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9.1.9. aplikovaná matematika

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Introduction

Discontinuous systems are used for modelling systems with instantaneous change of external forces or parameters of the system. They describe electrical circuits with switches, mechanical devices in which components impact with each other, problems with friction, sliding or squealing, models in the social and financial sciences, etc. Recently, there appeared many books [6, 9, 11, 26, 37, 41] and papers [3, 18, 34–36, 43, 44, 50] on developing the theory for discontinuous systems which is analogical to the classical smooth-systems theory such as Poincaré mapping, Melnikov method, continuation of periodic orbits, bifurcations.

In this thesis, we extend the classical theory on persistence of periodic orbits under small perturbations to perturbed piecewise-smooth nonlinear dynamical systems (PPSNDS) and hybrid systems, specifically hard impact oscillators. In our work, we use the method of Poincaré mapping (discontinuous, sliding, impact) to find the periodic solution in perturbed systems. We define corresponding distance function which zeros imply the existence of periodic orbits and then apply the Lyapunov-Schmidt reduction to find the roots. The results are stated in the terms of Poincaré-Andronov-Melnikov function to emphasise the analogy with the smooth case. By this method, we are allowed to study general n-dimensional systems as well as the local asymptotic properties of the persisting solution such as hyperbolicity, stability and instability. On the other side, we have to handle some technical difficulties concerning the calculation of derivatives of the Poincaré mapping. For the simplicity, we always assume that the original periodic solution or the family of periodic solutions either transversally cross the discontinuity boundary or slide on it. Of course, the idea of continuation of such a solution is not a new one, since it was already studied in 2-dimensional space in [3, 4]. However, by this time the extension to higher dimensions was not investigated. Moreover, our results on the local asymptotic properties are also unique of their type.

The thesis consists of two chapters. In the first one, we study the bifurcation of periodic orbits in PPSNDS. Subsequently in four sections, we study the following problems: forced periodic solution from a single periodic solution in discontinuous system, bifurcation of a single periodic solution from a family of periodic solutions or an isolated periodic solution of autonomous equation, periodic sliding solution of periodically perturbed discontinuous system. We also show how the things can be simplified in the special case of the family of periodic orbits or if the discontinuity boundary is linear. In addition, we study the mentioned asymptotic properties. The second chapter has only one section in which we investigate the periodically forced impact systems and the persistence of a periodic solution in them.

By this time, all results newly discovered in the thesis were submitted, accepted or already published in international mathematical journals [21–25].

Main results

Here we briefly state some of our results from the thesis where they are rigorously proved. We always assume that $\varepsilon, \alpha \in \mathbb{R}$, $\mu \in \mathbb{R}^p$, $p \geq 1$ are parameters and $\Omega_\pm := \{x \in \Omega \mid \pm h(x) > 0\}$, $\Omega_0 := \{x \in \Omega \mid h(x) = 0\}$ where h is sufficiently smooth function with the regular value 0 and $\Omega \subset \mathbb{R}^n$ is an open set in $\mathbb{R}^n$.

1 Periodically forced discontinuous systems

We begin with a nonautonomous perturbation of an autonomous discontinuous equation, i.e. we consider equation

$$\dot{x} = f_\pm(x) + \varepsilon g(x, t + \alpha, \varepsilon, \mu), \quad x \in \overline{\Omega}_\pm, \tag{1.1$_\varepsilon$}$$

with function g T-periodic in t. Let us assume

H1) Unperturbed equation $(1.1)_0$ has a T-periodic solution $\gamma(t)$ with initial point x_0, intersecting Ω_0 at $x_1 = \gamma(t_1)$, $x_2 = \gamma(t_2)$ and $0 < t_1 < t_2 < T$.

H2) Trajectory $\gamma(t)$ crosses the boundary Ω_0 transversally at x_1 and x_2.

H3) For the dimension of the null space of the corresponding operator it holds

$$\dim \mathcal{N}(\mathbb{I} - X_3(T)S_2X_2(t_2)S_1X_1(t_1)) = 1.$$

Here we denoted $X_1(t)$, $X_2(t)$, $X_3(t)$ the matrix solutions satisfying

$$\begin{array}{lll} \dot{X}_1(t) = \mathrm{D}f_+(\gamma(t))X_1(t) & \dot{X}_2(t) = \mathrm{D}f_-(\gamma(t))X_2(t) & \dot{X}_3(t) = \mathrm{D}f_+(\gamma(t))X_3(t) \\ X_1(0) = \mathbb{I}, & X_2(t_1) = \mathbb{I}, & X_3(t_2) = \mathbb{I}, \end{array}$$

respectively, and S_1, S_2 the saltation matrices [35, 42] defined as

$$S_1 = \mathbb{I} + \frac{(f_-(x_1) - f_+(x_1))\mathrm{D}h(x_1)}{\mathrm{D}h(x_1)f_+(x_1)}, \qquad S_2 = \mathbb{I} + \frac{(f_+(x_2) - f_-(x_2))\mathrm{D}h(x_2)}{\mathrm{D}h(x_2)f_-(x_2)}.$$

