\sqrt{a_1}(t-t_1)(b_1+b_2)}{\sqrt{y_0}} & \frac{\sqrt{a_1}(b_1+b_2)}{2a_2b_2\sqrt{y_0}} \\ 2(\sqrt{a_1y_0} - a_1b_2(t-t_1)) & -\frac{a_1}{a_2} \end{pmatrix} & \text{if } t \in [t_1,t_2), \\ \begin{pmatrix} 1 & 0 \\ 2(a_1b_1(t-t_2) - \sqrt{a_1y_0}) & 1 \end{pmatrix} & \text{if } t \in [t_2,T]. \end{cases} \tag{1.35}$$

It remains to verify the basic assumptions.

Proposition 1.20. *Conditions* H1), H2) *and* H3) *are satisfied.*

Proof. From Lemma 1.17 and using (1.34), conditions H1) and H2) are immediately satisfied.

Now let dim $\mathcal{N}(\mathbb{I} - \widetilde{P}_\xi(x_0, y_0, 0, \mu, \alpha)) > 1$. Then there exists

$$\bar{v} \in \mathcal{N}(\mathbb{I} - \widetilde{P}_\xi(x_0, y_0, 0, \mu, \alpha))$$

such that $\langle \bar{v}, f_+(x_0, y_0)\rangle = 0$ and we can write $\bar{v} = (0, v)^*$. Since

$$\widetilde{P}_\xi(x_0, y_0, 0, \mu, \alpha)\bar{v} = A(0)\bar{v} = \begin{pmatrix} -\frac{v(b_1+b_2)}{b_2\sqrt{a_1 y_0}} \\ v \end{pmatrix}$$

then $v = 0$, dim $\mathcal{N}(\mathbb{I} - \widetilde{P}_\xi(x_0, y_0, 0, \mu, \alpha)) = 1$ and the condition H3) is verified as well. □

Note that there is a lot of periodic trajectories in the neighbourhood of $\gamma(t)$ but none of them has the same period, because the period $T = 2\sqrt{\frac{y_0}{a_1}}\left(\frac{1}{b_1} + \frac{1}{b_2}\right)$ depends on the initial point (x_0, y_0).
We have

$$\mathcal{R}(\mathbb{I} - \widetilde{P}_\xi(x_0, y_0, 0, \mu, \alpha)) = \mathcal{R}(\mathbb{I} - A(0)) = \mathbb{R} \times \{0\}.$$

Accordingly, we set $\psi = (0, 1)^*$ and $A^*(t)\psi = \binom{a_{21}(t)}{a_{22}(t)}$, i.e. the second column of matrix $A(t)$. The assumptions of the Theorem 1.6 are equivalent to say that

$$\begin{aligned} M(\alpha) = \sin\omega\alpha &\left(\mu_1 \int_0^T a_{21}(t)\cos\omega t dt - \mu_2 \int_0^T a_{22}(t)\sin\omega t dt\right) \\ + \cos\omega\alpha &\left(\mu_1 \int_0^T a_{21}(t)\sin\omega t dt + \mu_2 \int_0^T a_{22}(t)\cos\omega t dt\right) \end{aligned}$$

has a simple root. It is easy to see, that this happens if and only if

$$\Phi(\omega) = \int_0^T \mathrm{e}^{-\imath\omega t}(\mu_1 a_{21}(t) - \imath\mu_2 a_{22}(t))dt \neq 0 \tag{1.36}$$

where $\imath = \sqrt{-1}$.

Similarly to [7], function $\Phi(\omega)$ is analytic for $\omega > 0$ and hence the following theorem holds (see [7, Theorem 4.2] and [49]).

Theorem 1.21. *When $\Phi(\omega)$ is not identically equal to zero, then there is at most a countable set $\{\omega_j\} \subset (0, \infty)$ with possible accumulating point at $+\infty$ such that for any $\omega \in (0, \infty)\backslash\{\omega_j\}$, the T-periodic orbit $\gamma(t)$ persists for $(1.33)_\varepsilon$ under perturbations for $\varepsilon \neq 0$ small.*

Because for general parameters, conditions on μ_1 and μ_2, so we could decide when $\Phi(\omega)$ is identically zero, or the set of roots is finite or countable, are too complicated, we rather provide an example with concrete numerical values of parameters.

Example 1.22. Consider system $(1.33)_\varepsilon$ with parameters

$$a_1 = a_2 = b_1 = b_2 = y_0 = 1. \tag{1.37}$$

Now, from (1.36) we have

$$\Phi(\omega) = -4\imath \frac{\mathrm{e}^{-2\imath\omega}}{\omega^2}(2\mu_1 + \omega\mu_2)\sin\omega(\cos\omega - 1).$$

Thence for $\omega \in (0, \infty)$ it holds: if $\omega = k\pi$ for some $k \in \mathbb{N}$ or $\omega = -\frac{2\mu_1}{\mu_2} > 0$ then $\Phi(\omega) = 0$. Applying Theorem 1.6 we arrive at the following result.

Corollary 1.23. *Consider* $(1.33)_\varepsilon$ *with parameters* (1.37). *If* $\omega > 0$ *is such that* $\omega \neq k\pi$ *for all* $k \in \mathbb{N}$ *and* $\omega \neq -\frac{2\mu_1}{\mu_2}$ *with* $\mu_2 \neq 0$ *then* T*-periodic orbit* $\gamma(t)$ *persists under perturbations for* $\varepsilon \neq 0$ *small.*

Finally, if $\Phi(\omega)$ is identically zero then higher order Poincaré-Andronov-Melnikov function must be derived [8]. We omit those computations in our case, because they are very awkward.

2 Bifurcation from family of periodic orbits in autonomous systems

In the previous section, we studied the discontinuous systems with time-periodic perturbation. Now, we move our attention to the case of autonomous perturbation. In this section, we investigate the persistence of a single periodic solution from a bunch of transverse periodic solutions of unperturbed system. In comparison to Section 1, here we seek the periodic solution with period close to the period of the original trajectory.

Let $\Omega \subset \mathbb{R}^n$ be an open set in $\mathbb{R}^n$ and $h(x)$ be a C^r-function on $\overline{\Omega}$, with $r \geq 3$. We set $\Omega_\pm := \{x \in \Omega \mid \pm h(x) > 0\}$, $\Omega_0 := \{x \in \Omega \mid h(x) = 0\}$. Let $f_\pm \in C_b^r(\overline{\Omega})$, $g \in C_b^r(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^p)$ and $h \in C_b^r(\overline{\Omega}, \mathbb{R})$. Let $\varepsilon \in \mathbb{R}$ and $\mu \in \mathbb{R}^p, p \geq 1$ be parameters. Furthermore, we suppose that 0 is a regular value of h.

We say that a function $x(t)$ is a solution of equation

$$\dot{x} = f_\pm(x) + \varepsilon g(x, \varepsilon, \mu), \quad x \in \overline{\Omega}_\pm, \tag{2.1}$$

if it is a solution of this equation in the sense analogical to Definition 1.1.

Let us assume

H1) For $\varepsilon = 0$ equation (2.1) has a smooth family of T^β-periodic orbits $\{\gamma(\beta, t)\}$ parametrized by $\beta \in V \subset \mathbb{R}^k$, $0 < k < n$, V is an open set in $\mathbb{R}^k$. Each of the orbits is uniquely determined by its initial point $x_0(\beta) \in \Omega_+$, $x_0 \in C_b^r$, and consists of three branches

$$\gamma(\beta, t) = \begin{cases} \gamma_1(\beta, t) & \text{if } t \in [0, t_1^\beta], \\ \gamma_2(\beta, t) & \text{if } t \in [t_1^\beta, t_2^\beta], \\ \gamma_3(\beta, t) & \text{if } t \in [t_2^\beta, T^\beta], \end{cases} \tag{2.2}$$

where $0 < t_1^\beta < t_2^\beta < T^\beta$, $\gamma_1(\beta, t) \in \Omega_+$ for $t \in [0, t_1^\beta)$, $\gamma_2(\beta, t) \in \Omega_-$ for $t \in (t_1^\beta, t_2^\beta)$ and $\gamma_3(\beta, t) \in \Omega_+$ for $t \in (t_2^\beta, T^\beta]$ and

$$\begin{aligned} x_1(\beta) &:= \gamma_1(\beta, t_1^\beta) = \gamma_2(\beta, t_1^\beta) \in \Omega_0, \\ x_2(\beta) &:= \gamma_2(\beta, t_2^\beta) = \gamma_3(\beta, t_2^\beta) \in \Omega_0, \\ x_0(\beta) &:= \gamma_3(\beta, T^\beta) = \gamma_1(\beta, 0) \in \Omega_+. \end{aligned} \tag{2.3}$$

We suppose in addition that vectors

$$\frac{\partial x_0(\beta)}{\partial \beta_1}, \ldots, \frac{\partial x_0(\beta)}{\partial \beta_k}, f_+(x_0(\beta))$$

are linearly independent whenever $\beta \in V$.

H2) Moreover, we also assume that

$$\mathrm{D}h(x_1(\beta))f_\pm(x_1(\beta)) < 0 \quad \text{and} \quad \mathrm{D}h(x_2(\beta))f_\pm(x_2(\beta)) > 0, \quad \beta \in V.$$

Note by H1), H2) and implicit function theorem (IFT) it can be shown that t_1^β, t_2^β and T^β are C_b^r-functions of β [1].

Now we study local bifurcations for $\gamma(\beta,t)$, so we fix $\beta_0 \in V$ and set $x_0^0 = x_0(\beta_0)$, $t_1^0 = t_1^{\beta_0}$, etc. Note by H1) $x_0(V)$ is an immersed C^r-submanifold of $\mathbb{R}^n$.

Let $x_+(\tau,\xi)(t,\varepsilon,\mu)$ and $x_-(\tau,\xi)(t,\varepsilon,\mu)$ denote the solution of initial value problem

$$\begin{aligned} \dot{x} &= f_\pm(x) + \varepsilon g(x,\varepsilon,\mu) \\ x(\tau) &= \xi \end{aligned} \tag{2.4}$$

with corresponding sign.

As in Lemma 1.2, conditions H1) and H2) establish the existence of a Poincaré mapping.

Lemma 2.1. *Assume* H1) *and* H2). *Then there exist $\varepsilon_0, r_0 > 0$, a neighbourhood $W \subset V$ of β_0 in $\mathbb{R}^k$ and a Poincaré mapping (cf. Fig. 2.1)*

$$P(\cdot,\beta,\varepsilon,\mu) : B(x_0^0, r_0) \to \Sigma_\beta$$

for all fixed $\beta \in W$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $\mu \in \mathbb{R}^p$, where

$$\Sigma_\beta = \{y \in \mathbb{R}^n \mid \langle y - x_0(\beta), f_+(x_0(\beta))\rangle = 0\}.$$

Moreover, $P : B(x_0^0, r_0) \times W \times (-\varepsilon_0, \varepsilon_0) \times \mathbb{R}^p \to \mathbb{R}^n$ is C^r-smooth in all arguments and $x_0(W) \subset B(x_0^0, r_0) \subset \Omega_+$.

Proof. IFT implies the existence of positive constants τ_1, r_1, δ_1, ε_1 and C^r-function

$$t_1(\cdot,\cdot,\cdot,\cdot) : (-\tau_1,\tau_1) \times B(x_0^0, r_1) \times (-\varepsilon_1,\varepsilon_1) \times \mathbb{R}^p \to (t_1^0 - \delta_1, t_1^0 + \delta_1)$$

such that $h(x_+(\tau,\xi)(t,\varepsilon,\mu)) = 0$ for $\tau \in (-\tau_1,\tau_1)$, $\xi \in B(x_0^0, r_1) \subset \Omega_+$, $\varepsilon \in (-\varepsilon_1,\varepsilon_1)$, $\mu \in \mathbb{R}^p$ and $t \in (t_1^0 - \delta_1, t_1^0 + \delta_1)$ if and only if $t = t_1(\tau,\xi,\varepsilon,\mu)$. Moreover, $t_1(0, x_0^0, 0, \mu) = t_1^0$ and

$$x_+(0, x_0(\beta))(t_1(0, x_0(\beta), 0, \mu), 0, \mu) \in \Omega_0 \cap \{\gamma(\beta,t) \mid t \in \mathbb{R}\},$$

thus $t_1(0, x_0(\beta), 0, \mu) = t_1^\beta$. Similarly, we derive functions t_2 and t_3 satisfying, respectively,

$$\begin{gathered} h(x_-(t_1(\tau,\xi,\varepsilon,\mu), x_+(\tau,\xi)(t_1(\tau,\xi,\varepsilon,\mu),\varepsilon,\mu))(t_2(\tau,\xi,\varepsilon,\mu),\varepsilon,\mu)) = 0, \\ \langle x_+(t_2(\tau,\xi,\varepsilon,\mu), x_-(t_1(\tau,\xi,\varepsilon,\mu), x_+(\tau,\xi)(t_1(\tau,\xi,\varepsilon,\mu),\varepsilon,\mu)) \\ (t_2(\tau,\xi,\varepsilon,\mu),\varepsilon,\mu))(t_3(\tau,\xi,\beta,\varepsilon,\mu),\varepsilon,\mu) - x_0(\beta), f_+(x_0(\beta))\rangle = 0. \end{gathered}$$

Moreover, we have $t_2(0, x_0(\beta), 0, \mu) = t_2^\beta$ and $t_3(0, x_0(\beta), \beta, 0, \mu) = T^\beta$. Poincaré mapping is then defined as

$$\begin{aligned} P(\xi,\beta,\varepsilon,\mu) = x_+(t_2(0,\xi,\varepsilon,\mu), x_-(t_1(0,\xi,\varepsilon,\mu), x_+(0,\xi)(t_1(0,\xi,\varepsilon,\mu),\varepsilon,\mu)) \\ (t_2(0,\xi,\varepsilon,\mu),\varepsilon,\mu))(t_3(0,\xi,\beta,\varepsilon,\mu),\varepsilon,\mu). \end{aligned} \tag{2.5}$$

∎

The next lemma describes some properties of derived Poincaré mapping.

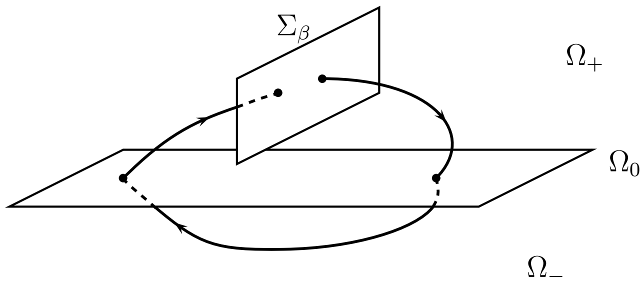

Fig. 2.1: Discontinuous Poincaré mapping

Lemma 2.2. *Let $P(\xi, \beta, \varepsilon, \mu)$ be defined by (2.5). Then*

$$P_\xi(x_0(\beta), \beta, 0, \mu) = (\mathbb{I} - \mathcal{S}_\beta)A(\beta, 0), \tag{2.6}$$

$$P_\beta(x_0(\beta), \beta, 0, \mu) = \mathcal{S}_\beta \mathrm{D}x_0(\beta), \tag{2.7}$$

$$P_\varepsilon(x_0(\beta), \beta, 0, \mu) = (\mathbb{I} - \mathcal{S}_\beta)\left(\int_0^{T^\beta} A(\beta, s)g(\gamma(\beta, s), 0, \mu)ds\right), \tag{2.8}$$

where P_ξ, P_β, P_ε are partial derivatives of P with respect to ξ, β, ε, respectively.

Here $\mathcal{S}_\beta$ is the orthogonal projection onto the 1-dimensional space $[f_+(x_0(\beta))]$ defined by

$$\mathcal{S}_\beta u = \frac{\langle u, f_+(x_0(\beta))\rangle f_+(x_0(\beta))}{\|f_+(x_0(\beta))\|^2} \tag{2.9}$$

and $A(\beta, t)$ is given by

$$A(\beta, t) = \begin{cases} X_3(\beta, T^\beta)S_2(\beta)X_2(\beta, t_2^\beta)S_1(\beta)X_1(\beta, t_1^\beta)X_1^{-1}(\beta, t) & \text{if } t \in [0, t_1^\beta), \\ X_3(\beta, T^\beta)S_2(\beta)X_2(\beta, t_2^\beta)X_2^{-1}(\beta, t) & \text{if } t \in [t_1^\beta, t_2^\beta), \\ X_3(\beta, T^\beta)X_3^{-1}(\beta, t) & \text{if } t \in [t_2^\beta, T^\beta], \end{cases} \tag{2.10}$$

where

$$S_1(\beta) = \mathbb{I} + \frac{(f_-(x_1(\beta)) - f_+(x_1(\beta)))\mathrm{D}h(x_1(\beta))}{\mathrm{D}h(x_1(\beta))f_+(x_1(\beta))}, \tag{2.11}$$

$$S_2(\beta) = \mathbb{I} + \frac{(f_+(x_2(\beta)) - f_-(x_2(\beta)))\mathrm{D}h(x_2(\beta))}{\mathrm{D}h(x_2(\beta))f_-(x_2(\beta))}, \tag{2.12}$$

and finally, $X_1(\beta, t)$, $X_2(\beta, t)$ and $X_3(\beta, t)$ solve the following linear initial value problems

$$\begin{aligned} \dot{X}_1(\beta, t) &= \mathrm{D}f_+(\gamma(\beta, t))X_1(\beta, t) \\ X_1(\beta, 0) &= \mathbb{I}, \end{aligned} \tag{2.13}$$

$$\begin{aligned} \dot{X}_2(\beta, t) &= \mathrm{D}f_-(\gamma(\beta, t))X_2(\beta, t) \\ X_2(\beta, t_1^\beta) &= \mathbb{I}, \end{aligned} \tag{2.14}$$

$$\begin{aligned} \dot{X}_3(\beta, t) &= \mathrm{D}f_+(\gamma(\beta, t))X_3(\beta, t) \\ X_3(\beta, t_2^\beta) &= \mathbb{I}, \end{aligned} \tag{2.15}$$

respectively.

Note saltation matrices $S_1(\beta)$, $S_2(\beta)$ are invertible (cf. [35]) with

$$S_1^{-1}(\beta) = \mathbb{I} + \frac{(f_+(x_1(\beta)) - f_-(x_1(\beta)))\mathrm{D}h(x_1(\beta))}{\mathrm{D}h(x_1(\beta))f_-(x_1(\beta))},$$
$$S_2^{-1}(\beta) = \mathbb{I} + \frac{(f_-(x_2(\beta)) - f_+(x_2(\beta)))\mathrm{D}h(x_2(\beta))}{\mathrm{D}h(x_2(\beta))f_+(x_2(\beta))}.$$

Considering the inner product $\langle a, b\rangle = b^* a$, it is possible to introduce matrix notation for operator $\mathcal{S}_\beta$ of (2.9):

$$S_\beta u = \frac{f_+(x_0(\beta))(f_+(x_0(\beta)))^*}{\|f_+(x_0(\beta))\|^2} u \tag{2.16}$$

which is symmetric, i.e. $S_\beta^* = S_\beta$. The derivative of mapping (2.5) has an important property:

Lemma 2.3. *For any $\xi \in B(x_0^0, r_0)$, $\beta \in W$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and $\mu \in \mathbb{R}^p$ $P_\xi(\xi, \beta, \varepsilon, \mu)$ has eigenvalue 0 with corresponding eigenvector $f_+(\xi) + \varepsilon g(\xi, \varepsilon, \mu)$, i.e.*

$$P_\xi(\xi, \beta, \varepsilon, \mu)[f_+(\xi) + \varepsilon g(\xi, \varepsilon, \mu)] = 0.$$

Proof. Similarly to unperturbed case in Lemma 1.3 (see also [35]) we have

$$\begin{aligned} &x_+(0, x_+(0, \xi)(t, \varepsilon, \mu))(t_1(0, x_+(0, \xi)(t, \varepsilon, \mu), \varepsilon, \mu), \varepsilon, \mu) \\ &\quad = x_+(0, \xi)(t_1(0, x_+(0, \xi)(t, \varepsilon, \mu), \varepsilon, \mu) + t, \varepsilon, \mu) \end{aligned}$$

as the first intersection point of the trajectory of perturbed system (2.1) and discontinuity boundary Ω_0. Hence

$$t_1(0, x_+(0, \xi)(t, \varepsilon, \mu), \varepsilon, \mu) + t = t_1(0, \xi, \varepsilon, \mu)$$

for any t sufficiently close to 0. Analogically we get

$$t_2(0, x_+(0, \xi)(t, \varepsilon, \mu), \varepsilon, \mu) + t = t_2(0, \xi, \varepsilon, \mu).$$

Next,

$$\begin{gathered} P(x_+(0, \xi)(t, \varepsilon, \mu), \beta, \varepsilon, \mu) = x_+(t_2(0, x_+(0, \xi)(t, \varepsilon, \mu), \varepsilon, \mu), \\ x_-(t_1(0, x_+(0, \xi)(t, \varepsilon, \mu), \varepsilon, \mu), x_+(0, x_+(0, \xi)(t, \varepsilon, \mu)) \\ (t_1(0, x_+(0, \xi)(t, \varepsilon, \mu), \varepsilon, \mu), \varepsilon, \mu))(t_2(0, x_+(0, \xi)(t, \varepsilon, \mu), \varepsilon, \mu), \varepsilon, \mu)) \\ (t_3(0, x_+(0, \xi)(t, \varepsilon, \mu), \beta, \varepsilon, \mu), \varepsilon, \mu) \\ = x_+(t_2(0, \xi, \varepsilon, \mu), x_-(t_1(0, \xi, \varepsilon, \mu), x_+(0, \xi)(t_1(0, \xi, \varepsilon, \mu), \varepsilon, \mu)) \\ (t_2(0, \xi, \varepsilon, \mu), \varepsilon, \mu))(t_3(0, x_+(0, \xi)(t, \varepsilon, \mu), \beta, \varepsilon, \mu) + t, \varepsilon, \mu) \end{gathered}$$

for any t close to 0. The most left-hand side of the latter equation is from Σ_β and the most right-hand side is a point of a trajectory of (2.1) with given ε starting at ξ, therefore it is a fixed point $\zeta \in \Sigma_\beta$ and $t_3(0, x_+(0, \xi)(t, \varepsilon, \mu), \beta, \varepsilon, \mu) + t$ is constant. Consequently,

$$P_\xi(\xi, \beta, \varepsilon, \mu)[f_+(\xi) + \varepsilon g(\xi, \varepsilon, \mu)] = \mathrm{D}_t P(x_+(0, \xi)(t, \varepsilon, \mu), \beta, \varepsilon, \mu)\big|_{t=0} = \mathrm{D}_t \zeta\big|_{t=0} = 0.$$

$\square$

Note if we take $\xi = x_0(\beta)$ and $\varepsilon = 0$ in the above lemma then $\zeta = x_0(\beta)$ and $t_3(0, x_+(0, x_0(\beta))(t, 0, \mu), \beta, 0, \mu) + t = T^\beta$.

For any $\xi \in \mathbb{R}^n$ we define orthogonal projection $\widetilde{\mathcal{S}}_\beta$ onto Σ_β

$$\widetilde{\mathcal{S}}_\beta : \xi \mapsto \xi - S_\beta(\xi - x_0(\beta)).$$

Denoting $F(\xi, \beta, \varepsilon, \mu) := \widetilde{\mathcal{S}}_\beta(\xi) - P(\xi, \beta, \varepsilon, \mu)$, ξ is an initial point from Σ_β of periodic orbit of perturbed system (2.1) if and only if it satisfies

$$F(\xi, \beta, \varepsilon, \mu) = 0, \qquad \xi \in \Sigma_\beta. \tag{2.17}$$

For $\xi \in \mathbb{R}^n$

$$F_\xi(x_0(\beta), \beta, 0, \mu)\xi = (\mathbb{I} - S_\beta)\xi - P_\xi(x_0(\beta), \beta, 0, \mu)\xi,$$

thus from Lemma 2.3

$$F_\xi(x_0(\beta), \beta, 0, \mu) f_+(x_0(\beta)) = 0, \tag{2.18}$$

where F_ξ is the partial derivative of F with respect to ξ. On the other side,

$$F(x_0(\beta), \beta, 0, \mu) = x_0(\beta) - P(x_0(\beta), \beta, 0, \mu) = 0, \quad \forall \beta \in W,$$

hence from (2.7)

$$P_\xi(x_0(\beta), \beta, 0, \mu)\mathrm{D}x_0(\beta) = (\mathbb{I} - S_\beta)\mathrm{D}x_0(\beta)$$

and therefore

$$F_\xi(x_0(\beta), \beta, 0, \mu)\mathrm{D}x_0(\beta) = 0, \qquad \forall \beta \in W.$$

Here we state the third condition:

H3) The set

$$\left\{ \frac{\partial x_0(\beta)}{\partial \beta_1}, \dots, \frac{\partial x_0(\beta)}{\partial \beta_k}, f_+(x_0(\beta)) \right\}$$

spans the null space of the operator $F_\xi(x_0(\beta), \beta, 0, \mu)$.

Note $\Sigma_\beta = [f_+(x_0(\beta))]^\perp + x_0(\beta)$. Let us denote

$$Z_\beta = \mathcal{N} F_\xi(x_0(\beta), \beta, 0, \mu) \cap [f_+(x_0(\beta))]^\perp, \quad Y_\beta = \mathcal{R} F_\xi(x_0(\beta), \beta, 0, \mu) \tag{2.19}$$

the restricted null space and the range of the corresponding operator, respectively. Now from condition H3) we have

$$Z_\beta = \left[\frac{\partial x_0(\beta_0)}{\partial \beta_1}, \dots, \frac{\partial x_0(\beta_0)}{\partial \beta_k}, f_+(x_0(\beta)) \right] \cap [f_+(x_0(\beta))]^\perp = (\mathbb{I} - S_\beta)\mathrm{D}x_0(\beta).$$

Using Gram-Schmidt orthogonalization we find an orthonormal basis $\{y_1, \dots, y_{n-k-1}\}$ for vector space $Z_\beta^\perp$ such that $Z_\beta \perp Z_\beta^\perp$ and $Z_\beta \oplus Z_\beta^\perp = [f_+(x_0(\beta))]^\perp$.

We can define orthogonal projections

$$\mathcal{Q}_\beta : \Sigma_\beta \to Y_\beta, \qquad \mathcal{P}_\beta : \Sigma_\beta \to Y_\beta^\perp, \tag{2.20}$$

where $Y_\beta^\perp$ is an orthogonal complement to Y_β in $[f_+(x_0(\beta))]^\perp$, and the decomposition for any z sufficiently close to manifold $x_0(W)$

$$z = x_0(\beta) + \xi \quad \text{for} \quad \beta \in W, \xi \in \left[\frac{\partial x_0(\beta)}{\partial \beta_1}, \dots, \frac{\partial x_0(\beta)}{\partial \beta_k}\right]^\perp.$$

The second condition of (2.17) gives another restriction on ξ, i.e.

$$\xi \in \left[\frac{\partial x_0(\beta)}{\partial \beta_1}, \dots, \frac{\partial x_0(\beta)}{\partial \beta_k}, f_+(x_0(\beta))\right]^\perp,$$

therefore

$$z = x_0(\beta) + \xi, \qquad \beta \in W,\, \xi \in Z_\beta^\perp. \tag{2.21}$$

Note Z_β, $\mathcal{Q}_\beta$ and $\mathcal{P}_\beta$ are C^{r-1}-smooth with respect to β. Consequently, applying Lyapunov-Schmidt reduction, equation (2.17) is equivalent to the next couple of equations

$$\mathcal{Q}_\beta F(x_0(\beta) + \xi, \beta, \varepsilon, \mu) = 0, \tag{2.22}$$

$$\mathcal{P}_\beta F(x_0(\beta) + \xi, \beta, \varepsilon, \mu) = 0. \tag{2.23}$$

Considering the first one as the equation for $\xi \in Z_\beta^\perp$, it can be solved via IFT, since $\mathcal{Q}_\beta F(x_0(\beta), \beta, 0, \mu) = 0$ and

$$\mathrm{D}_\xi \mathcal{Q}_\beta F(x_0(\beta) + \xi, \beta, 0, \mu)\big|_{\xi=0} = \mathcal{Q}_\beta F_\xi(x_0(\beta), \beta, 0, \mu)\big|_{Z_\beta^\perp},$$

where $\big|_{Z_\beta^\perp}$ denotes the restriction on $Z_\beta^\perp$, is an isomorphism $Z_\beta^\perp$ onto Y_β for all $\mu \in \mathbb{R}^p$. So there exist positive constants ε_2, r_2 and C^{r-1}-function

$$\xi(\cdot,\cdot,\cdot) : B(\beta_0, r_2) \times (-\varepsilon_2, \varepsilon_2) \times \mathbb{R}^p \to Z_\beta^\perp$$

such that $\mathcal{Q}_\beta F(x_0(\beta) + \xi, \beta, \varepsilon, \mu) = 0$ for $\beta \in B(\beta_0, r_2)$, $\varepsilon \in (-\varepsilon_2, \varepsilon_2)$ and $\mu \in \mathbb{R}^p$ if and only if $\xi = \xi(\beta, \varepsilon, \mu)$. Moreover $\xi(\beta, 0, \mu) = 0$, since $\mathcal{Q}_\beta F(x_0(\beta), \beta, 0, \mu) = 0$.

Equation (2.23) now has the form

$$G(\beta, \varepsilon, \mu) := \mathcal{P}_\beta F(x_0(\beta) + \xi(\beta, \varepsilon, \mu), \beta, \varepsilon, \mu) = 0 \tag{2.24}$$

for β close to β_0, ε to 0 and $\mu \in \mathbb{R}^p$. In unperturbed case, this is easily solved, i.e. $G(\beta, 0, \mu) = 0$. In order to have a persisting periodic orbit, equation (2.24) has to be satisfied for all ε sufficiently close to 0. The next condition follows

$$\begin{aligned} 0 = G_\varepsilon(\beta, 0, \mu) &= \mathcal{P}_\beta \mathrm{D}_\varepsilon F(x_0(\beta) + \xi(\beta, \varepsilon, \mu), \beta, \varepsilon, \mu)\big|_{\varepsilon=0} \\ &= \mathcal{P}_\beta \left[F_\xi(x_0(\beta), \beta, 0, \mu)\xi_\varepsilon(\beta, 0, \mu) + F_\varepsilon(x_0(\beta), \beta, 0, \mu)\right] \\ &= \mathcal{P}_\beta F_\varepsilon(x_0(\beta), \beta, 0, \mu) = -\mathcal{P}_\beta P_\varepsilon(x_0(\beta), \beta, 0, \mu), \end{aligned} \tag{2.25}$$

where G_ε, F_ε are the partial derivatives of G, F with respect to ε, respectively. We note [31] that there exists an orthogonal basis $\{\psi_1(\beta), \dots, \psi_k(\beta)\}$ of $Y_\beta^\perp$, i.e.

$Y_\beta^\perp = [\psi_1(\beta), \ldots, \psi_k(\beta)]$ for each $\beta \in W$ and ψ_i are C^{r-1}-smooth. Then projection $\mathcal{P}_\beta$ of (2.20) can be written in form

$$\mathcal{P}_\beta y = \sum_{i=1}^{k} \frac{\langle y, \psi_i(\beta)\rangle \psi_i(\beta)}{\|\psi_i(\beta)\|^2}.$$

Equation (2.25) can be rewritten as follows

$$\sum_{i=1}^{k} \frac{\langle P_\varepsilon(x_0(\beta), \beta, 0, \mu), \psi_i(\beta)\rangle\, \psi_i(\beta)}{\|\psi_i(\beta)\|^2} = 0. \tag{2.26}$$

Using linear independence of $\psi_1(\beta), \ldots, \psi_k(\beta)$ and Lemma 2.2 together with (2.26), we arrive at

$$M^\mu(\beta) = 0 \quad \text{if and only if} \quad G_\varepsilon(\beta, 0, \mu) = 0, \tag{2.27}$$

where

$$\begin{gathered} M^\mu(\beta) = (M_1^\mu(\beta), \ldots, M_k^\mu(\beta)), \\ M_i^\mu(\beta) = \int_0^{T^\beta} \langle g(\gamma(\beta, t), 0, \mu), A^*(\beta, t)\psi_i(\beta)\rangle dt, i = 1, \ldots, k. \end{gathered} \tag{2.28}$$

Note by (2.10), we have

$$A^*(\beta, t) = \begin{cases} X_1^{-1*}(\beta, t) X_1^*(\beta, t_1^\beta) S_1^*(\beta) X_2^*(\beta, t_2^\beta) S_2^*(\beta) X_3^*(\beta, T^\beta) & \text{if } t \in [0, t_1^\beta), \\ X_2^{-1*}(\beta, t) X_2^*(\beta, t_2^\beta) S_2^*(\beta) X_3^*(\beta, T^\beta) & \text{if } t \in [t_1^\beta, t_2^\beta), \\ X_3^{-1*}(\beta, t) X_3^*(\beta, T^\beta) & \text{if } t \in [t_2^\beta, T^\beta]. \end{cases} \tag{2.29}$$

We shall call the function M^μ defined by (2.28) as a Poincaré-Andronov-Melnikov function for discontinuous system (2.1) (see Remark 2.6 below).

We know [35] that the linearization of (2.1) with $\varepsilon = 0$ along T^β–periodic solution $\gamma(\beta, t)$ is given by

$$\dot{x} = \mathrm{D} f_\pm(\gamma(\beta, t))x \tag{2.30}$$

which splits into two unperturbed equations

$$\begin{aligned} \dot{x} &= \mathrm{D} f_+(\gamma(\beta, t))x \qquad && \text{if } t \in [0, t_1^\beta) \cup [t_2^\beta, T^\beta], \\ \dot{x} &= \mathrm{D} f_-(\gamma(\beta, t))x \qquad && \text{if } t \in [t_1^\beta, t_2^\beta) \end{aligned}$$

satisfying impulsive conditions

$$x(t_1^\beta+) = S_1(\beta)x(t_1^\beta-), \qquad x(t_2^\beta+) = S_2(\beta)x(t_2^\beta-) \tag{2.31}$$

and periodic condition

$$(\mathbb{I} - S_\beta)(x(T^\beta) - x(0)) = 0 \tag{2.32}$$

as well, where $x(t\pm) = \lim_{s\to t^\pm} x(s)$. Corresponding fundamental matrices are $X_1(\beta, t)$, $X_3(\beta, t)$ to plus-equation and $X_2(\beta, t)$ to minus-equation (cf. Lemma 2.2). It follows that the fundamental matrix solution of unperturbed variational equation (2.30) is given by

$$X(\beta,t)=\begin{cases}X_1(\beta,t) & \text{if } t\in[0,t_1^\beta),\\ X_2(\beta,t)S_1(\beta)X_1(\beta,t_1^\beta) & \text{if } t\in[t_1^\beta,t_2^\beta),\\ X_3(\beta,t)S_2(\beta)X_2(\beta,t_2^\beta)S_1(\beta)X_1(\beta,t_1^\beta) & \text{if } t\in[t_2^\beta,T^\beta].\end{cases} \tag{2.33}$$

Especially, $X(\beta,T^\beta)=A(\beta,0)$. To proceed, we need the following Fredholm like result for linear impulsive boundary value problems of the form (2.30), (2.31) and (2.32).

