Rényi relative entropies and noncommutative L_p-spaces

Anna Jenčová
Mathematical Institute, Slovak Academy of Sciences

Bedlewo, July 2018

Classical Rényi relative α-entropies

For p, q probability measures over a finite set X, $0 < \alpha \neq 1$:

$$D_\alpha(p\|q) := \frac{1}{\alpha - 1} \log \sum_x p(x)^\alpha q(x)^{1-\alpha}$$

- unique family of divergences satisfying a set of postulates
- relative entropy as a limit $\alpha \to 1$
- fundamental quantities appearing in many information - theoretic tasks

A. Rényi, *Proc. Symp. on Math., Stat. and Probability*, 1961.

Quantum extensions of Rényi relative α-entropies

ρ, σ density matrices, $0 < \alpha \neq 1$

Standard:

$$D_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \left(\mathrm{Tr}\, \rho^\alpha \sigma^{1-\alpha} \right)$$

D. Petz, *Rep. Math. Phys., 1984*

Sandwiched:

$$\tilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \mathrm{Tr} \left[\left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}} \right)^\alpha \right]$$

M. Müller-Lennert et al., *J. Math. Phys.*, 2013
M. M. Wilde et al., *Commun. Math. Phys.*, 2014

What is a "good" divergence?

Divergence $\equiv$ a measure of statistical "dissimilarity" of two states

Properties

- strict positivity: $D(\rho\|\sigma) \geq 0$ and $D(\rho\|\sigma) = 0$ iff $\rho = \sigma$
- data processing inequality:

$$D(\rho\|\sigma) \geq D(\Phi(\rho)\|\Phi(\sigma))$$

  for any quantum channel Φ
- $+$ other

Operational significance

- relation to performance of some procedures in information - theoretic tasks

Quantum Rényi relative α-entropies

Standard version D_α:

Properties[1]

- similar to classical, but not for all values of α
- data processing inequality only for $\alpha \in (0, 2]$

Operational significance[2,3]

- known only for $\alpha \in (0, 1)$: error exponents in quantum hypothesis testing

[1] D. Petz, *Rep. Math. Phys., 1984*

[2] K. M. R. Audenaert et al., *Commun. Math. Phys.*, 2008

[3] F. Hiai, M. Mosonyi, and T. Ogawa, *J. Math. Phys.*, 2008

Quantum Rényi relative α-entropies

Sandwiched version $\tilde{D}_\alpha$:

Properties[4]

- again similar to classical, but not for all α
- data processing inequality with respect to quantum channels, but only for $1/2 \leq \alpha \neq 1$

Operational significance[5]

- known only for $\alpha > 1$: strong converse exponents in quantum hypothesis testing

[4] R. L. Frank and E. H. Lieb, *J. Math. Phys.*, 2013

[5] M. Mosonyi, and T. Ogawa, *Commun. Math. Phys.*, 2017

Quantum relative entropy as limit value

For both D_α and $\tilde{D}_\alpha$, the quantum (Umegaki) relative entropy appears as a limit for $\alpha \to 1$:

$$\begin{aligned} \lim_{\alpha\to 1} D_\alpha(\rho\|\sigma) &= \lim_{\alpha\to 1} \tilde{D}_\alpha(\rho\|\sigma) \\ &= D_1(\rho\|\sigma) := \mathrm{Tr}\,\rho(\log(\rho) - \log(\sigma)) \end{aligned}$$

- a significant quantity in quantum information theory

Extension to von Neumann algebras

ρ, σ normal states on a von Neumann algebra $\mathcal{M}$

Standard: uses relative modular operator

$$D_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1} \log\langle\, \xi_\sigma, \Delta^\alpha_{\rho,\xi_\sigma} \xi_\sigma \,\rangle$$

D. Petz, Publ. RIMS, Kyoto Univ., 1985

- standard form $(\mathcal{M}, \mathcal{H}, J, \mathcal{H}^+)$, $\xi_\sigma \in \mathcal{H}^+$ vector representative
- good properties for $\alpha \in (0, 2]$
- error exponents in quantum hypothesis testing[6]

