$\sqrt{\cos(\pi(x^2+y^2))}$

Problém: odmocnina. Preto musí platiť, že $\cos(\pi(x^2+y^2)) \geq 0$. Funkcia cos je nezáporná na intervaloch tvaru $\langle -\frac{\pi}{2}+2k\pi, \frac{\pi}{2}+2k\pi\rangle, k \in Z$, preto $\pi(x^2+y^2) \in \langle -\frac{\pi}{2}+2k\pi, \frac{\pi}{2}+2k\pi\rangle, k \in Z$. Výraz $\pi(x^2+y^2) \geq 0$, preto má zmysel iba uvažovať intervaly pre $k \geq 1$ a $\langle 0, \frac{\pi}{2}\rangle$. Výsledkom sú kruh a medzikružia so stredom $(0,0)$. Kruh má polomer $\sqrt{\frac{1}{2}}$ a medzikružia s polomermi $\sqrt{\frac{4k-1}{2}}, \sqrt{\frac{4k+1}{2}}, k \geq 1$.

$$\boxed{D(f) = \{(x,y) \in R^2 : \tfrac{4k-1}{2} \leq x^2+y^2 \leq \tfrac{4k+1}{2}, k \geq 1\} \cup \{(x,y) : x^2+y^2 \leq \tfrac{1}{2}\}}$$

$\sqrt{x^2-y^2}$

Problém: odmocnina. Preto musí platiť: $x^2-y^2 \geq 0$, t.j. $x^2 \geq y^2$, resp. $|x| \geq |y|$. Oblasť ohraničená $y = |x|$ a $y = -|x|$, vrátane hraničných kriviek.

$$\boxed{D(f) = \{(x,y) \in R^2 : -|x| \leq y \leq |x|\}}$$

$\arcsin(x^2+y^2-2)$

Funkcia arcsin je definovaná len pre hodnoty z intervalu $\langle -1, 1\rangle$. Preto musí platiť: $-1 \leq x^2+y^2-2 \leq 1$. Po menšej úprave: $1 \leq x^2+y^2 \leq 3$. Oblasť medzikružia so stredom $(0,0)$ a polomermi 1 a 3 vrátane hraničných kružníc.

$$\boxed{D(f) = \{(x,y) \in R^2 : 1 \leq x^2+y^2 \leq 3\}}$$

$\frac{1}{\sqrt{1-xy}}$

Problémy: delenie 0 a odmocnina. Preto $\sqrt{1-xy} \neq 0$ a $1-xy \geq 0$. Z týchto podmienok dostaneme, že $1-xy > 0$, t.j. $xy < 1$. Oblasť medzi vetvami hyperboly $y = \frac{1}{x}$ vrátane asymptot $x = 0$ a $y = 0$.

$$\boxed{D(f) = \{(x,y) \in R^2 : xy < 1\}}$$

$\lim_{(x,y)\to(2,3)} 19x^2+6y-66$

Dosadenie 2 za x a 3 za y dostaneme výsledok. $19 \cdot 2^2 + 6 \cdot 3 - 66 = 28$

$$\boxed{28}$$

$\lim_{(x,y)\to(1,1)} \frac{x-y}{x^2-y^2}$

$$\lim_{(x,y)\to(1,1)} \frac{x-y}{x^2-y^2} = \lim_{(x,y)\to(1,1)} \frac{x-y}{(x-y)(x+y)} = \lim_{(x,y)\to(1,1)} \frac{1}{x+y} = \frac{1}{2}$$

$$\boxed{\frac{1}{2}}$$

$\lim_{(x,y)\to(2,-2)} \frac{x^2-y^2}{x^3+y^3}$

$$\lim_{(x,y)\to(2,-2)} \frac{x^2-y^2}{x^3+y^3} = \lim_{(x,y)\to(2,-2)} \frac{(x+y)(x-y)}{(x+y)(x^2-xy+y^2)} = \lim_{(x,y)\to(2,-2)} \frac{x-y}{x^2-xy+y^2} = \frac{4}{12} = \frac{1}{3}$$