Lemma 2.4. *Let $A(t)\in C([0,T],L(\mathbb{R}^n))$, $B_1,B_2,B_3\in L(\mathbb{R}^n)$, $0<t_1<t_2<T$ and $h\in\mathcal{C}:=C([0,t_1],\mathbb{R}^n)\cap C([t_1,t_2],\mathbb{R}^n)\cap C([t_2,T],\mathbb{R}^n)$. Then the nonhomogeneous problem*

$$\begin{gathered}\dot x=A(t)x+h(t),\\ x(t_i+)=B_ix(t_i-),\ i=1,2,\\ B_3(x(T)-x(0))=0\end{gathered} \tag{2.34}$$

has a solution $x\in\mathcal{C}^1:=C^1([0,t_1],\mathbb{R}^n)\cap C^1([t_1,t_2],\mathbb{R}^n)\cap C^1([t_2,T],\mathbb{R}^n)$ if and only if $\int_0^T\langle h(t),v(t)\rangle dt=0$ for any solution $v\in\mathcal{C}^1$ of the adjoint system given by

$$\begin{gathered}\dot v=-A^*(t)v,\\ v(t_i-)=B_i^*v(t_i+),\ i=1,2,\\ v(T)=v(0)\in\mathcal{N}B_3^\perp.\end{gathered} \tag{2.35}$$

Proof. If $h\in\mathcal{C}$ satisfies (2.34) for a $x\in\mathcal{C}^1$ then for any $v\in\mathcal{C}^1$ fulfilling (2.35), we derive

$$\begin{aligned}\int_0^T\langle h(t),v(t)\rangle dt&=\int_0^T\langle\dot x(t)-A(t)x(t),v(t)\rangle dt\\ &=\langle x(T),v(T)\rangle-\langle x(t_2+),v(t_2+)\rangle+\langle x(t_2-),v(t_2-)\rangle\\ -\langle x(t_1+),v(t_1+)\rangle&+\langle x(t_1-),v(t_1-)\rangle-\langle x(0),v(0)\rangle-\int_0^T\langle x(t),\dot v(t)+A^*(t)v(t)\rangle dt\\ &=\langle x(T)-x(0),v(T)\rangle+\langle x(0),v(T)-v(0)\rangle+\langle x(t_1-),v(t_1-)-B_1^*v(t_1+)\rangle\\ &\quad+\langle x(t_2-),v(t_2-)-B_2^*v(t_2+)\rangle-\int_0^T\langle x(t),\dot v(t)+A^*(t)v(t)\rangle dt=0.\end{aligned}$$

Reversely, we suppose that $h\in\mathcal{C}$ satisfies $\int_0^T\langle h(t),v(t)\rangle dt=0$ for any $v\in\mathcal{C}^1$ fulfilling (2.35). Let $X(t)$ be the fundamental solution of $\dot x=A(t)x$ with $X(0)=\mathbb{I}$. Then

$$v(t)=\begin{cases}X^{-1*}(t)X^*(t_1)B_1^*X^{-1*}(t_1)X^*(t_2)B_2^*X^{-1*}(t_2)X^*(T)v(T) & \text{if } t\in[0,t_1),\\ X^{-1*}(t)X^*(t_2)B_2^*X^{-1*}(t_2)X^*(T)v(T) & \text{if } t\in[t_1,t_2),\\ X^{-1*}(t)X^*(T)v(T) & \text{if } t\in[t_2,T]\end{cases}$$

and

$$X^*(t_1)B_1^*X^{-1*}(t_1)X^*(t_2)B_2^*X^{-1*}(t_2)X^*(T)v(T)=v(T)\in\mathcal{N}B_3^\perp=\mathcal{R}B_3^*\,. \tag{2.36}$$

So

$$
\begin{aligned}
0 = \int_0^T \langle h(t), v(t)\rangle dt = \Bigg\langle \int_{t_2}^T X(T)X^{-1}(s)h(s)ds \\
+ \int_{t_1}^{t_2} X(T)X^{-1}(t_2)B_2X(t_2)X^{-1}(s)h(s)ds \\
+ \int_0^{t_1} X(T)X^{-1}(t_2)B_2X(t_2)X^{-1}(t_1)B_1X(t_1)X^{-1}(s)h(s)ds, v(T)\Bigg\rangle.
\end{aligned} \tag{2.37}
$$

Next, by (2.36) and (2.37), we obtain $v(T) = B_3^*w$ for a $w \in \mathbb{R}^n$ satisfying

$$
\left(X^*(t_1)B_1^*X^{-1*}(t_1)X^*(t_2)B_2^*X^{-1*}(t_2)X^*(T) - \mathbb{I}\right) B_3^*w = 0 \tag{2.38}
$$

along with

$$
\begin{aligned}
0 = \langle B_3u, w\rangle, \\
u := \int_{t_2}^T X(T)X^{-1}(s)h(s)ds + \int_{t_1}^{t_2} X(T)X^{-1}(t_2)B_2X(t_2)X^{-1}(s)h(s)ds \\
+ \int_0^{t_1} X(T)X^{-1}(t_2)B_2X(t_2)X^{-1}(t_1)B_1X(t_1)X^{-1}(s)h(s)ds.
\end{aligned} \tag{2.39}
$$

But (2.38) and (2.39) imply the existence of $x_0 \in \mathbb{R}^n$ such that

$$
B_3\left(X(T)X^{-1}(t_2)B_2X(t_2)X^{-1}(t_1)B_1X(t_1)x_0 + u - x_0\right) = 0. \tag{2.40}
$$

Then using (2.40), we see that the function

$$
x(t) = \begin{cases}
X(t)x_0 + \int_0^t X(t)X^{-1}(s)h(s)ds & \text{if } t \in [0, t_1), \\
X(t)X^{-1}(t_1)B_1\Big(X(t_1)x_0 + \int_0^{t_1} X(t_1)X^{-1}(s)h(s)ds\Big) & \\
+ \int_{t_1}^t X(t)X^{-1}(s)h(s)ds & \text{if } t \in [t_1, t_2), \\
X(t)X^{-1}(t_2)B_2\Bigg(X(t_2)X^{-1}(t_1)B_1\Big(X(t_1)x_0 & \\
+ \int_0^{t_1} X(t_1)X^{-1}(s)h(s)ds\Big) + \int_{t_1}^{t_2} X(t_2)X^{-1}(s)h(s)ds\Bigg) & \\
+ \int_{t_2}^t X(t)X^{-1}(s)h(s)ds & \text{if } t \in [t_2, T]
\end{cases}
$$

is a solution of (2.34). The proof is finished. □

Applying the above lemma to (2.30), (2.31) and (2.32) we see that the adjoint variational system of (2.1) is given by the following linear impulsive boundary value problem

$$
\begin{aligned}
\dot{X} = -\mathrm{D}f_+^*(\gamma(\beta,t))X \qquad & \text{if } t \in [0, t_1^\beta], \\
\dot{X} = -\mathrm{D}f_-^*(\gamma(\beta,t))X \qquad & \text{if } t \in [t_1^\beta, t_2^\beta], \\
\dot{X} = -\mathrm{D}f_+^*(\gamma(\beta,t))X \qquad & \text{if } t \in [t_2^\beta, T^\beta], \\
X(t_i^\beta+) = S_i^*(\beta)^{-1}X(t_i^\beta-),\ i = 1, 2, & \\
X(T^\beta) = X(0) \in [f_+(x_0(\beta))]^\perp. &
\end{aligned} \tag{2.41}
$$

Note for each $i = 1, \dots, k$ and for any $\xi \in [f_+(x_0(\beta))]^\perp$ we have $S_\beta \psi_i(\beta) = 0$ and

$$
\begin{aligned}
0 &= \langle F_\xi(x_0(\beta), \beta, 0, \mu)\xi, \psi_i(\beta)\rangle = \langle (\mathbb{I} - P_\xi(x_0(\beta), \beta, 0, \mu))\xi, \psi_i(\beta)\rangle \\
&= \langle \xi, \psi_i(\beta) - A^*(\beta, 0)(\mathbb{I} - S_\beta)\psi_i(\beta)\rangle = \langle \xi, (\mathbb{I} - A^*(\beta, 0))\psi_i(\beta)\rangle
\end{aligned}
$$

and for $\xi \in [f_+(x_0(\beta))]$ (cf. (2.18))

$$
\begin{aligned}
0 &= \langle F_\xi(x_0(\beta), \beta, 0, \mu)\xi, \psi_i(\beta)\rangle = \langle (\mathbb{I} - S_\beta - P_\xi(x_0(\beta), \beta, 0, \mu))\xi, \psi_i(\beta)\rangle \\
&= \langle \xi, (\mathbb{I} - A^*(\beta, 0))(\mathbb{I} - S_\beta)\psi_i(\beta)\rangle = \langle \xi, (\mathbb{I} - A^*(\beta, 0))\psi_i(\beta)\rangle.
\end{aligned}
$$

Hence $A^*(\beta, 0)\psi_i(\beta) = \psi_i(\beta)$.

Next, from Lemma 1.5 it is easy to see that for each $i = 1, \dots, k$, $A^*(\beta, t)\psi_i(\beta)$ is the solution of (2.41).

Consequently, we can take in (2.28) any basis of T^β-periodic solutions of the adjoint variational equation.

Note, the condition $M^\mu(\beta) = 0$ is a necessary one for the persistence of periodic orbit.

The following theorem states the sufficient condition for the existence of a unique periodic solution of equation (2.1) with $\varepsilon \neq 0$:

Theorem 2.5. *Let conditions* H1), H2), H3) *be satisfied and* $M^\mu(\beta)$ *be defined by* (2.28). *If* $\beta_0 \in V$ *is a simple root of* M^{μ_0}, *i.e.*

$$
M^{\mu_0}(\beta_0) = 0, \qquad \det \mathrm{D}M^{\mu_0}(\beta_0) \neq 0,
$$

then there exist a neighbourhood U *of the point* $(0, \mu_0)$ *in* $\mathbb{R} \times \mathbb{R}^p$ *and a* C^{r-2}*-function* $\beta(\varepsilon, \mu)$, *with* $\beta(0, \mu_0) = \beta_0$, *such that perturbed equation* (2.1) *possesses a unique persisting closed trajectory. Moreover, it contains a point*

$$
x^*(\varepsilon, \mu) := x_0(\beta(\varepsilon, \mu)) + \xi(\beta(\varepsilon, \mu), \varepsilon, \mu) \in \Sigma_{\beta(\varepsilon,\mu)} \tag{2.42}
$$

and it has a period $t_3(0, x^*(\varepsilon, \mu), \beta(\varepsilon, \mu), \varepsilon, \mu)$.

Proof. We introduce the function

$$
H(\beta, \varepsilon, \mu) = \begin{cases} \frac{1}{\varepsilon} G(\beta, \varepsilon, \mu) & \text{if } \varepsilon \neq 0, \\ G_\varepsilon(\beta, \varepsilon, \mu) & \text{if } \varepsilon = 0. \end{cases}
$$

We recall that G_ε is the partial derivative of G with respect to ε. Clearly, H is C^{r-2}-smooth and $H(\beta, \varepsilon, \mu) = 0$ gives the desired periodic solution. Note that $H(\beta, 0, \mu) = -\Psi(\beta)M^\mu(\beta)$, where $\Psi(\beta)$ is $n \times k$ matrix with i-th column $\frac{\psi_i(\beta)}{\langle \psi_i(\beta), \psi_i(\beta)\rangle}$. Next, for the partial derivative H_β of H with respect to β, we derive

$$
H_\beta(\beta_0, 0, \mu_0) = -\mathrm{D}\Psi(\beta_0)M^{\mu_0}(\beta_0) - \Psi(\beta_0)\mathrm{D}M^{\mu_0}(\beta_0) = -\Psi(\beta_0)\mathrm{D}M^{\mu_0}(\beta_0),
$$

thus $H_\beta(\beta_0, 0, \mu_0)$ is an isomorphism, and from (2.27), IFT implies the existence of neighbourhood U and function $\beta(\varepsilon, \mu)$ from the statement of the theorem. Results on x^* and the period of the persisting orbit follow immediately from preceding arguments. ◻

Let us denote the persisting periodic trajectory from the latter theorem by $\gamma^*(\varepsilon, \mu, t)$. Then clearly $\gamma^*(0, \mu, t) = \gamma(\beta_0, t)$.

Note that if function g is discontinuous in x, the above theorem remains true.

Remark 2.6. If (2.1) is smooth, i.e. $f_\pm = f$, then $S_i(\beta) = \mathbb{I}$, $i = 1, 2$ and (2.41) has the form

$$\begin{aligned} \dot{X} &= -\mathrm{D}f^*(\gamma(\beta, t))X \qquad \text{if} \quad t \in [0, T^\beta], \\ X(T^\beta) &= X(0) \in [f_+(x_0(\beta))]^\perp. \end{aligned} \tag{2.43}$$

So the Poincaré-Andronov-Melnikov function (2.28) possesses the form

$$\begin{aligned} M^\mu(\beta) &= (M_1^\mu(\beta), \dots, M_k^\mu(\beta)), \\ M_i^\mu(\beta) &= \int_0^{T^\beta} \langle g(\gamma(\beta, t), 0, \mu), \psi_i(\beta, t)\rangle dt, i = 1, \dots, k \end{aligned} \tag{2.44}$$

for any smooth basis $\{\psi_i(\beta, t)\}_{i=1}^k$ of solutions of the adjoint periodic linear problem (2.43). Note (2.44) is the usual Poincaré-Andronov-Melnikov function [12, 14] for bifurcation of periodic orbits for NDS. This is a reason why we name (2.28) again as Poincaré-Andronov-Melnikov function for PPSNDS (2.1).

2.1 Geometric interpretation

In this section, we look at the investigated problem from a geometric point of view. For any $\beta, \widetilde{\beta} \in W$, we can solve $t(\beta, \widetilde{\beta})$ from

$$\begin{aligned} \langle \gamma(\widetilde{\beta}, t(\beta, \widetilde{\beta})) - x_0(\beta), f_+(x_0(\beta))\rangle = 0 \\ t(\beta, \beta) = 0. \end{aligned}$$

Note

$$\frac{\partial t(\beta, \beta)}{\partial \widetilde{\beta}} = -\frac{\langle \mathrm{D}x_0(\beta), f_+(x_0(\beta))\rangle}{\|f_+(x_0(\beta))\|^2}. \tag{2.45}$$

Set $\widetilde{x}_\beta(\widetilde{\beta}) = \gamma(\widetilde{\beta}, t(\beta, \widetilde{\beta}))$, $\gamma_\beta(\widetilde{\beta}, t) = \gamma(\widetilde{\beta}, t(\beta, \widetilde{\beta}) + t)$. So $\gamma_\beta(\widetilde{\beta}, 0) = \widetilde{x}_\beta(\widetilde{\beta})$, $\widetilde{x}_\beta(\beta) = x_0(\beta)$, $\gamma_\beta(\beta, t) = \gamma(\beta, t)$ and

$$F(\widetilde{x}_\beta(\widetilde{\beta}), \beta, 0, \mu) = 0. \tag{2.46}$$

Indeed, analogically to proof of Lemma 2.3 $t_i(0, \widetilde{x}_\beta(\widetilde{\beta}), 0, \mu) + t(\beta, \widetilde{\beta}) = t_i^{\widetilde{\beta}}$ for $i = 1, 2$, therefore

$$P(\widetilde{x}_\beta(\widetilde{\beta}), \beta, 0, \mu) = x_+(t_2^{\widetilde{\beta}}, x_2(\widetilde{\beta}))(t_3(0, \widetilde{x}_\beta(\widetilde{\beta}), \beta, 0, \mu) + t(\beta, \widetilde{\beta}), 0, \mu).$$

Since $\widetilde{x}_\beta(\widetilde{\beta}) \in \Sigma_\beta$, we obtain $t_3(0, \widetilde{x}_\beta(\widetilde{\beta}), \beta, 0, \mu) + t(\beta, \widetilde{\beta}) = T^{\widetilde{\beta}} + t(\beta, \widetilde{\beta})$ and consequently

$$\begin{aligned} F(\widetilde{x}_\beta(\widetilde{\beta}), \beta, 0, \mu) &= \widetilde{x}_\beta(\widetilde{\beta}) - P(\widetilde{x}_\beta(\widetilde{\beta}), \beta, 0, \mu) \\ = \widetilde{x}_\beta(\widetilde{\beta}) - x_+(t_2^{\widetilde{\beta}}, x_2(\widetilde{\beta}))(T^{\widetilde{\beta}} + t(\beta, \widetilde{\beta}), 0, \mu) &= \widetilde{x}_\beta(\widetilde{\beta}) - \gamma(\widetilde{\beta}, T^{\widetilde{\beta}} + t(\beta, \widetilde{\beta})) = 0. \end{aligned}$$

Equation (2.46) implies $F_\xi(x_0(\beta),\beta,0,\mu)\mathrm{D}\widetilde{x}_\beta(\beta)=0$. Note by (2.45) $\mathrm{D}\widetilde{x}_\beta(\beta)=(\mathbb{I}-S_\beta)\mathrm{D}x_0(\beta)$ and hence $F_\xi(x_0(\beta),\beta,0,\mu)(\mathbb{I}-S_\beta)\mathrm{D}x_0(\beta)=0$ what by (2.6) is equivalent to

$$(\mathbb{I}-S_\beta)\mathrm{D}x_0(\beta)=(\mathbb{I}-S_\beta)A(\beta,0)(\mathbb{I}-S_\beta)\mathrm{D}x_0(\beta).$$

Note $(\mathbb{I}-S_\beta)\mathrm{D}x_0(\beta)$ is k-dimensional.

Some solutions of (2.30) are described in the next lemma.

Lemma 2.7. *Vectors* $\frac{\partial\gamma(\beta,t)}{\partial\beta_1},\dots,\frac{\partial\gamma(\beta,t)}{\partial\beta_k},\frac{\partial\gamma(\beta,t)}{\partial t}$ *satisfy equation* (2.30) *as well as conditions* (2.31) *and* (2.32).

Proof. Using notation from (2.2) we obtain

$$\gamma(\beta,t)=\begin{cases}x_+(0,x_0(\beta))(t,0,\mu) & \text{if } t\in[0,t_1^\beta],\\ x_-(t_1^\beta,x_+(0,x_0(\beta))(t_1^\beta,0,\mu))(t,0,\mu) & \text{if } t\in[t_1^\beta,t_2^\beta],\\ x_+(t_2^\beta,x_-(t_1^\beta,x_+(0,x_0(\beta))(t_1^\beta,0,\mu))(t_2^\beta,0,\mu))(t,0,\mu) & \text{if } t\in[t_2^\beta,T^\beta].\end{cases} \tag{2.47}$$

Direct differentiation of (2.47) gives equation (2.30) with $x=\frac{\partial\gamma(\beta,t)}{\partial\beta_i}$ for $i=1,\dots,k$ or $x=\frac{\partial\gamma(\beta,t)}{\partial t}$. Relations (2.31) are also easily obtained from (2.47). Next, differentiating $\gamma(\beta,t+T^\beta)=\gamma(\beta,t)$ for any $t\in\mathbb{R}$ we derive $\dot\gamma(\beta,t+T^\beta)=\dot\gamma(\beta,t)$ and

$$\frac{\partial\gamma(\beta,t+T^\beta)}{\partial\beta_i}+\frac{\partial T^\beta}{\partial\beta_i}\dot\gamma(\beta,t+T^\beta)=\frac{\partial\gamma(\beta,t)}{\partial\beta_i}$$

what implies

$$(\mathbb{I}-S_\beta)\dot\gamma(\beta,T^\beta)=(\mathbb{I}-S_\beta)\dot\gamma(\beta,0)=(\mathbb{I}-S_\beta)f_+(x_0(\beta))=0$$

and

$$(\mathbb{I}-S_\beta)\left[\frac{\partial\gamma(\beta,T^\beta)}{\partial\beta_i}-\frac{\partial\gamma(\beta,0)}{\partial\beta_i}\right]=-\frac{\partial T^\beta}{\partial\beta_i}(\mathbb{I}-S_\beta)f_+(x_0(\beta))=0.$$

Consequently, (2.32) holds as well. The proof is finished. □

We get the following equivalence to condition H3):

Proposition 2.8. *Condition* H3) *is equivalent to say that the set*

$$\left\{\frac{\partial\gamma(\beta,t)}{\partial\beta_1},\dots,\frac{\partial\gamma(\beta,t)}{\partial\beta_k},\frac{\partial\gamma(\beta,t)}{\partial t}\right\}$$

is a basis of linearly independent solutions of (2.30), (2.31) *and* (2.32).

2.2 On the hyperbolicity of persisting orbit

Here we state the sufficient condition for the hyperbolicity of trajectory $\gamma^*(\varepsilon,\mu,t)$ with $\varepsilon\neq 0$. First, we recall the result from [17] (see also [48]):

Lemma 2.9. *Let* $E(\varepsilon)=\begin{pmatrix}A_\varepsilon & 0\\ 0 & B_\varepsilon\end{pmatrix}$ *and* $D(\varepsilon)$ *be continuous matrix functions* $\mathbb{R}^k\to\mathbb{R}^k$ *for* $\varepsilon\geq 0$ *such that* $\|A_\varepsilon\|\leq 1-c\varepsilon$, $\|B_\varepsilon^{-1}\|\leq 1-c\varepsilon$, *where* c *is a positive constant,* A_ε *and* B_ε *are* $k_1\times k_1$ *and* $(k-k_1)\times(k-k_1)$ *blocks, respectively, i.e.* $E(\varepsilon)$ *is strongly* 1*-hyperbolic. Then* $E(\varepsilon)+\varepsilon^2 D(\varepsilon)$ *has no eigenvalues on* S^1 *for* $\varepsilon\neq 0$ *sufficiently small.*

We shall also need the following lemma.

Lemma 2.10. *Let U be a neighbourhood of* 0 *in* $\mathbb{R}$ *and* $M \in C^1(U, L(\mathbb{R}^n))$, *i.e.* $M(\varepsilon)$ *is a real matrix of the form* $n \times n$ *with an eigenvalue* $\lambda(\varepsilon) = \exp(\alpha(\varepsilon) + \imath\beta(\varepsilon))$ *and the corresponding eigenvector* $u(\varepsilon)$. *Suppose that* $\lambda(0) \neq 0$ *is simple. Then*

$$\alpha'(0) = \Re\frac{(M'(0)u(0), u(0))_M}{\lambda(0)\|u(0)\|_M^2}, \qquad \beta'(0) = \Im\frac{(M'(0)u(0), u(0))_M}{\lambda(0)\|u(0)\|_M^2},$$

where $(\cdot,\cdot)_M$ *is an inner product in* $\mathbb{C}^n$ *such that* $(M(0)q, u(0))_M = 0$ *whenever* $(q, u(0))_M = 0$, *and* $\|q\|_M^2 = (q,q)_M$.

Proof. If we denote

$$v(\varepsilon) = \frac{u(\varepsilon)\|u(0)\|_M}{(u(\varepsilon), u(0))_M}$$

for each ε sufficiently small then $(v(\varepsilon), v(0))_M = 1$ and therefore $(v'(\varepsilon), v(0))_M = 0$. Differentiation of the identity $M(\varepsilon)v(\varepsilon) = \lambda(\varepsilon)v(\varepsilon)$ at $\varepsilon = 0$ gives

$$M'(0)v(0) + M(0)v'(0) = \lambda(0)[\alpha'(0) + \imath\beta'(0)]v(0) + \lambda(0)v'(0).$$

Applying inner product in form $(\cdot, v(0))_M$ on the above equality gives

$$(M'(0)v(0), v(0))_M = \lambda(0)[\alpha'(0) + \imath\beta'(0)].$$

When one returns to $u(0)$ the proof is finished. $\square$

Let (β_0, μ_0) determine the persisting trajectory $\gamma^*(\varepsilon,\mu,t)$ (see Theorem 2.5). We shall study the hyperbolicity of the orbit on the fixed hyperplane Σ_{β_0} instead of changing Σ_β with $\beta = \beta(\varepsilon,\mu)$. Take $w_0(\varepsilon,\mu) = x_+(0, x^*(\varepsilon,\mu))(t_4(\varepsilon,\mu),\varepsilon,\mu)$ where $t_4(\varepsilon,\mu)$ is the nearest return time ($|t_4(\varepsilon,\mu)|$ is small) such that $w_0(\varepsilon,\mu) \in \Sigma_{\beta_0}$. Note $t_4(0,\mu) = 0$ and $w_0(0,\mu) = x_0(\beta_0)$. Moreover, $x_+(0, w_0(\varepsilon,\mu))(\cdot,\varepsilon,\mu)$ is a unique persisting solution of (2.1) with $\varepsilon \neq 0$ small and μ close to μ_0, which is in a neighbourhood of $\gamma(\beta_0,\cdot)$. More precisely,

$$x_+(0, w_0(\varepsilon,\mu))(t,\varepsilon,\mu) = \gamma^*(\varepsilon,\mu,t_4(\varepsilon,\mu)+t).$$

Theorem 2.11. *Let* (β_0,μ_0) *be as in Theorem 2.5,* $\xi \in \mathbb{R}^n$ *and* C *be a regular* $n \times n$ *matrix such that*

$$C^{-1}P_\xi(x_0(\beta_0),\beta_0,0,\mu_0)C = \begin{pmatrix} \mathbb{I}_k & 0 \\ 0 & B \end{pmatrix} =: A,$$

where $\mathbb{I}_k$ *is the* $k \times k$ *identity matrix and* B *has simple eigenvalues* $\lambda_1,\ldots,\lambda_l \in S^1\backslash\{1\}$ *with corresponding eigenvectors* $v_1,\ldots,v_l$ *and none of other eigenvalues on* S^1. *Suppose that if we denote*

$$C^{-1}\mathrm{D}_\varepsilon\, P_\xi(w_0(\varepsilon,\mu_0),\beta_0,\varepsilon,\mu_0)\big|_{\varepsilon=0}\, C =: A_1 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} *is a* $k \times k$ *block then* A_{11} *has no eigenvalues on imaginary axis. Moreover, let*

$$\Re\frac{(A_{22}v_i, v_i)_B}{\lambda_i\|v_i\|_B^2} \neq 0, \qquad \forall i = 1,\ldots,l,$$

where $(\cdot,\cdot)_B$ *is an inner product in* $\mathbb{C}^n$ *such that* $(Bq, v_i)_B = 0$ *whenever* $(q, v_i)_B = 0$ *for each* $i = 1,\ldots,l$, *and* $\|q\|_B^2 = (q,q)_B$. *Then the persisting orbit* $\gamma^*(\varepsilon,\mu,t)$ *is hyperbolic.*

Proof. Let us denote $A^\mu(\varepsilon) = C^{-1}P_\xi(w_0(\varepsilon,\mu),\beta_0,\varepsilon,\mu),\varepsilon,\mu)C$ for ε small. Now from Taylor expansion we have $A^\mu(\varepsilon) = A+\varepsilon A_1^\mu + O(\varepsilon^2)$, where $A_1^{\mu 0} = A_1$ and the same for its parts (note if $\varepsilon = 0$, the dependency on μ is lost). If we take $P^\mu(\varepsilon) = \mathbb{I}+\varepsilon B_1^\mu + \varepsilon^2 B_2^\mu(\varepsilon)$ with continuous matrix function B_2^μ,

$$B_1^\mu = \begin{pmatrix} 0 & B_{12}^\mu \\ B_{21}^\mu & 0 \end{pmatrix}, \qquad B_{12}^\mu = A_{12}^\mu(\mathbb{I}-B)^{-1}, \qquad B_{21}^\mu = (B-\mathbb{I})^{-1}A_{21}^\mu$$

then

$$\widetilde{A}^\mu(\varepsilon) := P^\mu(\varepsilon)A^\mu(\varepsilon)P^{\mu-1}(\varepsilon) = \begin{pmatrix} \mathbb{I}_k & 0 \\ 0 & B \end{pmatrix} + \varepsilon \begin{pmatrix} A_{11}^\mu & 0 \\ 0 & A_{22}^\mu \end{pmatrix} + O(\varepsilon^2).$$

Similarly, without the change of eigenvalues, we could transform A_{11}^μ into form

$$\begin{pmatrix} A_{111}^\mu & 0 \\ 0 & A_{112}^\mu \end{pmatrix},$$

where A_{111}^μ is $k_1 \times k_1$ with $0 \le k_1 \le k$, $\Re\sigma(A_{111}^\mu) \subset (0,\infty)$ and $\Re\sigma(A_{112}^\mu) \subset (-\infty,0)$, hence we shall suppose that A_{11}^μ is already in this form. Consequently,

$$\begin{aligned}\widetilde{A}^\mu(\varepsilon) &= \begin{pmatrix} \mathbb{I}_{k_1}+\varepsilon A_{111}^\mu & 0 & 0 \\ 0 & \mathbb{I}_{k-k_1}+\varepsilon A_{112}^\mu & 0 \\ 0 & 0 & B+\varepsilon A_{22}^\mu \end{pmatrix} + O(\varepsilon^2) \\ &= \begin{pmatrix} E^\mu(\varepsilon) & 0 \\ 0 & B+\varepsilon A_{22}^\mu \end{pmatrix} + O(\varepsilon^2).\end{aligned}$$

It can be shown [17,48] that $E^\mu(\varepsilon)$ is strongly 1-hyperbolic for $\varepsilon > 0$ sufficiently small. Next, Lemma 2.10 applied on matrix function $\widetilde{A}_{22}^\mu(\varepsilon) = B + \varepsilon A_{22}^\mu$ with eigenvalues $\lambda_1^\mu(\varepsilon),\dots,\lambda_l^\mu(\varepsilon)$ such that $\lambda_1^\mu(0) = \lambda_1,\dots,\lambda_l^\mu(0) = \lambda_l \in S^1$, and eigenvectors $v_1,\dots,v_l$ corresponding to $\widetilde{A}_{22}^\mu(0)$ and $\lambda_1,\dots,\lambda_l$, respectively, implies that $\widetilde{A}_{22}^\mu(\varepsilon)$ is also strongly 1-hyperbolic for $\varepsilon > 0$ sufficiently small. Consequently, $\begin{pmatrix} E^\mu(\varepsilon) & 0 \\ 0 & B+\varepsilon A_{22}^\mu \end{pmatrix}$ is strongly 1-hyperbolic and Lemma 2.9 applies to $\widetilde{A}^\mu(\varepsilon)$.

Note from Lemma 2.3 $A^\mu(\varepsilon)$ has eigenvalue 0 with eigenvector $f_+(w_0(\varepsilon,\mu)) + \varepsilon g(w_0(\varepsilon,\mu),\varepsilon,\mu)$. In conclusion by Lemma 2.9, $A^\mu(\varepsilon)$ has no eigenvalues on S^1 for $\varepsilon > 0$ sufficiently small. Analogical result can be proved for $\varepsilon < 0$. That means that for $|\varepsilon| > 0$ sufficiently small perturbed trajectory $\gamma^*(\varepsilon,\mu,t)$ is hyperbolic. □

By special assumptions, the sufficient condition for stability of persisting periodic orbit may be easily obtained from the above-stated theorem.

Corollary 2.12. *Let the assumptions of Theorem 2.11 be fulfilled. Furthermore, let B have no eigenvalues outside the unit circle, A_{11} have all eigenvalues with negative real part and*

$$\Re\frac{(A_{22}v_i,v_i)_B}{\lambda_i\|v_i\|_B^2} < 0, \qquad \forall i = 1,\dots,l.$$

Then $\gamma^*(\varepsilon,\mu,t)$ is stable (repeller) for $\varepsilon > 0$ ($\varepsilon < 0$) small.

Generally, the formula for $\mathrm{D}_\varepsilon\, P_\xi(w_0(\varepsilon,\mu),\beta_0,\varepsilon,\mu)|_{\varepsilon=0}$ is really complicated. However, in concrete examples it may be much easier found using computer software. Now we describe how to do that.

Differentiating expansion

$$P(w,\beta_0,\varepsilon,\mu_0) = P(w,\beta_0,0,\mu_0) + \varepsilon P_1(w,\beta_0,0,\mu_0) + O(\varepsilon^2)$$

for any $w \in \Sigma_{\beta_0}$ gives

$$P_\xi(w,\beta_0,\varepsilon,\mu_0) = P_\xi(w,\beta_0,0,\mu_0) + \varepsilon P_{1\xi}(w,\beta_0,0,\mu_0) + O(\varepsilon^2),$$

thence

$$\begin{aligned} &\mathrm{D}_\varepsilon\, P_\xi(w_0(\varepsilon,\mu_0),\varepsilon,\mu_0)|_{\varepsilon=0} \\ &= P_{1\xi}(x_0(\beta_0),\beta_0,0,\mu_0) + P_{\xi\xi}(x_0(\beta_0),\beta_0,0,\mu_0)\frac{\partial w_0(0,\mu_0)}{\partial\varepsilon}, \end{aligned} \tag{2.48}$$

where $P_{1\xi}$ is the partial derivative of P_1 with respect to ξ, while $P_{\xi\xi}$ is the second partial derivative of P with respect to ξ.