Theorem 1.1. *Let*

$$A^*(t) = \begin{cases} X_1^{-1*}(t)X_1^*(t_1)S_1^*X_2^*(t_2)S_2^*X_3^*(T) & \text{if } t \in [0, t_1), \\ X_2^{-1*}(t)X_2^*(t_2)S_2^*X_3^*(T) & \text{if } t \in [t_1, t_2), \\ X_3^{-1*}(t)X_3^*(T) & \text{if } t \in [t_2, T] \end{cases}$$

and $\psi \in [\mathcal{R}(\mathbb{I} - A(0))]^\perp$ *be arbitrary and fixed. If* $\alpha_0 \in \mathbb{R}$, $\mu_0 \in \mathbb{R}^p$ *are such that*

$$\int_0^T \langle g(\gamma(t), t + \alpha_0, 0, \mu_0), A^*(t)\psi\rangle dt = 0,$$
$$\int_0^T \langle \mathrm{D}_t g(\gamma(t), t + \alpha_0, 0, \mu_0), A^*(t)\psi\rangle dt \neq 0$$

then there exists a neighbourhood U *of the point* $(0, \mu_0)$ *in* $\mathbb{R} \times \mathbb{R}^p$ *and a* C^{r-1}*-function* $\alpha(\varepsilon, \mu)$, *with* $\alpha(0, \mu_0) = \alpha_0$, *such that equation* $(1.1)_\varepsilon$ *with* $\alpha = \alpha(\varepsilon, \mu)$ *possesses a unique* T*-periodic piecewise* C^1*-smooth solution for each* $(\varepsilon, \mu) \in U$.

Nonlinear planar application

Consider the following piecewise nonlinear problem

$$\begin{aligned} &\left.\begin{aligned} \dot{x} &= (y-1) + \varepsilon\mu_1 \sin \omega t \\ \dot{y} &= -x \end{aligned}\right\} \quad \text{if } y > 0, \\ &\left.\begin{aligned} \dot{x} &= 2x + 5(y+1) + \left[x^2 + (y+1)^2\right][-x - (y+1)] + \varepsilon\mu_2(x+y) \\ \dot{y} &= -5x + 2(y+1) + \left[x^2 + (y+1)^2\right][x - (y+1)] \end{aligned}\right\} \quad \text{if } y < 0. \end{aligned} \tag{1.2$_\varepsilon$}$$

Proposition 1.2. *In system* $(1.2)_\varepsilon$, *if* μ_1, μ_2 *and* ω *satisfy*

$$|\mu_2| < \frac{1}{3}\frac{\mathrm{e}^{-2\pi}}{|\omega^2 - 1|}\frac{\sqrt{(\omega A + B)^2 + (\omega C + D)^2}}{E}|\mu_1|$$

where

$$A = 4\sqrt{2}\sin\left(\frac{3}{4}\pi\omega\right) + \left(3\mathrm{e}^{2\pi}\sqrt{2} + \sqrt{2}\right)\sin\left(\frac{5}{4}\pi\omega\right) + \left(3\mathrm{e}^{2\pi} - 3\right)\sin\left(2\pi\omega\right),$$

$$B=-5-3\mathrm{e}^{2\pi}-\left(\sqrt{2}+3\sqrt{2}\mathrm{e}^{2\pi}\right)\cos\left(\frac{3}{4}\pi\omega\right)+4\sqrt{2}\cos\left(\frac{5}{4}\pi\omega\right)+\left(5+3\mathrm{e}^{2\pi}\right)\cos\left(2\pi\omega\right),$$
$$C=3\mathrm{e}^{2\pi}-3-4\sqrt{2}\cos\left(\frac{3}{4}\pi\omega\right)-\left(\sqrt{2}+3\sqrt{2}\mathrm{e}^{2\pi}\right)\cos\left(\frac{5}{4}\pi\omega\right)+\left(3-3\mathrm{e}^{2\pi}\right)\cos\left(2\pi\omega\right),$$
$$D=-\left(\sqrt{2}+3\sqrt{2}\mathrm{e}^{2\pi}\right)\sin\left(\frac{3}{4}\pi\omega\right)+4\sqrt{2}\sin\left(\frac{5}{4}\pi\omega\right)+\left(5+3\mathrm{e}^{2\pi}\right)\sin\left(2\pi\omega\right),$$
$$E=\frac{\sqrt{2}}{975}\left(739-223\mathrm{e}^{-2\pi}\right),$$

then 2π-periodic orbit persists for $\varepsilon \neq 0$ small.

Piecewise linear planar application

Consider the system

$$\begin{aligned}
&\dot{x}=1+\varepsilon\mu_1\sin\omega t \\
&\dot{y}=-2x+\varepsilon\mu_2\cos\omega t
\end{aligned}\quad \text{if } y>0,$$
$$\begin{aligned}
&\dot{x}=-1+\varepsilon\mu_1\sin\omega t \\
&\dot{y}=-2x+\varepsilon\mu_2\cos\omega t
\end{aligned}\quad \text{if } y<0. \tag{1.3$_\varepsilon$}$$

Proposition 1.3. *In $(1.3)_\varepsilon$, if $\omega > 0$ is such that $\omega \neq k\pi$ for all $k \in \mathbb{N}$ and $\omega \neq -\frac{2\mu_1}{\mu_2}$ with $\mu_2 \neq 0$ then 4-periodic orbit*

$$\gamma(t)=\begin{cases}(t,1-t^2) & \text{if } t\in[0,1],\\ (2-t,(2-t)^2-1) & \text{if } t\in[1,3],\\ (t-4,1-(t-4)^2) & \text{if } t\in[3,4]\end{cases}$$

persists under perturbations for $\varepsilon \neq 0$ small.