$$\boxed{\frac{1}{3}}$$

$\lim_{(x,y)\to(0,0)} \frac{\sqrt{9-xy}-3}{xy}$

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{9-xy}-3}{xy} = \lim_{(x,y)\to(0,0)} \frac{\sqrt{9-xy}-3}{xy} \cdot \frac{\sqrt{9-xy}+3}{\sqrt{9-xy}+3} = \lim_{(x,y)\to(0,0)} \frac{-xy}{xy(\sqrt{9-xy}+3)} =$$
$$= -\lim_{(x,y)\to(0,0)} \frac{1}{\sqrt{9-xy}+3} = -\frac{1}{6}$$

$$\boxed{-\frac{1}{6}}$$

$\lim_{(x,y)\to(4,4)} \frac{y^2-xy}{\sqrt{y}-\sqrt{x}}$

$$\lim_{(x,y)\to(4,4)} \frac{y^2-xy}{\sqrt{y}-\sqrt{x}} = \lim_{(x,y)\to(4,4)} \frac{y(y-x)}{\sqrt{y}-\sqrt{x}} \cdot \frac{\sqrt{y}+\sqrt{x}}{\sqrt{y}+\sqrt{x}} = \lim_{(x,y)\to(4,4)} \frac{y(y-x)(\sqrt{y}+\sqrt{x})}{y-x} =$$
$$= \lim_{(x,y)\to(4,4)} y(\sqrt{y}+\sqrt{x}) = 16$$

$$\boxed{16}$$

$\lim_{(x,y)\to(1,1)} \frac{xy-x-2y+2}{1-y}$

$$\lim_{(x,y)\to(1,1)} \frac{xy-x-2y+2}{1-y} = \lim_{(x,y)\to(1,1)} \frac{(2-x)(1-y)}{1-y} = \lim_{(x,y)\to(1,1)} 2-x = 1$$

$\boxed{1}$

$\lim_{(x,y)\to(3,4)} \frac{y-x-1}{\sqrt{x+1}-\sqrt{y}}$

$\lim_{(x,y)\to(3,4)} \frac{y-x-1}{\sqrt{x+1}-\sqrt{y}} = \lim_{(x,y)\to(3,4)} \frac{y-x-1}{\sqrt{x+1}-\sqrt{y}} \cdot \frac{\sqrt{x+1}+\sqrt{y}}{\sqrt{x+1}+\sqrt{y}} = \lim_{(x,y)\to(3,4)} \frac{(y-x-1)(\sqrt{x+1}+\sqrt{y})}{x+1-y} =$

$= -\lim_{(x,y)\to(3,4)} \sqrt{x+1} + \sqrt{y} = -4$

$\boxed{-4}$

$\lim_{(x,y)\to(0,0)} (x^2+y^2)\cos\frac{1}{xy}$

Funkcia cos nadobúda hodnoty z intervalu $\langle -1, 1\rangle$, preto pre ľubovoľné x, y platí:

$-1 \cdot (x^2+y^2) \le (x^2+y^2)\cos\frac{1}{xy} \le 1 \cdot (x^2+y^2)$. Keďže lim zachováva monotónnosť, tak platí:

$\lim_{(x,y)\to(0,0)} -(x^2+y^2) \le \lim_{(x,y)\to(0,0)} (x^2+y^2)\cos\frac{1}{xy} \le \lim_{(x,y)\to(0,0)} x^2+y^2$. Prvá a tretia limita sa rovnajú a sú rovné 0, preto aj limita funkcie v "strede" je rová 0.

$\boxed{0}$

$\lim_{(x,y)\to(0,0)} \frac{\sin(x^4+y^4)}{x^4+y^4}$

Keď $(x,y) \to (0,0)$, tak výraz $x^4+y^4 \to 0$, preto možno transformovať limitu na limitu jednej funkcie. Označme $t := x^4+y^4$. Potom sa pôvodná limita rovná $\lim_{t\to 0} \frac{\sin t}{t} = 1$

$\boxed{1}$

$\lim_{()x,y)\to(0,0)} \frac{x+y}{x-y}$

Položme $x = ky$, potom limita je $\lim_{x\to 0} \frac{ky+y}{ky-y} = \frac{k+1}{k-1}$. Dosadením napr. $k=3$ a $k=4$, dostaneme 2 rôzne hodnoty, preto nemôže existovať limita v bode $(0,0)$.

