Sufficiency of quantum channels by Rényi divergences

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Quantum divergences

A (quantum) divergence is a dissimilarity measure for pairs of (quantum) states:

$$D : (\rho, \sigma) \mapsto D(\rho \| \sigma) \in \mathbb{R}^+$$

- strict positivity: $D(\rho\|\sigma) = 0$ iff $\rho = \sigma$;
- monotonicity (data processing inequality):

  $$D(\Phi(\rho)\|\Phi(\sigma)) \leq D(\rho\|\sigma)$$

  for any quantum channel Φ;
- operational significance: relation to performance of some procedures in information - theoretic tasks

Examples

- relative entropy:

$$S(\rho\|\sigma) = \mathrm{Tr}\,\rho(\log(\rho) - \log(\sigma));$$

- more general f-divergences;
- distinguishability measures from hypothesis testing:

$$P_t(\rho\|\sigma) = \|\rho - t\sigma\|_1, \qquad t > 0;$$

- distinguishability measures for n copies:

$$P_{t,n}(\rho\|\sigma) = \|\rho^{\otimes n} - t\sigma^{\otimes n}\|_1, \qquad t > 0,\ n \in \mathbb{N}.$$

Information loss and sufficient channels

Suppose Φ is a quantum channel.

Data processing inequality $\equiv$ information loss:

application of the channel Φ cannot increase the ability do distinguish ρ and σ.

Φ is sufficient with respect to a set of states $\mathcal{S}$: information is preserved for states in $\mathcal{S}$.

Sufficient quantum channels

How can we formulate this?

- equality in data processing inequality for some of the divergences and all states in $\mathcal{S}$;
- strongest definition - all states in $\mathcal{S}$ can be recovered:

Definition

We say that Φ is sufficient with respect to $\mathcal{S}$ if there is a channel Ψ (recovery map) such that

$$\Psi \circ \Phi(\rho) = \rho \qquad \forall \rho \in \mathcal{S}.$$

D. Petz, *Commun. Math. Phys.*, 1986

Sufficiency by divergences

Theorem

Let $\sigma \in \mathcal{S}$ be faithful. Φ is sufficient with respect to $\mathcal{S}$ if and only if

$$S(\Phi(\rho)\|\Phi(\sigma)) = S(\rho\|\sigma), \qquad \rho \in \mathcal{S}$$

(relative entropy determines sufficiency).

D. Petz, *Commun. Math. Phys.*, 1986

Theorem

The same holds for a large class of f-divergences, e.g. the standard Rényi divergences.

D. Petz, M. Mosonyi, F. Hiai

Classical Rényi divergences

For p, q probability measures over a finite set X, $0 < \alpha \neq 1$:

$$D_\alpha(p\|q) := \frac{1}{\alpha - 1} \log \sum_x p(x)^\alpha q(x)^{1-\alpha}$$

- unique family of divergences satisfying a set of postulates
- fundamental quantities appearing in many information - theoretic tasks
- relative entropy as a limit $\alpha \to 1$

A. Rényi, *Proc. Symp. on Math., Stat. and Probability*, 1961.

Quantum extensions of Rényi divergences

Standard:

$$D_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1}\log\left(\mathrm{Tr}\,\rho^\alpha\sigma^{1-\alpha}\right)$$

D. Petz, *Rep. Math. Phys., 1984*

Sandwiched:

$$\tilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1}\log\mathrm{Tr}\,\left[\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha\right]$$

M. Müller-Lennert et al., *J. Math. Phys.*, 2013

M. M. Wilde et al., *Commun. Math. Phys.*, 2014

Quantum Rényi divergences: properties

Standard version D_α,

- strict positivity, monotonicity: $\alpha \in (0, 2]$;
- Operational significance known for $\alpha \in (0, 1)$: error exponents in quantum hypothesis testing.[1,2]

Sandwiched version $\tilde{D}_\alpha$:

- strict positivity, monotonicity: $\alpha \in [1/2, 1) \cup (1, \infty]$;
- Operational significance known for $\alpha > 1$: strong converse exponents in quantum hypothesis testing.[3]

Both versions: relative entropy as a limit for $\alpha \to 1$.

