On the combinatorial structure of types of higher order quantum maps

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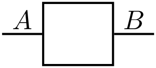
Introduction

Types of higher order quantum maps	$\simeq$	Boolean functions (type functions)	$\simeq$	Posets of subsets of system indices
$\downarrow$				$\downarrow$
causal structure		$\Longleftrightarrow$		order structure

Higher order quantum maps

Quantum channel:



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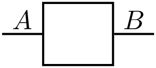
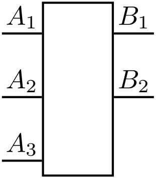
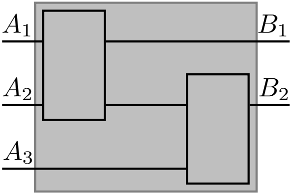


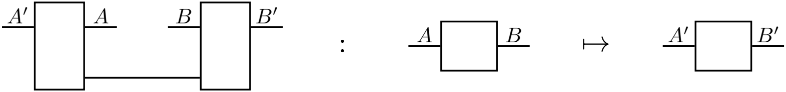
More complicated structure:





Higher order quantum maps

Quantum superchannels: transform channels to channels



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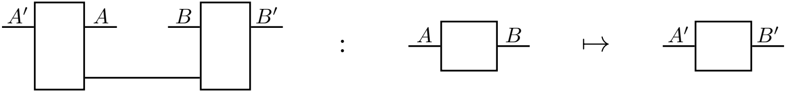

Higher order maps

- recursively built hierarchy of "transformations between transformations"

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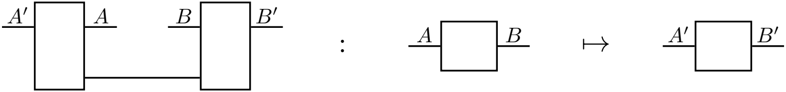
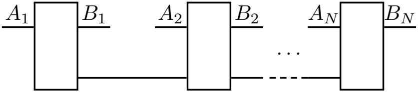


Higher order maps

- recursively built hierarchy of "transformations between transformations"
- most general quantum information protocols

Causal order

Quantum combs: a subclass of HOMs



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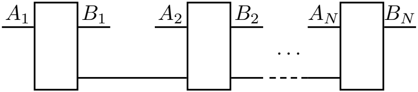

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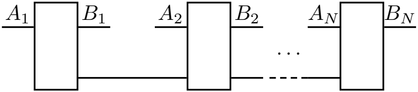

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- definite causal order of input and output "wires"

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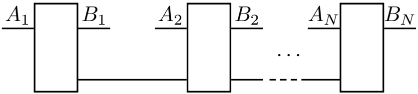

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Indefinite causal order:

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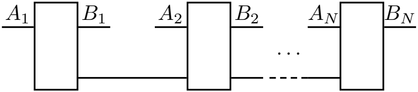

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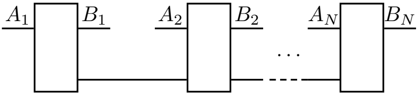

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Many different frameworks are being developed for HOM

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Type theory of higher order maps:

- elementary types $\equiv$ quantum systems (states): $A_1, A_2, \ldots$

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Bisio and Perinotti, 2019

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Any HOM type is a term over elementary types $A_1, \ldots, A_n$:

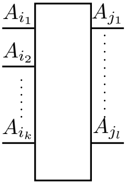
$$x = x(A_1, \ldots, A_n) = \overline{(A_{i_1} \otimes \bar{A}_{i_2})} \otimes (A_{i_3} \otimes \overline{(A_{i_4} \otimes \overline{(A_{i_5} \otimes ...)})})$$

Bisio and Perinotti, 2019

All HOMs of type x?