Now we note that $P_1(\xi,\beta,0,\mu)$ can be obtained by linearization of (2.4) as follows:

For the sense of simplicity, we shall omit some arguments. Let us denote $y_1(s)$, $y_2(s)$ and $y_3(s)$ the solutions of equations

$$\begin{aligned} \dot y_1 &= f_+(y_1) + \varepsilon g(y_1,\varepsilon,\mu) \quad \text{on } [0,s_1] \qquad & \dot y_2 &= f_-(y_2) + \varepsilon g(y_2,\varepsilon,\mu) \quad \text{on } [s_1,s_2] \\ y_1(0) &= \xi, & y_2(s_1) &= y_1(s_1), \end{aligned}$$

$$\begin{aligned} \dot y_3 &= f_+(y_3) + \varepsilon g(y_3,\varepsilon,\mu) \quad \text{on } [s_2,\infty) \\ y_3(s_2) &= y_2(s_2), \end{aligned}$$

respectively, where the first one has a general initial condition $\xi \in \Sigma_\beta$ and $s_1 < s_2$ satisfy $h(y_1(s_1)) = h(y_2(s_2)) = 0$. Moreover, let $s_3 > s_2$ be such that

$$\langle y_3(s_3) - x_0(\beta), f_+(x_0(\beta))\rangle = 0.$$

Taylor expansions with respect to ε

$$\begin{aligned} y_i(t) &= y_i^0(t) + \varepsilon y_i^1(t) + O(\varepsilon^2), \qquad i=1,2,3, \\ s_i &= s_i^0 + \varepsilon s_i^1 + O(\varepsilon^2), \qquad i=1,2,3 \end{aligned}$$

imply

$$\begin{aligned} \dot y_1^0 &= f_+(y_1^0) & \dot y_1^1 &= \mathrm{D}f_+(y_1^0)y_1^1 + g(y_1^0,0,\mu) \\ y_1^0(0) &= \xi \in \Sigma_\beta, & y_1^1(0) &= 0, \\ h(y_1^0(s_1^0)) &= 0, & s_1^1 &= -\tfrac{\mathrm{D}h(y_1^0(s_1^0))y_1^1(s_1^0)}{\mathrm{D}h(y_1^0(s_1^0))f_+(y_1^0(s_1^0))}, \\ \dot y_2^0 &= f_-(y_2^0) & \dot y_2^1 &= \mathrm{D}f_-(y_2^0)y_2^1 + g(y_2^0,0,\mu) \\ y_2^0(s_1^0) &= y_1^0(s_1^0), & y_2^1(s_1^0) &= \left[\mathbb{I} + \tfrac{(f_-(y_1^0(s_1^0)) - f_+(y_1^0(s_1^0)))\mathrm{D}h(y_1^0(s_1^0))}{\mathrm{D}h(y_1^0(s_1^0))f_+(y_1^0(s_1^0))}\right] y_1^1(s_1^0), \\ h(y_2^0(s_2^0)) &= 0, & s_2^1 &= -\tfrac{\mathrm{D}h(y_2^0(s_2^0))y_2^1(s_2^0)}{\mathrm{D}h(y_2^0(s_2^0))f_-(y_2^0(s_2^0))}, \end{aligned}$$

$$\dot{y}_3^0 = f_+(y_3^0) \qquad \dot{y}_3^1 = \mathrm{D}f_+(y_3^0)y_3^1 + g(y_3^0,0,\mu)$$
$$y_3^0(s_2^0) = y_2^0(s_2^0), \qquad y_3^1(s_2^0) = \left[\mathbb{I} + \tfrac{(f_+(y_2^0(s_2^0)) - f_-(y_2^0(s_2^0)))\mathrm{D}h(y_2^0(s_2^0))}{\mathrm{D}h(y_2^0(s_2^0))f_-(y_2^0(s_2^0))}\right] y_2^1(s_2^0),$$

$$y_3^0(s_3^0) \in \Sigma_\beta, \qquad s_3^1 = -\tfrac{\langle y_3^1(s_3^0), f_+(x_0(\beta))\rangle}{\langle f_+(y_3^0(s_3^0)), f_+(x_0(\beta))\rangle}.$$

Note that all y_i^j and s_i^j depend on ξ. Since

$$P(\xi,\beta,\varepsilon,\mu) = y_3(s_3) = y_3^0(s_3^0) + \varepsilon\left[\dot{y}_3^0(s_3^0)s_3^1 + y_3^1(s_3^0)\right] + O(\varepsilon^2),$$

we get

$$P_1(\xi,\beta,0,\mu) = f_+(y_3^0(s_3^0))s_3^1 + y_3^1(s_3^0). \tag{2.49}$$

Remark 2.13. If $\xi = x_0(\beta)$ then $s_1^0 = t_1^\beta$, $s_2^0 = t_2^\beta$, $s_3^0 = T^\beta$, $y_1^0(s_1^0) = x_1(\beta)$, $y_2^0(s_2^0) = x_2(\beta)$, $y_3^0(s_3^0) = x_0(\beta)$ and $P_1(x_0(\beta),\beta,0,\mu) = P_\varepsilon(x_0(\beta),\beta,0,\mu)$ (see (2.8)).

Hence the algorithm consists of subsequent computation of y_1^0, y_1^1, s_1^0, s_1^1, y_2^0, y_2^1, s_2^0, s_2^1, y_3^0, y_3^1, s_3^0, s_3^1, applying formula (2.49) and computing left over terms in (2.48).

2.3 The particular case of initial manifold

Here we consider special case of the manifold of initial points when $k = n-1$, i.e. $x_0(V)$ is immersed submanifold of codimension 1. Related problems have been studied in [47] for smooth dynamical systems. Then we can suppose that $x_0(\beta) = (\beta_1,\ldots,\beta_{n-1},0) = (\beta,0)$. Let us denote $\bar{\xi} = (\xi,0)$ for $\xi \in V$. We take $\Sigma = \mathbb{R}^{n-1}\times\{0\}\cap\Omega_+$ and a new Poincaré mapping $\widetilde{P} : V\times(-\varepsilon_0,\varepsilon_0)\times\mathbb{R}^p \to \Sigma$ defined as

$$\begin{gathered}\widetilde{P}(\xi,\varepsilon,\mu) = x_+(t_2(0,\bar{\xi},\varepsilon,\mu), x_-(t_1(0,\bar{\xi},\varepsilon,\mu), x_+(0,\bar{\xi})(t_1(0,\bar{\xi},\varepsilon,\mu),\varepsilon,\mu))\\ (t_2(0,\bar{\xi},\varepsilon,\mu),\varepsilon,\mu))(t_3(0,\xi,\varepsilon,\mu),\varepsilon,\mu),\end{gathered}$$

where $t_3(\cdot,\cdot,\cdot,\cdot)$ is a solution close to T^ξ of equation

$$\begin{gathered}\langle x_+(t_2(0,\bar{\xi},\varepsilon,\mu), x_-(t_1(0,\bar{\xi},\varepsilon,\mu), x_+(0,\bar{\xi})\\ (t_1(0,\bar{\xi},\varepsilon,\mu),\varepsilon,\mu))(t_2(0,\bar{\xi},\varepsilon,\mu),\varepsilon,\mu))(t,\varepsilon,\mu), e_n\rangle = 0\end{gathered}$$

with $e_n = (0,\ldots,0,1)\in\mathbb{R}^n$. Moreover $t_3(0,\xi,0,\mu) = T^\xi$. Then one can easily derive for the partial derivatives $\widetilde{P}_\xi$, $\widetilde{P}_\varepsilon$ of $\widetilde{P}$ with respect to ξ and ε, respectively, the following formulae

$$\widetilde{P}_\xi(\xi,0,\mu) = (\mathbb{I}-T_\xi)A(\xi,0),$$
$$\widetilde{P}_\varepsilon(\xi,0,\mu) = (\mathbb{I}-T_\xi)\left(\int_0^{T^\xi} A(\xi,s)g(\gamma(\xi,s),0,\mu)ds\right)$$

with $T_\xi u = \frac{\langle u,e_n\rangle f_+(\bar{\xi})}{\langle f_+(\bar{\xi}),e_n\rangle}$ and A given by (2.10). Note $\widetilde{P}(\xi,0,\mu) = \bar{\xi}\in\Sigma$ and

$$\widetilde{P}(\xi,\varepsilon,\mu) = \bar{\xi} + \varepsilon(\mathbb{I}-T_\xi)\int_0^{T^\xi} A(\xi,t)g(\gamma(\xi,t),0,\mu)dt + O(\varepsilon^2).$$

Consequently, we have the following theorem [17]:

Theorem 2.14. *Let conditions* H1), H2) *be satisfied and* $k = n-1$. *Let there be* $(\xi_0, \mu_0) \in V \times \mathbb{R}^p$ *such that* $M^{\mu_0}(\xi_0) = 0$ *and* $\det \mathrm{D}M^{\mu_0}(\xi_0) \neq 0$, *where*

$$M^{\mu}(\xi) = \left[(\mathbb{I} - T_\xi) \int_0^{T^\xi} A(\xi, t) g(\gamma(\xi, t), 0, \mu) dt \right]_{\mathbb{R}^{n-1}}$$

and the lower index $\mathbb{R}^{n-1}$ *denotes the restriction on first* $n-1$ *coordinates. Then there is a unique periodic solution* $x^*(\varepsilon, \mu, t)$ *near* $\gamma(\xi_0, t)$ *of* (2.1) *with* μ *close to* μ_0 *and* $\varepsilon \neq 0$ *small. Moreover, for* $\varepsilon > 0$ *small*

1. *if* $\Re\sigma(\mathrm{D}M^{\mu_0}(\xi_0)) \subset (-\infty, 0)$ *then* $x^*(\varepsilon, \mu, t)$ *is stable,*
2. *if* $\Re\sigma(\mathrm{D}M^{\mu_0}(\xi_0)) \cap (0, \infty) \neq \emptyset$ *then* $x^*(\varepsilon, \mu, t)$ *is unstable,*
3. *if* $0 \notin \Re\sigma(\mathrm{D}M^{\mu_0}(\xi_0))$ *then* $x^*(\varepsilon, \mu, t)$ *is hyperbolic with the same hyperbolicity type as* $\mathrm{D}M^{\mu_0}(\xi_0)$.

Proof. The existence part for $x^*(\varepsilon, \mu, t)$ follows like above. The local asymptotic properties for $x^*(\varepsilon, \mu, t)$ are derived from standard arguments of [17, 48]. ☐

2.4 3-dimensional piecewise-linear application

We shall consider the following problem

$$\begin{aligned} &\dot{x} = \varepsilon(z - x^n) \\ &\dot{y} = b_1 && \text{if } z > 0, \\ &\dot{z} = -2a_1 b_1 y + \varepsilon(\mu_1 - \mu_2 y^2) z \\ &\dot{x} = 0 \\ &\dot{y} = -b_2 && \text{if } z < 0 \\ &\dot{z} = -2a_2 b_2 y \end{aligned} \tag{2.50_ε}$$

with positive constants a_1, a_2, b_1, b_2; $n \in \mathbb{N}$ and vector $\mu = (\mu_1, \mu_2)$ of real parameters.

Here we have $\Omega_\pm = \{(x, y, z) \in \mathbb{R}^3 \mid \pm z > 0\}$, $\Omega_0 = \{(x, y, 0) \in \mathbb{R}^3\}$ and $h(x, y, z) = z$. Let $x_0(\beta) = (\beta_1, 0, \beta_2)$, $\beta = (\beta_1, \beta_2)$, $\beta_2 > 0$ be an initial point. Then we have $\Sigma = \{(x, 0, z) \in \mathbb{R}^3 \mid z > 0\}$. Due to linearity of problem $(2.50)_0$ some results may be easily obtained. These are concluded in the following lemma.

Lemma 2.15. *Unperturbed system* $(2.50)_0$ *possesses a* 2*-parametrized system* $\{\gamma(\beta, t) \mid \beta_2 > 0\}$ *of periodic orbits starting at* $(\beta_1, 0, \beta_2)$ *and given by (cf.* (2.2)*):*

$$\begin{aligned} \gamma_1(\beta, t) &= (\beta_1, b_1 t, -a_1 b_1^2 t^2 + \beta_2), \\ \gamma_2(\beta, t) &= (\beta_1, x_{12}(\beta) - b_2(t - t_1^\beta), a_2(x_{12}(\beta) - b_2(t - t_1^\beta))^2 - a_2 x_{12}^2(\beta)), \\ \gamma_3(\beta, t) &= (\beta_1, x_{22}(\beta) + b_1(t - t_2^\beta), -a_1(x_{22}(\beta) + b_1(t - t_2^\beta))^2 + a_1 x_{22}^2(\beta)), \end{aligned}$$

where

$$t_1^\beta = \frac{1}{b_1}\sqrt{\frac{\beta_2}{a_1}}, \qquad t_2^\beta = \frac{2}{b_2}\sqrt{\frac{\beta_2}{a_1}} + t_1^\beta, \qquad T^\beta = \frac{1}{b_1}\sqrt{\frac{\beta_2}{a_1}} + t_2^\beta,$$
$$x_i(\beta) = (x_{i1}(\beta), x_{i2}(\beta), x_{i3}(\beta)), \quad i = 1, 2,$$
$$x_1(\beta) = \left(\beta_1, \sqrt{\frac{\beta_2}{a_1}}, 0\right), \quad x_2(\beta) = \left(\beta_1, -\sqrt{\frac{\beta_2}{a_1}}, 0\right).$$

The corresponding fundamental matrices of (2.13), (2.14) *and* (2.15) *have now the forms*

$$X_1(\beta,t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2a_1b_1t & 1 \end{pmatrix}, \qquad X_2(\beta,t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2a_2b_2(t-t_1^\beta) & 1 \end{pmatrix},$$
$$X_3(\beta,t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2a_1b_1(t-t_2^\beta) & 1 \end{pmatrix}$$

and saltation matrices of (2.11) *and* (2.12) *are given by*

$$S_1(\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{\sqrt{a_1}(b_1+b_2)}{2a_1b_1\sqrt{\beta_2}} \\ 0 & 0 & \frac{a_2b_2}{a_1b_1} \end{pmatrix}, \qquad S_2(\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{\sqrt{a_1}(b_1+b_2)}{2a_2b_2\sqrt{\beta_2}} \\ 0 & 0 & \frac{a_1b_1}{a_2b_2} \end{pmatrix}.$$

Proof. The lemma can be proved exactly as Lemma 1.18. □

Another lemma shows that the derived theory can be applied to system $(2.50)_0$:

Lemma 2.16. *System* $(2.50)_0$ *satisfies conditions* H1) *and* H2).

Proof. Since $\frac{\partial x_0(\beta)}{\partial \beta_1} = (1,0,0)^*$, $\frac{\partial x_0(\beta)}{\partial \beta_2} = (0,0,1)^*$, $f_+(x_0(\beta)) = (0, b_1, 0)^*$ and using Lemma 2.15, H1) follows immediately. Since $\mathrm{D}h(x,y,z) = (0,0,1)$ and

$$f_+(x_1(\beta)) = \left(0, b_1, -2a_1b_1\sqrt{\frac{\beta_2}{a_1}}\right)^*, \qquad f_-(x_1(\beta)) = \left(0, -b_2, -2a_2b_2\sqrt{\frac{\beta_2}{a_1}}\right)^*,$$
$$f_+(x_2(\beta)) = \left(0, b_1, 2a_1b_1\sqrt{\frac{\beta_2}{a_1}}\right)^*, \qquad f_-(x_2(\beta)) = \left(0, -b_2, 2a_2b_2\sqrt{\frac{\beta_2}{a_1}}\right)^*,$$

H2) is also fulfilled. □

Using Lemma 2.15 we get

$$A(\beta,t)=\begin{cases}\begin{pmatrix}1 & 0 & 0\\ 0 & 1-\frac{2\sqrt{a_1}b_1t(b_1+b_2)}{b_2\sqrt{\beta_2}} & -\frac{b_1+b_2}{b_2\sqrt{a_1\beta_2}}\\ 0 & 2a_1b_1t & 1\end{pmatrix} & \text{if } t\in[0,t_1^\beta),\\ \begin{pmatrix}1 & 0 & 0\\ 0 & -1-\frac{2b_1}{b_2}+\frac{\sqrt{a_1}(t-t_1^\beta)(b_1+b_2)}{\sqrt{\beta_2}} & \frac{\sqrt{a_1}(b_1+b_2)}{2a_2b_2\sqrt{\beta_2}}\\ 0 & 2(\sqrt{a_1\beta_2}-a_1b_2(t-t_1^\beta)) & -\frac{a_1}{a_2}\end{pmatrix} & \text{if } t\in[t_1^\beta,t_2^\beta),\\ \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 2(a_1b_1(t-t_2^\beta)-\sqrt{a_1\beta_2}) & 1\end{pmatrix} & \text{if } t\in[t_2^\beta,T^\beta]\end{cases} \tag{2.51}$$

from (2.10). Consequently, the mapping $M^\mu(\beta)$ from Section 2.3 has the form

$$M^\mu(\beta)=\left(-\frac{2}{3}\frac{\sqrt{\beta_2}(3\beta_1^n-2\beta_2)}{\sqrt{a_1}b_1},\frac{4}{15}\frac{\beta_2^{3/2}(5\mu_1a_1-\mu_2\beta_2)}{a_1^{3/2}b_1}\right). \tag{2.52}$$

From Theorem 2.14 we obtain the following result.

Proposition 2.17. *For $\mu\in\mathbb{R}^2$ such that $\mu_1\mu_2\le 0$, $(\mu_1,\mu_2)\neq 0$ no periodic orbit persists. For $\mu_1\mu_2>0$ if $\varepsilon>0$ and*

1. *n is odd, the only persisting periodic trajectory $\gamma(\beta_0,t)$ of system $(2.50)_0$ is determined by $\beta_0=(\beta_{01},\beta_{02})$ with $\beta_{01}=\left(\frac{2}{3}\beta_{02}\right)^{1/n}$, $\beta_{02}=\frac{5\mu_1a_1}{\mu_2}$. Moreover, this trajectory is stable – it is a sink – for $\mu_1>0$ and unstable/hyperbolic for $\mu_1<0$,*
2. *n is even, there are exactly two persisting orbits γ_+, γ_- given by $\beta_{01}=\pm\left(\frac{2}{3}\beta_{02}\right)^{1/n}$, $\beta_{02}=\frac{5\mu_1a_1}{\mu_2}$ with corresponding sign in β_{01}. Moreover, if*
   (a) *$\mu_1>0$, then γ_+ is stable – it is a sink – and γ_- is unstable/hyperbolic,*
   (b) *$\mu_1<0$, then γ_+ is unstable/hyperbolic and γ_- is unstable – it is a source.*

If $\varepsilon<0$, the above statements remain true with sinks instead of sources and vice versa.

Proof. From (2.52) one can see that for $(\mu_1,\mu_2)\neq(0,0)$ the positive solution β_2 of

$$\frac{4}{15}\frac{\beta_2^{3/2}(5\mu_1a_1-\mu_2\beta_2)}{a_1^{3/2}b_1}=0$$

exists if and only if $\beta_2=\frac{5\mu_1a_1}{\mu_2}$ and $\mu_1\mu_2>0$. Then with respect to n we get one or two solutions β_1 of

$$-\frac{2}{3}\frac{\sqrt{\beta_2}(3\beta_1^n-2\beta_2)}{\sqrt{a_1}b_1}=0.$$

For arbitrary $n\in\mathbb{N}$ we have $\beta_{02}=\frac{5\mu_1a_1}{\mu_2}$ and

$$\mathrm{D}M^\mu\left(\left(\frac{2}{3}\beta_{02}\right)^{1/n},\beta_{02}\right)=\begin{pmatrix}-\frac{2\sqrt{5}\left(\frac{10}{3}\right)^{\frac{n-1}{n}}a_1^{\frac{3n-2}{2n}}n\left(\frac{\mu_1}{\mu_2}\right)^{\frac{3n-2}{2n}}}{\sqrt{a_1}b_1} & \frac{4\sqrt{5}}{3b_1}\sqrt{\frac{\mu_1}{\mu_2}}\\ 0 & -\frac{4\sqrt{5}\mu_1}{3b_1}\sqrt{\frac{\mu_1}{\mu_2}}\end{pmatrix}.$$

Therefore, this orbit persists. If n is even then

$$\mathrm{D}M^{\mu}\left(-\left(\frac{2}{3}\beta_{02}\right)^{1/n},\beta_{02}\right)=\begin{pmatrix}\dfrac{2\sqrt{5}\left(\frac{10}{3}\right)^{\frac{n-1}{n}}a_1^{\frac{3n-2}{2n}}n\left(\frac{\mu_1}{\mu_2}\right)^{\frac{3n-2}{2n}}}{\sqrt{a_1}b_1} & \frac{4\sqrt{5}}{3b_1}\sqrt{\frac{\mu_1}{\mu_2}}\\ 0 & -\frac{4\sqrt{5}\mu_1}{3b_1}\sqrt{\frac{\mu_1}{\mu_2}}\end{pmatrix}.$$

Hence it also persists. The statements on the stability of persisting trajectories follow directly from $\mathrm{D}M^{\mu}(\beta)$. □

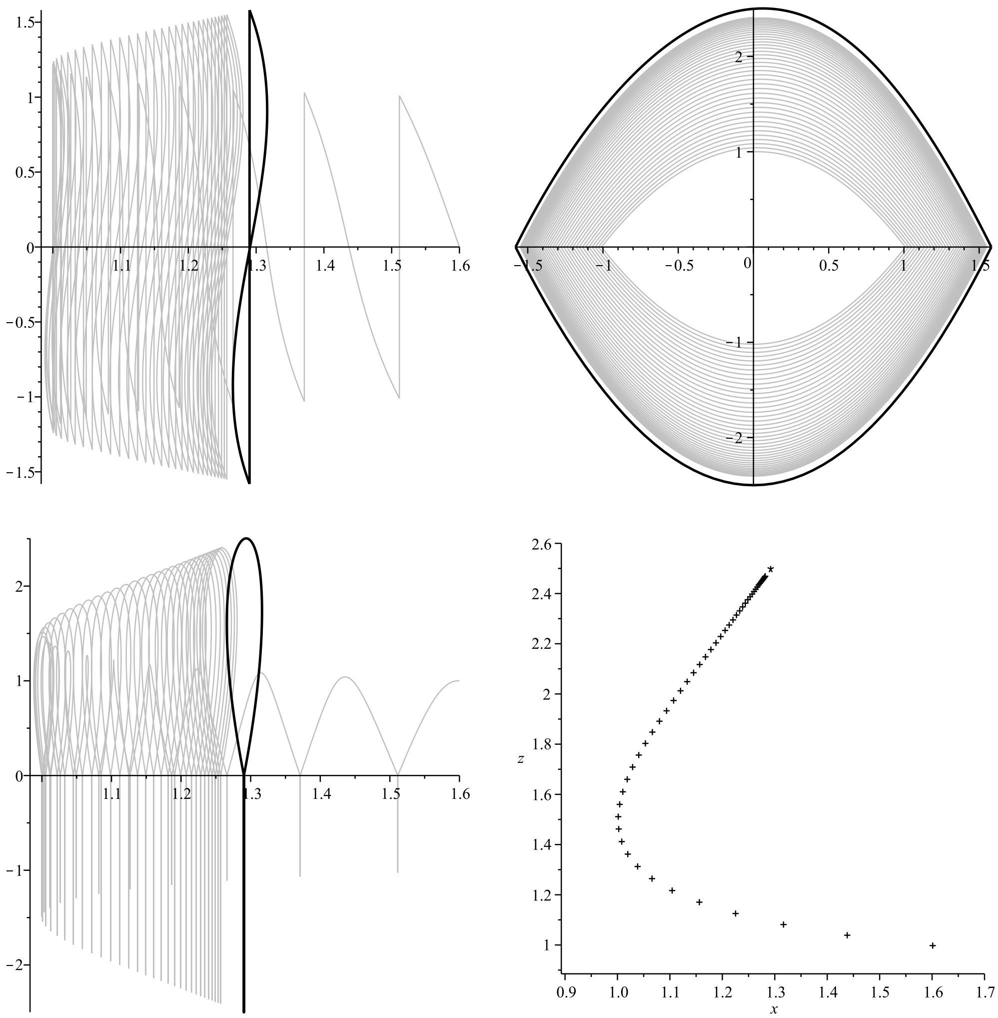

Fig. 2.2: Numerically computed trajectory projected onto xy-, yz- and xz-plane and the Poincaré mapping of the orbit of $(2.50)_{\varepsilon}$ with $a_1=a_2=b_1=b_2=1$, $\mu_1=1$, $\mu_2=2$, $n=2$, $\varepsilon=0.05$, $\beta=(1.6,1)$. Asterisk corresponds to persisting periodic orbit/stable limit cycle, denoted dark in previous figures

Remark 2.18. If $(\mu_1, \mu_2) = (0, 0)$ it is not possible to determine the persisting orbit via Theorem 2.14 since

$$M^{(0,0)}(\beta) = \left(-\frac{2}{3} \frac{\sqrt{\beta_2}(3\beta_1^n - 2\beta_2)}{\sqrt{a_1} b_1}, 0 \right).$$

However, we know that if there is a persisting trajectory, then there exists $\beta_2 > 0$ such that the trajectory contains $\left(\left(\frac{2}{3}\beta_2\right)^n, 0, \beta_2\right) \in \mathbb{R}^3$ if n is odd and $\left(\left(\frac{2}{3}\beta_2\right)^n, 0, \beta_2\right)$ or $\left(-\left(\frac{2}{3}\beta_2\right)^n, 0, \beta_2\right)$ if n is even. To find the persisting orbit, higher order Melnikov function has to be computed (cf. [8]).

2.5 Coupled Van der Pol and harmonic oscillators at 1-1 resonance

In this section we shall consider two weakly coupled oscillators at resonance, one of which is Van der Pol oscillator and the other harmonic oscillator, given by equations

$$\begin{aligned} \ddot{x} + \varepsilon(1 - x^2)\dot{x} + a_\pm^2 x + \varepsilon\mu(x - y) &= 0 \\ \ddot{y} + \varepsilon\dot{y} + \omega^2 y - \varepsilon\mu(x - y) &= 0 \end{aligned} \qquad \text{for } \pm x > 0 \tag{2.53$_\varepsilon$}$$

with positive constants a_+, a_-, ω such that

$$\frac{2}{\omega} = \frac{1}{a_+} + \frac{1}{a_-}, \tag{2.54}$$

where $\mu > 0$ is a fixed parameter and $\varepsilon \neq 0$ is small. After transforming $(2.53)_\varepsilon$ into a 4-dimensional system we get

$$\begin{aligned} \dot{x}_1 &= a_\pm x_2 \\ \dot{x}_2 &= -a_\pm x_1 - \varepsilon(1 - x_1^2)x_2 - \varepsilon\frac{\mu}{a_\pm}(x_1 - y_1) \\ \dot{y}_1 &= \omega y_2 \\ \dot{y}_2 &= -\omega y_1 - \varepsilon y_2 + \varepsilon\frac{\mu}{\omega}(x_1 - y_1) \end{aligned} \qquad \text{for } \pm x_1 > 0 \tag{2.55$_\varepsilon$}$$

and $h(x_1, x_2, y_1, y_2) = x_1$. Thus we have $\Omega_\pm = \{(x_1, x_2, y_1, y_2) \in \mathbb{R}^4 \mid \pm x_1 > 0\}$, $\Omega_0 = \{0\} \times \mathbb{R}^3$ and we take $\Sigma = \{(x_1, 0, y_1, y_2) \in \mathbb{R}^4 \mid x_1 > 0\}$ with initial point $x_0(\beta) = (\beta_1, 0, \beta_2, \beta_3)$, $\beta_1 > 0$. From linearity of unperturbed system $(2.55)_0$ the next result follows immediately and can be proved using matrix exponential (see e.g. Section 1.2).

Lemma 2.19. *System* $(2.55)_0$ *has a* 3*-parametrized family* $\{\gamma(\beta, t) \mid \beta_1 > 0\}$ *of periodic solutions such that* $\gamma(\beta, 0) = (\beta_1, 0, \beta_2, \beta_3)$ *given by (cf.* (2.2)*):*

$$\gamma_1(\beta, t) = (\beta_1 \cos a_+ t, -\beta_1 \sin a_+ t, \beta_2 \cos \omega t + \beta_3 \sin \omega t, -\beta_2 \sin \omega t + \beta_3 \cos \omega t),$$

$$\begin{aligned} \gamma_2(\beta, t) = \Big(&-\beta_1 \sin a_-(t - t_1^\beta), -\beta_1 \cos a_-(t - t_1^\beta), \beta_2 \cos \omega t + \beta_3 \sin \omega t, \\ &-\beta_2 \sin \omega t + \beta_3 \cos \omega t \Big), \end{aligned}$$

$$\begin{aligned} \gamma_3(\beta, t) = \Big(&\beta_1 \sin a_+(t - t_2^\beta), \beta_1 \cos a_+(t - t_2^\beta), \beta_2 \cos \omega t + \beta_3 \sin \omega t, \\ &-\beta_2 \sin \omega t + \beta_3 \cos \omega t \Big), \end{aligned}$$

where

$$t_1^\beta = \frac{\pi}{2a_+}, \qquad t_2^\beta = \frac{\pi}{a_-} + t_1^\beta, \qquad T^\beta = \frac{\pi}{2a_+} + t_2^\beta$$

and (cf. (2.3)*)*

$$\begin{aligned} \bar{x}_i(\beta) &= (\bar{x}_{i1}(\beta), \dots, \bar{x}_{i4}(\beta)), \quad i = 1, 2, \\ \bar{x}_1(\beta) &= \left(0, -\beta_1, \beta_2 \cos \omega t_1^\beta + \beta_3 \sin \omega t_1^\beta, -\beta_2 \sin \omega t_1^\beta + \beta_3 \cos \omega t_1^\beta\right), \\ \bar{x}_2(\beta) &= \left(0, \beta_1, \beta_2 \cos \omega t_2^\beta + \beta_3 \sin \omega t_2^\beta, -\beta_2 \sin \omega t_2^\beta + \beta_3 \cos \omega t_2^\beta\right). \end{aligned}$$

Using notation

$$X_\pm(t) = \begin{pmatrix} \cos a_\pm t & \sin a_\pm t & 0 & 0 \\ -\sin a_\pm t & \cos a_\pm t & 0 & 0 \\ 0 & 0 & \cos \omega t & \sin \omega t \\ 0 & 0 & -\sin \omega t & \cos \omega t \end{pmatrix}$$

the corresponding fundamental matrices of (2.13)*,* (2.14) *and* (2.15) *are* $X_1(t) = X_+(t)$, $X_2(t) = X_-(t - t_1^\beta)$ *and* $X_3(t) = X_+(t - t_2^\beta)$*, respectively. Saltation matrices of* (2.11) *and* (2.12) *are diagonal:*

$$S_1(\beta) = \operatorname{diag}\{a_-/a_+, 1, 1, 1\}, \qquad S_2(\beta) = \operatorname{diag}\{a_+/a_-, 1, 1, 1\}.$$

Note that the assumed relation (2.54) between a_+, a_- and ω means $\omega T^\beta = 2\pi$. That explains the name 1-1 resonance in $(2.53)_\varepsilon$.

Lemma 2.20. *System* $(2.55)_0$ *satisfies conditions* H1) *and* H2)*.*

Proof. Since $x_0(\beta) = (\beta_1, 0, \beta_2, \beta_3)$, $\mathrm{D}h(x, y, z) = (1, 0, 0, 0)$ and

$$\begin{aligned} f_+(x_0(\beta)) &= (0, -a_+\beta_1, \omega\beta_3, -\omega\beta_2)^*, \\ f_\pm(\bar{x}_1(\beta)) &= (-a_\pm\beta_1, 0, \omega\bar{x}_{14}(\beta), -\omega\bar{x}_{13}(\beta))^*, \\ f_\pm(\bar{x}_2(\beta)) &= (a_\pm\beta_1, 0, \omega\bar{x}_{24}(\beta), -\omega\bar{x}_{23}(\beta))^* \end{aligned}$$

both conditions are easy to verify. □

Of course, it is possible to continue with general values of $a_\pm$ and ω, but resulting formulae are rather awkward. Therefore we set $a_+ = 2$, $a_- = 6$, $\omega = 3$. Then following the procedure of Section 2.3 we derive the discontinuous Melnikov function $M^\mu(\beta) = (M_1^\mu(\beta), M_2^\mu(\beta), M_3^\mu(\beta))$ with

$$\begin{aligned} M_1^\mu(\beta) &= \frac{\pi}{12}(\beta_1^2 - 4)\beta_1 - \frac{43\sqrt{2}}{135}\mu\beta_3, \\ M_2^\mu(\beta) &= -\frac{\pi}{3}\beta_2 + \frac{43\sqrt{2}}{135}\mu\frac{\beta_2\beta_3}{\beta_1} - \frac{\pi}{12}\mu\beta_3, \\ M_3^\mu(\beta) &= \frac{\pi}{12}\mu\beta_2 - \frac{\pi}{3}\beta_3 - \frac{43\sqrt{2}}{135}\mu\frac{\beta_2^2}{\beta_1} + \frac{28\sqrt{2}}{135}\mu\beta_1. \end{aligned} \tag{2.56}$$

Equation $M_1^\mu(\beta)=0$ has a simple root

$$\beta_3(\mu,\beta_1)=\frac{45\sqrt{2}}{344}\frac{\pi(\beta_1^2-4)\beta_1}{\mu}. \tag{2.57}$$

Similarly

$$\beta_2(\mu,\beta_1)=\frac{45\sqrt{2}}{344}\frac{\pi(\beta_1^2-4)\beta_1}{\beta_1^2-8} \tag{2.58}$$

is a simple root of equation $M_2^\mu(\beta_1,\beta_2,\beta_3(\mu,\beta_1))=0$. Third equation $M_3^\mu(\beta)=0$ now has the form

$$\begin{aligned}\frac{\sqrt{2}\beta_1}{(\beta_1^2-8)^2\mu}&\left(-\frac{15}{344}\pi^2\beta_1^6+\left(\frac{75}{86}\pi^2+\frac{28}{135}\mu^2\right)\beta_1^4\right.\\ -\left(\frac{240}{43}\pi^2+\frac{15}{344}\pi^2\mu^2+\frac{448}{135}\mu^2\right)\beta_1^2&\left.+\frac{480}{43}\pi^2+\frac{15}{86}\pi^2\mu^2+\frac{1792}{135}\mu^2\right)=0.\end{aligned} \tag{2.59}$$

It is not possible to find solutions β_1 of the last equation analytically. Nevertheless, we can derive μ^2 as a rational function of β_1 given by

$$\begin{aligned}\mu^2(\beta_1)=F(\beta_1)&=\frac{p_1(\beta_1)}{p_2(\beta_2)},\\ p_1(\beta_1)&=2025\pi^2(\beta_1^2-4)(\beta_1^2-8)^2,\\ p_2(\beta_2)&=9632\beta_1^4-(2025\pi^2+154112)\beta_1^2+8100\pi^2+616448,\end{aligned} \tag{2.60}$$

which is plotted in Fig. 2.3.