2 Bifurcation from family of periodic orbits in autonomous systems

Now we state the sufficient condition for the bifurcation of a single periodic solution of an autonomous perturbed discontinuous equation

$$\dot{x}=f_\pm(x)+\varepsilon g(x,\varepsilon,\mu),\quad x\in\overline{\Omega}_\pm, \tag{2.1$_\varepsilon$}$$

from a nondegenerate family of periodic orbits of unperturbed equation $(2.1)_0$. Let us assume

H1) Equation $(2.1)_0$ has a smooth family of T^β-periodic solutions $\{\gamma(\beta,t)\}$ parametrized by $\beta \in V \subset \mathbb{R}^k$, $0 < k < n$, with initial points $x_0(\beta)$. Furthermore, vectors

$$\frac{\partial x_0(\beta)}{\partial\beta_1},\ldots,\frac{\partial x_0(\beta)}{\partial\beta_k},f_+(x_0(\beta))$$

are linearly independent whenever $\beta \in V$.

H2) $\gamma(\beta,t)$ intersects Ω_0 transversally at $x_1(\beta)=\gamma(\beta,t_1^\beta)$, $x_2(\beta)=\gamma(\beta,t_2^\beta)$ and $0<t_1^\beta<t_2^\beta<T^\beta$ for all $\beta \in V$.

H3) The set

$$\left\{\frac{\partial x_0(\beta)}{\partial\beta_1},\ldots,\frac{\partial x_0(\beta)}{\partial\beta_k},f_+(x_0(\beta))\right\}$$

spans the null space of the operator $(\mathbb{I}-S_\beta)(\mathbb{I}-A(\beta,0))$ where

$$S_\beta u=\frac{f_+(x_0(\beta))(f_+(x_0(\beta)))^*}{\|f_+(x_0(\beta))\|^2}u$$

is the orthogonal projection onto the linear space $[f_+(x_0(\beta))]$ and

$$A(\beta,t)=\begin{cases} X_3(\beta,T^\beta)S_2(\beta)X_2(\beta,t_2^\beta)S_1(\beta)X_1(\beta,t_1^\beta)X_1^{-1}(\beta,t) & \text{if } t\in[0,t_1^\beta),\\ X_3(\beta,T^\beta)S_2(\beta)X_2(\beta,t_2^\beta)X_2^{-1}(\beta,t) & \text{if } t\in[t_1^\beta,t_2^\beta),\\ X_3(\beta,T^\beta)X_3^{-1}(\beta,t) & \text{if } t\in[t_2^\beta,T^\beta]\end{cases}$$

with matrix solutions $X_1(\beta,t)$, $X_2(\beta,t)$, $X_3(\beta,t)$ satisfying

$$\begin{aligned}\dot X_1(\beta,t)&=\mathrm{D}f_+(\gamma(\beta,t))X_1(\beta,t)\\ X_1(\beta,0)&=\mathbb{I},\end{aligned}\qquad \begin{aligned}\dot X_2(\beta,t)&=\mathrm{D}f_-(\gamma(\beta,t))X_2(\beta,t)\\ X_2(\beta,t_1^\beta)&=\mathbb{I},\end{aligned}$$

$$\begin{aligned}\dot X_3(\beta,t)&=\mathrm{D}f_+(\gamma(\beta,t))X_3(\beta,t)\\ X_3(\beta,t_2^\beta)&=\mathbb{I},\end{aligned}$$

respectively, and saltation matrices

$$\begin{aligned}S_1^{-1}(\beta)&=\mathbb{I}+\frac{(f_+(x_1(\beta))-f_-(x_1(\beta)))\mathrm{D}h(x_1(\beta))}{\mathrm{D}h(x_1(\beta))f_-(x_1(\beta))},\\ S_2^{-1}(\beta)&=\mathbb{I}+\frac{(f_-(x_2(\beta))-f_+(x_2(\beta)))\mathrm{D}h(x_2(\beta))}{\mathrm{D}h(x_2(\beta))f_+(x_2(\beta))}.\end{aligned}$$

Let $\{\psi_1(\beta),\dots,\psi_k(\beta)\}$ be an orthogonal basis of $[\mathcal{R}((\mathbb{I}-S_\beta)(\mathbb{I}-A(\beta,0)))]^\perp$. Then we have a result on the sufficient condition for the persistence of a single solution from a bunch of periodic solutions.

Theorem 2.1. *If $\beta_0\in V$ is a simple root of M^{μ_0} for M^μ given by*

$$M^\mu(\beta)=(M_1^\mu(\beta),\dots,M_k^\mu(\beta)),$$
$$M_i^\mu(\beta)=\int_0^{T^\beta}\langle g(\gamma(\beta,t),0,\mu),A^*(\beta,t)\psi_i(\beta)\rangle\,dt, i=1,\dots,k,$$

i.e. $M^{\mu_0}(\beta_0)=0$, $\det \mathrm{D}M^{\mu_0}(\beta_0)\neq 0$, then there exist a neighbourhood U of the point $(0,\mu_0)$ in $\mathbb{R}\times\mathbb{R}^p$ and a C^{r-2}-function $\beta(\varepsilon,\mu)$, with $\beta(0,\mu_0)=\beta_0$, such that perturbed equation $(2.1)_\varepsilon$ with $(\varepsilon,\mu)\in U$ possesses a unique closed trajectory bifurcating from $\gamma(\beta_0,t)$.