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[1] K. M. R. Audenaert et al., *Commun. Math. Phys.*, 2008

[2] F. Hiai, M. Mosonyi, and T. Ogawa, *J. Math. Phys.*, 2008

[3] M. Mosonyi, and T. Ogawa, *Commun. Math. Phys.*, 2017

Sufficiency of channels by Rényi divergences

The standard Rényi divergence D_α determines sufficiency for all $\alpha \in (0, 2)$.

This talk

The same holds for the sandwiched Rényi divergence $\tilde{D}_\alpha$ with $\alpha \in (1/2, 1)$ and $\alpha > 1$.

AJ, Ann. H. Poincaré, 2018
AJ, arXiv:1707.00047

Sandwiched Rényi divergences and weighted L_p-spaces

$\mathcal{M}$ a von Neumann algebra, $L_p(\mathcal{M})$ - Haagerup L_p-space,
σ a faithful state, identified with an element in $L_1(\mathcal{M})$.

Kosaki L_p-spaces: complex interpolation

- continuous embedding

$$\mathcal{M} \to L_1(\mathcal{M}), \quad x \mapsto \sigma^{1/2} x \sigma^{1/2}$$

- interpolation spaces

$$L_p(\mathcal{M}, \sigma) := C_{1/p}(\mathcal{M}, L_1(\mathcal{M})) \text{ with norm } \|\cdot\|_{p,\sigma}, \quad 1 \le p \le \infty$$

- for $1/p + 1/q = 1$, the map

$$i_p : L_p(\mathcal{M}) \to L_1(\mathcal{M}), \qquad k \mapsto \sigma^{1/2q} k \sigma^{1/2q}$$

is an isometric isomorphism of $L_p(\mathcal{M})$ onto $L_p(\mathcal{M}, \sigma)$.

Sandwiched Rényi divergences and weighted L_p-spaces

Extension to non-faithful σ: by restriction to support $s(\sigma) = e$

$$L_p(\mathcal{M}, \sigma) = \{h \in L_1(\mathcal{M}),\ h = ehe \in L_p(e\mathcal{M}e, \sigma|_{e\mathcal{M}e})\}.$$

For normal states ρ, σ and $1 < \alpha < \infty$:

$$\tilde{D}_\alpha(\rho\|\sigma) = \begin{cases} \frac{\alpha}{\alpha-1}\log(\|\rho\|_{\alpha,\sigma}) & \text{if } \rho \in L_\alpha(\mathcal{M}, \sigma) \\ \infty & \text{otherwise.} \end{cases}$$

Extends the sandwiched Rényi divergence to von Neumann algebras.

Properties of $\tilde{D}_\alpha$ on von Neumann algebras

For $\alpha > 1$:

- strict positivity;
- (Araki) relative entropy as limit value:

$$\lim_{\alpha \to 1} \tilde{D}_\alpha(\rho \| \sigma) = S(\rho \| \sigma)$$

- monotonicity with respect to positive trace preserving maps: $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ restricts to a contraction

$$L_\alpha(\mathcal{M}, \sigma) \to L_\alpha(\mathcal{N}, \Phi(\sigma)), \qquad \alpha > 1$$

Corollary

S is monotone under positive trace preserving maps.

Sufficiency of channels by $\tilde{D}_\alpha$, $\alpha > 1$

$\mathcal{S}$ a set of normal states on $\mathcal{M}$, $\sigma \in \mathcal{S}$ faithful, Φ a channel, $\alpha > 1$.

Theorem

Assume that $\mathcal{S} \subseteq L_\alpha(\mathcal{M}, \sigma)$. Then Φ is sufficient with respect to $\mathcal{S}$ if and only if

$$\|\Phi(\rho)\|_{\alpha,\Phi(\sigma)} = \|\rho\|_{\alpha,\sigma}, \qquad \rho \in \mathcal{S}.$$

Easy proof for $\alpha = 2$:

- $L_2(\mathcal{M}, \sigma)$ is a Hilbert space, Φ a contraction, let $\Phi_\sigma := \Phi^*$;
- norm is preserved iff $\Phi_\sigma \circ \Phi(\rho) = \rho$;
- Φ_σ is a channel such that $\Phi_\sigma \circ \Phi(\sigma) = \sigma$.

Φ_σ - Petz dual, universal recovery map.

Universal recovery map

Theorem

Φ is sufficient with respect to $\mathcal{S}$ if and only if all $\rho \in \mathcal{S}$ are invariant states for the channel $\Phi_\sigma \circ \Phi$.

