In a class of general probabilistic theories (GPT), we characterize incompatibility of measurements, steering and Bell nonlocality as particular forms of entanglement. See [4] for details.

1. GPT: Basic definitions and assumptions

The basic ingredients in a GPT are states (preparations) and effects (yes/no experiments), assigning probabilities to each state [2]. We generally assume

- convexity:
  - a state space is a (compact) convex subset $K$ of a finite dimensional vector space
  - effects: affine maps $K \rightarrow [0,1]$ = channels (devices): affine maps $K \rightarrow K'$

- local tomography:
  - composite systems have tensor product structure: the composite state space $K_1 \otimes K_2$ satisfies $K_1 \otimes K_2 \subseteq K_1 \otimes K_2 \subseteq K_1 \otimes K_2$

= $K_1 \otimes K_2$ is the set of separable states

= $K_1 \otimes K_2$ is the maximal tensor product

= entangled states are elements of $K_1 \otimes K_2 \setminus K_1 \otimes K_2$

Some notation:

- $E(K)$ is the set of effects
- $A(K)$ = all affine maps $K \rightarrow R$
- $A(K)^+ = \text{convex cone of positive maps in } A(K)$
- $V(K) = \{ (\rho, \phi) \mid \phi \text{ is positive on } A(K) \}$

2. Examples

2.1 Classical state space

An $n$-dimensional simplex $\Delta_n$. If $K$ is a state space, a channel $f : K \rightarrow \Delta_n$, represents a measurement with values in $(\omega_0, \omega_1, ..., \omega_n)$:

$$ f(x) = \sum_i f_i(x) \omega_i, \quad f_i \in E(K(\omega_i)) $$

mapping $x \in K$ to outcome probabilities $f_i(x_1), ..., f_i(x_n)$.

2.2 Quantum state space

The set

$$ \Theta(H) = \{ \rho \in B(H)^+ \mid \text{dim } H = n \} $$

density operators on a Hilbert space, $\text{dim } H < \infty$.

2.3 Semiclassical state space

A product of simplices

$$ S = S_{\Delta_1} \times \cdots \times S_{\Delta_n} $$

This state space will be useful below. Elements of $S$ can be interpreted as conditional probabilities.

Observation: Each projection $\pi^i : S = \Delta_{n-1} \times \cdots \times \Delta_{n-1}$ is a measurement. The corresponding effects $m^i$ generate extreme rays of $A(S)$.

3. Incompatible measurements

Let $\mathcal{F} = \{ f_1, f_2 \}$ be a collection of measurements $f_j : K \rightarrow \Delta_{n-1}$. We say that $\mathcal{F}$ is compatible if all $f_j$ are marginals of a single joint measurement $g : K \rightarrow \Delta_{n-2}$:

$$ f_j = \sum_{i=1}^{n-1} \phi_i \delta_{\Delta_{n-1}} $$

Observation: $\mathcal{F}$ defines a channel $F = f_1 \times \cdots \times f_k : K \rightarrow S_{\Delta_{n-1}}$.

Theorem: $\mathcal{F}$ is compatible if and only if it is entanglement breaking.

3.1 Incompatibility witnesses

Let $\mathcal{F} = \{ f_1, ..., f_k \}$ be a basis of $A(K)^+$, $\nu_i \in A(K)$ a dual basis. For any linear map $\lambda : A(K)^+ \rightarrow A(K)^+$, put

$$ \lambda (\mathcal{F}) = \sum_i (\nu_i, \phi_i) $$

Theorem: $\mathcal{F}$ is incompatible if and only if there is an incompatibility witness: an affine map $\lambda : S \rightarrow V(K)$ such that

$$ \lambda (\mathcal{F}) = 0 $$

Example 1. Let $\mathcal{F} = \{ f_1, f_2 \}$. A witness $W$ maps $S_{\Delta_1} \times S_{\Delta_1}$:

$$ W = (f_1 \otimes f_2) $$

which satisfies

$$ W(f \otimes \rho) = f(W \otimes \rho) $$

Theorem: $\mathcal{F}$ is incompatible if and only if there is an incompatibility witness: an affine map $\lambda : S \rightarrow V(K)$ such that

$$ \lambda (\mathcal{F}) = 0 $$

4. Steering in GPT

Similarly to quantum steering [6], we have the following scenario: $A$ and $B$ share an unknown state $\rho_{AB} \in K_A \otimes K_B$. $A$ applies measurements from the collection $\mathcal{F}$, $B$ knows the post-measurement states

$$ \rho_B = (f_i \otimes id_{WKB}) \rho_{AB} $$

but not $\mathcal{F}$. The conditional states form an assemblage:

$$ \{ \rho_B \} \in K_B $$

We say that an assemblage certifies steering if it does not admit a local hidden state model

$$ \rho_B = \sum_x \lambda(x) \rho_B(x) $$

Observation: All assemblages can be identified with elements of $S(K_B)$. The assemblage $\{ \rho_B \}$ is obtained as $(\mathcal{F})^* \otimes id_{K_B}$.

Theorem: An assemblage $(\mathcal{F})^* \otimes id_{K_B}$ certifies steering if it is entangled.

5. Bell nonlocality

This time, both parties apply measurements from collections $\mathcal{F}_A$ and $\mathcal{F}_B$ to their respective systems and report the results. The state $\rho_{AB}$ is Bell nonlocal if the obtained conditional probabilities cannot be explained by a local hidden variable model

5.1 Bell witnesses and Bell's inequalities

A Bell witness is a steering witness in this special case: $W_{\rho_{AB}} = S \rightarrow A(K)^+ \rightarrow W(V(K))$. Bell's inequalities have the form

$$ Tr (T(B) \otimes F(A)) = Tr (T(B) \otimes F(A)) = 0 $$

Note that $W = T_A \otimes F_B = W_{\rho_{AB}} \otimes F(A)$ must be an incompatibility witness for violation of this inequality.

Example 2. For CHSH, $W_{\rho_{AB}}$ is an affine isomorphism $V(S_{\Delta_1} \times S_{\Delta_1}) \rightarrow A(S_{\Delta_1})^*$. Therefore, CHSH violation is equivalent to incompatibility if $A(K_B) = \Delta_{n-1}$ (cf. Th. 1).

In general, $V(S_{\Delta_1} \times S_{\Delta_1}) \rightarrow A(S_{\Delta_1})^*$.

This might explain the fact that some incompatibility cannot be detected by violation of Bell's inequalities [5].

References


Non-classical features in general probabilistic theories

Anna Jenčová
Mathematical Institute, Slovak Academy of Sciences
Bratislava, Slovakia
jencova@mat.savba.sk