3-dimensional piecewise-linear application

Consider the following problem

$$\begin{aligned}\dot x&=\varepsilon(z-x^n)\\ \dot y&=b_1\\ \dot z&=-2a_1b_1y+\varepsilon(\mu_1-\mu_2y^2)z\end{aligned}\quad \text{if } z>0,\qquad \begin{aligned}\dot x&=0\\ \dot y&=-b_2\\ \dot z&=-2a_2b_2y\end{aligned}\quad \text{if } z<0 \qquad (2.2)_\varepsilon$$

with positive constants a_1,a_2,b_1,b_2; $n\in\mathbb{N}$ and vector $\mu=(\mu_1,\mu_2)$ of real parameters. Let $x_0(\beta)=(\beta_1,0,\beta_2)$, $\beta=(\beta_1,\beta_2)$, $\beta_2>0$ be an initial point of $\gamma(\beta,t)$.

Proposition 2.2. *For $\mu\in\mathbb{R}^2$ such that $\mu_1\mu_2\le 0$, $(\mu_1,\mu_2)\neq 0$ no periodic orbit persists. For $\mu_1\mu_2>0$ if $\varepsilon>0$ and*

1. *n is odd, the only persisting periodic trajectory $\gamma(\beta_0,t)$ of system $(2.2)_0$ is determined by $\beta_0=(\beta_{01},\beta_{02})$ with $\beta_{01}=\left(\frac{2}{3}\beta_{02}\right)^{1/n}$, $\beta_{02}=\frac{5\mu_1a_1}{\mu_2}$. Moreover, this trajectory is stable – it is a sink – for $\mu_1>0$ and unstable/hyperbolic for $\mu_1<0$,*

2. *n is even, there are exactly two persisting orbits γ_+, γ_- given by $\beta_{01} = \pm\left(\frac{2}{3}\beta_{02}\right)^{1/n}$, $\beta_{02} = \frac{5\mu_1 a_1}{\mu_2}$ with corresponding sign in β_{01}. Moreover, if*
   (a) *$\mu_1 > 0$, then γ_+ is stable – it is a sink – and γ_- is unstable/hyperbolic,*
   (b) *$\mu_1 < 0$, then γ_+ is unstable/hyperbolic and γ_- is unstable – it is a source.*

If $\varepsilon < 0$, the above statements remain true with sinks instead of sources and vice versa.

3 Bifurcation from single periodic orbit in autonomous systems

For the case of degenerate family of periodic solutions of unperturbed equation $(2.1)_0$ we assume

H1) Equation $(2.1)_0$ has a unique periodic orbit $\gamma(t)$ of period T with initial point x_0.

H2) Trajectory $\gamma(t)$ crosses the boundary Ω_0 transversally at $x_1 = \gamma(t_1)$, $x_2 = \gamma(t_2)$ and $0 < t_1 < t_2 < T$.

Let $\gamma(\xi, t)$ denote the solution of $(2.1)_0$ with initial point ξ and $x_1(\xi) = \gamma(\xi, t_1(\xi))$, $x_2(\xi) = \gamma(\xi, t_2(\xi))$ be points close to x_1, x_2 where $\gamma(\xi, t)$ intersects Ω_0. Next, $x_3(\xi)$ is the image of ξ by the discontinuous Poincaré mapping and

$$S^\xi = \frac{f_+(x_3(\xi))f_+(x_0)^*}{\langle f_+(x_3(\xi)), f_+(x_0)\rangle}$$

is the projection onto $[f_+(x_3(\xi))]$ in the direction orthogonal to $f_+(x_0)$. We denote

$$A(\xi, t) = \begin{cases} X_3^\xi(t_3(\xi))S_2^\xi X_2^\xi(t_2(\xi))S_1^\xi X_1^\xi(t_1(\xi))X_1^\xi(t)^{-1} & \text{if } t \in [0, t_1(\xi)), \\ X_3^\xi(t_3(\xi))S_2^\xi X_2^\xi(t_2(\xi))X_2^\xi(t)^{-1} & \text{if } t \in [t_1(\xi), t_2(\xi)), \\ X_3^\xi(t_3(\xi))X_3^\xi(t)^{-1} & \text{if } t \in [t_2(\xi), t_3(\xi)] \end{cases}$$

where $X_1^\xi(t)$, $X_2^\xi(t)$, $X_3^\xi(t)$ are matrix solutions of

$$\begin{aligned} \dot X_1^\xi(t) &= \mathrm{D}f_+(\gamma(\xi,t))X_1^\xi(t) & \dot X_2^\xi(t) &= \mathrm{D}f_-(\gamma(\xi,t))X_2^\xi(t) \\ X_1^\xi(0) &= \mathbb{I}, & X_2^\xi(t_1(\xi)) &= \mathbb{I}, \end{aligned}$$

$$\begin{aligned} \dot X_3^\xi(t) &= \mathrm{D}f_+(\gamma(\xi,t))X_3^\xi(t) \\ X_3^\xi(t_2(\xi)) &= \mathbb{I}, \end{aligned}$$

respectively, and

$$S_1^\xi = \mathbb{I} + \frac{(f_-(x_1(\xi)) - f_+(x_1(\xi)))\mathrm{D}h(x_1(\xi))}{\mathrm{D}h(x_1(\xi))f_+(x_1(\xi))},$$

$$S_2^\xi = \mathbb{I} + \frac{(f_+(x_2(\xi)) - f_-(x_2(\xi)))\mathrm{D}h(x_2(\xi))}{\mathrm{D}h(x_2(\xi))f_-(x_2(\xi))}$$

are saltation matrices taken at general initial point ξ. By all this notation we can write the following result.