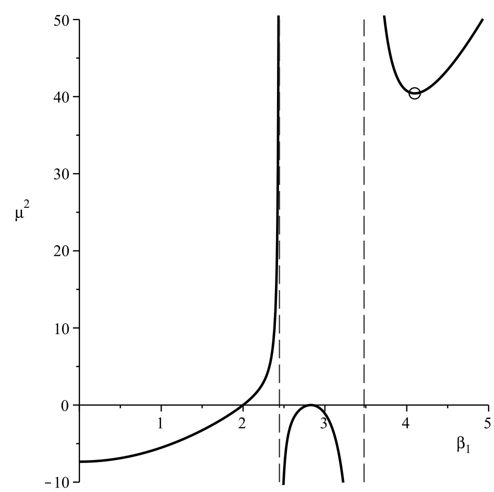

Fig. 2.3: The graph of the function $\mu^2(\beta_1)$ with respect to β_1. Asymptotes intersect β_1-axis at $\beta_{1-}\doteq 2.445$ and $\beta_{1+}\doteq 3.478$. The bifurcation point $(\bar\beta_1,F(\bar\beta_1))\doteq(4.097,40.423)$ is denoted by a circle

Note $p_2(\beta_1)=(\beta_1-\beta_{1-})(\beta_1+\beta_{1-})(\beta_1-\beta_{1+})(\beta_1+\beta_{1+})$ for

$$\beta_{1\pm}=\frac{1}{8}\sqrt{\frac{1}{301}\left(154112+2025\pi^2\pm 45\pi\sqrt{154112+2025\pi^2}\right)}$$

with $\beta_{1-} \doteq 2.445$ and $\beta_{1+} \doteq 3.478$. Clearly, only for positive values of μ^2 a periodic orbit can persist, which is determined by the equation $\mu^2 = F(\beta_1)$, $\beta_1 > 0$. Thus from now on we assume $\beta_1 > 2$. Moreover, from Fig. 2.3 one can see that for all $\mu > 0$ a periodic orbit can persist and for $\mu^2 > F(\bar{\beta}_1) \doteq 40.423$ there exist three possible persisting periodic solutions. Here $\bar{\beta}_1 \doteq 4.097$ is a unique solution of $F'(\beta_1) = 0$ greater than $2\sqrt{2} \doteq 2.828$. Note $F(2\sqrt{2}) = F'(2\sqrt{2}) = 0$. Since

$$\mu(\beta_1) = \sqrt{F(\beta_1)} > 0$$

for all

$$\begin{aligned} \beta_1 \in I &:= I_1 \cup I_2 \cup I_3, \\ I_1 &= (2, \beta_{1-}) \doteq (2, 2.445), \\ I_2 &= (\beta_{1+}, \bar{\beta}_1) \doteq (3.478, 4.097), \\ I_3 &= (\bar{\beta}_1, \infty) \doteq (4.097, \infty) \end{aligned} \tag{2.61}$$

is a simple root of equation (2.59) we know (see [20, Lemma 3.5.5]) even without calculating $\mathrm{D}M^\mu(\beta)$ that $(\beta_1, \beta_2(\mu(\beta_1), \beta_1), \beta_3(\mu(\beta_1), \beta_1))$ is a simple root of $M^{\mu(\beta_1)}(\beta)$ for all $\beta_1 \in I$. Hence using the first part of Theorem 2.14 we have just proved the following statement.

Proposition 2.21. *Let $\beta_2(\mu, \beta_1)$, $\beta_3(\mu, \beta_1)$ be defined by* (2.58), (2.57) *and I by* (2.61), *respectively. For $\mu \in (0, \sqrt{F(\bar{\beta}_1)}) \doteq (0, 6.358)$ there is a unique persisting periodic solution $x^*(\varepsilon, \mu, t)$ of system $(2.55)_\varepsilon$ for any $\varepsilon \neq 0$ small. Moreover*

$$x^*(0, \mu, 0) = (\beta^*, 0, \beta_2(\mu, \beta^*), \beta_3(\mu, \beta^*)), \quad \beta^* \in I_1$$

for $\beta^* = \beta^*(\mu)$ being a unique positive solution of equation (2.59).

For $\mu \in (\sqrt{F(\bar{\beta}_1)}, \infty) \doteq (6.358, \infty)$ system $(2.55)_\varepsilon$ possesses three persisting periodic solutions $x_i^*(\varepsilon, \mu, t)$, $i = 1, 2, 3$ for any $\varepsilon \neq 0$ small. In addition,

$$x_i^*(0, \mu, 0) = (\beta_i^*, 0, \beta_2(\mu, \beta_i^*), \beta_3(\mu, \beta_i^*)), \quad \beta_i^* \in I_i, \quad i = 1, 2, 3$$

for positive solutions $\beta_1^* = \beta_1^*(\mu)$, $\beta_2^* = \beta_2^*(\mu)$ and $\beta_3^* = \beta_3^*(\mu)$ of equation (2.59).

To study asymptotic properties of these periodic solutions, we need the following result.

Lemma 2.22. *The next statements hold for a cubic equation $\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$:*

(i) Its all roots have negative real parts if and only if $a_2 > 0$ and $a_1a_2 > a_0 > 0$.

(ii) Its all roots have positive real parts if and only if $a_2 < 0$ and $a_1a_2 < a_0 < 0$.

(iii) It has a zero root if and only if $a_0 = 0$, and this root is simple if $a_1 \neq 0$.

(iv) It has a nonzero pure imaginary root if and only if $a_1 > 0$ and $a_1a_2 = a_0$, and then it is given by $\pm\sqrt{a_1}\imath$.

Proof. (i) follows from the Routh-Hurwitz criterion [30]. (ii) follows from (i) by exchanging $\lambda \leftrightarrow -\lambda$. (iii) and (v) are elementary to prove. ☐

The Jacobian matrix of function $M^\mu(\beta)$ of (2.56) at $\beta_2 = \beta_2(\mu, \beta_1)$, $\beta_3 = \beta_3(\mu, \beta_1)$ is given by

$$\mathrm{D}M^\mu(\beta) = \begin{pmatrix} \frac{\pi}{4}\beta_1^2 - \frac{\pi}{3} & 0 & -\frac{43\sqrt{2}}{135}\mu \\ -\frac{15\sqrt{2}\pi^2}{1376}\frac{(\beta_1^2-4)^2}{\beta_1^2-8} & \frac{\pi}{12}(\beta_1^2 - 8) & \frac{\pi}{3}\frac{\mu}{\beta_1^2-8} \\ A & -\frac{\pi}{12}\frac{\mu\beta_1^2}{\beta_1^2-8} & -\frac{\pi}{3} \end{pmatrix},$$

$$A = \frac{\sqrt{2}}{185760}\frac{\mu((2025\pi^2 + 38528)\beta_1^4 - (16200\pi^2 + 616448)\beta_1^2 + 32400\pi^2 + 2465792)}{(\beta_1^2 - 8)^2}. \tag{2.62}$$

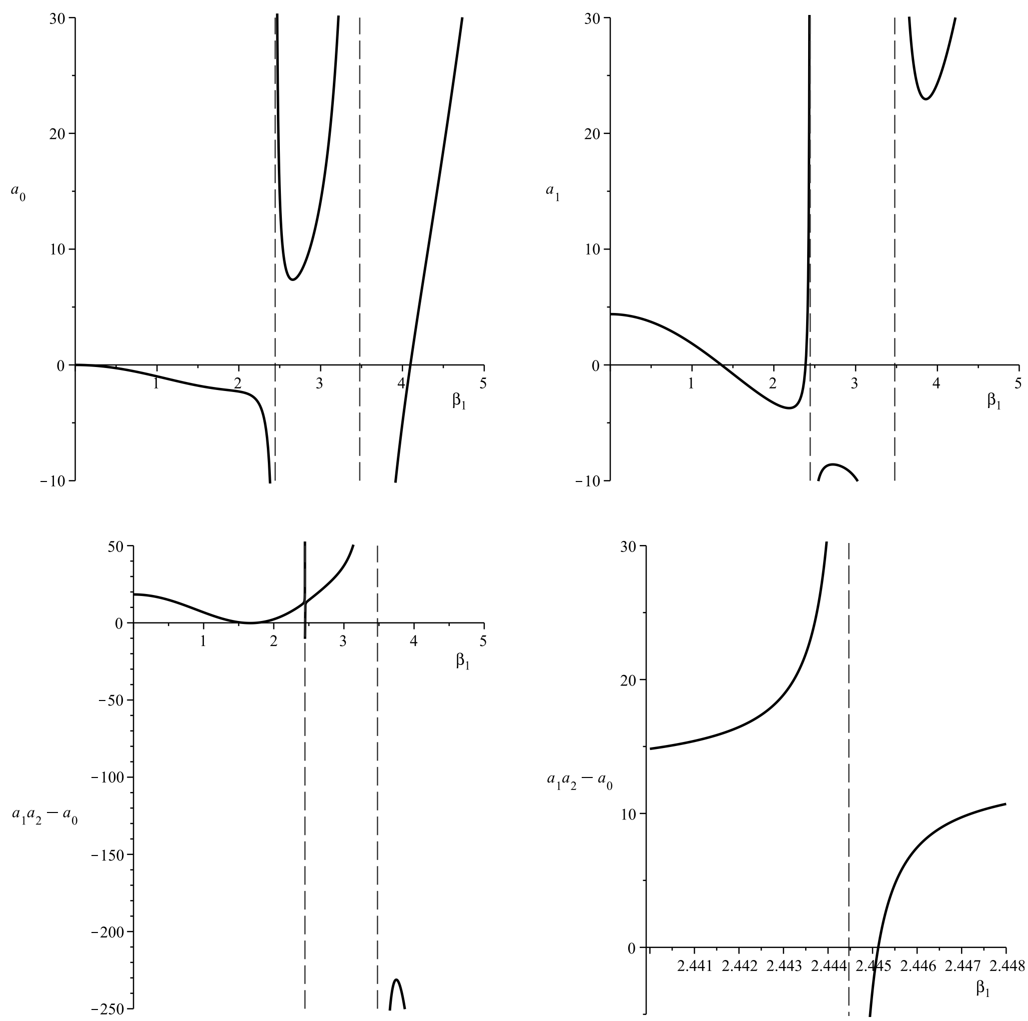

Fig. 2.4: Illustration of coefficients $a_0(\beta_1)$, $a_1(\beta_1)$ with respect to β_1 and the combination $a_1(\beta_1)a_2(\beta_1) - a_0(\beta_1)$ with detail in the neighbourhood of the asymptote

Consequently, the characteristic polynomial of matrix (2.62) has the form

$$
\begin{aligned}
P(\lambda) &= \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0,\\
a_0(\beta_1) &= \frac{\pi^3}{108B}\left(4816\beta_1^8 - (2025\pi^2 + 115584)\beta_1^6 + (16200\pi^2 + 924672)\beta_1^4\right.\\
&\qquad \left. -(32400\pi^2 + 2465792)\beta_1^2\right),\\
a_1(\beta_1) &= \frac{\pi^2}{72B}\left(14448\beta_1^8 - (2025\pi^2 + 423808)\beta_1^6 + (48600\pi^2 + 4315136)\beta^4\right.\\
&\qquad \left. -(226800\pi^2 + 17260544)\beta_1^2 + 259200\pi^2 + 19726336\right),\\
a_2(\beta_1) &= \frac{4\pi}{3} - \frac{\pi}{3}\beta_1^2,\\
B &= 9632\beta_1^4 - (2025\pi^2 + 154112)\beta_1^2 + 8100\pi^2 + 616448.
\end{aligned}
\tag{2.63}
$$

Dependence of $a_0(\beta_1)$, $a_1(\beta_1)$ and $a_1(\beta_1)a_2(\beta_1) - a_0(\beta_1)$ on β_1 is illustrated in Fig. 2.4. Now, a result on stability of persisting trajectories follows.

Proposition 2.23. *Let $x^*(\varepsilon, \mu, t)$, $x_i^*(\varepsilon, \mu, t)$, $i = 1, 2, 3$ be as in Proposition 2.21. Then for any $\varepsilon \neq 0$ small all of these solutions of system $(2.55)_\varepsilon$ are hyperbolic and for $\varepsilon > 0$ small none of them is stable and the only repeller is $x_2^*(\varepsilon, \mu, t)$.*

Proof. Using Theorem 2.14 and Lemma 2.22 applied on characteristic polynomial (2.63) we directly obtain the statement. The sign of $a_2(\beta_1)$ is obvious and those of $a_0(\beta_1)$, $a_1(\beta_1)$ and $a_1(\beta_1)a_2(\beta_1) - a_0(\beta_1)$ can be seen from Fig. 2.4 or computed from definitions in (2.63). □

Note, that from Lemma 2.22 in bifurcation point (cf. Fig. 2.4) $\bar{\beta}_1 \doteq 4.097$, $\bar{\mu} = \sqrt{F(\bar{\beta}_1)} \doteq 6.358$ the matrix $\mathrm{D}M^\mu(\beta)$ of (2.62) has an eigenvalue 0 of multiplicity 1. In this case we can not apply Theorem 2.14 and higher order Melnikov function (cf. [8]) has to be used to determine if the solution $\gamma(\bar{\beta}_1, \beta_2(\bar{\mu}, \bar{\beta}_1), \beta_3(\bar{\mu}, \bar{\beta}_1), t)$ even persists.

3 Bifurcation from single periodic orbit in autonomous systems

In this section, we consider degenerated case of the manifold of initial points from previous section, i.e. we shall assume, that the unperturbed equation possesses one isolated transverse periodic solution of period T and we look for sufficient condition on the perturbation such that the perturbed system has a periodic solution close to the original one with period close to T.

Let $\Omega \subset \mathbb{R}^n$ be an open set in $\mathbb{R}^n$ and $h(x)$ be a C^r-function on $\overline{\Omega}$, with $r \geq 3$. We set $\Omega_\pm := \{x \in \Omega \mid \pm h(x) > 0\}$, $\Omega_0 := \{x \in \Omega \mid h(x) = 0\}$. Let $f_\pm \in C_b^r(\overline{\Omega})$, $g \in C_b^r(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^p)$ and $h \in C_b^r(\overline{\Omega}, \mathbb{R})$. Let $\varepsilon \in \mathbb{R}$ and $\mu \in \mathbb{R}^p, p \geq 1$ be parameters. Furthermore, we suppose that 0 is a regular value of h.

We say that a function $x(t)$ is a solution of the equation

$$\dot{x} = f_\pm(x) + \varepsilon g(x, \varepsilon, \mu), \quad x \in \overline{\Omega}_\pm, \tag{3.1}$$

if it is a solution of this equation in the sense analogical to Definition 1.1.

Let us assume

H1) For $\varepsilon = 0$ equation (3.1) has a unique periodic orbit $\gamma(t)$ of period T. The orbit is given by its initial point $x_0 \in \Omega_+$ and consists of three branches

$$\gamma(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [0, t_1], \\ \gamma_2(t) & \text{if } t \in [t_1, t_2], \\ \gamma_3(t) & \text{if } t \in [t_2, T], \end{cases} \tag{3.2}$$

where $0 < t_1 < t_2 < T$, $\gamma_1(t) \in \Omega_+$ for $t \in [0, t_1)$, $\gamma_2(t) \in \Omega_-$ for $t \in (t_1, t_2)$ and $\gamma_3(t) \in \Omega_+$ for $t \in (t_2, T]$ and

$$\begin{aligned} x_1 &:= \gamma_1(t_1) = \gamma_2(t_1) \in \Omega_0, \\ x_2 &:= \gamma_2(t_2) = \gamma_3(t_2) \in \Omega_0, \\ x_0 &:= \gamma_3(T) = \gamma_1(0) \in \Omega_+. \end{aligned} \tag{3.3}$$

H2) Moreover, we also assume that

$$\mathrm{D}h(x_1) f_\pm(x_1) < 0 \qquad \text{and} \qquad \mathrm{D}h(x_2) f_\pm(x_2) > 0.$$

Let $x_+(\tau, \xi)(t, \varepsilon, \mu)$ and $x_-(\tau, \xi)(t, \varepsilon, \mu)$ denote the solution of initial value problem

$$\begin{aligned} \dot{x} &= f_\pm(x) + \varepsilon g(x, \varepsilon, \mu) \\ x(\tau) &= \xi \end{aligned} \tag{3.4}$$

with corresponding sign. Note

$$x_\pm(\tau, \xi)(t, \varepsilon, \mu) = x_\pm(0, \xi)(t - \tau, \varepsilon, \mu). \tag{3.5}$$

First, we slightly modify Lemma 1.2 for autonomous case.

Lemma 3.1. *Assume* H1) *and* H2). *Then there exist* $\varepsilon_0, r_0 > 0$ *and a Poincaré mapping (cf. Fig. 3.1)*

$$P(\cdot, \varepsilon, \mu) : B(x_0, r_0) \to \Sigma$$

for all fixed $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $\mu \in \mathbb{R}^p$, *where*

$$\Sigma = \{y \in \mathbb{R}^n \mid \langle y - x_0, f_+(x_0)\rangle = 0\}.$$

Moreover, $P : B(x_0, r_0) \times (-\varepsilon_0, \varepsilon_0) \times \mathbb{R}^p \to \mathbb{R}^n$ *is* C^r*-smooth in all arguments and* $B(x_0, r_0) \subset \Omega_+$.

Proof. The lemma can be easily proved as Lemma 1.2 using implicit function theorem (IFT). We obtain the existence of C^r-functions t_1, t_2 and t_3 satisfying, respectively,

$$h(x_+(\tau, \xi)(t_1(\tau, \xi, \varepsilon, \mu), \varepsilon, \mu)) = 0,$$

$$h(x_-(t_1(\tau, \xi, \varepsilon, \mu), x_+(\tau, \xi)(t_1(\tau, \xi, \varepsilon, \mu), \varepsilon, \mu))(t_2(\tau, \xi, \varepsilon, \mu), \varepsilon, \mu)) = 0$$

and

$$\begin{gathered}\langle x_+(t_2(\tau, \xi, \varepsilon, \mu), x_-(t_1(\tau, \xi, \varepsilon, \mu), x_+(\tau, \xi)(t_1(\tau, \xi, \varepsilon, \mu), \varepsilon, \mu)) \\ (t_2(\tau, \xi, \varepsilon, \mu), \varepsilon, \mu))(t_3(\tau, \xi, , \varepsilon, \mu), \varepsilon, \mu) - x_0, f_+(x_0)\rangle = 0\end{gathered}$$

for (τ, ξ, ε) close to $(0, x_0, 0)$ and $\mu \in \mathbb{R}^p$. Moreover, we have $t_1(0, x_0, 0, \mu) = t_1$, $t_2(0, x_0, 0, \mu) = t_2$ and $t_3(0, x_0, 0, \mu) = T$. Poincaré mapping is then defined as

$$\begin{aligned}P(\xi, \varepsilon, \mu) = x_+(t_2(0, \xi, \varepsilon, \mu), x_-(t_1(0, \xi, \varepsilon, \mu), x_+(0, \xi)(t_1(0, \xi, \varepsilon, \mu), \varepsilon, \mu)) \\ (t_2(0, \xi, \varepsilon, \mu), \varepsilon, \mu))(t_3(0, \xi, \varepsilon, \mu), \varepsilon, \mu).\end{aligned} \tag{3.6}$$

□

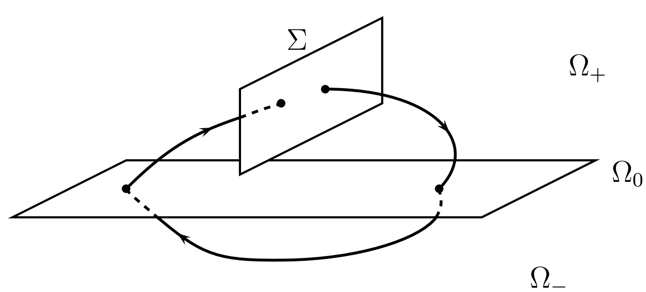

Fig. 3.1: Discontinuous Poincaré mapping

In contrast to Section 1 and Section 2, here we shall need to calculate second order derivative of Poincaré mapping with respect to ξ. In order to do this, we construct linearization of P at general point $(\xi, 0, \mu)$ with ξ sufficiently close to x_0. Note that in each of the next steps, dependence on μ is lost, since we set $\varepsilon = 0$ and μ occurred only as a parameter of g. Therefore we omit the dependence on ε and μ in the following linearizations and denote $x_\pm(\tau, \xi, t)$ the solution of unperturbed system

$$\begin{aligned}\dot{x} &= f_\pm(x) \\ x(\tau) &= \xi.\end{aligned} \tag{3.7}$$

For this time $t_i(\xi) = t_i(0, \xi, 0, \mu)$ for $i = 1, 2, 3$ and

$$
\begin{aligned}
x_1(\xi) &= x_+(0, \xi, t_1(\xi)), \\
x_2(\xi) &= x_-(t_1(\xi), x_1(\xi), t_2(\xi)), \\
x_3(\xi) &= x_+(t_2(\xi), x_2(\xi), t_3(\xi)).
\end{aligned}
\tag{3.8}
$$

Differentiating of (3.7) with respect to ξ we get

$$
\begin{aligned}
\dot{x}_{\pm\xi}(\tau, \xi, t) &= \mathrm{D}f_\pm(x_\pm(\tau, \xi, t))x_{\pm\xi}(\tau, \xi, t) \\
x_{\pm\xi}(\tau, \xi, \tau) &= \mathbb{I}
\end{aligned}
\tag{3.9}
$$

where the lower index ξ denotes the partial derivative with respect to ξ and $\mathbb{I}$ the $(n \times n)$-identity matrix. Let us denote

$$X_1^\xi(t) = x_{+\xi}(0, \xi, t), \tag{3.10}$$

$$X_2^\xi(t) = x_{-\xi}(t_1(\xi), x_1(\xi), t), \tag{3.11}$$

$$X_3^\xi(t) = x_{+\xi}(t_2(\xi), x_2(\xi), t). \tag{3.12}$$

From identities

$$h(x_1(\xi)) = 0, \qquad h(x_2(\xi)) = 0, \qquad \langle x_3(\xi) - x_0, f_+(x_0)\rangle = 0$$

using relations (3.8) we derive, respectively,

$$
t_{1\xi}(\xi) = -\frac{\mathrm{D}h(x_1(\xi))X_1^\xi(t_1(\xi))}{\mathrm{D}h(x_1(\xi))f_+(x_1(\xi))}, \qquad t_{2\xi}(\xi) = -\frac{\mathrm{D}h(x_2(\xi))X_2^\xi(t_2(\xi))S_1^\xi X_1^\xi(t_1(\xi))}{\mathrm{D}h(x_2(\xi))f_-(x_2(\xi))},
$$

$$
t_{3\xi}(\xi) = -\frac{\langle X_3^\xi(t_3(\xi))S_2^\xi X_2^\xi(t_2(\xi))S_1^\xi X_1^\xi(t_1(\xi))\cdot, f_+(x_0)\rangle}{\langle f_+(x_3(\xi)), f_+(x_0)\rangle},
$$

where

$$S_1^\xi = \mathbb{I} + \frac{(f_-(x_1(\xi)) - f_+(x_1(\xi)))\mathrm{D}h(x_1(\xi))}{\mathrm{D}h(x_1(\xi))f_+(x_1(\xi))}, \tag{3.13}$$

$$S_2^\xi = \mathbb{I} + \frac{(f_+(x_2(\xi)) - f_-(x_2(\xi)))\mathrm{D}h(x_2(\xi))}{\mathrm{D}h(x_2(\xi))f_-(x_2(\xi))} \tag{3.14}$$

are saltation matrices taken at general initial point ξ. Considering the inner product $\langle a, b\rangle = b^*a$, we can write

$$
t_{3\xi}(\xi) = -\frac{f_+(x_0)^* X_3^\xi(t_3(\xi))S_2^\xi X_2^\xi(t_2(\xi))S_1^\xi X_1^\xi(t_1(\xi))}{\langle f_+(x_3(\xi)), f_+(x_0)\rangle}.
$$

In view of these facts, we can state the lemma concluding some properties of Poincaré mapping.

Lemma 3.2. *Let $P(\xi, \varepsilon, \mu)$ be defined by* (3.6). *Then for ξ sufficiently close to x_0*

$$P_\xi(\xi, 0, \mu) = (\mathbb{I} - S^\xi)A(\xi, 0), \tag{3.15}$$

$$P_\varepsilon(x_0, 0, \mu) = (\mathbb{I} - S^{x_0})\left(\int_0^T A(x_0, s)g(\gamma(s), 0, \mu)ds\right), \tag{3.16}$$

where P_ξ, P_ε are partial derivatives of P with respect to ξ, ε, respectively. Here

$$S^\xi = \frac{f_+(x_3(\xi))f_+(x_0)^*}{\langle f_+(x_3(\xi)), f_+(x_0)\rangle} \tag{3.17}$$

is the projection onto $[f_+(x_3(\xi))]$ in the direction orthogonal to $f_+(x_0)$ (i.e. $S^\xi y = 0$ if and only if $\langle y, f_+(x_0)\rangle = 0$) and $A(\xi,t)$ is given by

$$A(\xi,t) = \begin{cases} X_3^\xi(t_3(\xi))S_2^\xi X_2^\xi(t_2(\xi))S_1^\xi X_1^\xi(t_1(\xi))X_1^\xi(t)^{-1} & \text{if } t \in [0, t_1(\xi)), \\ X_3^\xi(t_3(\xi))S_2^\xi X_2^\xi(t_2(\xi))X_2^\xi(t)^{-1} & \text{if } t \in [t_1(\xi), t_2(\xi)), \\ X_3^\xi(t_3(\xi))X_3^\xi(t)^{-1} & \text{if } t \in [t_2(\xi), t_3(\xi)] \end{cases} \tag{3.18}$$

with saltation matrices S_1^ξ of (3.13) *and S_2^ξ of* (3.14) *and X_1^ξ, X_2^ξ, X_3^ξ being defined by* (3.10), (3.11), (3.12), *respectively. In addition, $P_\xi(x_0,0,\mu)$ has an eigenvalue* 0 *with corresponding eigenvector $f_+(x_0)$, i.e.*

$$P_\xi(x_0,0,\mu)f_+(x_0) = 0.$$

Proof. Since $P(\xi,0,\mu) = x_3(\xi)$ where $x_3(\xi)$ is given by (3.8), result on $P_\xi(\xi,0,\mu)$ follows from preceding discussion. $P_\varepsilon(x_0,0,\mu)$ is obtained by differentiating (3.4) with respect to ε (cf. Lemma 1.4). Statement on the eigenvalue is proved in more general form in Lemma 2.3. $\square$

For simplicity, we shall drop the upper index ξ when $\xi = x_0$, i.e. $X_i(t) = X_i^{x_0}(t)$ for $i = 1,2,3$, $S_i = S_i^{x_0}$ for $i = 1,2$, $S = S^{x_0}$. Clearly $t_1(x_0) = t_1$, $t_2(x_0) = t_2$, $t_3(x_0) = T$, $x_1(x_0) = x_1$, $x_2(x_0) = x_2$, $x_3(x_0) = x_0$ and S is the orthogonal projection onto $[f_+(x_0)]$.

Now we want to find a persisting periodic solution of (3.1) for $\varepsilon \neq 0$, i.e. we are looking for periodic solution of perturbed equation in the neighbourhood of $\gamma(\cdot)$ such that if ε tends to 0 then the new solution tends to $\gamma(\cdot)$. This problem is equivalent to solve the next equation

$$F(x,\varepsilon,\mu) := x - P(x,\varepsilon,\mu) = 0 \tag{3.19}$$

for $(x,\varepsilon) \in \Sigma \times \mathbb{R}$ close to $(x_0, 0)$. Thus

$$F : (B(x_0,r_0) \cap \Sigma) \times (-\varepsilon_0, \varepsilon_0) \times \mathbb{R}^p \to [f_+(x_0)]^\perp \subset \mathbb{R}^n$$

and $F(x_0,0,\mu) = 0$. If we denote

$$Z = \mathcal{N}F_\xi(x_0,0,\mu), \qquad Y = \mathcal{R}F_\xi(x_0,0,\mu) \tag{3.20}$$

the null space and the range of the operator $F_\xi(x_0,0,\mu)$, then we take the orthogonal decomposition

$$[f_+(x_0)]^\perp = Z \oplus Z^\perp, \qquad [f_+(x_0)]^\perp = Y \oplus Y^\perp$$

with the orthogonal projections $\mathcal{Q} : [f_+(x_0)]^\perp \to Y$, $\mathcal{P} : [f_+(x_0)]^\perp \to Y^\perp$. Note $\Sigma = x_0 + [f_+(x_0)]^\perp$ and $\dim Z = \dim Y^\perp$. For $\xi_1 \in Z$, $\xi_2 \in Z^\perp$ we consider Taylor expansion

$$\begin{gathered} F(x_0 + \xi_1 + \xi_2, \varepsilon, \mu) = \mathrm{D}_\xi F(x_0,0,\mu)\xi_2 + \varepsilon \mathrm{D}_\varepsilon F(x_0,0,\mu) \\ + \frac{1}{2}\mathrm{D}_{\xi\xi}F(x_0,0,\mu)[\xi_1+\xi_2]^2 + \varepsilon \mathrm{D}_{\varepsilon\xi}F(x_0,0,\mu)[\xi_1+\xi_2] + \frac{1}{2}\varepsilon^2 \mathrm{D}_{\varepsilon\varepsilon}F(x_0,0,\mu) \\ + O(|\xi_1|^3 + |\xi_2|^3 + |\varepsilon|^3). \end{gathered}$$

Equation (3.19) has the form $F(x_0+\xi_1+\xi_2,\varepsilon,\mu)=0$ for $\xi_1\in Z$ and $\xi_2\in Z^\perp$ small. To solve it we apply Lyapunov-Schmidt decomposition

$$\mathcal{Q}F(x_0+\xi_1+\xi_2,\varepsilon,\mu)=0, \tag{3.21}$$
$$\mathcal{P}F(x_0+\xi_1+\xi_2,\varepsilon,\mu)=0. \tag{3.22}$$

First of these equations solved via IFT gives the existence of a unique C^r-function $\xi_2(\xi_1,\varepsilon,\mu)$ for ξ_1, ε small such that equation (3.21) is satisfied for ξ_1, ξ_2, ε small if and only if $\xi_2=\xi_2(\xi_1,\varepsilon,\mu)$. Moreover $\xi_2(0,0,\mu)=0$. Differentiating (3.21) for $\xi_2=\xi_2(\xi_1,\varepsilon,\mu)$ with respect to ξ_1 at $(\xi_1,\varepsilon)=(0,0)$ we derive $\mathrm{D}_{\xi_1}\xi_2(0,0,\mu)=0$. Therefore $\xi_2=O(\xi_1^2)+O(\varepsilon)$. So we scale

$$\xi_1\longleftrightarrow\varepsilon\xi_1,\qquad \varepsilon\longleftrightarrow\pm\varepsilon^2.$$

Then we get

$$\begin{aligned}0&=\frac{1}{\varepsilon^2}\mathcal{P}F(x_0+\varepsilon\xi_1+\xi_2(\varepsilon\xi_1,\pm\varepsilon^2,\mu),\pm\varepsilon^2,\mu)\\&=\pm\mathcal{P}\mathrm{D}_\varepsilon F(x_0,0,\mu)+\frac{1}{2}\mathcal{P}\mathrm{D}_{\xi\xi}F(x_0,0,\mu)\xi_1^2+O(\varepsilon)\end{aligned} \tag{3.23}$$

as an equivalent problem to equation (3.22). Let $\dim Z=k>0$ and $\{\psi_1,\dots,\psi_k\}$ be an orthogonal basis of $Y^\perp$. Then applying Lemma 3.2 equation (3.23) is

$$\begin{aligned}0=\sum_{i=1}^k\frac{\psi_i}{\|\psi_i\|^2}&\Bigg[\pm\int_0^T\langle g(\gamma(s)),0,\mu),A^*(x_0,s)\psi_i\rangle ds\\&+\frac{1}{2}\langle A^{-1}(x_0,0)\mathrm{D}_\xi((\mathbb{I}-S^\xi)A(\xi,0)\xi_1)_{\xi=x_0}\xi_1,A^*(x_0,0)\psi_i\rangle\Bigg]+O(\varepsilon).\end{aligned}$$

Linearization of equation (3.1) with $\varepsilon=0$ along T-periodic solution $\gamma(t)$ gives the variational equation

$$\dot{x}(t)=\mathrm{D}f_\pm(\gamma(t))x(t) \tag{3.24}$$

which splits into couple of equations

$$\begin{aligned}\dot{x}&=\mathrm{D}f_+(\gamma(t))x\qquad &&\text{if } t\in[0,t_1)\cup[t_2,T],\\ \dot{x}&=\mathrm{D}f_-(\gamma(t))x &&\text{if } t\in[t_1,t_2)\end{aligned}$$

satisfying impulsive conditions

$$x(t_1+)=S_1x(t_1-),\qquad x(t_2+)=S_2x(t_2-) \tag{3.25}$$

and periodic condition

$$(\mathbb{I}-S)(x(T)-x(0))=0 \tag{3.26}$$

as well, where $x(t\pm)=\lim_{s\to t^\pm}x(s)$. From definition of $X_i(t)$,

$$X(t)=\begin{cases}X_1(t) & \text{if } t\in[0,t_1),\\ X_2(t)S_1X_1(t_1) & \text{if } t\in[t_1,t_2),\\ X_3(t)S_2X_2(t_2)S_1X_1(t_1) & \text{if } t\in[t_2,T]\end{cases}$$

solves variational equation (3.24) and conditions (3.25). So does $X(t)c$ for any $c \in \mathbb{R}^n$. Moreover, $X(t)v$ is a solution of periodic condition (3.26) if and only if $v \in [f_+(x_0), Z]$. Indeed, from Lemma 3.2 and since $(\mathbb{I} - S)f_+(x_0) = 0$ we get

$$(\mathbb{I} - S)(\mathbb{I} - A(x_0, 0))f_+(x_0) = 0.$$

For $w \in Z \subset [f_+(x_0)]^\perp$

$$0 = \mathrm{D}_\xi F(x_0, 0, \mu)w = (\mathbb{I} - (\mathbb{I} - S)A(x_0, 0))w = (\mathbb{I} - S)(\mathbb{I} - A(x_0, 0))w$$

and for $w \in Z^\perp \subset [f_+(x_0)]^\perp$

$$0 \neq \mathrm{D}_\xi F(x_0, 0, \mu)w = (\mathbb{I} - S)(\mathbb{I} - A(x_0, 0))w.$$

From our result – Lemma 2.4 we know that the adjoint variational system of (3.1) with $\varepsilon = 0$ is given by the following linear impulsive boundary value problem

$$\begin{gathered}\dot{X} = -\mathrm{D}f_+^*(\gamma(t))X \quad \text{if } t \in [0, t_1],\\ \dot{X} = -\mathrm{D}f_-^*(\gamma(t))X \quad \text{if } t \in [t_1, t_2],\\ \dot{X} = -\mathrm{D}f_+^*(\gamma(t))X \quad \text{if } t \in [t_2, T],\\ X(t_i-) = S_i^* X(t_i+), \quad i = 1, 2,\\ X(T) = X(0) \in [f_+(x_0)]^\perp.\end{gathered} \tag{3.27}$$

Since

$$A^*(x_0, t) = X^{-1*}(t)X_1^*(t_1)S_1^*X_2^*(t_2)S_2^*X_3^*(T),$$

Lemma 1.5 implies that $A^*(x_0, t)\psi$ solves adjoint variational equation with impulsive conditions. If moreover $\psi \in Y^\perp$ then also boundary condition is satisfied, hence $A^*(x_0, t)\psi$ is a solution of adjoint variational system (3.27) whenever $\psi \in Y^\perp$. To see that $A^*(x_0, t)\psi$ satisfies boundary condition, we consider

$$\begin{aligned}0 &= \langle \mathrm{D}_\xi F(x_0, 0, \mu)\xi, \psi\rangle = \langle(\mathbb{I} - (\mathbb{I} - S)A(x_0, 0))\xi, \psi\rangle\\ &= \langle \xi, (\mathbb{I} - A^*(x_0, 0)(\mathbb{I} - S^*))\psi\rangle = \langle \xi, (\mathbb{I} - A^*(x_0, 0))\psi\rangle\end{aligned}$$

for all $\xi \in [f_+(x_0)]^\perp$ and if $\xi \in [f_+(x_0)]$, from Lemma 3.2 follows

$$\begin{aligned}0 = \langle(\mathbb{I} - S)\xi - P_\xi(x_0, 0, \mu)\xi, \psi\rangle &= \langle(\mathbb{I} - S)(\mathbb{I} - A(x_0, 0))\xi, \psi\rangle\\ &= \langle \xi, (\mathbb{I} - A^*(x_0, 0))\psi\rangle\end{aligned}$$

taking P_ξ as a partial derivative of P with respect to ξ. In conclusion, we get the following theorem.