[6] V. Jaksic et al., *Rev. Math. Phys., 2012*

Extension to von Neumann algebras

Sandwiched: uses weighted L_p-norms

$$\tilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \|\eta(\rho)\|_{p,\sigma}^p,$$

where $\|\cdot\|_{p,\sigma}$ is the

- Araki-Masuda L_p-norm and $p = 2\alpha$, $\alpha \in (1/2, 1) \cup (1, \infty)$[7] (Araki-Masuda divergences)
- Kosaki L_p-norm and $p = \alpha > 1$[8]

Operational significance

- Conjecture: strong converse exponents

---

[7] M. Berta, V. B. Scholz, and M. Tomamichel, Ann. H. Poincaré, 2018

[8] AJ, arXiv:1609.08462, to appear in Ann. H. Poincaré, 2018

Haagerup L_p-spaces

For $1 \leq p \leq \infty$, $L_p(\mathcal{M})$ - Haagerup L_p-space:

- Banach space of (unbounded) operators;
- duality, Hölder inequality, ...;
- $\mathcal{M} \simeq L_\infty(\mathcal{M})$;
- the predual $\mathcal{M}_* \simeq L_1(\mathcal{M})$: $\rho \mapsto h_\rho$, $\mathrm{Tr}\, h_\rho = \rho(1)$;
- $L_2(\mathcal{M})$ a Hilbert space: $\langle\, h, k \,\rangle = \mathrm{Tr}\, k^* h$

Standard form: $(\lambda(\mathcal{M}), L_2(\mathcal{M}), J, L_2(\mathcal{M})^+)$:

$$\lambda(x)h = xh,\ \ Jh = h^*, \qquad x \in \mathcal{M},\ \ h \in L_2(\mathcal{M}).$$

$h_\rho^{1/2}$ - (unique) vector representative of $\rho \in \mathcal{M}_*^+$ in $L_2(\mathcal{M})^+$.

Kosaki L_p-spaces with respect to a faithful normal state

Let σ be a faithful normal state. We use complex interpolation:

- continuous embedding

$$\mathcal{M} \to L_1(\mathcal{M}), \quad x \mapsto h_\sigma^{1/2} x h_\sigma^{1/2}$$

- interpolation spaces

$$L_p(\mathcal{M}, \sigma) := C_{1/p}(\mathcal{M}, L_1(\mathcal{M})) \text{ with norm } \|\cdot\|_{p,\sigma}, \quad 1 \leq p \leq \infty$$

- for $1/p + 1/q = 1$, the map

$$i_p : L_p(\mathcal{M}) \to L_1(\mathcal{M}), \qquad k \mapsto h_\sigma^{1/2q} k h_\sigma^{1/2q}$$

  is an isometric isomorphism of $L_p(\mathcal{M})$ onto $L_p(\mathcal{M}, \sigma)$.

A definition of $\tilde{D}_\alpha$, $\alpha > 1$

Extension to non-faithful σ: by restriction to support $s(\sigma) = e$

$$L_p(\mathcal{M}, \sigma) = \{h \in L_1(\mathcal{M}),\ h = ehe \in L_p(e\mathcal{M}e, \sigma|_{e\mathcal{M}e})\}.$$

For normal states ρ, σ and $1 < \alpha < \infty$:

$$\tilde{D}_\alpha(\rho\|\sigma) = \begin{cases} \frac{\alpha}{\alpha-1}\log(\|h_\rho\|_{\alpha,\sigma}) & \text{if } h_\rho \in L_\alpha(\mathcal{M}, \sigma) \\ \infty & \text{otherwise.} \end{cases}$$

Properties of $\tilde{D}_\alpha$, $\alpha > 1$

- Extension: for density matrices, $\tilde{D}_\alpha$ coincides with the sandwiched Rényi relative entropy
- Strict positivity:

$$\tilde{D}_\alpha(\rho\|\sigma) \geq 0, \text{ with equality if and only if } \rho = \sigma.$$

- Monotonicity: if $\rho \neq \sigma$ and $\tilde{D}_\alpha(\rho\|\sigma) < \infty$, then

$$\alpha' \mapsto \tilde{D}_{\alpha'}(\rho\|\sigma) \text{ is strictly increasing for } \alpha' \in (1, \alpha].$$

