Comparison of quantum channels and quantum statistical experiments

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Ordering of quantum channels

Let Φ and Ψ be two channels with the same input space.

Ordering by degradability: $\Phi \preceq \Psi$ if

$$\Phi = \alpha \circ \Psi \text{ for some channel } \alpha.$$

Applications:

- (anti)degradable channels
- for classical-to-quantum channels:

$$\alpha : \{\rho_1, \ldots, \rho_n\} \to \{\sigma_1, \ldots, \sigma_n\}$$

- Correctable subsystems/subalgebras for a quantum channel

Deficiency of quantum channels

Approximate version: deficiency

$$\delta(\Phi, \Psi) = \inf_{\alpha} \|\Phi - \alpha \circ \Psi\|_{\diamond}$$

- ϵ-versions of the previous applications

The aim

The aim is to express $\delta(\Phi, \Psi)$ in terms of success probabilities for some ensembles.

The classical case: comparison of statistical experiments

Statistical experiment: $(X, \Sigma, \{p_\theta, \ \theta \in \Theta\})$

- probability distributions over sample space (X, Σ)

- in our setting: X, Θ finite sets

- Comparison of statistical experiments: (Blackwell, 1951)
  $\mathcal{S} = (X, \{p_\theta, \theta \in \Theta\}), \quad \mathcal{T} = (Y, \{q_\theta, \theta \in \Theta\})$

$$\mathcal{S} \preceq \mathcal{T} \text{ if there is a channel } \alpha : q_\theta \mapsto p_\theta, \qquad \forall \theta$$

$$\equiv \mathcal{S} \text{ is a randomization of } \mathcal{T}$$

- Deficiency of statistical experiments: (Le Cam, 1969)

$$\delta(\mathcal{S}, \mathcal{T}) = \inf_\alpha \sup_\theta \|p_\theta - \alpha(q_\theta)\|_1$$

- Le Cam distance: $\Delta(\mathcal{S}, \mathcal{T}) = \max\{\delta(\mathcal{S}, \mathcal{T}), \delta(\mathcal{T}, \mathcal{S})\}$.

BSS theorem

(Blackwell, Sherman, Stein, 1951)

$\mathcal{S} \preceq \mathcal{T} \iff$

for any decision space (D, g):

- D a finite set of decisions
- $g : \Theta \times D \to \mathbb{R}^+$ a payoff function,

and any decision rule μ for $\mathcal{S}$:

- a channel with input X and output D,

there is a decision rule ν for $\mathcal{T}$ such that the average payoffs satisfy:

$$\sum_{x,d} \mu(x|d) p_\theta(x) g(\theta, d) \leq \sum_{y,d} \nu(y|d) q_\theta(y) g(\theta, d), \quad \forall \theta$$

Randomization criterion

(Le Cam 1969)

$\delta(\mathcal{S}, \mathcal{T}) \leq \epsilon \iff$

for any decision space (D, g)

and any decision rule μ for $\mathcal{S}$,

there is a decision rule ν for $\mathcal{T}$ such that for all θ:

$$\sum_{x,d} \mu(x|d) p_\theta(x) g(\theta, d) \leq \sum_{y,d} \nu(y|d) q_\theta(y) g(\theta, d) + \frac{\epsilon}{2} \sup_d g(\theta, d)$$

Quantum case: statistical experiments

- Quantum statistical experiments: $(\mathcal{H}, \{\rho_\theta, \theta \in \Theta\})$
  - $\mathcal{H}$ a Hilbert space, $\dim(\mathcal{H}) < \infty$
  - $\rho_\theta \in \mathfrak{S}(\mathcal{H})$ density operators: $\rho_\theta \geq 0$, $\mathrm{Tr}\,\rho_\theta = 1$.
- (classical) decision problems: (D, g)
  - decision rules: POVMs $\{M_d, d \in D\}$, $M_d \geq 0$, $\sum_d M_d = I$.
  - average payoff:

$$\sum_d g(\theta, d)\mathrm{Tr}\,[M_d \rho_\theta]$$

- A straightforward generalization of BSS theorem by comparing the average payoffs - not true (Matsumoto, 2014)