A map of type x:

- $A_1, \ldots, A_n$ - elementary types



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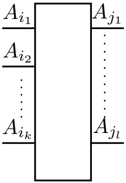

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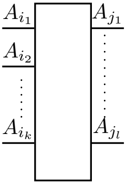

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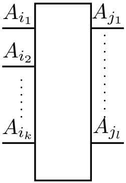

Theorem (Combinatorial description of HOM types)

The set of all HOMs of type x can be described by a unique Boolean function = type function:

$$f : \{0,1\}^n \to \{0,1\}, \quad f(0 \ldots 0) = 1.$$

Bisio and Perinotti, 2019

The type functions

Set of type functions:

$$\mathcal{T}_n \subseteq \mathcal{F}_n := \{f : \{0,1\}^n \to \{0,1\} \mid f(0\ldots0) = 1\}$$

Questions

- Characterize $\mathcal{T}_n$?
- Properties of type funcions $\equiv$ properties of HOMs?

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- Characterize $\mathcal{T}_n$? (no real answers)
- Properties of type funcions $\equiv$ properties of HOMs? (some examples)

Construction of type functions

Types $\equiv$ terms over variables $A_1, \dots, A_n$:

$$x = \overline{(A_{i_1} \otimes \bar{A}_{i_2})} \otimes (A_{i_3} \otimes \overline{(A_{i_4} \otimes \overline{(A_{i_5} \otimes ...)})})$$

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- terms are linear (variables are not repeating)

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- if $x \mapsto f$ then

$$\bar{x} \mapsto f^* \text{ - complement of } f \text{ in } \mathcal{F}_n.$$

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Möbius transform: the unique function $\hat{f} : 2^{[n]} \to \mathbb{R}$ such that

$$f(s) = \sum_{T \subseteq [n]} \hat{f}(T) p_T(s).$$

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We define the poset of subsets of $[n]$:

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Let $f \in \mathcal{T}_n$. Then $\mathcal{P}_f \subseteq 2^{[n]}$ is a graded poset with even rank and f is determined by $\mathcal{P}_f$:

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where ρ is the rank function of $\mathcal{P}_f$.

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Not all functions of this form are in $\mathcal{T}_n$!

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Type postes are constructed from singletons using products and complements.

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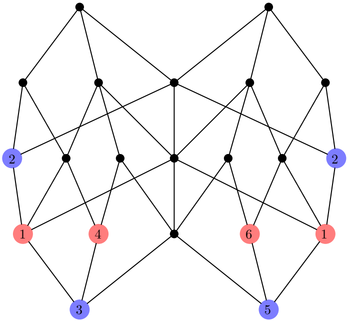

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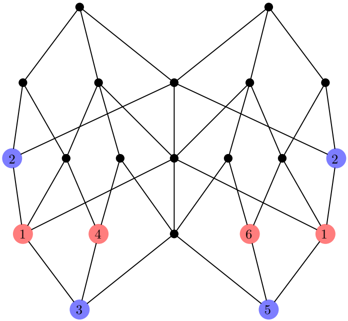

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$$\Longleftrightarrow$$

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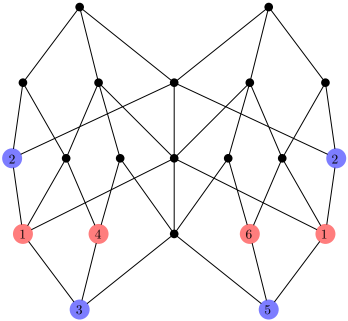

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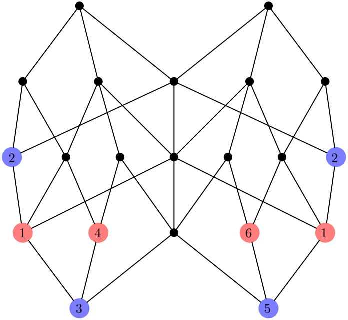
  $$\iff$$

  $i \in \ell_T$, $\rho(T)$ is even
- $i \in [n]$ is not a label $\equiv$ free output