$\boxed{\text{Neexistuje}}$

$\lim_{(x,y)\to(3,3)} \frac{x+y}{x-y}$

Nech $x = 3+t$ a $y = 3+s$. Zámenou premenných transformujeme pôvodnú limitu na $\lim_{(s,t)\to(0,0)} \frac{3+t+3+s}{3+t-(3+s)} = \lim_{(s,t)\to(0,0)} \frac{6+t+s}{t-s}$. Teraz stačí položiť $s = 2t$ - limita po takejto krivke je sprava je $\lim_{t\to 0^+} \frac{6+3t}{-t} = -\infty$ a zľava $\lim_{t\to 0^-} \frac{6+3t}{-t} = +\infty$. Preto celková limita nemôv ze existovať.

$\boxed{\text{Neexistuje}}$

$\lim_{(x,y)\to(0,0)} \frac{y^2}{y^2+x^4}$

Neexistuje: $y = kx^2$, potom prejde limita na $\lim_{x\to 0} \frac{k^2x^4}{k^2x^4+x^4} = \lim_{x\to 0} \frac{k^2x^4}{(1+k^2)x^4} = \frac{k^2}{1+k^2}$. Dosadením hodnôt za k napr. 1 a 2, dostaneme, že existujú postupnosti, ktoré sa blížia k $(0,0)$, a majú rôzne limity.

$\boxed{\text{Neexistuje}}$

$f(x,y) = (\sin^2 x - 3\cos^2 y)^{19}$

$\frac{\partial f}{\partial x} = 19(\sin^2 x - 3\cos^2 y)^{18}(2\sin x\cos x) = 19(\sin^2 x - 3\cos^2 y)^{18}\sin 2x$

$\frac{\partial f}{\partial y} = 19(\sin^2 x - 3\cos^2 y)^{18}(6\cos y\sin y) = 57(\sin^2 x - 3\cos^2 y)^{18}\sin 2x$

$$\boxed{\frac{\partial f}{\partial x} = 19(\sin^2 x - 3\cos^2 y)^{18}\sin 2x, \quad \frac{\partial f}{\partial y} = 57(\sin^2 x - 3\cos^2 y)^{18}\sin 2x}$$

$f(x,y) = \sqrt{x(3y^3-x^2)}$

$\frac{\partial f}{\partial x} = \frac{1}{2}\frac{1}{\sqrt{x(3y^3-x^2)}} \cdot (3y^3 - x^2 + x(-2x)) = \frac{3}{2}\frac{y^3-x^2}{\sqrt{x(3y^3-x^2)}}$

$\frac{\partial f}{\partial y} = \frac{1}{2}\frac{1}{\sqrt{x(3y^3-x^2)}} \cdot (9xy^2) = \frac{9}{2}\frac{\sqrt{x}y^2}{\sqrt{3y^3-x^2}}$

$$\boxed{\frac{\partial f}{\partial x} = \frac{3}{2}\frac{y^3-x^2}{\sqrt{x(3y^3-x^2)}}, \quad \frac{\partial f}{\partial y} = \frac{9}{2}\frac{\sqrt{x}y^2}{\sqrt{3y^3-x^2}}}$$

$f(x,y) = \operatorname{arctg}\frac{x-y}{1+xy}$

$$\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{x-y}{1+xy}\right)^2} \cdot \frac{1+xy-y(x-y)}{(1+xy)^2} = \frac{(1+xy)^2}{(1+xy)^2+(x-y)^2} \cdot \frac{1+y^2}{(1+xy)^2} = \frac{1+y^2}{1+2xy+x^2y^2+x^2+y^2-2xy} = \frac{1+y^2}{(1+x^2)(1+y^2)} = \frac{1}{1+x^2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{x-y}{1+xy}\right)^2} \cdot \frac{-(1+xy)-x(x-y)}{(1+xy)^2} = -\frac{(1+xy)^2}{(1+xy)^2+(x-y)^2} \cdot \frac{1+x^2}{(1+xy)^2} = -\frac{1+x^2}{(1+x^2)(1+y^2)} = -\frac{1}{1+y^2}$$

$$\boxed{\frac{\partial f}{\partial x} = \frac{1}{1+x^2}, \quad \frac{\partial f}{\partial y} = -\frac{1}{1+y^2}}$$