Quantum case

- Quantum BSS theorem:
  - requires additional entanglement (Shmaya, 2005)
  - or tensoring with a complete set of states (Buscemi, 2012)
- Quantum randomization criterion: (Matsumoto, 2010)
  - uses quantum decision spaces, unclear interpretation
- Ordering of quantum channels: (Chefles, 2009)
- Deficiency of quantum channels: the main result
  - we also obtain a version of the randomization criterion for quantum experiments

Randomization criterion and MHT

Ensemble: $\mathcal{E} = \{\lambda_i, \pi_i\}_{i=1}^k$,

$\pi_1, \ldots, \pi_k$ probability distributions, $\lambda_1, \ldots, \lambda_k$ - prior probabilities

Optimal success probability:

$$P_{succ}(\mathcal{E}) = \sup_\mu \sum_i \lambda_i \sum_x \mu(x|i)\pi_i(x)$$

Randomization criterion: $\delta(\mathcal{S}, \mathcal{T}) \leq \epsilon \iff$

for any ensemble $\mathcal{E} = \{\lambda_i, \pi_i\}$, $\pi_i \in \mathcal{P}(\Theta)$:

$$P_{succ}(\{\lambda_i, \sum_\theta \pi_i(\theta)p_\theta\}) \leq P_{succ}(\{\lambda_i, \sum_\theta \pi_i(\theta)q_\theta\}) + \frac{\epsilon}{2}P_{succ}(\mathcal{E})$$

Randomization criterion and MHT

For classical channels:

$\delta(\Phi, \Psi) \leq \epsilon \iff$

for any ensemble $\mathcal{E} = \{\lambda_i, \pi_i\}$ on the input space,

$$P_{succ}(\{\lambda_i, \Phi(\pi_i)\}) \leq P_{succ}(\{\lambda_i, \Psi(\pi_i)\}) + \frac{\epsilon}{2} P_{succ}(\mathcal{E})$$

Deficiency of quantum channels

- Quantum channel: completely positive trace preserving map

$$\Phi : B(\mathcal{H}) \to B(\mathcal{K}),$$

  $\mathcal{H}$, $\mathcal{K}$ finite dimensional Hilbert spaces.

- $\mathcal{C}(\mathcal{H},\mathcal{K})$ - the set of all channels
- Deficiency: $\Phi \in \mathcal{C}(\mathcal{H},\mathcal{K})$, $\Psi \in \mathcal{C}(\mathcal{H},\mathcal{K}')$,

$$\delta(\Phi,\Psi) = \min_{\alpha \in \mathcal{C}(\mathcal{K}',\mathcal{K})} \|\Phi - \alpha \circ \Psi\|_\diamond$$

- Quantum ensemble: $\mathcal{E} = \{\lambda_i, \tau_i\}$, τ_i are density operators.
  Optimal success probability:

$$P_{succ}(\mathcal{E}) = \max_M \sum_i \lambda_i \mathrm{Tr}\,[M_i \tau_i]$$

Randomization criterion for quantum channels

Theorem

Let $\Phi \in \mathcal{C}(\mathcal{H}, \mathcal{K})$, $\Psi \in \mathcal{C}(\mathcal{H}, \mathcal{K}')$.

$\delta(\Phi, \Psi) \leq \epsilon \iff$

for any ensemble $\mathcal{E} = \{\lambda_i, \sigma_i\}_{i=1}^{\dim(\mathcal{K})^2}$ *on* $\mathcal{H} \otimes \mathcal{K}$,

$$P_{succ}((\Phi \otimes id_{\mathcal{K}})(\mathcal{E})) \leq P_{succ}((\Psi \otimes id_{\mathcal{K}})(\mathcal{E})) + \frac{\epsilon}{2} P_{succ}(\mathcal{E})$$

The case $\epsilon = 0$: (Chefles, 2009)

The main tools

- properties of the diamond norm $\| \cdot \|_\diamond$ and its dual: $\| \cdot \|^\diamond$
  - relation of the norms to the order structure of completely positive maps
  - relation of $\| \cdot \|^\diamond$ to optimal success probabilities
- the minimax theorem (as in the classical case)