- Order relations: extension to $\mathcal{M}_*^+$ satisfies:
  if $\rho_0 \leq \rho$ and $\sigma_0 \leq \sigma$, then

$$\tilde{D}_\alpha(\rho_0\|\sigma) \leq \tilde{D}_\alpha(\rho\|\sigma), \quad \tilde{D}_\alpha(\rho\|\sigma_0) \geq \tilde{D}_\alpha(\rho\|\sigma).$$

- Joint lower semicontinuity on $\mathcal{M}_*^+$

Properties of $\tilde{D}_\alpha$, $\alpha > 1$

- Generalized mean: if $\rho = \rho_1 \oplus \rho_2$, $\sigma = \sigma_1 \oplus \sigma_2$, then

$$\exp\{(\alpha - 1)\tilde{D}_\alpha(\rho\|\sigma)\} = \exp\{(\alpha - 1)\tilde{D}_\alpha(\rho_1\|\sigma_1)\} + \exp\{(\alpha - 1)\tilde{D}_\alpha(\rho_1\|\sigma_1)\}.$$

- Joint quasi-convexity: $(\rho, \sigma) \mapsto \exp\{(\alpha - 1)\tilde{D}_\alpha(\rho, \sigma)\}$ is jointly convex.

Relation to the standard version D_α

For $s(\rho) \leq s(\sigma)$, $p > 1$,

$$\rho(1)^{1-p} \|\Delta_{\rho,\sigma}^{1-1/2p} h_\sigma^{1/2}\|_2^{2p} \leq \|h_\rho\|_{p,\sigma}^p \leq \|\Delta_{\rho,\sigma}^{p/2} h_\sigma^{1/2}\|_2^2.$$

(the upper bound is an extension of the Araki-Lieb-Thirring inequality[9,10]). Using this, we obtain:

For normal states ρ, σ, $\alpha > 1$:

$$D_{2-1/\alpha}(\rho\|\sigma) \leq \tilde{D}_\alpha(\rho\|\sigma) \leq D_\alpha(\rho\|\sigma).$$

[9] H. Kosaki, *Proc. Amer. Math. Soc.*, 1992

[10] M. Berta, V. B. Scholz, and M. Tomamichel, Ann. H. Poincaré, 2018

Limit values

$$\lim_{\alpha\to 1} \tilde{D}_\alpha(\rho\|\sigma) = D_1(\rho\|\sigma)$$

$$\lim_{\alpha\to\infty} \tilde{D}_\alpha(\rho\|\sigma) = \tilde{D}_\infty(\rho\|\sigma)$$

Araki relative entropy:

$$D_1(\rho\|\sigma) = \begin{cases} \langle h_\rho^{1/2}, \log(\Delta_{\rho,\sigma}) h_\rho^{1/2} \rangle, & \text{if } s(\rho) \leq s(\sigma) \\ \infty & \text{otherwise} \end{cases}$$

relative max entropy:

$$\tilde{D}_\infty(\rho\|\sigma) := \inf\{\lambda > 0,\ \rho \leq 2^\lambda \sigma\}$$

Positive trace-preserving maps

$\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ positive, trace-preserving. Let $\sigma_0 = \Phi(\sigma)$.

- Φ is a contraction $L_1(\mathcal{M}) \to L_1(\mathcal{N})$.
- $\Phi(h_\sigma^{1/2} x h_\sigma^{1/2}) = h_{\sigma_0}^{1/2} y h_{\sigma_0}^{1/2}$ for some $y \in \mathcal{N}$ and

$$\Phi_\rho^* : x \mapsto y$$

  is a positive unital normal map $\mathcal{M} \to \mathcal{N}$ - Petz dual[11]
- Φ restricts to a contraction $L_\infty(\mathcal{M}, \sigma) \to L_\infty(\mathcal{N}, \sigma_0)$.
- By Riesz-Thorin:

  Φ restricts to a contraction $L_p(\mathcal{M}, \sigma) \to L_p(\mathcal{N}, \sigma_0)$ for all $1 \leq p \leq \infty$.