$f(x,y) = \arcsin\sqrt{\frac{x-y}{x+y}}$

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{1-(\sqrt{\frac{x-y}{x+y}})^2}} \cdot \frac{1}{2}\sqrt{\frac{x+y}{x-y}} \cdot \frac{x+y-(x-y)}{(x+y)^2} = \frac{1}{\sqrt{\frac{x+y-(x-y)}{x+y}}} \cdot \frac{1}{2} \cdot \sqrt{\frac{x+y}{x-y}} \cdot \frac{2y}{(x+y)^2} = \sqrt{\frac{x+y}{2y}} \cdot \sqrt{\frac{x+y}{x-y}} \cdot \frac{y}{(x+y)^2} =$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{x-y}} \cdot \frac{\sqrt{y}}{x+y}$$

$$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{1-(\sqrt{\frac{x-y}{x+y}})^2}} \cdot \frac{1}{2}\sqrt{\frac{x+y}{x-y}} \cdot \frac{-(x+y)-(x-y)}{(x+y)^2} = \sqrt{\frac{x+y}{2y}} \cdot \sqrt{\frac{x+y}{x-y}} \cdot \frac{-x}{(x+y)^2} = -\frac{1}{\sqrt{2y}} \cdot \frac{1}{\sqrt{x-y}} \cdot \frac{x}{x+y}$$

$$\boxed{\frac{\partial f}{\partial x} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{x-y}} \cdot \frac{\sqrt{y}}{x+y}, \quad \frac{\partial f}{\partial y} = -\frac{1}{\sqrt{2y}} \cdot \frac{1}{\sqrt{x-y}} \cdot \frac{x}{x+y}}$$

$f(x,y,z) = \frac{x}{y} + \frac{y}{z} - \frac{z}{x}$

$$\frac{\partial f}{\partial x} = \frac{1}{y} + \frac{z}{x^2}, \quad \frac{\partial f}{\partial y} = -\frac{x}{y^2} + \frac{1}{z}, \quad \frac{\partial f}{\partial z} = -\frac{y}{z^2} - \frac{1}{x}$$

$$\boxed{\frac{\partial f}{\partial x} = \frac{1}{y} + \frac{z}{x^2}, \quad \frac{\partial f}{\partial y} = -\frac{x}{y^2} + \frac{1}{z}, \quad \frac{\partial f}{\partial z} = -\frac{y}{z^2} - \frac{1}{x}}$$

$f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$

$$\frac{\partial f}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}} \cdot 2x = -\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

Zo symetrie: $\frac{\partial f}{\partial y} = -\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \quad \frac{\partial f}{\partial z} = -\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}$

$$\boxed{\frac{\partial f}{\partial x} = -\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \quad \frac{\partial f}{\partial y} = -\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \quad \frac{\partial f}{\partial z} = -\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}}$$

$f(x,y,z) = \sqrt{y\cos z + x\sin z}$

$$\frac{\partial f}{\partial x} = \frac{1}{2}\frac{\sin z}{\sqrt{y\cos z+x\sin z}}, \quad \frac{\partial f}{\partial y} = \frac{1}{2}\frac{\cos z}{\sqrt{y\cos z+x\sin z}}, \quad \frac{\partial f}{\partial z} = \frac{1}{2}\frac{x\cos z-y\sin z}{\sqrt{y\cos z+x\sin z}}$$

$$\boxed{\frac{\partial f}{\partial x} = \frac{1}{2}\frac{\sin z}{\sqrt{y\cos z+x\sin z}}, \quad \frac{\partial f}{\partial y} = \frac{1}{2}\frac{\cos z}{\sqrt{y\cos z+x\sin z}}, \quad \frac{\partial f}{\partial z} = \frac{1}{2}\frac{x\cos z-y\sin z}{\sqrt{y\cos z+x\sin z}}}$$