The space of Hermitian maps

$\mathcal{L}(\mathcal{H},\mathcal{K})$ - space of Hermitian maps $\phi : B(\mathcal{H}) \to B(\mathcal{K})$,

$$\phi(X^*) = \phi(X)^*$$

- A positive cone $\mathcal{L}(\mathcal{H},\mathcal{K})^+$ - completely positive maps
- A compact convex subset $\mathcal{C}(\mathcal{H},\mathcal{K}) \subset \mathcal{L}(\mathcal{H},\mathcal{K})^+$ of channels
- The dual space $\equiv \mathcal{L}(\mathcal{K},\mathcal{H})$, with duality

$$\langle \phi, \psi \rangle = \mathrm{Tr}\, C(\phi) C(\psi^*)$$

  $C(\phi)$ is the Choi matrix of ϕ: $C(\phi) = \sum_{i,j} \phi(|i\rangle\langle j|) \otimes |i\rangle\langle j|$.
- the dual positive cone: $\mathcal{L}(\mathcal{K},\mathcal{H})^+$

Diamond norm and its dual

- diamond norm in $\mathcal{L}(\mathcal{H},\mathcal{K})$ (Kitaev, 1997)

$$\|\phi\|_\diamond = \sup_{\rho\in\mathfrak{S}(\mathcal{H}\otimes\mathcal{H})} \|(\phi\otimes id_{\mathcal{H}})(\rho)\|_1$$

- the dual norm $\|\cdot\|^\diamond$ in $\mathcal{L}(\mathcal{K},\mathcal{H})$
- alternative form, using the order structure:

$$\|\phi\|_\diamond = \inf_{\alpha\in\mathcal{C}(\mathcal{H},\mathcal{K})} \inf\{\lambda>0,\ -\lambda\alpha\le\phi\le\lambda\alpha\}$$

$$\|\psi\|^\diamond = \inf_{\sigma\in\mathfrak{S}(\mathcal{H})} \inf\{\lambda>0,\ -\lambda\phi_\sigma\le\psi\le\lambda\phi_\sigma\}$$

erasure channels $\phi_\sigma : \mathfrak{S}(\mathcal{K}) \ni \rho \mapsto \sigma$, $\sigma\in\mathfrak{S}(\mathcal{H})$.

Properties of the norms

- If $\phi \in \mathcal{L}(\mathcal{H}, \mathcal{K})^+$, then

$$\|\phi\|_\diamond = \sup_{\sigma \in \mathfrak{S}(\mathcal{H})} \operatorname{Tr}[\phi(\sigma)], \quad \|\phi\|^\diamond = \sup_{\alpha \in \mathcal{C}(\mathcal{K}, \mathcal{H})} \langle \alpha, \phi \rangle.$$

- If $\phi, \psi \in \mathcal{C}(\mathcal{H}, \mathcal{K})$, then

$$\|\phi - \psi\|_\diamond = 2 \sup_{\gamma \geq 0, \|\gamma\|^\diamond \leq 1} \langle \gamma, \phi - \psi \rangle$$

- If χ and ξ are channels, then the maps

$$\phi \mapsto \chi \circ \phi, \qquad \phi \mapsto \phi \circ \xi$$

are contractions with respect to both norms.

The dual norm and guessing probabilities

Let $\mathcal{E} = \{\lambda_i, \rho_i\}_{i=1}^k$ be an ensemble. It is an easy observation that

$$P_{succ}(\mathcal{E}) = \|\gamma_{\mathcal{E}}\|^{\diamond},$$

where $\gamma_{\mathcal{E}} : X \mapsto \sum_i X_{ii}\lambda_i\rho_i \in \mathcal{L}(\mathbb{C}^k, \mathcal{H})^+$.

Theorem

Let $\gamma \in \mathcal{L}(\mathcal{K}, \mathcal{H})^+$. Then there is an ensemble $\mathcal{E}_\gamma$ on $\mathcal{H} \otimes \mathcal{K}$ such that

$$\|\gamma\|^{\diamond} = \dim(\mathcal{K})\mathrm{Tr}\left[\gamma(I)\right]P_{succ}(\mathcal{E}_\gamma)$$

Moreover, if Φ is a channel, we have

$$\mathcal{E}_{\Phi\circ\gamma} = (\Phi \otimes id)(\mathcal{E}_\gamma)$$

Proof of the randomization criterion (sketch)

$\Longrightarrow$: Assume $\delta(\Phi, \Psi) \leq \epsilon$.