[11] D. Petz, *Quart. J. Math. Oxford*, 1988

Data processing inequality

For $\alpha > 1$, normal states ρ, σ, positive, trace-preserving Φ:

$$\tilde{D}_\alpha(\rho\|\sigma) \geq \tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma))$$

Consequently, by the limit $\alpha \to 1$:

For normal states ρ, σ,

$$D_1(\rho\|\sigma) \geq D_1(\Phi(\rho)\|\Phi(\sigma))$$

holds for any positive trace-preserving map Φ.

---

first observed for $\mathcal{M} = B(\mathcal{H})$, $\mathcal{H}$ separable, by A. Müller-Hermes, D. Reeb, Ann. H. Poincaré, 2017

The Araki-Masuda norms

The Araki-Masuda L_p-norm:

- for $2 \leq p \leq \infty$, $\xi \in L_2(\mathcal{M})^+$,

$$\|\xi\|_{p,\sigma}^{AM} = \sup_{\omega \in \mathcal{M}_*^+, \omega(1)=1} \|\Delta_{\omega,\sigma}^{1/2-1/p} \xi\|_2$$

  if $s(\omega_\xi) \leq s(\sigma)$ and is infinite otherwise

- for $1 \leq p < 2$,

$$\|\xi\|_{p,\sigma}^{AM} = \inf_{\omega \in \mathcal{M}_*^+, \omega(1)=1, s(\omega) \geq s(\omega_\xi)} \|\Delta_{\omega,\sigma}^{1/2-1/p} \xi\|_2$$

M. Berta, V. B. Scholz, and M. Tomamichel, Ann. H. Poincaré, 2018

The Araki-Masuda norms

- can be defined for any *-representation of $\mathcal{M}$ on a Hilbert space $\mathcal{H}$ and any vector $\xi \in \mathcal{H}$
- depends only on the vector state

$$\omega_\xi = \langle \cdot\xi, \xi \rangle$$

- duality relation: for $1/p + 1/q = 1$

$$|\langle \eta, \xi \rangle| \leq \|\eta\|_{p,\sigma}^{AM} \|\xi\|_{q,\sigma}^{AM}, \qquad \xi, \eta \in \mathcal{H}$$

- if $1 < p \leq 2$, there is a (unique) element $\eta_0 \in \mathcal{H}$ such that $\|\eta_0\|_{q,\sigma}^{AM} = 1$ and

$$\langle \eta, \eta_0 \rangle = \|\eta\|_{p,\sigma}^{AM}$$

M. Berta, V. B. Scholz, and M. Tomamichel, Ann. H. Poincaré, 2018

The Araki-Masuda divergences

Our setting: the standard form $(\lambda(\mathcal{M}), L_2(\mathcal{M}), *, L_2(\mathcal{M})^+)$:

For normal states ρ, σ and $\alpha \in [1/2, 1) \cup (1, \infty)$:

$$\tilde{D}_\alpha^{AM}(\rho\|\sigma) = \frac{2\alpha}{\alpha - 1} \log(\|h_\rho^{1/2}\|_{2\alpha,\sigma}^{AM})$$

M. Berta, V. B. Scholz, and M. Tomamichel, Ann. H. Poincaré, 2018

$\tilde{D}_\alpha$ and $\tilde{D}_\alpha^{AM}$

- For $1 < \alpha < \infty$:

$$\tilde{D}_\alpha(\rho\|\sigma) = \tilde{D}_\alpha^{AM}(\rho\|\sigma)$$

- For $1/2 < \alpha < 1$: $h_\sigma^{\frac{1-\alpha}{2\alpha}} h_\rho^{1/2} \in L_{2\alpha}(\mathcal{M})$ and

$$\tilde{D}_\alpha(\rho\|\sigma) := \tilde{D}_\alpha^{AM}(\rho\|\sigma) = \frac{2\alpha}{\alpha - 1} \log \| h_\sigma^{\frac{1-\alpha}{2\alpha}} h_\rho^{1/2} \|_{2\alpha}$$

Data processing inequality for $\tilde{D}_\alpha$, $\alpha \in [1/2, 1)$

We have to assume that $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ is trace preserving and completely positive (= a quantum channel).