Theorem 3.3. *Let $\{\psi_1, \ldots, \psi_k\}$ be an orthogonal basis of $Y^\perp$ with Y given by* (3.20) *and $A(\xi, t)$ be defined by* (3.18). *If ξ_1^0 is a simple root of function $M_\pm^{\mu_0}(\xi_1)$ where $M_\pm^\mu(\xi_1) = (M_{1\pm}^\mu(\xi_1), \ldots, M_{k\pm}^\mu(\xi_1))$ and*

$$M_{i\pm}^\mu(\xi_1) = \pm\int_0^T \langle g(\gamma(s), 0, \mu), A^*(x_0, s)\psi_i\rangle ds$$
$$+\frac{1}{2}\left\langle A^{-1}(x_0, 0)\mathrm{D}_\xi((\mathbb{I} - S^\xi)A(\xi, 0)\xi_1)_{\xi=x_0}\xi_1, A^*(x_0, 0)\psi_i\right\rangle$$

for $i = 1, \dots, k$ *with "+" or "−" sign, i.e.* $M_+^{\mu_0}(\xi_1^0) = 0$, $\det \mathrm{D}_{\xi_1} M_+^{\mu_0}(\xi_1^0) \neq 0$ *or* $M_-^{\mu_0}(\xi_1^0) = 0$, $\det \mathrm{D}_{\xi_1} M_-^{\mu_0}(\xi_1^0) \neq 0$, *then there exists a unique (for each sign)* C^r*-function* $\xi_1(\epsilon, \mu)$ *with* $\epsilon \sim 0$ *small and* $\mu \sim \mu_0$ *such that there is a periodic solution of equation* (3.1) *with* $\varepsilon = \pm\epsilon^2 \neq 0$ *sufficiently small and* μ *close to* μ_0*. This solution has an initial point*

$$x^* = x_0 + \epsilon\xi_1(\epsilon, \mu) + \xi_2(\epsilon\xi_1(\epsilon, \mu), \pm\epsilon^2, \mu)$$

and period $t_3(0, x^*, \pm\epsilon^2, \mu)$*. Note* $\xi_1(0, \mu_0) = \xi_1^0$*.*

3.1 The special case – linear switching manifold

If the function h has the form $h(x) = \langle a, x\rangle + c$ for given $a \in \mathbb{R}^n$, $c \in \mathbb{R}$, some of our results can be simplified. In this case we can take $\Sigma \subset \Omega_0$ and derive another Poincaré mapping $P(\cdot, \varepsilon, \mu) : B(x_0, r_0) \subset \Sigma \to \Sigma$ given by (cf. (3.6))

$$P(\xi, \varepsilon, \mu) = x_-(x_+(\xi)(t_1(\xi, \varepsilon, \mu), \varepsilon, \mu))(t_2(\xi, \varepsilon, \mu) - t_1(\xi, \varepsilon, \mu), \varepsilon, \mu) \tag{3.28}$$

where we omitted the dependence on τ, since we always assume $\tau = 0$ and equation (3.1) is autonomous. This time

$$t_2(\xi, \varepsilon, \mu) = t_3(\xi, \varepsilon, \mu), \qquad x_2(\xi) = x_-(t_1(\xi), x_1(\xi), t_2(\xi)) = x_3(\xi)$$

(see (3.8)) and we have only one saltation matrix S_1^ξ. Hence the assumption H1) has this time the form:

H1') For $\varepsilon = 0$ equation (3.1) has a unique periodic orbit $\gamma(t)$ of period T. The orbit is given by its initial point $x_0 \in \Sigma \subset \Omega_0$ and consists of two branches

$$\gamma(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [0, t_1], \\ \gamma_2(t) & \text{if } t \in [t_1, T], \end{cases}$$

where $0 < t_1 < T$, $\gamma_1(t) \in \Omega_+$ for $t \in (0, t_1)$, $\gamma_2(t) \in \Omega_-$ for $t \in (t_1, T)$ and

$$x_1 := \gamma_1(t_1) = \gamma_2(t_1) \in \Omega_0,$$
$$x_0 := \gamma_2(T) = \gamma_1(0) \in \Omega_0.$$

The new Poincaré mapping has slightly different properties.

Lemma 3.4. *Let* $h(x) = \langle a, x\rangle + c$ *for given* $a \in \mathbb{R}^n$, $c \in \mathbb{R}$ *and* $P(\xi, \varepsilon, \mu)$ *be defined by* (3.28)*. Then for* $\xi \in \Sigma$ *sufficiently close to* x_0

$$P_\xi(\xi, 0, \mu) = (\mathbb{I} - S^\xi)A(\xi, 0),$$
$$P_\varepsilon(x_0, 0, \mu) = (\mathbb{I} - S^{x_0})\left(\int_0^T A(x_0, s)g(\gamma(s), 0, \mu)ds\right),$$

where P_ξ, P_ε *are partial derivatives of* P *with respect to* ξ, ε*, respectively,*

$$S^\xi w = \frac{\langle a, w\rangle}{\langle a, f_-(x_2(\xi))\rangle} f_-(x_2(\xi)) \qquad w \in \mathbb{R}^n, \tag{3.29}$$

$$A(\xi, t) = \begin{cases} X_2^\xi(t_2(\xi))S_1^\xi X_1^\xi(t_1(\xi))X_1^\xi(t)^{-1} & \text{if } t \in [0, t_1(\xi)), \\ X_2^\xi(t_2(\xi))X_2^\xi(t)^{-1} & \text{if } t \in [t_1(\xi), t_2(\xi)] \end{cases} \tag{3.30}$$

with saltation matrix S_1^ξ of (3.13) *now given by*

$$S_1^\xi w = w + \frac{\langle a, w\rangle}{\langle a, f_+(x_1(\xi))\rangle}(f_-(x_1(\xi)) - f_+(x_1(\xi))) \qquad w \in \mathbb{R}^n \tag{3.31}$$

and X_1^ξ, X_2^ξ being defined by (3.10), (3.11), *respectively.*

We consider equation (3.19) with P given by (3.28). By the procedure described in the preceding section we derive Poincaré-Andronov-Melnikov function and the following theorem analogical to Theorem 3.3.

Theorem 3.5. *Let $h(x) = \langle a, x\rangle + c$ for given $a \in \mathbb{R}^n$, $c \in \mathbb{R}$, $\{\psi_1, \dots, \psi_k\}$ be an orthogonal basis of $Y^\perp$ with Y given by* (3.20) *and $A(\xi, t)$ be defined by* (3.30) *and S^ξ by* (3.29). *If ξ_1^0 is a simple root of function $M_\pm^{\mu_0}(\xi_1)$ where*

$$M_\pm^\mu(\xi_1) = (M_{1\pm}^\mu(\xi_1), \dots, M_{k\pm}^\mu(\xi_1))$$

and

$$M_{i\pm}^\mu(\xi_1) = \pm\int_0^T \langle g(\gamma(s), 0, \mu), A^*(x_0, s)\psi_i\rangle ds$$
$$+\frac{1}{2}\left\langle A^{-1}(x_0, 0)\mathrm{D}_\xi((\mathbb{I} - S^\xi)A(\xi, 0)\xi_1)_{\xi=x_0}\xi_1, A^*(x_0, 0)\psi_i\right\rangle$$

for $i = 1, \dots, k$ with "$+$" or "$-$" sign, i.e. $M_+^{\mu_0}(\xi_1^0) = 0$, $\det \mathrm{D}_{\xi_1} M_+^{\mu_0}(\xi_1^0) \neq 0$ or $M_-^{\mu_0}(\xi_1^0) = 0$, $\det \mathrm{D}_{\xi_1} M_-^{\mu_0}(\xi_1^0) \neq 0$, then there exists a unique (for each sign) C^r-function $\xi_1(\epsilon, \mu)$ with $\epsilon \sim 0$ and $\mu \sim \mu_0$ such that there is a periodic solution of equation (3.1) *with $\varepsilon = \pm\epsilon^2 \neq 0$ sufficiently small and μ close to μ_0. This solution has an initial point*

$$x^* = x_0 + \epsilon\xi_1(\epsilon, \mu) + \xi_2(\epsilon\xi_1(\epsilon, \mu), \pm\epsilon^2, \mu)$$

and period $t_2(x^*, \pm\epsilon^2, \mu)$. Note $\xi_1(0, \mu_0) = \xi_1^0$.

The method of Poincaré mapping can be used to determine the hyperbolicity and stability of the persisting orbit, similarly to Section 2.2. For the degenerate case when $\dim Z = n - 1$ for Z defined in (3.20), i.e. $Y = \{0\} = Z^\perp$ and $Y^\perp = [\psi_1, \dots, \psi_k] = Z$, we have the next result.

Theorem 3.6. *Let μ_0, ξ_1^0 be as in Theorem 3.5, $h(x) = \langle a, x\rangle + c$ for given $a \in \mathbb{R}^n$, $c \in \mathbb{R}$ and P be defined by* (3.28). *Assume that $\mathrm{D}_{\xi\xi}\ P(x_0, 0, \mu_0)\xi_1^0|_{[a]^\perp}$ has no eigenvalues on imaginary axis. Then the persisting trajectory is hyperbolic.*

Proof. From Taylor expansion with respect to ε at $\varepsilon = 0$ for the derivative of Poincaré mapping at general point x

$$\mathrm{D}_\xi P(x, \pm\varepsilon^2, \mu)v = \mathrm{D}_\xi P(x, 0, \mu)v + O(\varepsilon^2)$$

we have at $x_0 + \varepsilon\xi_1(\varepsilon, \mu)$

$$A^\mu(\varepsilon)v := \mathrm{D}_\xi P(x_0 + \varepsilon\xi_1(\varepsilon, \mu), \pm\varepsilon^2, \mu)v = \mathrm{D}_\xi P(x_0 + \varepsilon\xi_1(\varepsilon, \mu), 0, \mu)v + O(\varepsilon^2)$$
$$= \mathrm{D}_\xi P(x_0, 0, \mu)v + \varepsilon\mathrm{D}_{\xi\xi}P(x_0, 0, \mu)\xi_1^0 v + O(\varepsilon^2) = v + \varepsilon\mathrm{D}_{\xi\xi}P(x_0, 0, \mu)\xi_1^0 v + O(\varepsilon^2).$$

Hence $A^\mu(\varepsilon) = \mathbb{I} + \varepsilon A_1^\mu + O(\varepsilon^2)$ where $A_1^\mu = \mathrm{D}_{\xi\xi}\, P(x_0, 0, \mu)\xi_1^0|_{[a]^\perp}$. Let k_1 be the number of all eigenvalues of A_1^μ with negative real parts and $k_2 := n - k_1 - 1$ with positive. Then there exists a regular matrix $P^\mu(\varepsilon)$ such that

$$\widetilde{A}^\mu(\varepsilon) := P^\mu(\varepsilon)A^\mu(\varepsilon)P^{\mu-1}(\varepsilon) = \mathbb{I} + \varepsilon \begin{pmatrix} A_{11}^\mu & 0 \\ 0 & A_{22}^\mu \end{pmatrix} + O(\varepsilon^2) = E^\mu(\varepsilon) + O(\varepsilon^2)$$

where A_{11}^μ, A_{22}^μ are $(k_1 \times k_1)$-, $(k_2 \times k_2)$-blocks, respectively, and $\Re\sigma(A_{11}^\mu) \subset (-\infty, 0)$, $\Re\sigma(A_{22}^\mu) \subset (0, \infty)$. It can be shown [17, 48] that $E^\mu(\varepsilon)$ is strongly 1-hyperbolic. Consequently, the statement follows from Lemma 2.9. $\square$

We can slightly modify the assumptions of the last theorem to obtain a stability criterion.

Corollary 3.7. *Let the assumptions of Theorem 3.6 be fulfilled and moreover all eigenvalues of* $\mathrm{D}_{\xi\xi}\, P(x_0, 0, \mu^0)\xi_1^0|_{[a]^\perp}$ *have negative real parts. Then the persisting periodic orbit is stable (repeller) for* $\varepsilon > 0$ $(\varepsilon < 0)$ *sufficiently small.*

3.2 Planar application

In this section we consider the following system

$$\left.\begin{aligned} \dot{x} &= y + \delta + \varepsilon x(2 - \mu_1 x^2 - \mu_2 y^2) \\ \dot{y} &= -x + \varepsilon(x + y(x - y^2)) \end{aligned}\right\} \quad \text{if } y > 0,$$

$$\left.\begin{aligned} \dot{x} &= x + y - \delta + (x^2 + (y-\delta)^2)(-x - (y-\delta)) \\ &\quad + (x^2 + (y-\delta)^2)^2(x/4 + (y-\delta)/2) \\ \dot{y} &= -x + y - \delta + (x^2 + (y-\delta)^2)(x - (y-\delta)) \\ &\quad + (x^2 + (y-\delta)^2)^2(-x/2 + (y-\delta)/4) \end{aligned}\right\} \quad \text{if } y < 0 \qquad (3.32)_\varepsilon$$

with parameters $\mu_1, \mu_2 \in \mathbb{R}$ and constant $0 < \delta < \sqrt{2}$. We investigate the persistence of periodic orbit under perturbation using the method described in Section 3.1. In this case we have $h(x, y) = y$, $\Omega_\pm = \{(x, y) \in \mathbb{R}^2 \mid \pm y > 0\}$, $\Omega_0 = \mathbb{R} \times \{0\}$ and take $\Sigma = \{(x, 0) \in \mathbb{R}^2 \mid x < 0\}$. Phase portrait of the first part of unperturbed problem $(3.32)_0$ considered in whole plane consists of concentric circles with the common center at $(0, -\delta)$. In affine polar coordinates $x = \rho\cos\phi$, $y = \rho\sin\phi + \delta$, the second part of $(3.32)_0$ has the form

$$\dot{\rho} = \frac{1}{4}\rho(\rho^2 - 2)^2, \quad \dot{\phi} = -1 + \rho^2 - \frac{\rho^4}{2},$$

so it possesses only one periodic solution which is described in the next statement.

Lemma 3.8. *Unperturbed system* $(3.32)_0$ *possesses a unique periodic solution given by*

$$\gamma(t) = \begin{cases} (-\sqrt{2-\delta^2}\cos t + \delta\sin t, -\delta + \sqrt{2-\delta^2}\sin t + \delta\cos t) & \text{if } t \in [0, t_1], \\ (\sqrt{2-\delta^2}\cos(t - t_1) - \delta\sin(t - t_1), & \\ \quad \delta - \sqrt{2-\delta^2}\sin(t - t_1) - \delta\cos(t - t_1)) & \text{if } t \in [t_1, T] \end{cases} \qquad (3.33)$$

with

$$t_1 = \arccos(\delta^2 - 1), \qquad T = 2t_1$$

and

$$\bar{x}_0 = (-\sqrt{2-\delta^2}, 0), \qquad \bar{x}_1 = (\sqrt{2-\delta^2}, 0).$$

Now we should calculate fundamental matrices $X_1^\xi(t)$, $X_2^\xi(t)$, saltation matrix $S_1(\xi)$ and projection S^ξ for general $\xi \in \Sigma$ close to $\bar{x}_0$. However, it is sufficient to derive the formula $\mathrm{D}_\xi((\mathbb{I} - S^\xi)A(\xi, 0)\xi_1)\xi_1$ at $\xi = \bar{x}_0$ and for $\xi_1 \in \Sigma$. This formula is rather awkward and can be found at the end of this section. We underline that it is enough to derive $X_1(t)$, $X_2(t)$, S_1, and S, i.e. the matrices evaluated at $\xi = \bar{x}_0$. Nevertheless, Diliberto's Theorem 1.9 has to be used to obtain fundamental matrix $X_2(t)$. Applying this theorem we get the mentioned matrices.

Lemma 3.9. *System* $(3.32)_0$ *has fundamental matrices* $X_1(t)$*,* $X_2(t)$ *along* $\gamma(t)$ *of* (3.33)*, satisfying* (3.10), (3.11) *for* $\xi = \bar{x}_0$*, respectively, given by*

$$X_1(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \qquad X_2(t) = \widetilde{X}_2(t)\widetilde{X}_2^{-1}(t_1)$$

where

$$\widetilde{X}_2(t) = \begin{pmatrix} -\sqrt{2-\delta^2}\sin(t-t_1) - \delta\cos(t-t_1) & A\sin(t-t_1) + B\cos(t-t_1) \\ -\sqrt{2-\delta^2}\cos(t-t_1) + \delta\sin(t-t_1) & A\cos(t-t_1) - B\sin(t-t_1) \end{pmatrix},$$
$$A = -4\sqrt{2-\delta^2}(t-t_1) - \delta, \qquad B = \sqrt{2-\delta^2} - 4\delta(t-t_1).$$

Saltation matrix S_1 *of* (3.31) *at* $\xi = \bar{x}_0$ *has the form*

$$S_1 = \begin{pmatrix} 1 & \frac{2\delta}{\sqrt{2-\delta^2}} \\ 0 & 1 \end{pmatrix}.$$

Projection S *defined by* (3.29) *at* $\xi = \bar{x}_0$ *is*

$$S = \begin{pmatrix} 0 & -\frac{\delta}{\sqrt{2-\delta^2}} \\ 0 & 1 \end{pmatrix}.$$

Proof. Matrices S_1 and S are obtained directly from their definitions. Since the first part of $(3.32)_0$ is linear, $X_1(t)$ is the matrix solutions of this system. Using Theorem 1.9 we derive matrix $\widetilde{X}_2(t)$ and consequently $X_2(t)$ which has to fulfil $X_2(t_1) = \mathbb{I}$. $\square$

Using the above Lemma 3.9, we derive

$$(\mathbb{I} - S^{\bar{x}_0})A(\bar{x}_0, 0) = \begin{pmatrix} 1 & -\frac{\delta}{\sqrt{2-\delta^2}} \\ 0 & 0 \end{pmatrix}, \quad \psi_1 = (1, 0)$$

so (cf. Lemma 3.4)

$$P_\xi(\bar{x}_0, 0, \mu) = (\mathbb{I} - S^{\bar{x}_0})A(\bar{x}_0, 0)|_{T_{\bar{x}_0}\Sigma} = 1$$

for the derivative of the Poincaré mapping and $|_{T_{\bar{x}_0}\Sigma}$ denoting the restriction onto the tangent space to Σ at $\bar{x}_0$.

For arbitrary $0 < \delta < \sqrt{2}$ the Poincaré-Andronov-Melnikov function is rather awkward. Therefore, we fix parameter $\delta = 1$, so after some algebra we get

$$\begin{gathered} M_{\pm}^{\mu}(u) = \pm G(\mu) - \pi u^2, \quad \mu = (\mu_1, \mu_2), \\ G(\mu) = \frac{68 - 135\pi + 30\pi^2}{24} + \frac{8 - 19\pi + 6\pi^2}{8}\mu_1 + \frac{28 - 65\pi + 18\pi^2}{24}\mu_2. \end{gathered} \tag{3.34}$$

For the simplicity we shortened

$$M_{\pm}^{\mu} : T_{\bar{x}_0}\Sigma = \mathbb{R} \times \{0\} \to T_{\bar{x}_0}\Sigma, \qquad \xi_1 = (u, 0) \mapsto M_{\pm}^{\mu}(\xi_1)$$

to

$$M_{\pm}^{\mu} : \mathbb{R} \to \mathbb{R}, \qquad u \mapsto M_{\pm}^{\mu}(u).$$

Theorem 3.5 implies the existence of periodic solutions after perturbation.

Proposition 3.10. *Let $\mu^0 = (\mu_1^0, \mu_2^0)$ be such that $G(\mu^0) \neq 0$ for G given by* (3.34). *Then equation* $(3.32)_\varepsilon$ *with $\delta = 1$ has exactly two (zero) periodic solutions orbitally close to γ for $\varepsilon \neq 0$ sufficiently small with $G(\mu^0)\varepsilon > 0$ ($G(\mu^0)\varepsilon < 0$) and μ close to μ^0.*

Proof. Let $G(\mu^0) > 0$. Then

$$u_{\pm} = \pm\sqrt{\frac{G(\mu^0)}{\pi}}$$

are simple roots of $M_{+}^{\mu^0}(u) = 0$ for M given by (3.34). Similarly, if $G(\mu^0) < 0$ then

$$v_{\pm} = \pm\sqrt{-\frac{G(\mu^0)}{\pi}}$$

are simple roots of $M_{-}^{\mu^0}(u) = 0$. Consequently, the statement follows from Theorem 3.5. $\square$

One can compute that in this case $\mathrm{D}_{\xi\xi}P(x_0, 0, \mu)uv = -2\pi uv$. So, from Corollary 3.7 we get the result on stability of the persisting orbits (cf. Fig. 3.2). For the simplicity, we denote $\gamma_{u_\pm}^{\varepsilon}(t)$ and $\gamma_{v_\pm}^{\varepsilon}(t)$ the persisting solutions containing points $x_0 + \varepsilon(u_\pm, 0) + O(\varepsilon^2)$ and $x_0 + \varepsilon(v_\pm, 0) + O(\varepsilon^2)$, respectively.

Proposition 3.11. *Let μ^0 be such that $G(\mu^0) > 0$ ($G(\mu^0) < 0$). Then $\gamma_{u_+}^{\varepsilon}(t)$ is stable and $\gamma_{u_-}^{\varepsilon}(t)$ is repelling ($\gamma_{v_+}^{\varepsilon}(t)$ is repelling and $\gamma_{v_-}^{\varepsilon}(t)$ is stable).*

Formula for the second derivative

We calculate

$$\begin{gathered} \mathrm{D}_\xi((\mathbb{I} - S^\xi)A(\xi, 0)u)_{\xi = x_0}u \\ = \mathrm{D}_\xi\left((\mathbb{I} - S^\xi)X_3^\xi(t_3(\xi))S_2^\xi X_2^\xi(t_2(\xi))S_1^\xi X_1^\xi(t_1(\xi))u\right)_{\xi = x_0} u \end{gathered}$$

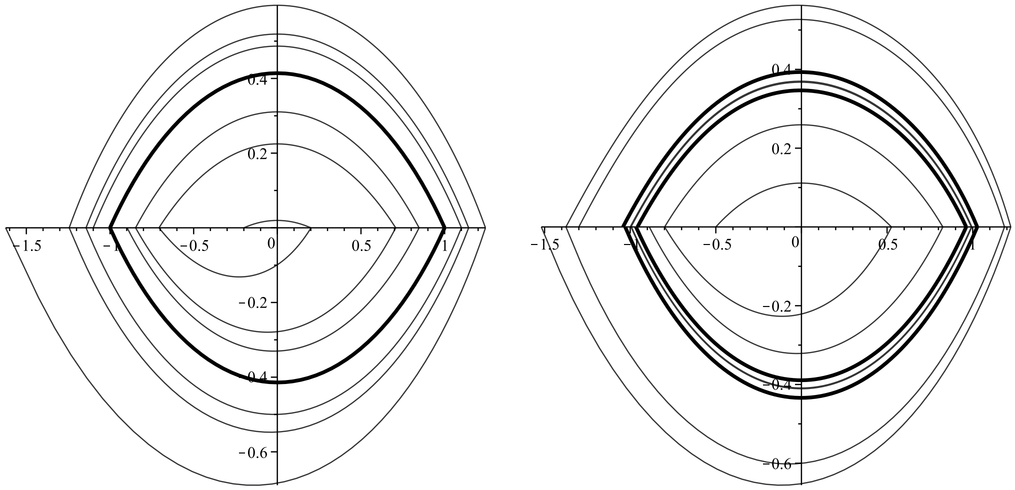

Fig. 3.2: Numerically computed trajectory of $(3.32)_\varepsilon$ with $\delta = 1$, $\mu_1 = 3$, $\mu_2 = 4$ and $\varepsilon = 0$ on the left, $\varepsilon = 0.1$ on the right. Periodic orbits are denoted with dark. In perturbed case $G(\mu^0) \doteq 0.563$, $u_+ \doteq -0.958$, $u_- \doteq -1.042$

as the derivative of a product, where

$$\begin{aligned}
\mathrm{D}_\xi \left[X_1^\xi(t_1(\xi))\right]_{x_0} u &= Y_1(t_1)u + \mathrm{D}f_+(x_1)X_1(t_1)\mathrm{D}t_1(x_0)u,\\
\mathrm{D}_\xi \left[X_2^\xi(t_2(\xi))\right]_{x_0} u &= Y_2(t_2)S_1X_1(t_1)u - X_2(t_2)\mathrm{D}f_-(x_1)\mathrm{D}t_1(x_0)u\\
&\quad + \mathrm{D}f_-(x_2)X_2(t_2)\mathrm{D}t_2(x_0)u,\\
\mathrm{D}_\xi \left[X_3^\xi(t_3(\xi))\right]_{x_0} u &= Y_3(T)S_2X_2(t_2)S_1X_1(t_1)u - X_3(T)\mathrm{D}f_+(x_2)\mathrm{D}t_2(x_0)u\\
&\quad + \mathrm{D}f_+(x_0)X_3(T)\mathrm{D}t_3(x_0)u,
\end{aligned}$$

$$\begin{aligned}
\mathrm{D}\ S_1^\xi\Big|_{x_0} u = \frac{1}{(\mathrm{D}h(x_1)f_+(x_1))^2}\big[&[(\mathrm{D}f_-(x_1) - \mathrm{D}f_+(x_1))\mathrm{D}x_1(x_0)u\mathrm{D}h(x_1)\\
&+ (f_-(x_1) - f_+(x_1))\mathrm{D}^2h(x_1)\mathrm{D}x_1(x_0)u]\mathrm{D}h(x_1)f_+(x_1)\\
&- (f_-(x_1) - f_+(x_1))\mathrm{D}h(x_1)[\mathrm{D}^2h(x_1)\mathrm{D}x_1(x_0)uf_+(x_1)\\
&+ \mathrm{D}h(x_1)\mathrm{D}f_+(x_1)\mathrm{D}x_1(x_0)u]\big],
\end{aligned}$$

$$\begin{aligned}
\mathrm{D}\ S_2^\xi\Big|_{x_0} u = \frac{1}{(\mathrm{D}h(x_2)f_-(x_2))^2}\big[&[(\mathrm{D}f_+(x_2) - \mathrm{D}f_-(x_2))\mathrm{D}x_2(x_0)u\mathrm{D}h(x_2)\\
&+ (f_+(x_2) - f_-(x_2))\mathrm{D}^2h(x_2)\mathrm{D}x_2(x_0)u]\mathrm{D}h(x_2)f_-(x_2)\\
&- (f_+(x_2) - f_-(x_2))\mathrm{D}h(x_2)[\mathrm{D}^2h(x_2)\mathrm{D}x_2(x_0)uf_-(x_2)\\
&+ \mathrm{D}h(x_2)\mathrm{D}f_-(x_2)\mathrm{D}x_2(x_0)u]\big],
\end{aligned}$$

$$\mathrm{D}\ S^\xi\big|_{x_0} u = \frac{(\|f_+(x_0)\|^2\mathbb{I} - f_+(x_0)f_+(x_0)^*)\mathrm{D}f_+(x_0)\mathrm{D}x_3(x_0)uf_+(x_0)^*}{\|f_+(x_0)\|^4},$$

$$
\begin{gathered}
\mathrm{D}x_1(x_0)u = X_1(t_1)u + f_+(x_1)\mathrm{D}t_1(x_0)u, \\
\mathrm{D}x_2(x_0)u = X_2(t_2)S_1X_1(t_1)u + f_-(x_2)\mathrm{D}t_2(x_0)u, \\
\mathrm{D}x_3(x_0)u = (\mathbb{I} - S)A(x_0, 0)u,
\end{gathered}
$$

$$
\mathrm{D}t_1(x_0)u = -\frac{\mathrm{D}h(x_1)X_1(t_1)u}{\mathrm{D}h(x_1)f_+(x_1)}, \qquad \mathrm{D}t_2(x_0)u = -\frac{\mathrm{D}h(x_2)X_2(t_2)S_1X_1(t_1)u}{\mathrm{D}h(x_2)f_-(x_2)},
$$

$$
\mathrm{D}t_3(x_0)u = -\frac{f_+(x_0)^*X_3(T)S_2X_2(t_2)S_1X_1(t_1)u}{\|f_+(x_0)\|^2},
$$

$$
\begin{aligned}
Y_1(t)uv &= X_1(t)\int_0^t X_1^{-1}(s)\mathrm{D}^2 f_+(\gamma(s))X_1(s)uX_1(s)vds, \quad t \in [0, t_1], \\
Y_2(t)uv &= X_2(t)\int_{t_1}^t X_2^{-1}(s)\mathrm{D}^2 f_-(\gamma(s))X_2(s)uX_2(s)vds, \quad t \in [t_1, t_2], \\
Y_3(t)uv &= X_3(t)\int_{t_2}^t X_3^{-1}(s)\mathrm{D}^2 f_+(\gamma(s))X_3(s)uX_3(s)vds, \quad t \in [t_2, T].
\end{aligned}
$$

4 Sliding solution of periodically perturbed systems

Until now, we always assumed, that the periodic trajectory of unperturbed system transversally crosses the discontinuity boundary and later returns back, again transversally through the boundary. This time, we shall investigate the persistence of periodic trajectories that after a transverse impact remain on the boundary for some time and then return into the original region. We consider a discontinuous differential equation with a time periodic nonautonomous perturbation as in Section 1.

Let $\Omega \subset \mathbb{R}^n$ be an open set in $\mathbb{R}^n$ and $h(x)$ be a C^r-function on $\overline{\Omega}$, with $r \geq 3$. We set $\Omega_\pm := \{x \in \Omega \mid \pm h(x) > 0\}$, $\Omega_0 := \{x \in \Omega \mid h(x) = 0\}$ for a regular value 0 of h. Let $f_\pm \in C_b^r(\overline{\Omega})$, $g_\pm \in C_b^r(\overline{\Omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^p)$ and $h \in C_b^r(\overline{\Omega}, \mathbb{R})$. Moreover, let $g_\pm$ be T-periodic in t. Let $\varepsilon \in \mathbb{R}$ and $\mu \in \mathbb{R}^p, p \geq 1$ be parameters.

Definition 4.1. We say that a function $x(t)$ is a sliding solution of the equation

$$\dot{x} = f_\pm(x) + \varepsilon g_\pm(x, t + \alpha, \varepsilon, \mu), \quad x \in \overline{\Omega}_\pm, \tag{4.1}$$

if it is continuous, piecewise C^1, satisfies equation (4.1) on $\Omega_\pm$, equation

$$\dot{x} = F_0(x, t + \alpha, \varepsilon, \mu) \tag{4.2}$$

on Ω_0, where

$$\begin{gathered} F_0(x, t, \varepsilon, \mu) = (1 - \beta(x, t, \varepsilon, \mu))F_-(x, t, \varepsilon, \mu) + \beta(x, t, \varepsilon, \mu)F_+(x, t, \varepsilon, \mu), \\ F_\pm(x, t, \varepsilon, \mu) = f_\pm(x) + \varepsilon g_\pm(x, t, \varepsilon, \mu), \\ \beta(x, t, \varepsilon, \mu) = \frac{\mathrm{D}h(x)F_-(x, t, \varepsilon, \mu)}{\mathrm{D}h(x)(F_-(x, t, \varepsilon, \mu) - F_+(x, t, \varepsilon, \mu))} \end{gathered}$$

(see [27]) and, moreover, the following holds: if $x(t_0) \in \Omega_0$ and there exists $\rho_1 > 0$ such that for any $0 < \rho < \rho_1$ we have $x(t_0 - \rho) \in \Omega_\pm$, then there exists $\rho_2 > 0$ such that $x(t_0 + \rho) \in \Omega_0$ for any $0 < \rho < \rho_2$ (the solution remains on Ω_0 for some nonzero time).

Note that the "sliding" of a sliding solution is assured since $\mathrm{D}h(x)F_0(x, t+\alpha, \varepsilon, \mu) = 0$ for $x \in \Omega_0$.