D. Petz, *Quart. J. Math. Oxford*, 1988

Mean ergodic theorem: there is a faithful normal conditional expectation E such that

$$\Phi_\sigma \circ \Phi(\rho) = \rho \iff \rho \circ E = \rho.$$

Sufficiency of channels by $\tilde{D}_\alpha$, $\alpha > 1$

Let $\rho \in \mathcal{S} \subset L_\alpha(\mathcal{M}, \sigma)$, then

$$\rho = t\sigma^{1/2\beta}\tau^{1/\alpha}\sigma^{1/2\beta}$$

for a normal state τ, $t > 0$, $\alpha^{-1} + \beta^{-1} = 1$. Introduce the family:

$$\rho_{\alpha'} := t_{\alpha'}\sigma^{1/2\beta'}\tau^{1/\alpha'}\sigma^{1/2\beta'} \in L_{\alpha'}(\mathcal{M}, \sigma), \qquad \alpha' > 1$$

By interpolation:

If $\|\Phi(\rho_{\alpha'})\|_{\alpha',\Phi(\sigma)} = \|\rho_{\alpha'}\|_{\alpha',\sigma}$ for some $\alpha' > 1$, then for all.

By properties of conditional expectations:

If $\Phi_\sigma \circ \Phi(\rho_{\alpha'}) = \rho_{\alpha'}$ for some $\alpha' > 1$, then for all.

The case $\alpha \in (1/2, 1)$

For $\alpha \in (1/2, 1)$, we have $\sigma^{\frac{1-\alpha}{2\alpha}} \rho^{1/2} \in L_{2\alpha}(\mathcal{M})$. Put

$$\tilde{D}_\alpha(\rho\|\sigma) = \frac{2\alpha}{\alpha - 1} \log \|\sigma^{\frac{1-\alpha}{2\alpha}} \rho^{1/2}\|_{2\alpha}$$

- can be obtained using Araki-Masuda L_p-norms;
- strict positivity;
- relative entropy as a limit $\alpha \to 1$;
- monotonicity with respect to channels;
- a duality between $\tilde{D}_\alpha$ and $\tilde{D}_{\alpha^*}$, $\alpha^* := \frac{\alpha}{2\alpha - 1}$.

M. Berta, V. B. Scholz, and M. Tomamichel, Ann. H. Poincaré, 2018

Sufficiency of channels by $\tilde{D}_\alpha$, $\alpha \in (1/2, 1)$

Let $\alpha \in (1/2, 1)$, then by polar decomposition:

$$\sigma^{\frac{1-\alpha}{2\alpha}} \rho^{1/2} = \tau^{1/2\alpha} u \in L_{2\alpha}(\mathcal{M})$$

for some $\tau \in L_1(\mathcal{M})^+$ and $u \in \mathcal{M}$ partial isometry.
By duality, we can show that

$$\tilde{D}_\alpha(\rho\|\sigma) \geq \tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma)) + \tilde{D}_{\alpha^*}(\omega\|\sigma) - \tilde{D}_{\alpha^*}(\Phi(\omega)\|\Phi(\sigma))$$

with

$$\omega = t\sigma^{1/2\beta^*} \tau^{1/\alpha^*} \sigma^{1/2\beta^*} \in L_{\alpha^*}(\mathcal{M}, \sigma).$$

Sufficiency of channels by $\tilde{D}_\alpha$, $\alpha \in (1/2, 1)$

From

$$\tilde{D}_\alpha(\rho\|\sigma) \geq \tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma)) + \tilde{D}_{\alpha^*}(\omega\|\sigma) - \tilde{D}_{\alpha^*}(\Phi(\omega)\|\Phi(\sigma))$$

we see that

$$\tilde{D}_\alpha(\rho\|\sigma) = \tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma)) \implies \tilde{D}_{\alpha^*}(\omega\|\sigma) = \tilde{D}_{\alpha^*}(\Phi(\omega)\|\Phi(\sigma))$$

- Since $\alpha^* > 1$, this implies that $\Phi_\sigma \circ \Phi(\omega) = \omega$;
- the same is true for τ;
- using properties of conditional expectations, we get that also $\Phi_\sigma \circ \Phi(\rho) = \rho$.