Theorem 3.1. *Let $\{\psi_1, \ldots, \psi_k\}$ be an orthogonal basis of $[\mathcal{R}(\mathbb{I} - (\mathbb{I} - S^{x_0})A(x_0, 0))]^\perp$. If ξ_1^0 is a simple root of function $M_\pm^{\mu_0}(\xi_1)$ where $M_\pm^\mu(\xi_1) = (M_{1\pm}^\mu(\xi_1), \ldots, M_{k\pm}^\mu(\xi_1))$ and*

$$\begin{aligned} M_{i\pm}^\mu(\xi_1) = \pm &\int_0^T \langle g(\gamma(s), 0, \mu), A^*(x_0, s)\psi_i\rangle ds \\ &+ \frac{1}{2}\left\langle A^{-1}(x_0, 0)\mathrm{D}_\xi((\mathbb{I} - S^\xi)A(\xi, 0)\xi_1)_{\xi = x_0}\xi_1, A^*(x_0, 0)\psi_i\right\rangle \end{aligned}$$

for $i=1,\dots,k$ with "+" or "−" sign, i.e., $M_+^{\mu_0}(\xi_1^0)=0$, $\det \mathrm{D}_{\xi_1} M_+^{\mu_0}(\xi_1^0)\neq 0$ or $M_-^{\mu_0}(\xi_1^0)=0$, $\det \mathrm{D}_{\xi_1} M_-^{\mu_0}(\xi_1^0)\neq 0$, then there exists a unique (for each sign) C^r-function $\xi_1(\epsilon,\mu)$ with $\epsilon\sim 0$ small and $\mu\sim\mu_0$ such that there is a periodic solution of equation $(2.1)_\varepsilon$ with $\varepsilon=\pm\epsilon^2\neq 0$ sufficiently small and μ close to μ_0.

Function $\xi_1(\epsilon,\mu)$ affects the initial point of the persisting solution which is exactly expressed in the thesis.

Planar application

Consider the following system

$$\begin{aligned}
&\begin{aligned}\dot x &= y+1+\varepsilon x(2-\mu_1 x^2-\mu_2 y^2)\\ \dot y &= -x+\varepsilon(x+y(x-y^2))\end{aligned} && \text{if } y>0,\\
&\begin{aligned}\dot x &= x+y-1+(x^2+(y-1)^2)(-x-(y-1))\\ &\quad +(x^2+(y-1)^2)^2(x/4+(y-1)/2)\\ \dot y &= -x+y-1+(x^2+(y-1)^2)(x-(y-1))\\ &\quad +(x^2+(y-1)^2)^2(-x/2+(y-1)/4)\end{aligned} && \text{if } y<0
\end{aligned} \tag{3.1$_\varepsilon$}$$

with parameters $\mu_1,\mu_2\in\mathbb{R}$.

Proposition 3.2. *Let $\mu^0=(\mu_1^0,\mu_2^0)$ be such that $G(\mu^0)\neq 0$ for G given by*

$$G(\mu)=\frac{68-135\pi+30\pi^2}{24}+\frac{8-19\pi+6\pi^2}{8}\mu_1+\frac{28-65\pi+18\pi^2}{24}\mu_2.$$

Then equation $(3.1)_\varepsilon$ has exactly two (zero) periodic solutions orbitally close to

$$\gamma(t)=\begin{cases}(-\cos t+\sin t,-1+\sin t+\cos t) & \text{if } t\in[0,\pi/2],\\ (\cos(t-\pi/2)-\sin(t-\pi/2),1-\sin(t-\pi/2)-\cos(t-\pi/2)) & \text{if } t\in[\pi/2,\pi]\end{cases}$$

for $\varepsilon\neq 0$ sufficiently small with $G(\mu^0)\varepsilon>0$ ($G(\mu^0)\varepsilon<0$) and μ close to μ^0.

4 Sliding solution of periodically perturbed systems

Here we consider T-periodically forced autonomous equation

$$\dot x = f_\pm(x)+\varepsilon g_\pm(x,t+\alpha,\varepsilon,\mu),\quad x\in\overline{\Omega}_\pm, \tag{4.1$_\varepsilon$}$$

and seek the persisting sliding T-periodic solution, i.e. such a solution that remains on the boundary for some nonzero time. That means a sliding solution satisfies equation (cf. [27])

$$\dot x = F_0(x,t+\alpha,\varepsilon,\mu)=f_0(x)+\varepsilon g_0(x,t+\alpha,\varepsilon,\mu) \tag{4.2$_\varepsilon$}$$

on Ω_0, where

$$\begin{gathered}
F_0(x,t,\varepsilon,\mu)=(1-\beta(x,t,\varepsilon,\mu))F_-(x,t,\varepsilon,\mu)+\beta(x,t,\varepsilon,\mu)F_+(x,t,\varepsilon,\mu),\\
F_\pm(x,t,\varepsilon,\mu)=f_\pm(x)+\varepsilon g_\pm(x,t,\varepsilon,\mu),\\
\beta(x,t,\varepsilon,\mu)=\frac{\mathrm{D}h(x)F_-(x,t,\varepsilon,\mu)}{\mathrm{D}h(x)(F_-(x,t,\varepsilon,\mu)-F_+(x,t,\varepsilon,\mu))}.
\end{gathered}$$

So we assume

H1) Equation $(4.1)_0$ has a T-periodic solution $\gamma(t)$ with initial point $x_0\in\Omega_+$.