Let us assume

H1) For $\varepsilon = 0$ equation (4.1) has a T-periodic orbit $\gamma(t)$. The orbit is given by its initial point $x_0 \in \Omega_+$ and consists of three branches

$$\gamma(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [0, t_1], \\ \gamma_2(t) & \text{if } t \in [t_1, t_2], \\ \gamma_3(t) & \text{if } t \in [t_2, T], \end{cases} \tag{4.3}$$

where $0 < t_1 < t_2 < T$, $\gamma_1(t) \in \Omega_+$ for $t \in [0, t_1)$, $\gamma_2(t) \in \Omega_0$ for $t \in [t_1, t_2]$, $\gamma_3(t) \in \Omega_+$ for $t \in (t_2, T]$ and

$$\begin{aligned} x_1 &:= \gamma_1(t_1) = \gamma_2(t_1) \in \Omega_0, \\ x_2 &:= \gamma_2(t_2) = \gamma_3(t_2) \in \Omega_0, \\ x_0 &:= \gamma_3(T) = \gamma_1(0) \in \Omega_+. \end{aligned} \tag{4.4}$$

H2) Moreover, we also assume that

$$\begin{aligned} \mathrm{D}h(x)(f_-(x)-f_+(x)) > 0 \qquad & \text{if } x\in\Omega_0,\\ \mathrm{D}h(\gamma(t))f_+(\gamma(t)) < 0 \qquad & \text{if } t\in[t_1,t_2),\\ \mathrm{D}h(\gamma(t))f_-(\gamma(t)) > 0 \qquad & \text{if } t\in[t_1,t_2],\end{aligned}$$
$$\mathrm{D}h(x_2)f_+(x_2)=0, \qquad \mathrm{D}_s^2[h(\gamma(t_2+s))]_{s=0^+}\neq 0.$$

Later, it will be seen that it is sufficient to assume the first inequality in H2) to be satisfied only in the neighbourhood of $\{\gamma(t) \mid t\in[t_1,t_2]\}$. Next, from H2) it follows that $\mathrm{D}h(x)(F_-(x,t,\varepsilon,\mu)-F_+(x,t,\varepsilon,\mu))>0$ for any $\varepsilon\neq 0$ sufficiently close to 0. Hence $F_0(x,t,\varepsilon,\mu)$ is well-defined and we get

$$F_0(x,t,\varepsilon,\mu)=f_0(x)+\varepsilon g_0(x,t,\varepsilon,\mu)$$

where

$$f_0(x)=\frac{f_+(x)\mathrm{D}h(x)f_-(x)-f_-(x)\mathrm{D}h(x)f_+(x)}{\mathrm{D}h(x)(f_-(x)-f_+(x))}$$

and

$$\begin{aligned} g_0(x,t,\varepsilon,\mu) &= \frac{1}{[\mathrm{D}h(x)(f_-(x)-f_+(x))]^2}[(f_+(x)-f_-(x))\\ &\times(\mathrm{D}h(x)g_+(x,t,0,\mu)\mathrm{D}h(x)f_-(x)-\mathrm{D}h(x)g_-(x,t,0,\mu)\mathrm{D}h(x)f_+(x))\\ &+(g_+(x,t,0,\mu)\mathrm{D}h(x)f_-(x)-g_-(x,t,0,\mu)\mathrm{D}h(x)f_+(x))\mathrm{D}h(x)(f_-(x)-f_+(x))]+O(\varepsilon).\end{aligned}$$

Denote

$$K_{\varepsilon,\mu,\alpha}=\{x\in\Omega_0 \mid x\sim x_2, \mathrm{D}h(x)F_+(x,\varepsilon,\mu,\alpha)=0\}.$$

Remark 4.2.

1. We obtain that $K=K_{0,\mu,\alpha}$ is a C^{r-1}-submanifold of Ω_0 of codimension 1 in a neighbourhood of x_2. So is $K_{\varepsilon,\mu,\alpha}$ for ε sufficiently small.

2. Since $\dot\gamma(t_2^-)=f_0(x_2)=f_+(x_2)=\dot\gamma(t_2^+)$ where $\dot\gamma(t^\pm)=\lim_{s\to t^\pm}\dot\gamma(s)$, then $\gamma(t)$ is C^1-smooth at $t=t_2$.

3. From identity

$$\begin{aligned} &\mathrm{D}_s^2[h(\gamma(t_2+s))]_{s=0^+}=\mathrm{D}_s[\mathrm{D}h(\gamma(t_2+s))f_+(\gamma(t_2+s))]_{s=0^+}\\ &=\mathrm{D}^2h(x_2)f_+(x_2)f_+(x_2)+\mathrm{D}h(x_2)\mathrm{D}f_+(x_2)f_+(x_2)=\mathrm{D}_x[\mathrm{D}h(x)f_+(x)]_{x=x_2}f_0(x_2)\end{aligned}$$

   we get that $\gamma(t)$ crosses K transversally. Then clearly the solution of perturbed equation, which is close to $\gamma(t)$ crosses $K_{\varepsilon,\mu,\alpha}$ transversally. Moreover, assumptions H1), H2) imply that

$$\mathrm{D}_s^2[h(\gamma(t_2+s))]_{s=0^+}>0.$$

Let $x_i(\tau,\xi)(t,\varepsilon,\mu,\alpha)$ denote the solution of initial value problem

$$\begin{aligned}\dot x&=f_i(x)+\varepsilon g_i(x,t+\alpha,\varepsilon,\mu)\\ x(\tau)&=\xi\end{aligned} \tag{4.5}$$

with i being an element of a set of lower indices $\{+,-,0\}$. First, we modify Lemma 1.2 for the case of a sliding trajectory and show the existence of a sliding Poincaré mapping.

Lemma 4.3. *Assume* H1) *and* H2). *Then there exist* $\varepsilon_0, r_0 > 0$ *and a Poincaré mapping (cf. Fig. 4.1)*

$$P(\cdot, \varepsilon, \mu, \alpha) : B(x_0, r_0) \to \Sigma$$

for all fixed $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $\mu \in \mathbb{R}^p$, $\alpha \in \mathbb{R}$, *where*

$$\Sigma = \{y \in \mathbb{R}^n \mid \langle y - x_0, f_+(x_0)\rangle = 0\}.$$

Moreover, $P : B(x_0, r_0) \times (-\varepsilon_0, \varepsilon_0) \times \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^n$ *is* C^{r-1}*-smooth in all arguments and* $B(x_0, r_0) \subset \Omega_+$.

Proof. Implicit function theorem (IFT) yields the existence of positive constants τ_1, r_1, δ_1, ε_1 and C^r-function

$$t_1(\cdot,\cdot,\cdot,\cdot,\cdot) : (-\tau_1, \tau_1) \times B(x_0, r_1) \times (-\varepsilon_1, \varepsilon_1) \times \mathbb{R}^p \times \mathbb{R} \to (t_1 - \delta_1, t_1 + \delta_1)$$

such that $h(x_+(\tau, \xi)(t, \varepsilon, \mu, \alpha)) = 0$ for $\tau \in (-\tau_1, \tau_1)$, $\xi \in B(x_0, r_1) \subset \Omega_+$, $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$, $\mu \in \mathbb{R}^p$, $\alpha \in \mathbb{R}$ and $t \in (t_1 - \delta_1, t_1 + \delta_1)$ if and only if $t = t_1(\tau, \xi, \varepsilon, \mu, \alpha)$. Moreover, $t_1(0, x_0, 0, \mu, \alpha) = t_1$. Now, since

$$\begin{aligned}
&\mathrm{D}_s h(x_0(t_1(0, x_0, 0, \mu, \alpha), x_+(0, x_0)(t_1(0, x_0, 0, \mu, \alpha), 0, \mu, \alpha))(t_2 + s, 0, \mu, \alpha))_{s=0^+} = 0,\\
&\mathrm{D}_s^2 h(x_0(t_1(0, x_0, 0, \mu, \alpha), x_+(0, x_0)(t_1(0, x_0, 0, \mu, \alpha), 0, \mu, \alpha))(t_2 + s, 0, \mu, \alpha))_{s=0^+} > 0,
\end{aligned}$$

IFT gives the existence of positive constants τ_2, r_2, δ_2, ε_2 and C^{r-1}-function

$$t_2(\cdot,\cdot,\cdot,\cdot,\cdot) : (-\tau_2, \tau_2) \times B(x_0, r_2) \times (-\varepsilon_2, \varepsilon_2) \times \mathbb{R}^p \times \mathbb{R} \to (t_2 - \delta_2, t_2 + \delta_2)$$

such that

$$x_0(t_1(\tau, \xi, \varepsilon, \mu, \alpha), x_+(\tau, \xi)(t_1(\tau, \xi, \varepsilon, \mu, \alpha), \varepsilon, \mu, \alpha))(t, \varepsilon, \mu, \alpha) \in K_{\varepsilon,\mu,\alpha},$$

i.e.

$$\begin{aligned}
&\mathrm{D}_s h(x_0(t_1(\tau, \xi, \varepsilon, \mu, \alpha), x_+(\tau, \xi)(t_1(\tau, \xi, \varepsilon, \mu, \alpha), \varepsilon, \mu, \alpha))(t + s, \varepsilon, \mu, \alpha))_{s=0^+} = 0,\\
&\mathrm{D}_s^2 h(x_0(t_1(\tau, \xi, \varepsilon, \mu, \alpha), x_+(\tau, \xi)(t_1(\tau, \xi, \varepsilon, \mu, \alpha), \varepsilon, \mu, \alpha))(t + s, \varepsilon, \mu, \alpha))_{s=0^+} > 0
\end{aligned}$$

for $\tau \in (-\tau_2, \tau_2)$, $\xi \in B(x_0, r_2) \subset B(x_0, r_1)$, $\varepsilon \in (-\varepsilon_2, \varepsilon_2)$, $\mu \in \mathbb{R}^p$, $\alpha \in \mathbb{R}$ and $t \in (t_2 - \delta_2, t_2 + \delta_2)$ if and only if $t = t_2(\tau, \xi, \varepsilon, \mu, \alpha)$. Moreover, $t_2(0, x_0, 0, \mu, \alpha) = t_2$. Similarly to $t_1(\tau, \xi, \varepsilon, \mu, \alpha)$, we obtain positive constants τ_0, r_0, δ_0, ε_0 and C^{r-1}-function

$$t_3(\cdot,\cdot,\cdot,\cdot,\cdot) : (-\tau_0, \tau_0) \times B(x_0, r_0) \times (-\varepsilon_0, \varepsilon_0) \times \mathbb{R}^p \times \mathbb{R} \to (T - \delta_0, T + \delta_0)$$

satisfying

$$\begin{gathered}
\langle x_+(t_2(\tau, \xi, \varepsilon, \mu, \alpha), x_0(t_1(\tau, \xi, \varepsilon, \mu, \alpha), x_+(\tau, \xi)(t_1(\tau, \xi, \varepsilon, \mu, \alpha), \varepsilon, \mu, \alpha))\\
(t_2(\tau, \xi, \varepsilon, \mu, \alpha), \varepsilon, \mu, \alpha))(t_3(\tau, \xi, \varepsilon, \mu, \alpha), \varepsilon, \mu, \alpha) - x_0, f_+(x_0)\rangle = 0
\end{gathered}$$

for any $\tau \in (-\tau_0, \tau_0)$, $\xi \in B(x_0, r_0) \subset B(x_0, r_2)$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $\mu \in \mathbb{R}^p$, $\alpha \in \mathbb{R}$. In addition, $t_3(0, x_0, 0, \mu, \alpha) = T$. Consequently, the Poincaré mapping is defined as

$$\begin{aligned}
P(\xi, \varepsilon, \mu, \alpha) = x_+(t_2(0, \xi, \varepsilon, \mu, \alpha), x_0(t_1(0, \xi, \varepsilon, \mu, \alpha), x_+(0, \xi)(t_1(0, \xi, \varepsilon, \mu, \alpha), \varepsilon, \mu, \alpha))\\
(t_2(0, \xi, \varepsilon, \mu, \alpha), \varepsilon, \mu, \alpha))(t_3(0, \xi, \varepsilon, \mu, \alpha), \varepsilon, \mu, \alpha).
\end{aligned}$$

$\square$

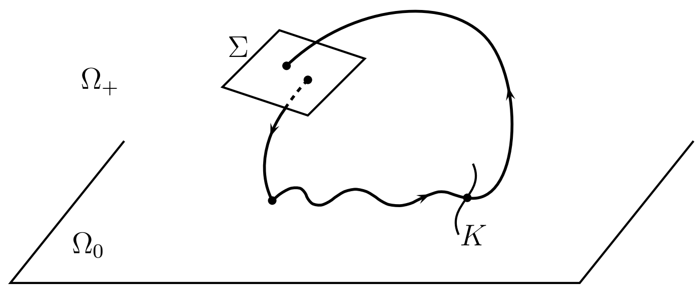

Fig. 4.1: Sliding Poincaré mapping

To find T-periodic solutions of perturbed equation (4.1) close to $\gamma(\cdot)$, we solve a couple of equations

$$P(\xi,\varepsilon,\mu,\alpha)=\xi,$$
$$t_3(0,\xi,\varepsilon,\mu,\alpha)=T.$$

We reduce this problem to a single equation

$$F(x,\varepsilon,\mu,\alpha):=x-\widetilde{P}(x,\varepsilon,\mu,\alpha)=0 \tag{4.6}$$

for $(x,\alpha)\in\Sigma\times\mathbb{R}$, $x\sim x_0$ and parameters $\varepsilon\sim 0$, $\mu\in\mathbb{R}^p$ by introducing the stroboscopic Poincaré mapping (cf. (1.6))

$$\begin{aligned}\widetilde{P}(\xi,\varepsilon,\mu,\alpha)=x_+(t_2(0,\xi,\varepsilon,\mu,\alpha),x_0(t_1(0,\xi,\varepsilon,\mu,\alpha),x_+(0,\xi)(t_1(0,\xi,\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha))\\ (t_2(0,\xi,\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha))(T,\varepsilon,\mu,\alpha).\end{aligned} \tag{4.7}$$

Properties of the mapping are concluded in the following lemma.

Lemma 4.4. *Let $\widetilde{P}(\xi,\varepsilon,\mu,\alpha)$ be defined by (4.7). Then its derivatives fulfil*

$$\mathrm{D}_\xi\widetilde{P}(x_0,0,\mu,\alpha)=A(0), \tag{4.8}$$
$$\mathrm{D}_\varepsilon\widetilde{P}(x_0,0,\mu,\alpha)=\int_0^T A(s)g(\gamma(s),s+\alpha,\mu)ds, \tag{4.9}$$

where

$$g(x,t,\mu)=\begin{cases}g_+(x,t,0,\mu) & \text{if } x\in\Omega_+,\\ g_0(x,t,0,\mu) & \text{if } x\in\Omega_0,\end{cases} \tag{4.10}$$

$A(t)$ is given by

$$A(t)=\begin{cases}X_3(T)X_2(t_2)SX_1(t_1)X_1(t)^{-1} & \text{if } t\in[0,t_1),\\ X_3(T)X_2(t_2)X_2(t)^{-1} & \text{if } t\in[t_1,t_2),\\ X_3(T)X_3(t)^{-1} & \text{if } t\in[t_2,T]\end{cases} \tag{4.11}$$

with saltation matrix

$$S=\mathbb{I}+\frac{(f_0(x_1)-f_+(x_1))\mathrm{D}h(x_1)}{\mathrm{D}h(x_1)f_+(x_1)} \tag{4.12}$$

and fundamental matrix solutions $X_1(t)$, $X_2(t)$, $X_3(t)$ satisfying, respectively,

$$\begin{array}{lll}\dot{X}_1(t) = \mathrm{D}f_+(\gamma(t))X_1(t) & \dot{X}_2(t) = \mathrm{D}f_0(\gamma(t))X_2(t) & \dot{X}_3(t) = \mathrm{D}f_+(\gamma(t))X_3(t) \\ X_1(0) = \mathbb{I}, & X_2(t_1) = \mathbb{I}, & X_3(t_2) = \mathbb{I}.\end{array} \tag{4.13}$$

In addition, $\mathrm{D}_\xi \widetilde{P}(x_0, 0, \mu, \alpha)$ has an eigenvalue 1 with corresponding eigenvector $f_+(x_0)$, i.e.

$$\mathrm{D}_\xi \widetilde{P}(x_0, 0, \mu, \alpha) f_+(x_0) = f_+(x_0).$$

Proof. Analogically to Section 1 we derive the following identities

$$\mathrm{D}_\xi x_+(0, x_0)(t, 0, \mu, \alpha) = X_1(t),$$
$$\mathrm{D}_\varepsilon x_+(0, x_0)(t, 0, \mu, \alpha) = \int_0^t X_1(t)X_1^{-1}(s)g_+(\gamma(s), s+\alpha, 0, \mu)ds$$

for $t \in [0, t_1]$,

$$\mathrm{D}_\xi x_0(t_1, x_1)(t, 0, \mu, \alpha) = X_2(t), \qquad \mathrm{D}_\tau x_0(t_1, x_1)(t, 0, \mu, \alpha) = -X_2(t)f_0(x_1),$$
$$\mathrm{D}_\varepsilon x_0(t_1, x_1)(t, 0, \mu, \alpha) = \int_{t_1}^t X_2(t)X_2^{-1}(s)g_0(\gamma(s), s+\alpha, 0, \mu)ds$$

for $t \in [t_1, t_2]$,

$$\mathrm{D}_\xi x_+(t_2, x_2)(t, 0, \mu, \alpha) = X_3(t), \qquad \mathrm{D}_\tau x_+(t_2, x_2)(t, 0, \mu, \alpha) = -X_3(t)f_+(x_2),$$
$$\mathrm{D}_\varepsilon x_+(t_2, x_2)(t, 0, \mu, \alpha) = \int_{t_2}^t X_3(t)X_3^{-1}(s)g_+(\gamma(s), s+\alpha, 0, \mu)ds$$

for $t \in [t_2, T]$ and for times

$$\mathrm{D}_\xi t_1(0, x_0, 0, \mu, \alpha) = -\frac{\mathrm{D}h(x_1)X_1(t_1)}{\mathrm{D}h(x_1)f_+(x_1)},$$
$$\mathrm{D}_\varepsilon t_1(0, x_0, 0, \mu, \alpha) = -\frac{\mathrm{D}h(x_1)\int_0^{t_1} X_1(t_1)X_1^{-1}(s)g_+(\gamma(s), s+\alpha, 0, \mu)ds}{\mathrm{D}h(x_1)f_+(x_1)},$$

$$\mathrm{D}_\xi t_2(0, x_0, 0, \mu, \alpha) = -\frac{(\mathrm{D}^2h(x_2)f_0(x_2) + \mathrm{D}h(x_2)\mathrm{D}f_0(x_2))X_2(t_2)SX_1(t_1)}{\mathrm{D}^2h(x_2)f_0(x_2)f_0(x_2) + \mathrm{D}h(x_2)\mathrm{D}f_0(x_2)f_0(x_2)},$$
$$\mathrm{D}_\varepsilon t_2(0, x_0, 0, \mu, \alpha) = -\frac{1}{\mathrm{D}^2h(x_2)f_0(x_2)f_0(x_2) + \mathrm{D}h(x_2)\mathrm{D}f_0(x_2)f_0(x_2)}$$
$$\times\Bigg[(\mathrm{D}^2h(x_2)f_0(x_2) + \mathrm{D}h(x_2)\mathrm{D}f_0(x_2))\bigg(\int_{t_1}^{t_2} X_2(t_2)X_2^{-1}(s)g_0(\gamma(s), s+\alpha, 0, \mu)ds$$
$$+\int_0^{t_1} X_2(t_2)SX_1(t_1)X_1^{-1}(s)g_+(\gamma(s), s+\alpha, 0, \mu)ds\bigg) + \mathrm{D}h(x_2)g_0(x_2, t_2+\alpha, 0, \mu)\Bigg].$$

Differentiating (4.7) with respect to ξ, respectively ε, we get the statements on the derivatives. Now let t be sufficiently small. Since

$$x_+(0, x_+(0, x_0)(t, 0, \mu, \alpha))(t_1(0, x_+(0, x_0)(t, 0, \mu, \alpha), 0, \mu, \alpha), 0, \mu, \alpha)$$
$$= x_+(0, x_0)(t_1(0, x_+(0, x_0)(t, 0, \mu, \alpha), 0, \mu, \alpha) + t, 0, \mu, \alpha)$$

is an element of Ω_0 and as well of $\{\gamma(t) \mid t \sim t_1\}$ we have

$$t_1(0, x(0, x_0)(t, 0, \mu, \alpha), 0, \mu, \alpha) + t = t_1$$

for all t close to 0. Consequently,

$$x_+(0, x_0)(t_1(0, x_+(0, x_0)(t, 0, \mu, \alpha), 0, \mu, \alpha) + t, 0, \mu, \alpha) = x_1.$$

Similarly, the left-hand side of the following identity is an element of K and the right-hand side is from $\{\gamma(t) \mid t \sim t_2\}$

$$\begin{gathered}
x_0(t_1(0, x_+(0, x_0)(t, 0, \mu, \alpha), 0, \mu, \alpha), x_+(0, x_+(0, x_0)(t, 0, \mu, \alpha)) \\
(t_1(0, x_+(0, x_0)(t, 0, \mu, \alpha), 0, \mu, \alpha), 0, \mu, \alpha))(t_2(0, x_+(0, x_0)(t, 0, \mu, \alpha), 0, \mu, \alpha), 0, \mu, \alpha) \\
= x_0(t_1(0, x_+(0, x_0)(t, 0, \mu, \alpha), 0, \mu, \alpha), x_1)(t_2(0, x_+(0, x_0)(t, 0, \mu, \alpha), 0, \mu, \alpha), 0, \mu, \alpha) \\
= x_0(t_1 - t, x_1)(t_2(0, x_+(0, x_0)(t, 0, \mu, \alpha), 0, \mu, \alpha), 0, \mu, \alpha) \\
= x_0(t_1, x_1)(t_2(0, x_+(0, x_0)(t, 0, \mu, \alpha), 0, \mu, \alpha) + t, 0, \mu, \alpha).
\end{gathered}$$

Therefore,

$$t_2(0, x_+(0, x_0)(t, 0, \mu, \alpha), 0, \mu, \alpha) + t = t_2$$

for all t close to 0 and

$$x_0(t_1, x_1)(t_2(0, x_+(0, x_0)(t, 0, \mu, \alpha), 0, \mu, \alpha) + t, 0, \mu, \alpha) = x_2.$$

Finally,

$$\begin{gathered}
x_+(t_2(0, x_+(0, x_0)(t, 0, \mu, \alpha), 0, \mu, \alpha), x_0(t_1(0, x_+(0, x_0)(t, 0, \mu, \alpha), 0, \mu, \alpha), \\
x_+(0, x_+(0, x_0)(t, 0, \mu, \alpha))(t_1(0, x_+(0, x_0)(t, 0, \mu, \alpha), 0, \mu, \alpha), 0, \mu, \alpha)) \\
(t_2(0, x_+(0, x_0)(t, 0, \mu, \alpha), 0, \mu, \alpha), 0, \mu, \alpha))(T, 0, \mu, \alpha) \\
= x_+(t_2(0, x_+(0, x_0)(t, 0, \mu, \alpha), 0, \mu, \alpha), x_2)(T, 0, \mu, \alpha) \\
= x_+(t_2 - t, x_2)(T, 0, \mu, \alpha) = x_+(t_2, x_2)(T + t, 0, \mu, \alpha).
\end{gathered}$$

Hence,

$$\begin{aligned}
\mathrm{D}_\xi \widetilde{P}(x_0, 0, \mu, \alpha) f_+(x_0) &= \mathrm{D}_t[\widetilde{P}(x_+(0, x_0)(t, 0, \mu, \alpha), 0, \mu, \alpha)]_{t=0} \\
&= \mathrm{D}_t[x_+(t_2, x_2)(T + t, 0, \mu, \alpha)]_{t=0} = f_+(x_0)
\end{aligned}$$

and the proof is finished. □

Note that in the case of sliding the second saltation matrix (see (1.14), (2.12), (3.14)) S_2 has the form

$$S_2 = \mathbb{I} + \frac{(f_+(x_2) - f_0(x_2))(\mathrm{D}^2 h(x_2) f_0(x_2) + \mathrm{D}h(x_2)\mathrm{D}f_0(x_2))}{\mathrm{D}^2 h(x_2) f_0(x_2) f_0(x_2) + \mathrm{D}h(x_2)\mathrm{D}f_0(x_2) f_0(x_2)} = \mathbb{I}$$

since $f_+(x_2) = f_0(x_2)$. This corresponds to the regularity of $\gamma(t)$ at t_2 (see Remark 4.2).

Now, we solve equation (4.6) using Lyapunov-Schmidt reduction method. We denote

$$Z = \mathcal{N}\mathrm{D}_\xi F(x_0, 0, \mu, \alpha), \qquad Y = \mathcal{R}\mathrm{D}_\xi F(x_0, 0, \mu, \alpha) \tag{4.14}$$

the null space and the range of the operator $\mathrm{D}_\xi F(x_0, 0, \mu, \alpha)$ and

$$\mathcal{Q} : \mathbb{R}^n \to Y, \qquad \mathcal{P} : \mathbb{R}^n \to Y^\perp \tag{4.15}$$

orthogonal projections onto Y and $Y^\perp$, respectively, where $Y^\perp$ is the orthogonal complement to Y in $\mathbb{R}^n$. From Lemma 4.4 we know that $f_+(x_0) \in Z$. For the simplicity, we take the third assumption, so-called non-degeneracy condition

H3) $\mathcal{N}\mathrm{D}_\xi F(x_0, 0, \mu, \alpha) = [f_+(x_0)]$.

Using the orthogonal projections, we split equation (4.6) into couple of equations

$$\begin{aligned} \mathcal{Q}F(\xi, \varepsilon, \mu, \alpha) = 0, \\ \mathcal{P}F(\xi, \varepsilon, \mu, \alpha) = 0. \end{aligned} \tag{4.16}$$

The first one of these can be solved using IFT, since

$$\mathcal{Q}F(x_0, 0, \mu, \alpha) = 0$$

and $\mathcal{Q}\mathrm{D}_\xi F(x_0, 0, \mu, \alpha)$ is an isomorphism $[f_+(x_0)]^\perp$ onto Y for all $(\mu, \alpha) \in \mathbb{R}^p \times \mathbb{R}$. Thus we get the existence of a unique C^{r-1}-function $\xi = \xi(\varepsilon, \mu, \alpha)$ for ε close to 0 and $(\mu, \alpha) \in \mathbb{R}^p \times \mathbb{R}$ satisfying $\mathcal{Q}F(\xi(\varepsilon, \mu, \alpha), \varepsilon, \mu, \alpha) = 0$ for all such $(\varepsilon, \mu, \alpha)$ and $\xi(0, \mu, \alpha) = x_0$. The second equation is so-called bifurcation equation for $\alpha \in \mathbb{R}$

$$\mathcal{P}F(\xi(\varepsilon, \mu, \alpha), \varepsilon, \mu, \alpha) = 0. \tag{4.17}$$

Let $Y^\perp = [\psi]$ for arbitrary and fixed ψ. Then we can write

$$\mathcal{P}u = \frac{\langle u, \psi \rangle \psi}{\|\psi\|^2}$$

and bifurcation equation (4.17) gets the form

$$G(\varepsilon, \mu, \alpha) := \frac{\langle F(\xi(\varepsilon, \mu, \alpha), \varepsilon, \mu, \alpha), \psi \rangle \psi}{\|\psi\|^2} = 0. \tag{4.18}$$

Since $\xi(0, \mu, \alpha) = x_0$ and x_0 is a fixed point of $\widetilde{P}(\cdot, 0, \mu, \alpha)$, $G(0, \mu, \alpha) = 0$ for all $(\mu, \alpha) \in \mathbb{R}^p \times \mathbb{R}$. Next, we want the periodic orbit to persist for all $\varepsilon \neq 0$ small, so we set $\mathrm{D}_\varepsilon G(0, \mu, \alpha) = 0$, i.e.

$$\begin{aligned} \mathrm{D}_\varepsilon G(0, \mu, \alpha) &= \frac{\langle (\mathrm{D}_\xi F(x_0, 0, \mu, \alpha) \mathrm{D}_\varepsilon \xi(0, \mu, \alpha) + \mathrm{D}_\varepsilon F(x_0, 0, \mu, \alpha)), \psi \rangle \psi}{\|\psi\|^2} \\ &= \frac{\langle \mathrm{D}_\varepsilon F(x_0, 0, \mu, \alpha), \psi \rangle \psi}{\|\psi\|^2} = -\frac{\langle \mathrm{D}_\varepsilon \widetilde{P}(x_0, 0, \mu, \alpha), \psi \rangle \psi}{\|\psi\|^2} = 0. \end{aligned}$$

We define the sliding Poincaré-Andronov-Melnikov function as

$$M^\mu(\alpha) = \int_0^T \langle g(\gamma(s), s + \alpha, \mu), A^*(s)\psi \rangle ds \tag{4.19}$$

where $g(x,t,\mu)$ is given by (4.10) and

$$A^*(t) = \begin{cases} X_1^{-1*}(t)X_1^*(t_1)S^*X_2^*(t_2)X_3^*(T) & \text{if } t \in [0,t_1), \\ X_2^{-1*}(t)X_2^*(t_2)X_3^*(T) & \text{if } t \in [t_1,t_2), \\ X_3^{-1*}(t)X_3^*(T) & \text{if } t \in [t_2,T]. \end{cases} \tag{4.20}$$

We note that $A^*(t)\psi = X_1^{-1*}(t)\psi$ for any $t \in [0,t_1)$ and $\mathrm{D}_\varepsilon G(0,\mu,\alpha) = -\frac{M^\mu(\alpha)\psi}{\|\psi\|^2}$. Linearization of equations (4.1), (4.2) with $\varepsilon = 0$ along $\gamma(t)$ gives the variational equation

$$\begin{aligned} \dot{x}(t) &= \mathrm{D}f_+(\gamma(t))x(t) \quad \text{if } t \in [0,t_1) \cup [t_2,T], \\ \dot{x}(t) &= \mathrm{D}f_0(\gamma(t))x(t) \quad \text{if } t \in [t_1,t_2) \end{aligned} \tag{4.21}$$

with impulsive condition

$$x(t_1^+) = Sx(t_1^-) \tag{4.22}$$

and periodic condition

$$B(x(0) - x(T)) = 0 \tag{4.23}$$

where $B = \frac{\psi\psi^*}{\|\psi\|^2}$ is the orthogonal projection onto $Y^\perp$. Note that C^1-smoothness of $\gamma(t)$ at t_2 corresponds to the second "impulsive" condition $x(t_2^+) = \mathbb{I}x(t_2^-)$. Due to definitions of $X_1(t)$, $X_2(t)$, $X_3(t)$ in (4.13), it is obvious that

$$X(t) = \begin{cases} X_1(t) & \text{if } t \in [0,t_1), \\ X_2(t)SX_1(t_1) & \text{if } t \in [t_1,t_2), \\ X_3(t)X_2(t_2)SX_1(t_1) & \text{if } t \in [t_2,T] \end{cases}$$

satisfies variational equation (4.21) together with conditions (4.22), (4.23). Using the classical result – Lemma 1.5 and our result – Lemma 2.4 we shall derive the adjoint variational equation to (4.1), (4.2) with $\varepsilon = 0$ along $\gamma(t)$. One can simply set $B_1 = S$, $B_2 = \mathbb{I}$, $B_3 = B$ in Lemma 2.4 to see, that the adjoint variational system is given by the following linear impulsive boundary value problem

$$\begin{aligned} \dot{x}(t) &= -\mathrm{D}f_+^*(\gamma(t))x(t) \quad \text{if } t \in [0,t_1) \cup [t_2,T], \\ \dot{x}(t) &= -\mathrm{D}f_0^*(\gamma(t))x(t) \quad \text{if } t \in [t_1,t_2), \\ x(t_1^-) &= S^*x(t_1^+), \\ x(T) &= x(0) \in Y^\perp. \end{aligned} \tag{4.24}$$

From definition of $A(t)$ in (4.11) it is easy to see that $A^{-1}(t)$ solves the variational equation (4.21) with the impulsive condition (4.22). Then Lemma 1.5 yields that $A^*(t)\psi$ solves the adjoint variational equation with corresponding impulsive condition. In fact, it satisfies the boundary condition as well. Indeed, from Lemma 4.4 for any $\xi \in [f_+(x_0)]$

$$0 = \langle(\mathbb{I} - A(0))\xi, \psi\rangle = \langle\xi, (\mathbb{I} - A^*(0))\psi\rangle$$

and the same holds in the orthogonal complement to $[f_+(x_0)]$, i.e. if $\xi \in [f_+(x_0)]^\perp$ then

$$0 = \langle \mathrm{D}_\xi F(x_0,0,\mu,\alpha)\xi, \psi\rangle = \langle(\mathbb{I} - A(0))\xi, \psi\rangle = \langle\xi, (\mathbb{I} - A^*(0))\psi\rangle.$$

Consequently, we can take in (4.19) any solution of the adjoint variational system (4.24) instead of $A^*(t)\psi$. In conclusion, we get the main result.

Theorem 4.5. *Let Y, $M^{\mu}(\alpha)$, g, $A^*(t)$ be given by (4.14), (4.19), (4.10), (4.20), respectively, and $\psi \in Y^{\perp}$ be arbitrary and fixed. If α_0 is a simple root of M^{μ_0}, i.e.*

$$\int_0^T \langle g(\gamma(s), s+\alpha_0, \mu_0), A^*(s)\psi\rangle ds = 0,$$
$$\int_0^T \langle \mathrm{D}_t g(\gamma(s), s+\alpha_0, \mu_0), A^*(s)\psi\rangle ds \neq 0$$

then there exists a unique C^{r-2}-function $\alpha(\varepsilon, \mu)$ for $\varepsilon \sim 0$, $\mu \sim \mu_0$ such that $\alpha(0, \mu_0) = \alpha_0$ and there is a unique T-periodic solution $x(\varepsilon, \mu)(t)$ of equation (4.1) with parameters ε, μ and $\alpha = \alpha(\varepsilon, \mu)$, which solves equation (4.2) on Ω_0 and is orbitally close to $\gamma(t)$, i.e. $|x(\varepsilon, \mu)(t) - \gamma(t - \alpha(\varepsilon, \mu))| = O(\varepsilon)$ for any $t \in \mathbb{R}$.

4.1 Piecewise linear application

In this section we shall consider the following three dimensional piecewise linear problem

$$\begin{aligned}
&\dot{x} = -\delta_3 x + \varepsilon \cos \mu_1(t+\alpha) \\
&\dot{y} = \delta_2 y - \omega(z-\delta_1) + \varepsilon \sin \mu_2(t+\alpha) \qquad \text{if } z > 0, \\
&\dot{z} = \omega y + \delta_2(z-\delta_1) \\
&\dot{x} = -\delta_3 x + u \\
&\dot{y} = \delta_2(y+\delta) \qquad \text{if } z < 0 \\
&\dot{z} = \omega(y+\delta)
\end{aligned} \tag{4.25$_\varepsilon$}$$

with constants $\delta_{1,2,3} > 0$, $\delta > -y_1$ (y_1 will be determined later, for this time we consider δ sufficiently large), $u \in \mathbb{R}$ and parameters $\alpha \in \mathbb{R}$, $\mu_1, \mu_2 > 0$, $\varepsilon \sim 0$. We shall shorten the notation $\mu = (\mu_1, \mu_2)$ for vector of parameters.