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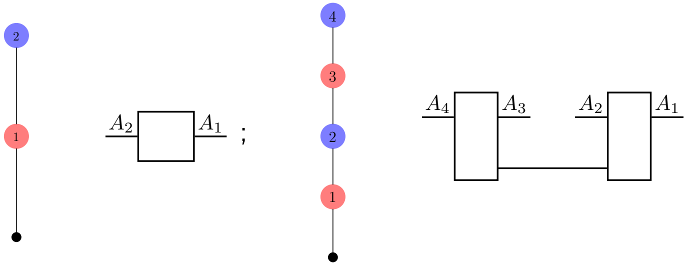
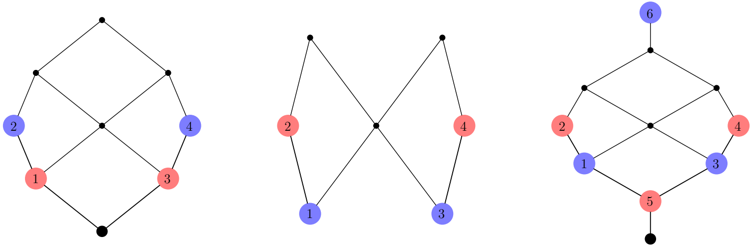
Examples

With $n = 2$ and $n = 4$:



Further examples



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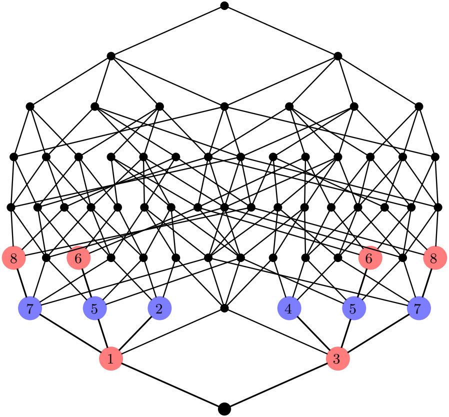
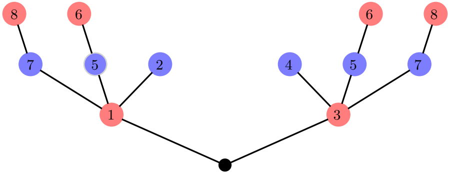


The restricted type poset

Let $\mathcal{P}_f^0$ be the subposet consisting of labeled elements and $\emptyset$ (if present in $\mathcal{P}_f$):

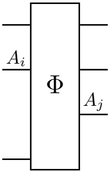


Theorem

Let $f \in \mathcal{T}_n$, then f is fully determined by $\mathcal{P}_f^0 \subseteq 2^{[n]}$.

No-signaling relations

For a channel Φ,
input index i, output index j:



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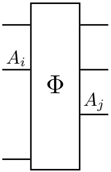

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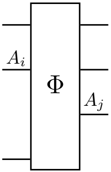

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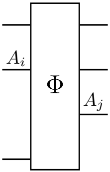

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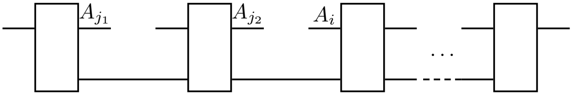
  $(i \not\rightsquigarrow_\Phi j)$ for all Φ of a type with type function f
- Combs are characterized by no-signaling conditions:

$$i \not\rightsquigarrow j_1$$
$$i \not\rightsquigarrow j_2$$



No-signaling in $\mathcal{P}_f^0$

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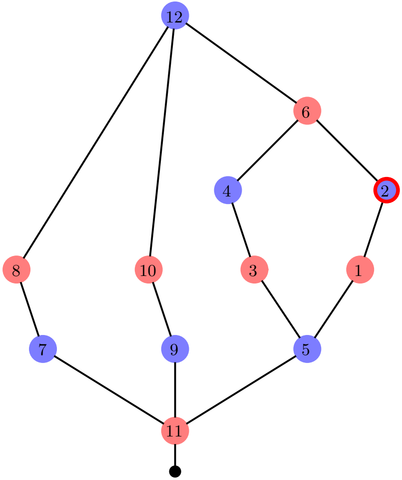
- j is not a label,