H2) $\gamma(t)$ transversally hits Ω_0 at $x_1 = \gamma(t_1)$ and tangentially leaves at $x_2 = \gamma(t_2)$ back to Ω_+.

H3) It holds $\mathcal{N}(\mathbb{I} - A(0)) = [f_+(x_0)]$ where $A(t)$ is given by

$$A(t) = \begin{cases} X_3(T)X_2(t_2)SX_1(t_1)X_1(t)^{-1} & \text{if } t \in [0, t_1), \\ X_3(T)X_2(t_2)X_2(t)^{-1} & \text{if } t \in [t_1, t_2), \\ X_3(T)X_3(t)^{-1} & \text{if } t \in [t_2, T] \end{cases}$$

with saltation matrix

$$S = \mathbb{I} + \frac{(f_0(x_1) - f_+(x_1))\mathrm{D}h(x_1)}{\mathrm{D}h(x_1)f_+(x_1)}$$

and fundamental matrix solutions $X_1(t)$, $X_2(t)$, $X_3(t)$ satisfying, respectively,

$$\begin{array}{lll} \dot{X}_1(t) = \mathrm{D}f_+(\gamma(t))X_1(t) & \dot{X}_2(t) = \mathrm{D}f_0(\gamma(t))X_2(t) & \dot{X}_3(t) = \mathrm{D}f_+(\gamma(t))X_3(t) \\ X_1(0) = \mathbb{I}, & X_2(t_1) = \mathbb{I}, & X_3(t_2) = \mathbb{I}. \end{array}$$

Theorem 4.1. *Let $\psi \in [\mathcal{R}(\mathbb{I} - A(0))]^\perp$ be arbitrary and fixed and*

$$g(x, t, \mu) := \begin{cases} g_+(x, t, 0, \mu) & \text{if } x \in \Omega_+, \\ g_0(x, t, 0, \mu) & \text{if } x \in \Omega_0. \end{cases}$$

If $\alpha_0 \in \mathbb{R}$, $\mu_0 \in \mathbb{R}^p$ are such that

$$\int_0^T \langle g(\gamma(s), s + \alpha_0, \mu_0), A^*(s)\psi\rangle ds = 0,$$
$$\int_0^T \langle \mathrm{D}_t g(\gamma(s), s + \alpha_0, \mu_0), A^*(s)\psi\rangle ds \neq 0$$

then there exists a unique C^{r-2}-function $\alpha(\varepsilon, \mu)$ for $\varepsilon \sim 0$, $\mu \sim \mu_0$ such that $\alpha(0, \mu_0) = \alpha_0$ and there is a unique T-periodic solution $x(\varepsilon, \mu)(t)$ of equation $(4.1)_\varepsilon$ with parameters ε, μ and $\alpha = \alpha(\varepsilon, \mu)$, which solves equation $(4.2)_\varepsilon$ on Ω_0 and is orbitally close to $\gamma(t)$, i.e.

$$|x(\varepsilon, \mu)(t) - \gamma(t - \alpha(\varepsilon, \mu))| = O(\varepsilon)$$

for any $t \in \mathbb{R}$.

Piecewise linear application

Consider the following three dimensional piecewise linear problem

$$\begin{aligned} \dot{x} &= -x + \varepsilon \cos \mu_1(t + \alpha) \\ \dot{y} &= y/2 - (z - 1) + \varepsilon \sin \mu_2(t + \alpha) && \text{if } z > 0, \\ \dot{z} &= y + (z - 1)/2 \\ \\ \dot{x} &= -x + u \\ \dot{y} &= y/2 + 5 && \text{if } z < 0 \\ \dot{z} &= y + 10 \end{aligned} \tag{4.3$_\varepsilon$}$$

with δ sufficiently large, $u \in \mathbb{R}$ and parameters $\alpha \in \mathbb{R}$, $\mu_1, \mu_2 > 0$, $\varepsilon \sim 0$. T-periodic sliding solution $\gamma(t)$ of unperturbed equation $(4.3)_0$ hits and leaves the boundary at t_1 and t_2, respectively. These times have to be computed numerically, which can be found in the thesis.