Clearly, we have $h(x,y,z) = z$, $\Omega_{\pm} = \{(x,y,z) \in \mathbb{R}^3 \mid \pm z > 0\}$, $\Omega_0 = \mathbb{R}^2 \times \{0\}$. Consequently, we obtain equation on Ω_0

$$\begin{aligned}
&\dot{x} = Ax + By + C + \varepsilon G \cos \mu_1(t+\alpha) \\
&\dot{y} = Dy + E + \varepsilon G \sin \mu_2(t+\alpha) \\
&\dot{z} = 0
\end{aligned} \tag{4.26$_\varepsilon$}$$

where

$$A = -\delta_3, \qquad B = -\frac{\omega u}{\omega\delta + \delta_1\delta_2}, \qquad C = \frac{\delta_1\delta_2 u}{\omega\delta + \delta_1\delta_2},$$
$$D = \frac{\delta_1(\delta_2^2 + \omega^2)}{\omega\delta + \delta_1\delta_2}, \qquad E = \delta D, \qquad G = \frac{\omega(y+\delta)}{\omega\delta + \delta_1\delta_2}.$$

Since both equations – the first part of $(4.25)_0$ and $(4.26)_0$ are linear, one can easily derive their flows in Ω_+ and Ω_0, respectively,

$$\begin{aligned}
\varphi_+(x,y,z,t) = \big(&x\mathrm{e}^{-\delta_3 t}, \mathrm{e}^{\delta_2 t}(y\cos\omega t - (z-\delta_1)\sin\omega t), \\
&\mathrm{e}^{\delta_2 t}(y \sin\omega t + (z-\delta_1)\cos\omega t) + \delta_1\big),
\end{aligned}$$
$$\varphi_0(x,y,0,t) = \left(x\mathrm{e}^{At} - \frac{(C-\delta B)(1-\mathrm{e}^{At})}{A} + \frac{B(y+\delta)(\mathrm{e}^{Dt} - \mathrm{e}^{At})}{D-A}, -\delta + (y+\delta)\mathrm{e}^{Dt}, 0\right).$$

Now we need to find points $\bar{x}_i = (x_i, y_i, z_i)$, $i = 0, 1, 2$ and then we set

$$\gamma_1(t) = \varphi_+(\bar{x}_0, t), \qquad \gamma_2(t) = \varphi_0(\bar{x}_1, t - t_1), \qquad \gamma_3(t) = \varphi_+(\bar{x}_2, t - t_2). \tag{4.27}$$

At grazing point $\bar{x}_2$ two conditions are satisfied

$$z_2 = 0 \qquad \text{and} \qquad \omega y_2 + \delta_2(z_2 - \delta_1) = 0$$

due to the assumption H2). Thus we get $K = \mathbb{R} \times \left\{\frac{\delta_1\delta_2}{\omega}\right\} \times \{0\}$ and $\bar{x}_2 = (x_2, \frac{\delta_1\delta_2}{\omega}, 0)$. Following γ_3 we reach the point $\gamma_3(T) = \bar{x}_0$. If we set $\bar{x}_0 = (x_0, 0, z_0)$, we get an equation for the period T

$$\mathrm{e}^{\delta_2(T-t_2)}\left(\frac{\delta_1\delta_2}{\omega}\cos\omega(T-t_2) + \delta_1\sin\omega(T-t_2)\right) = 0.$$

Denoting $c = \arctan\frac{\delta_2}{\omega} \in (0, \pi/2)$ we have $T = t_2 + (\pi - c)/\omega$ as the time of first intersection of $\{\gamma_3(t) \mid t > t_2\}$ with $\mathbb{R} \times \{0\} \times \mathbb{R}$. Then the other coordinates of $\bar{x}_0$ are

$$x_0 = x_2\mathrm{e}^{-\delta_3(\pi-c)/\omega} \quad \text{and} \quad z_0 = \frac{\delta_1}{\omega}\mathrm{e}^{\delta_2(\pi-c)/\omega}\sqrt{\delta_2^2 + \omega^2} + \delta_1.$$

Note that $z_0 > 2\delta_1$. The relation $\gamma_1(t_1) = \bar{x}_1 \in \Omega_0$ yields the implicit equation for t_1:

$$\mathrm{e}^{\delta_2 t_1}\cos\omega t_1 = -\frac{\delta_1}{z_0 - \delta_1} \tag{4.28}$$

where we look for the smallest positive root. Note that the right-hand side of the last identity is from $(-1, 0)$. Therefore $t_1 \in (\pi/(2\omega), \pi/\omega)$. Next we obtain x_1, y_1 and, finally, we connect $\bar{x}_1$ with $\bar{x}_2$ via γ_2. In conclusion, we have the following lemma.

Lemma 4.6. *Unperturbed system* $(4.25)_0$, $(4.26)_0$ *possesses a T-periodic sliding solution $\gamma(t)$ of* (4.3) *given by* (4.27). *Moreover, $\bar{x}_i = (x_i, y_i, z_i)$, $i = 0, 1, 2$ where*

$$x_0 = \frac{\mathrm{e}^{\delta_3 t_2}}{\mathrm{e}^{\delta_3(t_2+(\pi-c)/\omega)} - 1}\left[\frac{B(y_1+\delta)(\mathrm{e}^{D(t_2-t_1)} - \mathrm{e}^{A(t_2-t_1)})}{D-A} - \frac{(C-\delta B)(1-\mathrm{e}^{A(t_2-t_1)})}{A}\right],$$

$$y_0 = 0, \qquad z_0 = \frac{\delta_1}{\omega}\mathrm{e}^{\delta_2(\pi-c)/\omega}\sqrt{\delta_2^2+\omega^2} + \delta_1,$$

$$\bar{x}_1 = \left(x_0\mathrm{e}^{-\delta_3 t_1}, -\mathrm{e}^{\delta_2 t_1}(z_0-\delta_1)\sin\omega t_1, 0\right), \quad \bar{x}_2 = \left(x_0\mathrm{e}^{\delta_3(\pi-c)/\omega}, \frac{\delta_1\delta_2}{\omega}, 0\right),$$

t_1 is given by (4.28) *and*

$$t_2 = t_1 + \frac{1}{D}\ln\frac{y_2+\delta}{y_1+\delta}, \qquad T = t_2 + (\pi - c)/\omega.$$

Fundamental matrices $X_1(t)$, $X_2(t)$, $X_3(t)$ of (4.13) *have, respectively, the form*

$$X_1(t) = \begin{pmatrix} \mathrm{e}^{-\delta_3 t} & 0 & 0 \\ 0 & \mathrm{e}^{\delta_2 t}\cos\omega t & -\mathrm{e}^{\delta_2 t}\sin\omega t \\ 0 & \mathrm{e}^{\delta_2 t}\sin\omega t & \mathrm{e}^{\delta_2 t}\cos\omega t \end{pmatrix},$$

$$X_2(t) = \begin{pmatrix} \mathrm{e}^{A(t-t_1)} & \frac{B\left(\mathrm{e}^{D(t-t_1)} - \mathrm{e}^{A(t-t_1)}\right)}{D-A} & 0 \\ 0 & \mathrm{e}^{D(t-t_1)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad X_3(t) = X_1(t - t_2)$$

and the saltation matrix of (4.12) *is*

$$S = \begin{pmatrix} 1 & 0 & -\frac{u}{\omega\delta+\delta_1\delta_2} \\ 0 & 1 & \frac{\delta_1\omega-\delta\delta_2}{\omega\delta+\delta_1\delta_2} \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. The statement on the periodic trajectory and the mentioned points follow from the preceding discussion. The fundamental matrices are easily obtained due to linearity of unperturbed systems $(4.25)_0$ and $(4.26)_0$, matrix S from its definition (4.12). □

Now, we verify the basic assumptions of this section.

Lemma 4.7. *System* $(4.25)_0$, $(4.26)_0$ *satisfies conditions* H1), H2).

Proof. We have constructed $\gamma(t)$ such that the first condition would be satisfied. To verify the second one, we estimate

$$\mathrm{D}h(\bar{x})(f_-(\bar{x}) - f_+(\bar{x})) = \omega\delta + \delta_1\delta_2 > 0$$

for $\bar{x} \in \Omega_0$,

$$\begin{aligned} \mathrm{D}h(\gamma(t))f_+(\gamma(t)) &= -\delta_1\delta_2 + \omega\left(-\delta + (y_1+\delta)\mathrm{e}^{D(t-t_1)}\right) \\ &< -\delta_1\delta_2 + \omega\left(-\delta + (y_1+\delta)\mathrm{e}^{D(t_2-t_1)}\right) = 0 \end{aligned}$$

for $t \in [t_1, t_2)$ and

$$\mathrm{D}h(\bar{x}_2)f_+(\bar{x}_2) = \omega y_2 - \delta_1\delta_2 = 0.$$

Furthermore,

$$\mathrm{D}h(\gamma(t))f_-(\gamma(t)) = \omega(y_1+\delta)\mathrm{e}^{D(t-t_1)} \geq \omega(y_1+\delta) > 0$$

for $t \in [t_1, t_2]$ and

$$\mathrm{D}_s^2[h(\gamma(t_2+s))]_{s=0^+} = \mathrm{D}h(\bar{x}_2)\mathrm{D}f_+(\bar{x}_2)f_+(\bar{x}_2) = \omega(\delta_2 y_2 + \delta_1\omega) > 0.$$

Moreover, note that for $t \in (t_2, T]$ we have

$$\mathrm{D}h(\gamma(t)) = \frac{\delta_1(\delta_2^2+\omega^2)}{\omega}\mathrm{e}^{\delta_2(t-t_2)}\sin\omega(t-t_2) > 0.$$

Hence $h(\gamma(t)) > 0$ for all $t \in (t_2, T]$. This completes the proof. □

Since we can not express explicitly t_1 from equation (4.28), we proceed numerically. From now on, we consider fixed values of $\delta_{1,2,3}$, δ and ω. We set

$$\delta_1 = \delta_3 = 1, \qquad \delta_2 = 1/2, \qquad \delta = 10, \qquad \omega = 1. \tag{4.29}$$

From Lemma 4.6 we get

$$\begin{aligned} \bar{x}_0 &\doteq (0.007u, 0, 5.265), & t_1 &\doteq 1.673, \\ \bar{x}_1 &\doteq (0.001u, -9.793, 0), & t_2 &\doteq 34.642, \\ \bar{x}_2 &\doteq (0.106u, 0.5, 0), & T &\doteq 37.320. \end{aligned}$$

When parameters are fixed, we verify the third assumption.

Lemma 4.8. *System* $(4.25)_\varepsilon$, $(4.26)_\varepsilon$ *with parameters* (4.29) *satisfies condition* H3) *if* $u \neq 0$.

Proof. From Lemma 4.4, $\dim Z \geq 1$ for Z given by (4.14). Let us suppose that $\dim Z > 1$. Then there exists a vector $v \in Z$ such that $\langle v, f_+(\bar{x}_0)\rangle = 0$, i.e. there exist constants $a, b \in \mathbb{R}$ such that $|a| + |b| > 0$ and

$$v = a \begin{pmatrix} \delta_2(z_0 - \delta_1) \\ 0 \\ \delta_3 x_0 \end{pmatrix} + b \begin{pmatrix} -\omega(z_0 - \delta_1) \\ \delta_3 x_0 \\ 0 \end{pmatrix}$$

since $[f_+(\bar{x}_0)]^\perp = [(\delta_2(z_0 - \delta_1), 0, \delta_3 x_0)^*, (-\omega(z_0 - \delta_1), \delta_3 x_0, 0)^*]$. On substituting parameters (4.29) we obtain

$$v \doteq a \begin{pmatrix} 2.133 \\ 0 \\ 0.007u \end{pmatrix} + b \begin{pmatrix} -4.265 \\ 0.007u \\ 0 \end{pmatrix}.$$

Consequently,

$$(\mathbb{I} - A(0))v \doteq \begin{pmatrix} (-0.005a - 0.002b)u^2 + 2.133a - 4.265b \\ (-2.787a - 1.394b)u \\ (1.401a + 0.701b)u \end{pmatrix}$$

what is equal to zero for $u \neq 0$ if and only if $a = b = 0$ and we get a contradiction with the dimension of Z. $\square$

Note that in the case $u = 0$ the condition H3) is broken and the considered system possesses a degenerate sliding periodic solution which remains in the yz-plane for all the time.

Using Fredholm alternative we get

$$Y^\perp = \mathcal{R}(\mathrm{D}_\xi F(\bar{x}_0, 0, \mu, \alpha))^\perp = \mathcal{N}(\mathrm{D}_\xi F(\bar{x}_0, 0, \mu, \alpha))^* = [(0, 1, 1.98957)^*]$$

where the last equality was derived numerically. So we choose $\psi = (0, 1, 1.98957)^*$. Then the Poincaré-Andronov-Melnikov function defined by (4.19) has the form

$$M^\mu(\alpha) \doteq \int_0^{t_1} (\cos s - 1.98957 \sin s)\mathrm{e}^{-s/2} \sin \mu_2(s + \alpha) ds$$
$$- \int_{t_1}^{t_2} 0.0178 \sin \mu_2(s + \alpha) ds + \int_{t_2}^{T} 10^7(2.458 \cos s - 28.186 \sin s)\mathrm{e}^{-s/2} \sin \mu_2(s + \alpha) ds. \tag{4.30}$$

This can be easily transformed to

$$M^\mu(\alpha) = K \sin \mu_2 \alpha + L \cos \mu_2 \alpha$$

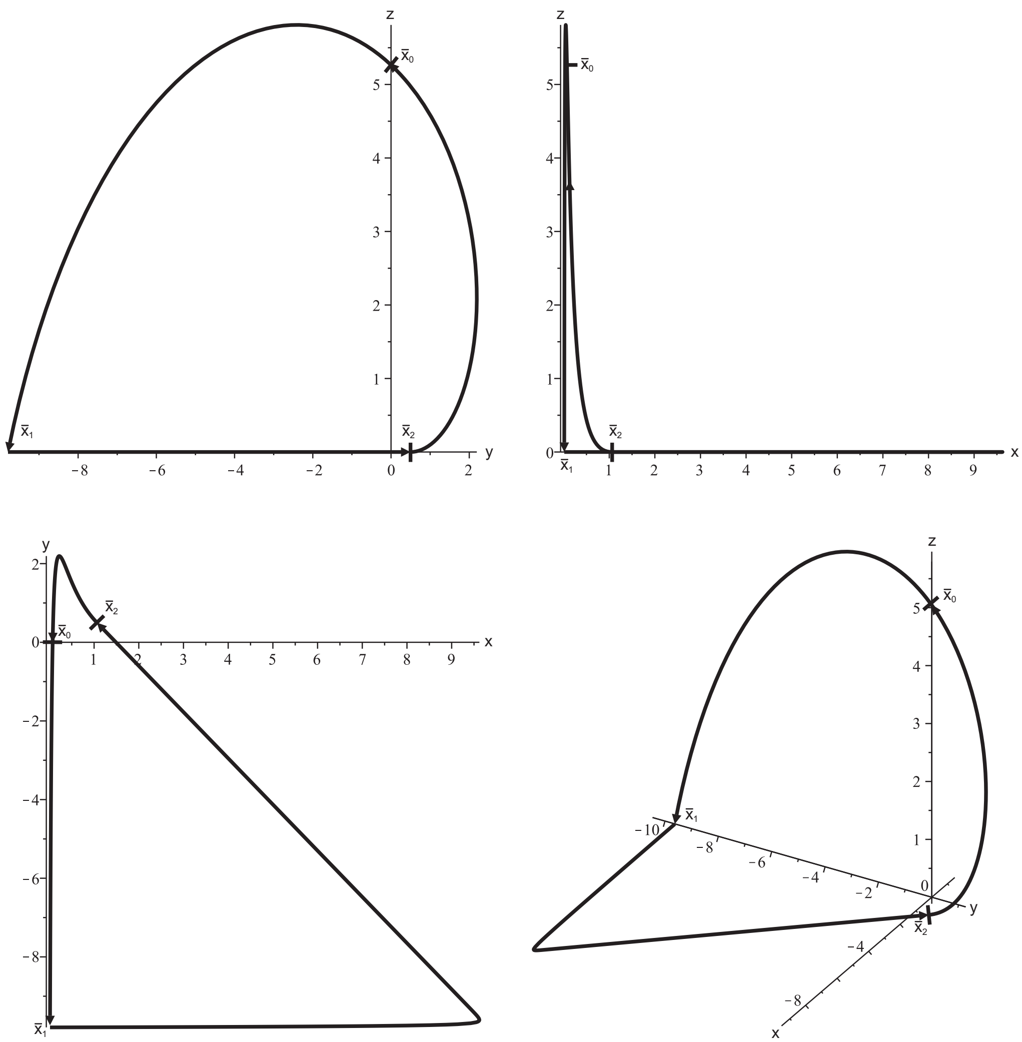

Fig. 4.2: Sliding periodic orbit in unperturbed system $(4.25)_0$, $(4.26)_0$ with parameters (4.29) and $u = 10$ projected onto yz-, xz- and xy-plane and the 3-dimensional overview

where

$$\begin{aligned}
K &= \int_0^{t_1} (\cos s - 1.98957 \sin s) \mathrm{e}^{-s/2} \cos \mu_2 s ds - \int_{t_1}^{t_2} 0.0178 \cos \mu_2 s ds \\
&\quad + \int_{t_2}^{T} 10^7 (2.458 \cos s - 28.186 \sin s) \mathrm{e}^{-s/2} \cos \mu_2 s ds, \\
L &= \int_0^{t_1} (\cos s - 1.98957 \sin s) \mathrm{e}^{-s/2} \sin \mu_2 s ds - \int_{t_1}^{t_2} 0.0178 \sin \mu_2 s ds \\
&\quad + \int_{t_2}^{T} 10^7 (2.458 \cos s - 28.186 \sin s) \mathrm{e}^{-s/2} \sin \mu_2 s ds.
\end{aligned} \tag{4.31}$$

Clearly, $M^\mu(\alpha)$ has a simple root if and only if $K^2+L^2 \neq 0$ or, alternatively, if function

$$\Phi(\mu_2) = \int_0^T \varphi(s)\mathrm{e}^{\imath\mu_2 s}ds$$

with

$$\varphi(s) = \begin{cases} (\cos s - 1.98957\sin s)\mathrm{e}^{-s/2} & \text{if } s \in [0,t_1), \\ -0.0178 & \text{if } s \in [t_1,t_2), \\ 10^7(2.458\cos s - 28.186\sin s)\mathrm{e}^{-s/2} & \text{if } s \in [t_2,T] \end{cases}$$

and $\imath = \sqrt{-1}$, is nonzero. After integrating we obtain

$$\begin{aligned}\Phi(\mu_2) = \frac{1}{\mu_2(4\mu_2^2+4\imath\mu_2-5)}\big[&5.958\mu_2 + 4\imath\mu_2^2 + (0.089\imath - 3.106\mu_2 + 3.536\imath\mu_2^2)\mathrm{e}^{\imath t_1\mu_2} \\ &-(0.089\imath + 34.016\mu_2)\mathrm{e}^{\imath t_2\mu_2} - (5.958\mu_2 + 4\imath\mu_2^2)\mathrm{e}^{\imath T\mu_2}\big]. \end{aligned} \tag{4.32}$$

Since the denominator is nonzero for any $\mu_2 > 0$, function $\Phi(\mu_2)$ is well-defined and it is enough to investigate the numerator. Now we apply the condition on the period of the perturbation function, more precisely

$$(\cos\mu_1 T, \sin\mu_2 T, 0) = (1,0,0),$$

i.e. $\mu_1 = 2k_1\pi/T$, $\mu_2 = 2k_2\pi/T$ for some $k_1,k_2 \in \mathbb{N}$. For such μ_2 the numerator in (4.32) has the form

$$(0.089\imath - 3.106\mu_2 + 3.536\imath\mu_2^2)\mathrm{e}^{\imath t_1\mu_2} - (0.089\imath + 34.016\mu_2)\mathrm{e}^{\imath t_2\mu_2}.$$

For the simplicity, we denote it by $\varphi(\mu_2)$. Then

$$|\varphi(\mu_2)| \geq 3.536\mu_2^2 - 37.123\mu_2 - 0.178$$

what is greater than zero for μ_2 positive if $\mu_2 > 11$. Hence the function $\Phi(\mu_2)$ is nonzero for such $\mu_2 \in (2\pi/T)\mathbb{N}$. Moreover, it can be numerically shown that $\varphi(\mu_2)$ is nonzero for $\mu_2 \in (0,11]$ as well (cf. Fig. 4.3). Consequently, $\Phi(2k_2\pi/T)$ is nonzero for each $k_2 \in \mathbb{N}$ and Theorem 4.5 can be applied.

Proposition 4.9. *Let $u \neq 0$, $\mu_1 = 2k_1\pi/T$, $\mu_2 = 2k_2\pi/T$ for given $k_1,k_2 \in \mathbb{N}$. Then for each $k \in R$ where*

$$R = \{r \in \mathbb{Z} \mid r\pi - \lambda \in [0, 2k_2\pi)\}$$

and λ is such that

$$\cos\lambda = \frac{K}{\sqrt{K^2+L^2}}, \qquad \sin\lambda = \frac{L}{\sqrt{K^2+L^2}}$$

for K, L defined by (4.31), there exists a unique T-periodic sliding solution $x_k(\varepsilon)(t)$ of system $(4.25)_\varepsilon$ with $\varepsilon \neq 0$ sufficiently small and

$$\alpha = \alpha_k(\varepsilon) = \frac{k\pi - \lambda}{\mu_2} + O(\varepsilon)$$

such that

$$|x_k(\varepsilon)(t) - \gamma(t-\alpha)| = O(\varepsilon)$$

for any $t \in \mathbb{R}$. So for each $u \neq 0$, $k_1,k_2 \in \mathbb{N}$ there are at least as many different T-periodic sliding solutions as the number of elements of R.

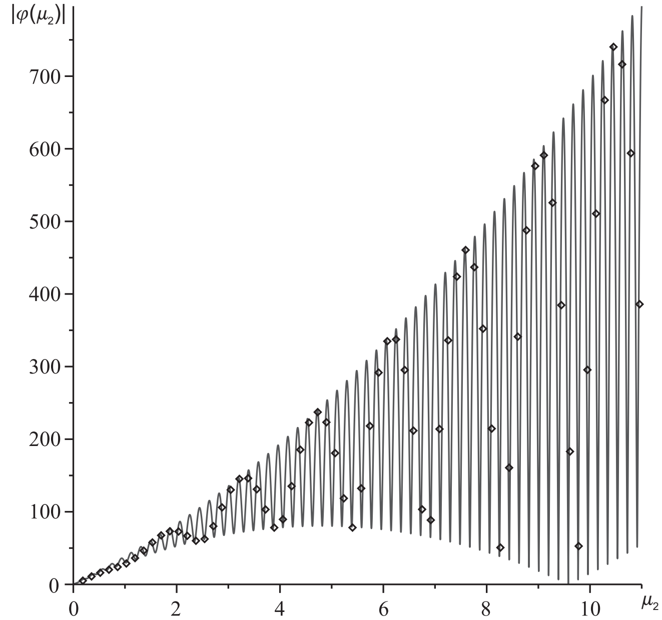

Fig. 4.3: The dependence of $|\varphi(\mu_2)|$ on μ_2. Diamonds depict the values at $2k_2\pi/T$ for $k_2 \in \{1, \ldots, 65\}$

Proof. First, the period matching condition $T = 2k_1\pi/\mu_1 = 2k_2\pi/\mu_2$ for $k_1, k_2 \in \mathbb{N}$ has to be satisfied. Next, for μ_2 such that $\Phi(\mu_2) \neq 0$, the root α_0 of Poncaré-Andronov-Melnikov function $M^\mu(\alpha)$ of (4.30) satisfies $\sin(\mu_2\alpha + \lambda) = 0$. Therefore, $\alpha_0 = (k\pi - \lambda)/\mu_2$ for $k \in \mathbb{Z}$. Moreover, the T-periodicity of functions g_+, g_- (and consequently of function g_0) in t means that only for $\alpha_0 \in [0, T)$, i.e. for $k \in R$ we get the different solutions. The rest follows from Theorem 4.5. $\square$

Chapter II

Hybrid systems

Combination of differential and difference equation is known as hybrid system (cf. [9]). In this system, a mapping given by the difference equation is applied on a solution $x(t)$ of the differential equation at appropriate times, which leads to time-switching system or impacting hybrid system (hard-impact oscillator) where the switching depends on the position $x(t)$ not on time t. In this chapter we study the second case. For a better imagination we introduce a motivation example.

Consider a motion of a single particle in one spatial dimension described by the position $x(t)$ and the velocity $\dot{x}(t)$. We suppose that it is moving under a linear spring, it is weakly damped and forced. So its position satisfies the ordinary differential equation

$$\ddot{x} + \varepsilon\zeta\dot{x} + x = \varepsilon\mu\cos\omega t \quad \text{if} \quad x(t) < \sigma, \tag{0.1}$$

where $\varepsilon\zeta$ measures the viscous damping, $\varepsilon\mu$ is the magnitude of forcing, ε is a small parameter, and we suppose that the motion is free to move in the region $x < \sigma$ for $\sigma > 0$, until some time $t = t_0$ at which $x = \sigma$ where there is an impact with a rigid obstacle. Then, at $t = t_0$, we suppose that $(x(t_0^-), \dot{x}(t_0^-))$ is mapped in zero time via an impact law to

$$x(t_0^+) = x(t_0^-) \quad \text{and} \quad \dot{x}(t_0^+) = -(1+\varepsilon r)\dot{x}(t_0^-),$$

where $x(t^\pm) = \lim_{s\to t^\pm} x(s)$, $\dot{x}(t^\pm) = \lim_{s\to t^\pm} \dot{x}(s)$ and $0 < 1 + \varepsilon r \leq 1$ is the Newton coefficient of restriction [9]. Another well-known piecewise linear impact system is a weakly damped and forced inverted pendulum given by equations [26]

$$\begin{gathered} \ddot{x} + \varepsilon\zeta\dot{x} - x = \varepsilon\mu\cos\omega t \quad \text{if} \quad |x(t)| < \sigma, \\ x(t^+) = x(t^-) \quad \text{and} \quad \dot{x}(t^+) = -(1+\varepsilon r)\dot{x}(t^-) \quad \text{if} \quad |x(t^-)| = \sigma. \end{gathered} \tag{0.2}$$

We can study coupled (0.1) and (0.2) to get higher dimensional impact system.

So a system for weakly forced impact oscillator consists of an ordinary differential equation

$$\ddot{x} = f_1(x, \dot{x}) + \varepsilon g_1(x, \dot{x}, t, \varepsilon, \mu) \tag{0.3}$$

and an impact condition

$$\dot{x}(t^+) = f_2(\dot{x}(t^-)) + \varepsilon g_2(x(t^-), \dot{x}(t^-), t, \varepsilon, \mu) \quad \text{if} \quad h(x(t^-)) = 0. \tag{0.4}$$

The above equation rewritten as an evolution system has a form

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_1(x_1, x_2) + \varepsilon g_1(x_1, x_2, t, \varepsilon, \mu), \\
x_2(t^+) &= f_2(x_2(t^-)) + \varepsilon g_2(x_1(t^-), x_2(t^-), t, \varepsilon, \mu) \quad \text{if} \quad h(x_1(t^-)) = 0.
\end{aligned}
$$

In fact, we shall investigate a more general case (see (1.1), (1.2)).

1 Periodically forced impact systems

In this section we investigate the persistence of a single T-periodic orbit of autonomous system with impact under nonautonomous perturbation and derive a sufficient condition.

Let $\Omega \subset \mathbb{R}^n$ be an open set in $\mathbb{R}^n$ and $h(x)$ be a C^r-function on $\overline{\Omega}$, with $r \geq 3$. We set $\Omega_0 := \{x \in \Omega \mid h(x) = 0\}$, $\Omega_1 := \Omega \backslash \Omega_0$. Let $f_1 \in C_b^r(\overline{\Omega})$, $f_2 \in C_b^r(\Omega_0, \Omega_0)$, $g_1 \in C_b^r(\overline{\Omega} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^p)$, $g_2 \in C_b^r(\Omega_0 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^p)$ and $h \in C_b^r(\overline{\Omega}, \mathbb{R})$. Furthermore, we suppose that $g_{1,2}$ are T-periodic in $t \in \mathbb{R}$ and 0 is a regular value of h. Let $\varepsilon, \alpha \in \mathbb{R}$ and $\mu \in \mathbb{R}^p, p \geq 1$ be parameters.

Definition 1.1. We say that a function $x(t)$ is a solution of an impact system

$$\dot{x} = f_1(x) + \varepsilon g_1(x, t, \varepsilon, \mu), \quad x \in \Omega_1, \tag{1.1}$$

$$x(t^+) = f_2(x(t^-)) + \varepsilon g_2(x(t^-), t, \varepsilon, \mu) \quad \text{if} \quad h(x(t^-)) = 0, \tag{1.2}$$

if it is piecewise C^1-smooth satisfying equation (1.1) on Ω_1, equation (1.2) on Ω_0 and, moreover, the following holds: if for some t_0 we have $x(t_0) \in \Omega_0$, then there exists $\rho > 0$ such that for any $t \in (t_0 - \rho, t_0)$, $s \in (t_0, t_0 + \rho)$ we have $h(x(t))h(x(s)) > 0$.

For modelling problem given by (1.1), (1.2) we assume

H1) The unperturbed equation

$$\dot{x} = f_1(x) \tag{1.3}$$

has a T-periodic orbit $\gamma(t)$ which is discontinuous at $t = t_1 \in (0, T)$ where it satisfies impact condition

$$x(t^+) = f_2(x(t^-)) \quad \text{if} \quad h(x(t^-)) = 0. \tag{1.4}$$

The orbit is given by its initial point $x_0 \in \Omega_1$ and consists of two branches

$$\gamma(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [0, t_1), \\ \{x_1, x_2\} & \text{if } t = t_1, \\ \gamma_2(t) & \text{if } t \in (t_1, T], \end{cases} \tag{1.5}$$

where $0 < t_1 < T$, $\gamma(t) \in \Omega_1$ for $t \in [0, t_1) \cup (t_1, T]$, $\gamma(t_1) \subset \Omega_0$ and

$$
\begin{aligned}
x_1 &:= \gamma_1(t_1^-) \in \Omega_0, \\
x_2 &:= \gamma_2(t_1^+) \in \Omega_0, \\
x_0 &:= \gamma_2(T) = \gamma_1(0) \in \Omega_1.
\end{aligned} \tag{1.6}
$$

H2) Moreover, we also assume that

$$\mathrm{D}h(x_1)f_1(x_1)\mathrm{D}h(x_2)f_1(x_2) < 0.$$

The geometric meaning of assumption H2) is that the impact periodic solution $\gamma(t)$ from H1) transversally hits and leaves the impact surface Ω_0 at t_1^- and t_1^+, respectively.

Next, since impact system (1.3), (1.4) is autonomous, $\gamma(t-\alpha)$ is also its solution for any $\alpha \in \mathbb{R}$. So we are looking for a forced T-periodic solution $x(t)$ of the perturbed impact system (1.1), (1.2) which is orbitally close to γ, i.e. $x(t) \sim \gamma(t-\alpha)$ for some α depending on $\varepsilon \neq 0$ small. For this reason, by shifting the time, we study a shifted (1.1), (1.2) of the form

$$\dot{x} = f_1(x) + \varepsilon g_1(x, t+\alpha, \varepsilon, \mu), \quad x \in \Omega_1, \tag{1.7}$$

$$x(t^+) = f_2(x(t^-)) + \varepsilon g_2(x(t^-), t+\alpha, \varepsilon, \mu) \quad \text{if} \quad h(x(t^-)) = 0 \tag{1.8}$$

with additional parameter $\alpha \in \mathbb{R}$.

Let $x(\tau,\xi)(t,\varepsilon,\mu,\alpha)$ denote the solution of initial value problem

$$\begin{aligned} \dot{x} &= f_1(x) + \varepsilon g_1(x, t+\alpha, \varepsilon, \mu) \\ x(\tau) &= \xi. \end{aligned} \tag{1.9}$$

First, we modify Lemma 1.2 from Chapter I for impact system (1.7), (1.8).

Lemma 1.2. *Assume* H1) *and* H2). *Then there exist* $\varepsilon_0, r_0 > 0$ *and a Poincaré mapping (cf. Fig. 1.1)*

$$P(\cdot,\varepsilon,\mu,\alpha) : B(x_0,r_0) \to \Sigma$$

for all fixed $\varepsilon \in (-\varepsilon_0,\varepsilon_0)$, $\mu \in \mathbb{R}^p$, $\alpha \in \mathbb{R}$, *where* $B(x,r)$ *is a ball in* $\mathbb{R}^n$ *with center at* x *and radius* r, *and*

$$\Sigma = \{y \in \mathbb{R}^n \mid \langle y - x_0, f_1(x_0)\rangle = 0\}.$$

Moreover, $P : B(x_0,r_0) \times (-\varepsilon_0,\varepsilon_0) \times \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^n$ *is* C^r*-smooth in all arguments and* $B(x_0,r_0) \subset \Omega_1$.

Proof. The lemma follows from implicit function theorem (IFT) [16]. We obtain the existence of positive constants τ_1, r_1, δ_1, ε_1 and C^r-function

$$t_1(\cdot,\cdot,\cdot,\cdot,\cdot) : (-\tau_1,\tau_1) \times B(x_0,r_1) \times (-\varepsilon_1,\varepsilon_1) \times \mathbb{R}^p \times \mathbb{R} \to (t_1 - \delta_1, t_1 + \delta_1)$$

such that $h(x(\tau,\xi)(t,\varepsilon,\mu,\alpha)) = 0$ for $\tau \in (-\tau_1,\tau_1)$, $\xi \in B(x_0,r_1) \subset \Omega_1$, $\varepsilon \in (-\varepsilon_1,\varepsilon_1)$, $\mu \in \mathbb{R}^p$, $\alpha \in \mathbb{R}$ and $t \in (t_1 - \delta_1, t_1 + \delta_1)$ if and only if $t = t_1(\tau,\xi,\varepsilon,\mu,\alpha)$. Moreover, $t_1(0,x_0,0,\mu,\alpha) = t_1$.