- Note that
$$\delta(\Phi \otimes id_{\mathcal{K}_0}, \Psi \otimes id_{\mathcal{K}_0}) \leq \delta(\Psi, \Phi),$$
for any $\mathcal{K}_0$.
- Use the properties of the norms to show that $\delta(\Phi \otimes id_{\mathcal{K}}, \Psi \otimes id_{\mathcal{K}}) \leq \epsilon$ implies
$$\|(\Phi \otimes id_{\mathcal{K}}) \circ \gamma\|^\diamond \leq \|(\Psi \otimes id_{\mathcal{K}}) \circ \gamma\|^\diamond + \frac{\epsilon}{2}\|\gamma\|^\diamond$$
for any $\gamma \in \mathcal{L}(\mathcal{K}, \mathcal{H} \otimes \mathcal{K})^+$.
- For any ensemle $\mathcal{E}$, put $\gamma = \gamma_{\mathcal{E}}$ and use $\|\gamma_{\mathcal{E}}\|^\diamond = P_{succ}(\mathcal{E})$.

$\Longleftarrow$:

- By definition and properties of $\|\cdot\|_\diamond$:

$$\delta(\Phi,\Psi) = 2 \min_{\alpha\in\mathcal{C}(\mathcal{K}',\mathcal{K})} \left\{ \max_{\substack{\gamma\in\mathcal{L}(\mathcal{K},\mathcal{H})^+,\\ \|\gamma\|^\diamond\leq 1}} \langle\gamma, \Phi - \alpha\circ\Psi\rangle \right\}$$

- use minimax theorem and get

$$\begin{aligned}\delta(\Phi,\Psi) &= 2\max_\gamma \min_\alpha \langle\gamma, \Phi - \alpha\circ\Psi\rangle \\ &\leq 2\max_\gamma \left\{\|\Phi\circ\gamma\|^\diamond - \|\Psi\circ\gamma\|^\diamond\right\}.\end{aligned}$$

- use the relation of $\|\cdot\|^\diamond$ and P_{succ} to see that the last expression is less than ϵ.

Randomization criterion for quantum experiments

By one more use of the minimax theorem, we obtain:

Theorem

Let $\mathcal{S} = (\mathcal{H}, \{\rho_\theta,\ \theta \in \Theta\})$ and $\mathcal{T} = (\mathcal{K}, \{\sigma_\theta,\ \theta \in \Theta\})$ be quantum experiments, $\epsilon \geq 0$. The following are equivalent:

(i) *There is some channel α such that*

$$\sup_{\theta \in \Theta} \|\sigma_\theta - \alpha(\rho_\theta)\|_1 \leq \epsilon$$

(ii) *For any $\{\theta_1, \ldots, \theta_n\} \subseteq \Theta$ and any ensemble $\mathcal{E} = \{\lambda_i, \tau_i\}_{i=1}^k$ on $\mathbb{C}^n \otimes \mathcal{K}$ with block-diagonal states $\tau_i = \sum |e_j\rangle\langle e_j| \otimes \tau_i^j$,*

$$P_{succ}(\{\lambda_i, \sum_j \sigma_{\theta_j} \otimes \tau_i^j\}) \leq P_{succ}(\{\lambda_i, \sum_j \rho_{\theta_j} \otimes \tau_i^j\}) + \frac{\epsilon}{2} P_{succ}(\mathcal{E})$$

Further research

The proof relies on the order structure of completely positive maps and the trace-preserving condition

- this suggests that more general situations can be treated similarly:

- more specific quantum protocols (quantum networks)
- more general types of deficiency: pre-processings, pre- and post-processings, correlations:
  - In the classical case: (Shannon, 1958)

$$\Phi = \sum_i \lambda_i \alpha_i \circ \Psi \circ \beta_i$$

Approximate version: (Raginsky, 2011)

- different distance measures