- Stinespring representation: $\Phi^* = T^*\pi(\cdot)T$, π a normal *-representation, T isometry
- if $\rho = \omega_\eta$, DPI for is equivalent to

$$\|T\eta\|^{AM}_{2\alpha,\Phi(\sigma)} \geq \|\eta\|^{AM}_{2\alpha,\sigma}$$

for $\alpha \in [1/2, 1)$ and

$$\|T\eta\|^{AM}_{2\alpha,\Phi(\sigma)} \leq \|\eta\|^{AM}_{2\alpha,\sigma}$$

for $\alpha > 1$.

M. Berta, V. B. Scholz, and M. Tomamichel, Ann. H. Poincaré, 2018

Data processing inequality for $\tilde{D}_\alpha$, $\alpha \in [1/2, 1)$

Let $p = 2\alpha$, $1/p + 1/q = 1$. Let $\eta_0 \in \mathcal{H}$ be such that $\|\eta_0\|^{AM}_{q,\sigma} = 1$ and $\|\eta\|^{AM}_{p,\sigma} = \langle \eta, \eta_0 \rangle$, then

$$\begin{aligned} \|\eta\|^{AM}_{p,\sigma} = \langle T\eta, T\eta_0 \rangle &\leq \|T\eta\|^{AM}_{p,\Phi(\sigma)} \|T\eta_0\|^{AM}_{q,\Phi(\sigma)} \\ &\leq \|T\eta\|^{AM}_{p,\Phi(\sigma)} \end{aligned}$$

since $\|T\eta_0\|^{AM}_{q,\sigma} \leq 1$ by DPI for $\alpha > 1$.

M. Berta, V. B. Scholz, and M. Tomamichel, Ann. H. Poincaré, 2018

Sufficient (reversible) channels

Let

- $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ be a channel
- ρ, σ normal states, with $s(\rho) \leq s(\sigma)$.

Definition

Φ is sufficient with respect to $\{\rho, \sigma\}$ if there exists a recovery map:
a channel $\Psi : L_1(\mathcal{N}) \to L_1(\mathcal{M})$, such that

$$\Psi \circ \Phi(h_\rho) = h_\rho, \qquad \Psi \circ \Phi(h_\sigma) = h_\sigma.$$

D. Petz., *Commun. Math. Phys.*, 1986

Characterizations of sufficient channels

- Universal recovery map:
  Let $\Phi_\sigma := (\Phi_\sigma^*)_*$ (Petz dual), then $\Phi_\sigma \circ \Phi(h_\sigma)$.

Φ is sufficient with respect to $\{\rho, \sigma\}$ if and only if

$$\Phi_\sigma \circ \Phi(h_\rho) = h_\rho$$

D. Petz, *Quart. J. Math. Oxford*, 1988

- A conditional expectation:

There exists a conditional expectation $E : \mathcal{M} \to \mathcal{M}$ such that $\sigma \circ E = \sigma$ and Φ is sufficient with respect to $\{\rho, \sigma\}$ if and only if

$$\rho \circ E = \rho.$$

Characterization of sufficient channels by divergences

A divergence D characterizes sufficiency if

$$D(\Phi(\rho)\|\Phi(\sigma)) = D(\rho\|\sigma) < \infty$$

implies that Φ is sufficient with respect to $\{\rho, \sigma\}$.

The following divergences characterize sufficiency:

- D_1 (Araki relative entropy)
- D_α for $\alpha \in (0, 1)$ (standard Rényi relative entropies)

D. Petz, *Commun. Math. Phys.*, 1986
AJ, D. Petz, *IDAQP*, 2006

Characterizations of sufficient channels by $\tilde{D}_\alpha$

The sandwiched Rényi relative entropies $\tilde{D}_\alpha$ characterize sufficiency, for $\alpha \in (1/2, 1) \cup (1, \infty)$.

AJ, arXiv:1609.08462, to appear in Ann. H. Poincaré, 2018
AJ, arXiv:1707.00047

- For $\alpha > 1$: the assumption

$$\tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma)) = \tilde{D}_\alpha(\rho\|\sigma) < \infty \implies$$

  $h_\rho \in L_\alpha(\mathcal{M}, \sigma)$ and Φ is a contraction preserving its norm;
- For $\alpha \in (1/2, 1)$: Stinespring representation, duality relations.