Proposition 4.2. *Let $u \neq 0$, $\mu_1 = 2k_1\pi/T$, $\mu_2 = 2k_2\pi/T$ for given $k_1, k_2 \in \mathbb{N}$. Then for each $k \in R$ where*

$$R = \{r \in \mathbb{Z} \mid r\pi - \lambda \in [0, 2k_2\pi)\}$$

and λ is such that

$$\cos\lambda = \frac{K}{\sqrt{K^2+L^2}}, \qquad \sin\lambda = \frac{L}{\sqrt{K^2+L^2}},$$

$$K = \int_0^{t_1} (\cos s - 1.98957\sin s)\mathrm{e}^{-s/2}\cos\mu_2 s ds - \int_{t_1}^{t_2} 0.0178\cos\mu_2 s ds + \int_{t_2}^{T} 10^7(2.458\cos s - 28.186\sin s)\mathrm{e}^{-s/2}\cos\mu_2 s ds,$$

$$L = \int_0^{t_1} (\cos s - 1.98957\sin s)\mathrm{e}^{-s/2}\sin\mu_2 s ds - \int_{t_1}^{t_2} 0.0178\sin\mu_2 s ds + \int_{t_2}^{T} 10^7(2.458\cos s - 28.186\sin s)\mathrm{e}^{-s/2}\sin\mu_2 s ds,$$

there exists a unique T-periodic sliding solution $x_k(\varepsilon)(t)$ of system $(4.3)_\varepsilon$ with $\varepsilon \neq 0$ sufficiently small and

$$\alpha = \alpha_k(\varepsilon) = \frac{k\pi - \lambda}{\mu_2} + O(\varepsilon)$$

such that

$$|x_k(\varepsilon)(t) - \gamma(t-\alpha)| = O(\varepsilon)$$

for any $t \in \mathbb{R}$. So for each $u \neq 0$, $k_1, k_2 \in \mathbb{N}$ there are at least as many different T-periodic sliding solutions as the number of elements of R.

5 Periodically forced impact systems

Finally, we investigate a problem of continuation of T-periodic orbit in periodically forced impact system given by

$$\dot{x} = f_1(x) + \varepsilon g_1(x, t, \varepsilon, \mu), \quad x \in \Omega_1, \tag{5.1$_\varepsilon$}$$

$$x(t^+) = f_2(x(t^-)) + \varepsilon g_2(x(t^-), t, \varepsilon, \mu) \quad \text{if} \quad h(x(t^-)) = 0, \tag{5.2$_\varepsilon$}$$

where $\Omega_0 := \{x \in \Omega \mid h(x) = 0\}$ and $\Omega_1 := \Omega \setminus \Omega_0$. Let us assume

- H1) Unperturbed equation $(5.1)_0$ has a T-periodic orbit $\gamma(t)$ with initial point $x_0 \in \Omega_1$, which is discontinuous at $t = t_1 \in (0, T)$ where it satisfies impact condition $(5.2)_0$.
- H2) Solution $\gamma(t)$ hits and leaves boundary Ω_0 transversally at $x_1 = \gamma(t_1^-)$ and $x_2 = \gamma(t_1^+)$, respectively, where $\gamma(t_1^\pm) = \lim_{s \to t^\pm} \gamma(s)$.
- H3) It holds $\mathcal{N}(\mathbb{I} - A(0)) = [f_1(x_0)]$ for $A(t)$ given by

$$A(t) = \begin{cases} X_2(T)SX_1(t_1)X_1^{-1}(t) & \text{if } t \in [0, t_1), \\ X_2(T)X_2^{-1}(t) & \text{if } t \in [t_1, T] \end{cases}$$

with impact saltation matrix

$$S = \mathrm{D}f_2(x_1) + \frac{(f_1(x_2) - \mathrm{D}f_2(x_1)f_1(x_1))\mathrm{D}h(x_1)}{\mathrm{D}h(x_1)f_1(x_1)}$$

and fundamental matrix solutions $X_1(t)$, $X_2(t)$ satisfying, respectively,

$$\dot{X}_1(t) = \mathrm{D}f_1(\gamma(t))X_1(t) \qquad \dot{X}_2(t) = \mathrm{D}f_1(\gamma(t))X_2(t)$$
$$X_1(0) = \mathbb{I}, \qquad X_2(t_1) = \mathbb{I}.$$

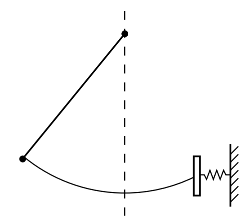
Theorem 5.1. *Let $\psi \in [\mathcal{R}(\mathbb{I} - A(0))]^\perp$ be arbitrary and fixed. If $\alpha_0 \in \mathbb{R}$, $\mu_0 \in \mathbb{R}^p$ are such that*

$$\int_0^T \langle g_1(\gamma(s), s+\alpha_0, 0, \mu_0), A^*(s)\psi\rangle ds + \langle X_2(T) g_2(x_1, t_1+\alpha_0, 0, \mu_0), \psi\rangle = 0,$$
$$\int_0^T \langle \mathrm{D}_t g_1(\gamma(s), s+\alpha_0, 0, \mu_0), A^*(s)\psi\rangle ds + \langle X_2(T) \mathrm{D}_t g_2(x_1, t_1+\alpha_0, 0, \mu_0), \psi\rangle \neq 0$$

then there exists a unique C^{r-1}-function $\alpha(\varepsilon, \mu)$ for $\varepsilon \sim 0$ small and $\mu \sim \mu_0$ such that $\alpha(0, \mu_0) = \alpha_0$ and there is a unique T-periodic solution $x_{\varepsilon,\mu}(t)$ of equation $(5.1)_\varepsilon$ with parameters $\varepsilon \neq 0$ sufficiently small, μ close to μ_0 and $\alpha = \alpha(\varepsilon, \mu)$, which satisfies condition $(5.2)_\varepsilon$ and

$$|x_{\varepsilon,\mu}(t) - \gamma(t - \alpha(\varepsilon, \mu))| = O(\varepsilon).$$

Pendulum hitting moving obstacle



Consider a mathematical pendulum which impacts an oscillating wall. The horizontal distance between the wall and the center of the pendulum is $\delta + \varepsilon \sin \mu t$ where δ is a positive constant. We denote x the angle and l the length of the massless cord.