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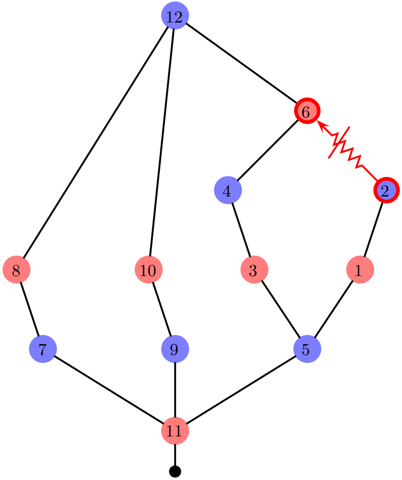
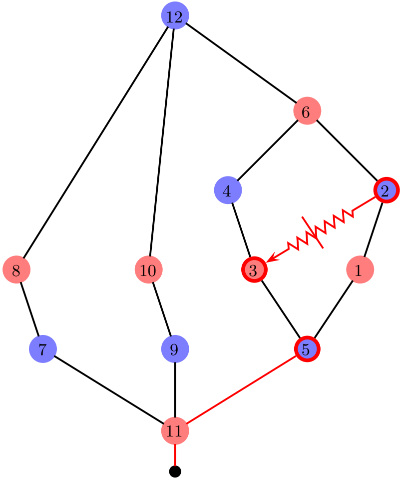


No-signaling in $\mathcal{P}_f^0$

Let $i \in \ell_S$ be an input label. Then $i \not\rightsquigarrow_f j$ for an output j iff one of the following holds:

- j is not a label,
- $j \in \ell_T$, $S \leq T$,
- $j \in \ell_T$, $S \wedge T$ exists and has even height.



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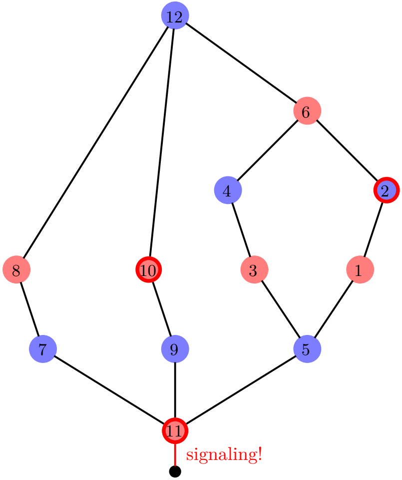

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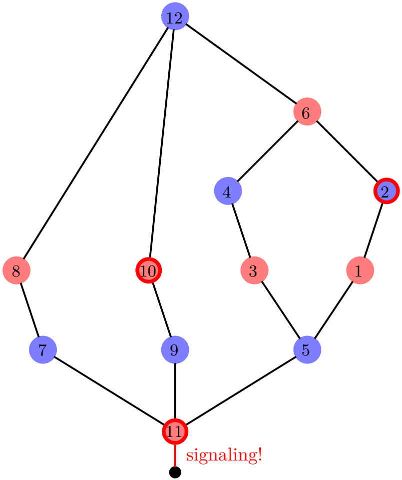

Repeated labels - similar conditions with some quantifiers

The normal form of a type function

Theorem

For any $f \in \mathcal{T}_n$, there are chains (combs) $\beta_{ab} \in \mathcal{T}_n$, $a \in A$, $b \in B$, with the same inputs and outputs as f, such that

$$f = \bigvee_{a \in A} \bigwedge_{b \in B} \beta_{ab}$$

The normal form of a type function

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- infimum of chains - definite causal order

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Question

Is there a better way to obtain this?

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