The case $\alpha = 2$

An easy proof for $\tilde{D}_2$:

- $L_2(\mathcal{M}, \sigma)$ is a Hilbert space
- Φ_σ is the adjoint of $\Phi : L_2(\mathcal{M}, \sigma) \to L_2(\mathcal{N}, \Phi(\sigma))$

By well known properties of contractions on Hilbert spaces:

$$\Phi_\sigma \circ \Phi(h_\rho) = h_\rho.$$

The case $\alpha > 1$: Two lemmas

Let τ be a normal state, $s(\tau) \leq s(\sigma)$. Put

$$h_\tau(z) = h_\sigma^{(1-z)/2} h_\tau^z h_\sigma^{(1-z)/2}, \quad 0 \leq Re(z) \leq 1,$$

$h_\tau(1/p) \in L_p(\mathcal{M}, \sigma)$ for $p > 1$.

If $\|\Phi(h_\tau(1/p))\|_{p,\Phi(\sigma)} = \|h_\tau(1/p)\|_{p,\sigma}$ for some $p = p_0 > 1$, then the equality holds for all $p > 1$.

Let $h_\rho = th_\tau(1/p)$, $p > 1$, $t > 0$. Then Φ is sufficient with respect to $\{\rho, \sigma\}$ if and only if it is sufficient with respect to $\{\tau, \sigma\}$.

The case $\alpha > 1$: Proof

By assumption,

- $h_\rho = t h_\tau(1/\alpha)$ for some $t > 0$ and a normal state τ
- $\|\Phi(h_\tau(1/\alpha))\|_{\alpha,\Phi(\sigma)} = \|h_\tau(1/\alpha)\|_{\alpha,\sigma}$

Then

- $\|\Phi(h_\tau(1/2))\|_{2,\Phi(\sigma)} = \|h_\tau(1/2)\|_{2,\sigma}$

$$\Longrightarrow$$

- Φ is sufficient with respect to $\{\xi, \sigma\}$, $h_\xi = s h_\tau(1/2)$

$$\Longrightarrow$$

- Φ is sufficient with respect to $\{\tau, \sigma\}$

$$\Longrightarrow$$

- Φ is sufficient with respect to $\{\rho, \sigma\}$.

The case $\alpha \in (1/2, 1)$

Proof: Put $p := 2\alpha > 1$.

- $h_\sigma^{1/p-1/2} h_\rho^{1/2} \in L_p(\mathcal{M})$, so that

$$h_\sigma^{1/p-1/2} h_\rho^{1/2} = h_\tau^{1/p} u$$

for some $\tau \in \mathcal{M}_*^+$ and $u \in \mathcal{M}$ (polar decomposition)

- put $\eta = h_\rho^{1/2}$, $\eta_0 = h_\sigma^{1/p-1/2} h_\tau^{1/q} \tau(1)^{-1/q}$, then

$$\begin{aligned} \|\eta\|_{p,\sigma}^{AM} = \langle \eta, \eta_0 \rangle &\leq \| T\eta\|_{p,\Phi(\sigma)}^{AM} \| T\eta_0\|_{q,\Phi(\sigma)}^{AM} \\ &\leq \| T\eta\|_{p,\Phi(\sigma)}^{AM} = \|\eta\|_{p,\sigma}^{AM} \end{aligned}$$

so that

$$\| T\eta_0\|_{q,\Phi(\sigma)}^{AM} = \|\eta_0\|_{q,\sigma}^{AM} (= 1)$$

The case $\alpha \in (1/2, 1)$

- this implies

$$\tilde{D}_{\alpha^*}(\Phi(\omega)\|\Phi(\sigma)) = \tilde{D}_{\alpha^*}(\omega\|\sigma) < \infty$$

  for $\omega = \omega_{\eta_0}$ and $\alpha^* = q/2 > 1$
- hence Φ is sufficient with respect to $\{\omega, \sigma\}$, and also $\{\tau, \sigma\}$ since $h_\omega = th_\tau(1/\alpha^*)$
- we infer $\rho \circ E = \rho$ from

$$h_\sigma^{1/p-1/2} h_\rho^{1/2} = h_\tau^{1/p} u.$$