Then x satisfies the dimensionless equation

$$\ddot{x} = -\omega^2 x$$

with a given frequency $\omega > 0$ and impact condition

$$\dot{x}(t^+) = -\dot{x}(t^-) + \varepsilon\mu \cos \mu t^- \frac{\sqrt{l^2 - (\delta + \varepsilon \sin \mu t^-)^2}}{l}$$

whenever

$$x(t^-) - \arcsin \frac{\delta + \varepsilon \sin \mu t^-}{l} = 0.$$

After transformation into the form of $(5.1)_\varepsilon$, $(5.2)_\varepsilon$ we obtain

$$\begin{aligned} \dot{u}(t) &= \omega v(t) - \varepsilon \frac{\mu \cos \mu(t+\alpha)}{\sqrt{l^2 - (\delta + \varepsilon \sin \mu(t+\alpha))^2}} \\ \dot{v}(t) &= -\omega u(t) - \varepsilon \frac{\omega \sin \mu(t+\alpha)}{\sqrt{l^2 - \delta^2}} + O(\varepsilon^2) \end{aligned} \qquad (5.3)_\varepsilon$$

with impact condition

$$\begin{aligned} u(t^+) &= u(t^-) \\ v(t^+) &= -v(t^-) + \varepsilon\mu \cos \mu(t^- + \alpha) \frac{\sqrt{l^2 - (\delta + \varepsilon \sin \mu(t^- + \alpha))^2}}{\omega l} \\ &\text{if} \quad h(u(t^-), v(t^-)) = 0 \end{aligned} \qquad (5.4)_\varepsilon$$

where

$$h(u, v) = u - \arcsin \frac{\delta}{l}.$$

Lemma 5.2. *System $(5.3)_0$, $(5.4)_0$ possesses a family of periodic orbits $\gamma^u(t)$ parametrized by $u < -\hat{u}$, $\hat{u} = \arcsin \frac{\delta}{l}$ such that*

$$\gamma^u(t) = \begin{cases} (u \cos \omega t, -u \sin \omega t) & \text{if } t \in [0, t_1), \\ \{(u_1, v_1), (u_2, v_2)\} & \text{if } t = t_1, \\ (u \cos \omega(T-t), u \sin \omega(T-t)) & \text{if } t \in (t_1, T] \end{cases}$$

where

$$t_1 = \frac{1}{\omega} \arccos \frac{\hat{u}}{u}, \qquad (u_1, v_1) = (u \cos \omega t_1, -u \sin \omega t_1) = (\hat{u}, \sqrt{u^2 - \hat{u}^2}),$$
$$T = 2t_1, \qquad (u_2, v_2) = (u_1, -v_1) = (\hat{u}, -\sqrt{u^2 - \hat{u}^2}).$$

Proposition 5.3. *Let $0 < \omega$, $0 < \mu$ and $k \in \mathbb{N}$ be such that $k\omega < \mu < 2k\omega$. Then for each $r \in \{0, 1, \cdots, 2k-1\}$, there exists a unique $2k\pi/\mu$-periodic solution $x_{k,r,\varepsilon}(t)$ of system $(5.3)_\varepsilon$, $(5.4)_\varepsilon$ with $\varepsilon \neq 0$ sufficiently small and*

$$\alpha = \alpha_{k,r}(\varepsilon) = \frac{\pi(2r+1)}{2\mu} + O(\varepsilon)$$

such that

$$|x_{k,r,\varepsilon}(t) - \gamma^u(t - \alpha)| = O(\varepsilon)$$

for any $t \in \mathbb{R}$ and $u = u(k) = \frac{\hat{u}}{\cos \frac{k\omega\pi}{\mu}}$. So there are at least $2 \sum_{k \in (\frac{\mu}{2\omega}, \frac{\mu}{\omega}) \cap \mathbb{N}} k$ different impact periodic solutions.

Conclusion

In the thesis, we have studied the bifurcation of a single periodic solution from an isolated periodic solution or a nondegenerate family of periodic orbits in discontinuous autonomous system under nonautonomous or autonomous perturbation. This was done by the use of a discontinuous Poincaré mapping and the construction of the corresponding distance function. Its roots correspond to periodic solutions in perturbed system and were found using Lyapunov-Schmidt reduction method. So we stated the sufficient conditions for the persistence of a periodic solution in terms of a Poincaré-Andronov-Melnikov function. Later, we proved analogical results for periodically forced sliding solution of a discontinuous system and periodically forced solution of an impact system with the aid of a sliding Poincaré mapping and an impact Poincaré mapping, respectively. In addition, we investigated the local asymptotic properties of the persisting solution such as hyperbolicity, stability and instability.

Due to no restrictions on the dimension of the spatial variable and parameters our results are original and bring new possibilities for further research. For example, on can weaker one of our basic assumptions – transversality condition H2) or non-degeneracy condition H3). We considered the second case for the bifurcation from a single periodic solution under autonomous perturbation (Section 3 of Chapter I). The first case yields so-called grazing bifurcation (see [9]).

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