Analogically, we derive function t_2 satisfying

$$\begin{gathered} \langle x(t_1(\tau,\xi,\varepsilon,\mu,\alpha), f_2(x(\tau,\xi)(t_1(\tau,\xi,\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha)) \\ +\varepsilon g_2(x(\tau,\xi)(t_1(\tau,\xi,\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha), t_1(\tau,\xi,\varepsilon,\mu,\alpha)+\alpha,\varepsilon,\mu)) \\ (t_2(\tau,\xi,\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha) - x_0, f_1(x_0)\rangle = 0. \end{gathered}$$

Moreover, we have $t_2(0,x_0,0,\mu,\alpha) = T$. Poincaré mapping is then defined as

$$\begin{gathered} P(\xi,\varepsilon,\mu,\alpha) = x(t_1(0,\xi,\varepsilon,\mu,\alpha), f_2(x(0,\xi)(t_1(0,\xi,\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha)) \\ +\varepsilon g_2(x(0,\xi)(t_1(0,\xi,\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha), t_1(0,\xi,\varepsilon,\mu,\alpha)+\alpha,\varepsilon,\mu))(t_2(0,\xi,\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha). \end{gathered} \tag{1.10}$$

The proof is finished. $\square$

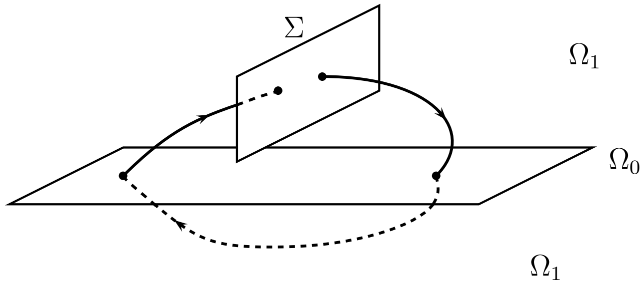

Fig. 1.1: Impact Poincaré mapping

Since we are looking for a persisting periodic orbit with fixed period T, we have to solve a couple of equations

$$\begin{aligned} P(\xi,\varepsilon,\mu,\alpha) &= \xi, \\ t_2(0,\xi,\varepsilon,\mu,\alpha) &= T. \end{aligned}$$

Therefore, we define the stroboscopic Poincaré mapping (cf. (1.6), (4.7) in Chapter I)

$$\begin{aligned} \widetilde{P}(\xi,\varepsilon,\mu,\alpha) = x(t_1(0,\xi,\varepsilon,\mu,\alpha), f_2(x(0,\xi)(t_1(0,\xi,\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha)) \\ +\varepsilon g_2(x(0,\xi)(t_1(0,\xi,\varepsilon,\mu,\alpha),\varepsilon,\mu,\alpha), t_1(0,\xi,\varepsilon,\mu,\alpha)+\alpha,\varepsilon,\mu))(T,\varepsilon,\mu,\alpha) \end{aligned} \tag{1.11}$$

and solve a single equation

$$F(\xi,\varepsilon,\mu,\alpha) := \xi - \widetilde{P}(\xi,\varepsilon,\mu,\alpha) = 0 \quad \text{for} \quad \xi \in \Sigma. \tag{1.12}$$

Now, we calculate the linearization of $\widetilde{P}$ at $(\xi,\varepsilon) = (x_0, 0)$. We obtain the next result.

Lemma 1.3. *Let $\widetilde{P}(\xi,\varepsilon,\mu,\alpha)$ be defined by* (1.11)*. Then for all $\mu \in \mathbb{R}^p$, $\alpha \in \mathbb{R}$*

$$\widetilde{P}_\xi(x_0,0,\mu,\alpha) = A(0), \tag{1.13}$$

$$\widetilde{P}_\varepsilon(x_0,0,\mu,\alpha) = \int_0^T A(s) g_1(\gamma(s), s+\alpha, 0, \mu) ds + X_2(T) g_2(x_1, t_1+\alpha, 0, \mu), \tag{1.14}$$

where $\widetilde{P}_\xi$, $\widetilde{P}_\varepsilon$ are partial derivatives of $\widetilde{P}$ with respect to ξ, ε, respectively. Here $A(t)$ is given by

$$A(t) = \begin{cases} X_2(T) S X_1(t_1) X_1^{-1}(t) & \text{if } t \in [0,t_1), \\ X_2(T) X_2^{-1}(t) & \text{if } t \in [t_1,T] \end{cases} \tag{1.15}$$

with impact saltation matrix

$$S = \mathrm{D}f_2(x_1) + \frac{(f_1(x_2) - \mathrm{D}f_2(x_1)f_1(x_1))\mathrm{D}h(x_1)}{\mathrm{D}h(x_1)f_1(x_1)} \tag{1.16}$$

and fundamental matrix solutions $X_1(t)$, $X_2(t)$ satisfying, respectively,

$$\begin{aligned} &\dot{X}_1(t) = \mathrm{D}f_1(\gamma(t))X_1(t) && \dot{X}_2(t) = \mathrm{D}f_1(\gamma(t))X_2(t) \\ &X_1(0) = \mathbb{I}, && X_2(t_1) = \mathbb{I}. \end{aligned} \tag{1.17}$$

In addition, $\widetilde{P}_\xi(x_0, 0, \mu, \alpha)$ *has an eigenvalue* 1 *with corresponding eigenvector* $f_1(x_0)$, *i.e.*

$$\widetilde{P}_\xi(x_0, 0, \mu, \alpha) f_1(x_0) = f_1(x_0).$$

Proof. The statement on the derivatives of $\widetilde{P}$ follows from its definition with the aid of the following identities

$$\begin{aligned} \mathrm{D}_\xi x(0, x_0)(t, 0, \mu, \alpha) &= X_1(t), \\ \mathrm{D}_\varepsilon x(0, x_0)(t, 0, \mu, \alpha) &= \int_0^t X_1(t) X_1^{-1}(s) g_1(\gamma(s), s + \alpha, 0, \mu) ds \end{aligned}$$

for $t \in [0, t_1]$,

$$\begin{gathered} \mathrm{D}_\xi x(t_1, x_2)(t, 0, \mu, \alpha) = X_2(t), \qquad \mathrm{D}_\tau x(t_1, x_2)(t, 0, \mu, \alpha) = -X_2(t) f_1(x_2), \\ \mathrm{D}_\varepsilon x(t_1, x_2)(t, 0, \mu, \alpha) = \int_{t_1}^t X_2(t) X_2^{-1}(s) g_1(\gamma(s), s + \alpha, 0, \mu) ds \end{gathered}$$

for $t \in [t_1, T]$ and

$$\begin{aligned} \mathrm{D}_\xi t_1(0, x_0, 0, \mu, \alpha) &= -\frac{\mathrm{D}h(x_1) X_1(t_1)}{\mathrm{D}h(x_1) f_1(x_1)}, \\ \mathrm{D}_\varepsilon t_1(0, x_0, 0, \mu, \alpha) &= -\frac{\mathrm{D}h(x_1) \int_0^{t_1} X_1(t_1) X_1^{-1}(s) g_1(\gamma(s), s + \alpha, 0, \mu) ds}{\mathrm{D}h(x_1) f_1(x_1)}. \end{aligned}$$

To proceed with the proof, we note $\varepsilon = 0$ results $x(\tau, \xi)(t, 0, \mu, \alpha) = x(\tau, \xi)(t)$, the solution of $\dot{x} = f_1(x)$, $x(\tau) = \xi$, and it is independent of (μ, α). Analogically $t_1(\tau, \xi, 0, \mu, \alpha) = t_1(\tau, \xi)$, $t_2(\tau, \xi, 0, \mu, \alpha) = t_2(\tau, \xi)$, $P(\xi, 0, \mu, \alpha) = P(\xi)$ and $\widetilde{P}(\xi, 0, \mu, \alpha) = \widetilde{P}(\xi)$.

Now, since for all $t > 0$ small, it holds that

$$x(0, \gamma_1(t))(t_1(0, \gamma_1(t))) = x(0, x_0)(t + t_1(0, \gamma_1(t))) = \gamma_1(t + t_1(0, \gamma_1(t)))$$

is an element of Ω_0 and as well of $\{\gamma(t) \mid t \in \mathbb{R}\}$ we have

$$t + t_1(0, \gamma_1(t)) = t_1.$$

Consequently

$$x(0, \gamma_1(t))(t_1(0, \gamma_1(t))) = x_1.$$

Then we obtain

$$\begin{aligned} \widetilde{P}(\gamma_1(t)) &= x(t_1(0, \gamma_1(t)), f_2(x(0, \gamma_1(t))(t_1(0, \gamma_1(t)))))(T) \\ &= x(t_1(0, \gamma_1(t)), f_2(x_1))(T) = x(t_1 - t, x_2)(T) = x(t_1, x_2)(T + t) = \gamma_1(t). \end{aligned}$$

Hence

$$\widetilde{P}_\xi(x_0, 0, \mu, \alpha) f_1(x_0) = \mathrm{D}_t[\widetilde{P}(\gamma_1(t))]_{t=0^+} = \mathrm{D}_t[\gamma_1(t)]_{t=0^+} = f_1(x_0).$$

The proof is finished. $\square$

We solve equation (1.12) for $(\xi, \alpha) \in \Sigma \times \mathbb{R}$ with parameters ε, μ using Lyapunov-Schmidt reduction method. Obviously $F(x_0, 0, \mu, \alpha) = 0$ for all $(\mu, \alpha) \in \mathbb{R}^p \times \mathbb{R}$. Let us denote

$$Z = \mathcal{N}\mathrm{D}_\xi F(x_0, 0, \mu, \alpha), \qquad Y = \mathcal{R}\mathrm{D}_\xi F(x_0, 0, \mu, \alpha) \tag{1.18}$$

the null space and the range of the corresponding operator and

$$\mathcal{Q} : \mathbb{R}^n \to Y, \qquad \mathcal{P} : \mathbb{R}^n \to Y^\perp \tag{1.19}$$

orthogonal projections onto Y and $Y^\perp$, respectively, where $Y^\perp$ is the orthogonal complement to Y in $\mathbb{R}^n$. Here we take the third assumption

H3) $\mathcal{N}\mathrm{D}_\xi F(x_0, 0, \mu, \alpha) = [f_1(x_0)]$.

Equation (1.12) is split into couple of equations

$$\begin{aligned} \mathcal{Q}F(\xi, \varepsilon, \mu, \alpha) &= 0, \\ \mathcal{P}F(\xi, \varepsilon, \mu, \alpha) &= 0 \end{aligned} \tag{1.20}$$

where the first one can be solved using IFT, since

$$\mathcal{Q}F(x_0, 0, \mu, \alpha) = 0$$

and $\mathcal{Q}\mathrm{D}_\xi F(x_0, 0, \mu, \alpha)$ is an isomorphism from $[f_1(x_0)]^\perp$ onto Y for all $(\mu, \alpha) \in \mathbb{R}^p \times \mathbb{R}$. Thus we get the existence of a C^r-function $\xi = \xi(\varepsilon, \mu, \alpha)$ for ε close to 0 and $(\mu, \alpha) \in \mathbb{R}^p \times \mathbb{R}$ such that $\mathcal{Q}F(\xi(\varepsilon, \mu, \alpha), \varepsilon, \mu, \alpha) = 0$ for all such $(\varepsilon, \mu, \alpha)$ and $\xi(0, \mu, \alpha) = x_0$. The second equation is so-called persistence equation for $\alpha \in \mathbb{R}$

$$\mathcal{P}F(\xi(\varepsilon, \mu, \alpha), \varepsilon, \mu, \alpha) = 0. \tag{1.21}$$

Let $\psi \in Y^\perp$ be arbitrary and fixed. Then we can write

$$\mathcal{P}u = \frac{\langle u, \psi \rangle \psi}{\|\psi\|^2}$$

and persistence equation (1.21) has the form

$$G(\varepsilon, \mu, \alpha) := \frac{\langle F(\xi(\varepsilon, \mu, \alpha), \varepsilon, \mu, \alpha), \psi \rangle \psi}{\|\psi\|^2} = 0. \tag{1.22}$$

Clearly, $G(0, \mu, \alpha) = 0$ for all $(\mu, \alpha) \in \mathbb{R}^p \times \mathbb{R}$. Moreover, we want the periodic orbit to persist, so we need to solve $G(\varepsilon, \mu, \alpha) = 0$ for $\varepsilon \neq 0$ small. But since $G(\varepsilon, \mu, \alpha) = \mathrm{D}_\varepsilon G(0, \mu, \alpha)\varepsilon + o(\varepsilon)$, the equality $\mathrm{D}_\varepsilon G(0, \mu_0, \alpha_0) = 0$ is a necessary condition for a point $(0, \mu_0, \alpha_0)$ to be a starting persistence value, this means if there is a sequence $\{(\varepsilon_n, \mu_n, \alpha_n)\}_{n \in \mathbb{N}}$ such that $\varepsilon_n \neq 0$, $(\varepsilon_n, \mu_n, \alpha_n) \to (0, \mu_0, \alpha_0)$ for $n \to \infty$ and $G(\varepsilon_n, \mu_n, \alpha_n) = 0$, then $\mathrm{D}_\varepsilon G(0, \mu_0, \alpha_0) = 0$. So we derive

$$\begin{aligned} \mathrm{D}_\varepsilon G(0, \mu, \alpha) &= \frac{\langle (\mathrm{D}_\xi F(x_0, 0, \mu, \alpha)\mathrm{D}_\varepsilon \xi(0, \mu, \alpha) + \mathrm{D}_\varepsilon F(x_0, 0, \mu, \alpha)), \psi \rangle \psi}{\|\psi\|^2} \\ &= \frac{\langle \mathrm{D}_\varepsilon F(x_0, 0, \mu, \alpha), \psi \rangle \psi}{\|\psi\|^2} = -\frac{\langle \mathrm{D}_\varepsilon \widetilde{P}(x_0, 0, \mu, \alpha), \psi \rangle \psi}{\|\psi\|^2}. \end{aligned}$$

We denote

$$M^{\mu}(\alpha) = \int_0^T \langle g_1(\gamma(s), s+\alpha, 0, \mu), A^*(s)\psi\rangle ds + \langle X_2(T)g_2(x_1, t_1+\alpha, 0, \mu), \psi\rangle \quad (1.23)$$

the impact Poincaré-Andronov-Melnikov function, where

$$A^*(t) = \begin{cases} X_1^{-1*}(t)X_1^*(t_1)S^*X_2^*(T) & \text{if } t \in [0, t_1), \\ X_2^{-1*}(t)X_2^*(T) & \text{if } t \in [t_1, T]. \end{cases} \quad (1.24)$$

Note that $\mathrm{D}_\varepsilon G(0, \mu, \alpha) = -\frac{M^{\mu}(\alpha)\psi}{\|\psi\|^2}$.

Linearization of unperturbed impact system (1.3), (1.4) along T-periodic solution $\gamma(t)$ gives the variational equation

$$\dot{x}(t) = \mathrm{D}f_1(\gamma(t))x(t) \quad (1.25)$$

with impulsive condition

$$x(t_1^+) = Sx(t_1^-) \quad (1.26)$$

and periodic condition

$$B(x(0) - x(T)) = 0 \quad (1.27)$$

where $B = \frac{\psi\psi^*}{\|\psi\|^2}$ is the orthogonal projection onto $Y^\perp$. From definition of $X_1(t), X_2(t)$,

$$X(t) = \begin{cases} X_1(t) & \text{if } t \in [0, t_1), \\ X_2(t)SX_1(t_1) & \text{if } t \in [t_1, T] \end{cases}$$

solves variational equation (1.25) and conditions (1.26), (1.27).

Now, on letting $B_1 = S$, $B_2 = \mathbb{I}$, $B_3 = B$ in Lemma 2.4 one can see that the adjoint variational system of (1.3) and impact condition (1.4) (i.e. adjoint system of (1.25), (1.26), (1.27)) is given by the following linear impulsive boundary value problem

$$\begin{aligned} \dot{x}(t) &= -\mathrm{D}f_1^*(\gamma(t))x(t), \qquad t \in [0, T], \\ x(t_1^-) &= S^*x(t_1^+), \\ x(T) &= x(0) \in Y^\perp. \end{aligned} \quad (1.28)$$

From (1.24) we know that $A^*(t)\psi$ solves adjoint variational equation with impulsive condition. To see that it satisfies the boundary condition as well, we consider

$$0 = \langle \mathrm{D}_\xi F(x_0, 0, \mu, \alpha)\xi, \psi\rangle = \langle((\mathbb{I} - A(0))\xi, \psi\rangle = \langle \xi, (\mathbb{I} - A^*(0))\psi\rangle$$

for all $\xi \in [f_1(x_0)]^\perp$ and if $\xi \in [f_1(x_0)]$, from Lemma 1.3 follows

$$0 = \langle \xi - \xi, \psi\rangle = \langle(\mathbb{I} - A(0))\xi, \psi\rangle = \langle \xi, (\mathbb{I} - A^*(0))\psi\rangle.$$

As a consequence, we can take in (1.23) any solution of the adjoint variational system (1.28). Summarizing, we get the main result.

Theorem 1.4. *Let $\psi \in Y^\perp$ be arbitrary and fixed, Y be given by* (1.18) *and $A^*(t)$, $M^\mu(\alpha)$ be defined by* (1.24), (1.23), *respectively. If α_0 is a simple root of function $M^{\mu_0}(\alpha)$, i.e.*

$$\int_0^T \langle g_1(\gamma(s), s+\alpha_0, 0, \mu_0), A^*(s)\psi\rangle ds + \langle X_2(T) g_2(x_1, t_1+\alpha_0, 0, \mu_0), \psi\rangle = 0,$$
$$\int_0^T \langle \mathrm{D}_t g_1(\gamma(s), s+\alpha_0, 0, \mu_0), A^*(s)\psi\rangle ds + \langle X_2(T) \mathrm{D}_t g_2(x_1, t_1+\alpha_0, 0, \mu_0), \psi\rangle \neq 0$$

then there exists a unique C^{r-1}-function $\alpha(\varepsilon, \mu)$ for $\varepsilon \sim 0$ small and $\mu \sim \mu_0$ such that $\alpha(0, \mu_0) = \alpha_0$ and there is a unique T-periodic solution $x_{\varepsilon,\mu}(t)$ of equation (1.1) *with parameters $\varepsilon \neq 0$ sufficiently small, μ close to μ_0 and $\alpha = \alpha(\varepsilon, \mu)$, which satisfies condition* (1.2) *and $|x_{\varepsilon,\mu}(t) - \gamma(t - \alpha(\varepsilon, \mu))| = O(\varepsilon)$.*

Proof. We set
$$H(\varepsilon, \mu, \alpha) = \begin{cases} \mathrm{D}_\varepsilon G(0, \mu, \alpha) & \text{for } \varepsilon = 0, \\ G(\varepsilon, \mu, \alpha)/\varepsilon & \text{for } \varepsilon \neq 0. \end{cases}$$

Then H is C^{r-1}-smooth. Assumptions of our theorem imply $H(0, \mu_0, \alpha_0) = 0$ and $\mathrm{D}_\alpha H(0, \mu_0, \alpha_0) \neq 0$. By IFT there exists a unique C^{r-1}-function $\alpha(\varepsilon, \mu)$ for $\varepsilon \sim 0$ small and $\mu \sim \mu_0$ such that $\alpha(0, \mu_0) = \alpha_0$, and $H(\varepsilon, \mu, \alpha(\varepsilon, \mu)) = 0$. But this means that $x(0, \xi(\varepsilon, \mu, \alpha(\varepsilon, \mu)))(t, \varepsilon, \mu, \alpha(\varepsilon, \mu))$ is a solution of (1.7), (1.8) with $\alpha = \alpha(\varepsilon, \mu)$ and it satisfies
$$|x(0, \xi(\varepsilon, \mu, \alpha(\varepsilon, \mu)))(t, \varepsilon, \mu, \alpha(\varepsilon, \mu)) - \gamma(t)| = O(\varepsilon).$$
Then
$$x_{\varepsilon,\mu}(t) = x(0, \xi(\varepsilon, \mu, \alpha(\varepsilon, \mu)))(t - \alpha(\varepsilon, \mu), \varepsilon, \mu, \alpha(\varepsilon, \mu))$$
is the desired solution of (1.1), (1.2). The proof is completed. □

1.1 Pendulum hitting moving obstacle

Here we provide an application of derived theory to problem of mathematical pendulum which impacts an oscillating wall (see Fig. 1.2). The horizontal distance between the wall and the center of the pendulum is $\delta + \varepsilon z(t, \varepsilon, \mu)$ where z is periodic in t and δ is a positive constant. We denote x the angle and l the length of the massless cord.

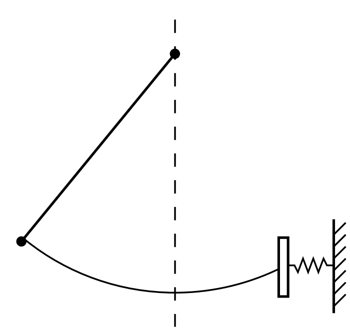

Fig. 1.2: Impacting pendulum

Then x satisfies the dimensionless equation

$$\ddot{x} = -\omega^2 x$$

with a given frequency $\omega > 0$ and impact condition

$$\dot{x}(t^+) = -\dot{x}(t^-) + \varepsilon \dot{z}(t^-, \varepsilon, \mu) \frac{\sqrt{l^2 - (\delta + \varepsilon z(t^-, \varepsilon, \mu))^2}}{l}$$

whenever

$$x(t^-) - \arcsin \frac{\delta + \varepsilon z(t^-, \varepsilon, \mu)}{l} = 0,$$

which follows from actual position of the wall and its speed projected onto tangent line to the trajectory of bob. Writing as a system we get

$$\begin{aligned} \dot{x} &= \omega y \\ \dot{y} &= -\omega x \end{aligned}$$

and

$$\begin{aligned} x(t^+) &= x(t^-) \\ y(t^+) &= -y(t^-) + \varepsilon \dot{z}(t^-, \varepsilon, \mu) \frac{\sqrt{l^2 - (\delta + \varepsilon z(t^-, \varepsilon, \mu))^2}}{\omega l} \end{aligned}$$

if

$$x(t^-) - \arcsin \frac{\delta + \varepsilon z(t^-, \varepsilon, \mu)}{l} = 0.$$

To obtain a problem in form of (1.7), (1.8) we introduce parameter α and transform the variables

$$u = x - \arcsin \frac{\delta + \varepsilon z(t + \alpha, \varepsilon, \mu)}{l} + \arcsin \frac{\delta}{l}, \qquad v = y.$$

So we get

$$\begin{aligned} \dot{u}(t) &= \omega v(t) - \varepsilon \frac{\dot{z}(t + \alpha, \varepsilon, \mu)}{\sqrt{l^2 - (\delta + \varepsilon z(t + \alpha, \varepsilon, \mu))^2}} \\ \dot{v}(t) &= -\omega u(t) - \varepsilon \frac{\omega z(t + \alpha, 0, \mu)}{\sqrt{l^2 - \delta^2}} + O(\varepsilon^2) \end{aligned} \tag*{$(1.29)_\varepsilon$}$$

with impact condition

$$\begin{aligned} u(t^+) &= u(t^-) \\ v(t^+) &= -v(t^-) + \varepsilon \dot{z}(t^- + \alpha, \varepsilon, \mu) \frac{\sqrt{l^2 - (\delta + \varepsilon z(t^- + \alpha, \varepsilon, \mu))^2}}{\omega l} \\ &\text{if} \quad h(u(t^-), v(t^-)) = 0 \end{aligned} \tag*{$(1.30)_\varepsilon$}$$

where

$$h(u, v) = u - \arcsin \frac{\delta}{l}.$$

Note the dependence on ε in notation, i.e. $(1.29)_0$, $(1.30)_0$ denotes the unperturbed system and the unperturbed impact condition, respectively. Moreover, we denote $\hat{u} = \arcsin \frac{\delta}{l}$.

In this case we have $\Omega_0 = \{(u, v) \in \mathbb{R}^2 \mid u = \hat{u}\}$, $\Sigma = \{(u, 0) \in \mathbb{R}^2 \mid u < -\hat{u}\}$ and the following lemma describing the unperturbed problem.

Lemma 1.5. *System* $(1.29)_0$, $(1.30)_0$ *possesses a family of periodic orbits* $\gamma^u(t)$ *parametrized by* $u < -\hat{u}$ *such that*

$$\gamma^u(t) = \begin{cases} (u\cos\omega t, -u\sin\omega t) & \text{if } t \in [0, t_1), \\ \{(u_1, v_1), (u_2, v_2)\} & \text{if } t = t_1, \\ (u\cos\omega(T-t), u\sin\omega(T-t)) & \text{if } t \in (t_1, T] \end{cases}$$

where

$$\begin{aligned} t_1 &= \frac{1}{\omega}\arccos\frac{\hat{u}}{u}, & (u_1, v_1) &= (u\cos\omega t_1, -u\sin\omega t_1) = (\hat{u}, \sqrt{u^2 - \hat{u}^2}), \\ T &= 2t_1, & (u_2, v_2) &= (u_1, -v_1) = (\hat{u}, -\sqrt{u^2 - \hat{u}^2}). \end{aligned}$$

The fundamental matrices defined by (1.17) *and impact saltation matrix of* (1.16) *have the form*

$$X_1(t) = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}, \qquad X_2(t) = X_1(t - t_1), \qquad S = \begin{pmatrix} -1 & 0 \\ -\frac{2u_1}{v_1} & -1 \end{pmatrix},$$

respectively.

Proof. Due to linearity of $(1.29)_0$, taken $(u, 0) \in \Sigma$ as an initial point of $\gamma^u(t)$ one can easily compute $\gamma_1^u(t)$ (see (1.5)). Time of impact t_1 is the first intersection point of $\{\gamma_1^u(t) \mid t > 0\}$ with Ω_0 and is obtained from identity $h(\gamma_1^u(t_1)) = 0$. Accordingly, we get $(u_1, v_1) = \gamma_1^u(t_1)$ and $(u_2, v_2) = f_2(u_1, v_1)$ where $f_2(u, v) = (u, -v)$ is the right-hand side of $(1.30)_0$. Analogically to $\gamma_1^u(t)$ we get

$$\gamma_2^u(t) = (u_2\cos\omega(t - t_1) + v_2\sin\omega(t - t_1), -u_2\sin\omega(t - t_1) + v_2\cos\omega(t - t_1)).$$

From periodicity of $\gamma^u(t)$ we get T as a solution of equation

$$\gamma_2^u(T) = (u, 0)$$

or equivalently of a couple of equations

$$\begin{aligned} u_2\cos\omega(T - t_1) + v_2\sin\omega(T - t_1) &= u \\ -u_2\sin\omega(T - t_1) + v_2\cos\omega(T - t_1) &= 0. \end{aligned}$$

We have

$$T = t_1 + \frac{1}{\omega}\operatorname{arccot}\frac{u_2}{v_2} = 2t_1.$$

Therefore using trigonometric sum identities

$$\gamma_2^u(t) = (u\cos\omega(T - t), u\sin\omega(T - t)).$$

Matrices $X_1(t)$, $X_2(t)$ and S are obtained directly from their definitions since $(1.29)_0$ is linear. ∎

Now we verify the basic assumptions.

Lemma 1.6. *System* $(1.29)_0$, $(1.30)_0$ *satisfies conditions* H1), H2) *and* H3).

Proof. From construction of $\gamma^u(t)$, H1) is immediately verified. So is H2) since

$$\mathrm{D}h(u_1,v_1)f_1(u_1,v_1)=\omega v_1>0, \qquad \mathrm{D}h(u_2,v_2)f_1(u_2,v_2)=\omega v_2<0.$$

From Lemma 1.5 we get

$$\mathrm{D}_\xi F(u,0,0,\mu,\alpha)=\mathbb{I}-X_2(T)SX_1(t_1)=\begin{pmatrix}0 & 0\\ \frac{2u_1}{v_1} & 0\end{pmatrix}.$$

Hence, it is easy to see, that $f_1(u,0)=(0,-\omega u)\in Z$ for Z of (1.18). Suppose that $\dim Z>1$. Then there exists $w\in Z$ such that $\langle w, f_1(u,0)\rangle=0$, i.e. $w=(\zeta,0)$. Next

$$\mathrm{D}_\xi F(u,0,0,\mu,\alpha)w=\begin{pmatrix}0\\ \frac{2\zeta u_1}{v_1}\end{pmatrix}.$$

Thus $\zeta=0$ and the verification of the last condition is completed. $\square$

According to (1.15), by multiplying the corresponding matrices from Lemma 1.5 we derive

$$A(t)=\begin{cases}\begin{pmatrix}\cos\omega t & -\sin\omega t\\ -\frac{2u_1}{v_1}\cos\omega t+\sin\omega t & \frac{2u_1}{v_1}\sin\omega t+\cos\omega t\end{pmatrix} & \text{if } t\in[0,t_1),\\ \begin{pmatrix}\cos\left(\omega t-2\arccos\frac{\hat u}{u}\right) & -\sin\left(\omega t-2\arccos\frac{\hat u}{u}\right)\\ \sin\left(\omega t-2\arccos\frac{\hat u}{u}\right) & \cos\left(\omega t-2\arccos\frac{\hat u}{u}\right)\end{pmatrix} & \text{if } t\in[t_1,T].\end{cases}$$

Fredholm alternative yields

$$\mathcal{R}(\mathbb{I}-A(0))^\perp=\mathcal{N}(\mathbb{I}-A^*(0))=\mathcal{N}\begin{pmatrix}0 & \frac{2u_1}{v_1}\\ 0 & 0\end{pmatrix}=[(1,0)]$$

thus we take $\psi=(1,0)^*$.

Let us consider $z(t,\varepsilon,\mu)=\sin\mu t$. It is sufficient to assume $\mu>0$ (the case $\mu<0$ is covered by parameter $\alpha\in\mathbb{R}$). Then Poincaré-Andronov-Melnikov function defined by (1.23) has the form

$$\begin{aligned}M^\mu(\alpha)=\frac{1}{2\sqrt{l^2-\delta^2}}\int_0^{t_1}&(\omega-\mu)\cos(\omega t-\mu(t+\alpha))\\ &-(\omega+\mu)\cos(\omega t+\mu(t+\alpha))dt\\ +\frac{1}{2\sqrt{l^2-\delta^2}}\int_{t_1}^{T}&(\omega-\mu)\cos\left(\omega t-2\arccos\frac{\hat u}{u}-\mu(t+\alpha)\right)\\ -(\omega+\mu)\cos&\left(\omega t-2\arccos\frac{\hat u}{u}+\mu(t+\alpha)\right)dt-\frac{\sqrt{l^2-\delta^2}}{\omega l}\frac{\mu v_1}{u}\cos\mu(t_1+\alpha).\end{aligned}$$

After some algebra we get

$$M^\mu(\alpha)=\frac{\nu(u)\cos\mu(t_1+\alpha)}{\sqrt{l^2-\delta^2}}$$

where

$$\nu(u)=\frac{\mu(l^2-\delta^2)}{\omega l}\sqrt{1-\left(\frac{\hat u}{u}\right)^2}-2\sin\mu t_1.$$

Function $M^\mu(\alpha)$ can be easily differentiated with respect to α and one can apply Theorem 1.4.

Proposition 1.7. *Let $0 < \omega$, $0 < \mu$ and $k \in \mathbb{N}$ be such that $k\omega < \mu < 2k\omega$. Then for each $r \in \{0, 1, \cdots, 2k-1\}$, there exists a unique $2k\pi/\mu$-periodic solution $x_{k,r,\varepsilon}(t)$ of system $(1.29)_\varepsilon$, $(1.30)_\varepsilon$ with $\varepsilon \neq 0$ sufficiently small and*

$$\alpha = \alpha_{k,r}(\varepsilon) = \frac{\pi(2r+1)}{2\mu} + O(\varepsilon)$$

such that

$$|x_{k,r,\varepsilon}(t) - \gamma^u(t-\alpha)| = O(\varepsilon)$$

for any $t \in \mathbb{R}$ and $u = u(k) = \frac{\hat{u}}{\cos\frac{k\omega\pi}{\mu}}$. So there are at least $2\sum_{k\in(\frac{\mu}{2\omega},\frac{\mu}{\omega})\cap\mathbb{N}} k$ different impact periodic solutions.

Proof. Since the forcing $\sin\mu t$ has periods $2k\pi/\mu$, $k \in \mathbb{Z}$, we need the period matching condition $T = 2k\pi/\mu$ for some $k \in \mathbb{N}$. This gives

$$2k\pi/\mu = T = 2t_1 = 2\frac{1}{\omega}\arccos\frac{\hat{u}}{u}.$$

Since $\arccos\frac{\hat{u}}{u} \in (\pi/2, \pi)$ we get the assumption $k\omega < \mu < 2k\omega$. Then

$$u = u(k) = \frac{\hat{u}}{\cos\frac{k\omega\pi}{\mu}} < -\hat{u}.$$

Hence

$$\nu(u(k)) = \frac{\mu(l^2-\delta^2)}{\omega l}\sin\frac{k\omega\pi}{\mu} > 0.$$

So clearly, $M^\mu(\alpha_0) = 0$ if and only if

$$\alpha_0 = \frac{\pi(2s+1)}{2\mu} - t_1 = \frac{\pi(2(s-k)+1)}{2\mu}$$

for $s \in \mathbb{Z}$. In the period interval $[0, T] = [0, 2k\pi/\mu]$, we have $2k$ different

$$\alpha_0 \in \left\{\frac{(2r+1)\pi}{2\mu}, \quad r \in \{0, 1, \cdots, 2k-1\}\right\}.$$

Obviously, each α_0 is a simple root of $M^\mu(\alpha)$. The rest follows from Theorem 1.4. ∎

Conclusion

The aim of this thesis was to establish sufficient conditions for the bifurcation of a single periodic solution under perturbation in discontinuous systems in general n-dimensional space. This was done by the construction of a discontinuous Poincaré mapping and the corresponding distance function which roots imply the existence of a periodic solution close to the original one for perturbed system. Then the roots were found by Lyapunov-Schmidt reduction method and the results were stated in the terms of Poincaré-Andronov-Melnikov function. So the aim was successfully achieved in Chapter I, more specifically in Section 1 for nonautonomous perturbation and in Sections 2 and 3 for autonomous perturbation assuming that the unperturbed autonomous system possessed a nondegenerate family of periodic solutions or isolated periodic solution, respectively. In the case of autonomous perturbation we also stated conditions for the hyperbolicity, stability and instability of the persisting solution. These results can be analogically shown for the other problems investigated in the work. Next, we have studied the persistence of a forced sliding periodic solution in perturbed piecewise-smooth nonlinear dynamical systems and impact periodic solution in impact systems, in Section 4 and Section 1 of Chapter II, respectively. Moreover, after each theoretical part we provided one or more applications of the preceding results on planar and more-dimensional examples. One can see that in concrete examples the computations were nontrivial and sometimes the only possibility to finish them was with the aid of computer. Of course, the solutions of nonlinear problems in illustrations were obtained numerically.

Throughout the work, one of our basic assumption was the transversality condition. It means that the original solution hits and leaves the discontinuity boundary transversally, except the sliding solution which always leaves the boundary tangentially. Subsequently, this led to construction of discontinuous Poincaré mappings. In further research, this assumption can be make weaker or even omitted, e.g. for the study of grazing bifurcations where the solution of the unperturbed problem “touches” the boundary and after perturbation it can transversally cross the boundary, again hit it tangentially or stay in its original part of the phase space.

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