On the properties of spectral effect algebras

Anna Jenčová
Mathematical Institute, Slovak Academy of Sciences

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Dedicated to Sylvia Pulmannová and Anatolij Dvurečenskij

Effect algebras

An effect algebra is a system $(E, 0, 1, \oplus)$, where $0, 1 \in E$ are constants, $\oplus$ is a partial binary operation on E such that:

(E1) if $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$;

(E2) if $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $a \oplus (b \oplus c)$ is defined and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$;

(E3) for every $a \in E$ there is unique $a' \in E$ such that $a \oplus a' = 1$;

(E4) if $a \oplus 1 \in E$, then $a = 0$.

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Partial order: $a \leq b$ if there is some $c \in E$, such that $b = a \oplus c$

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An algebraic model of the algebra of Hilbert space effects:

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Fundamental questions:

- What are the properties that characterize the algebra of Hilbert space effects?
- How to derive quantum theory from some basic operational postulates?

States and effects

Basic concepts of an operational theory:

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Under reasonable conditions on F, $\mathcal{E}$ is representated as an effect algebra .

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Convex structure on $\mathcal{S}$ and $\mathcal{E}$: states/effects can be chosen at random from a given finite set:

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$\mathcal{E}$ is represented as a convex effect algebra.

Convex effect algebras

A convex effect algebra (or an effect module): an effect algebra $(E, 0, 1, \oplus)$ with an EA-bimorphism

$$[0,1] \times E \to E, \qquad (\lambda, a) \mapsto \lambda a,$$

such that

$$\mu(\lambda a) = \lambda(\mu a), \qquad 1a = a.$$

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Representation of CEA

Every convex effect algebra is affinely isomorphic to an interval $[0, u]$ in an ordered vector space (V, V^+) with an order unit u.

Gudder & Pulmannová, 1998

Contexts

Let E be a convex effect algebra. Then $a \in E$ is

- sharp if $a \in E$ such that $a \wedge a' = 0$
- indecomposable if $b \leq a$ implies that $b = ta$ for some $t \in [0, 1]$

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A context is a finite set $A = \{a_1, \ldots, a_n\} \in \mathcal{S}_1(E)$ such that

$$a_1 \oplus \cdots \oplus a_n = 1$$

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Spectral decomposition of $a \in E$:

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Spectral effect algebra

A convex effect algebra $(E, 0, 1, \oplus)$ where any $a \in E$ has a spectral decomposition.

Gudder, 2018

Basic examples

Classical theory

Let $\Omega = \{\omega_1, \ldots, \omega_n\}$, $E = \{f : \Omega \to [0,1]\}$ (fuzzy events). E is specified as spectral effect algebra with a single context:

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Quantum theory

Let $\mathcal{H}$ be a Hilbert space, $\dim(\mathcal{H}) = n$, $E = \mathcal{E}(\mathcal{H})$ - Hilbert space effects. Then E is a spectral effect algebra with uncountably many contexts having the same number of elements:

- sets $\{e_1, \ldots, e_n\}$ of orthogonal minimal projections

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- a contradiction, since a, b are not summable
- hence $C_\lambda \neq C_\mu$ if $\lambda \neq \mu \in [0, 1]$ - uncountably many contexts.

Further examples: effects over a convex set

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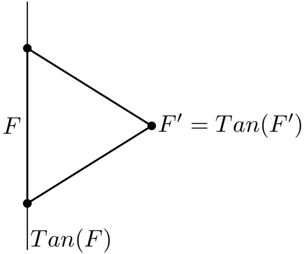

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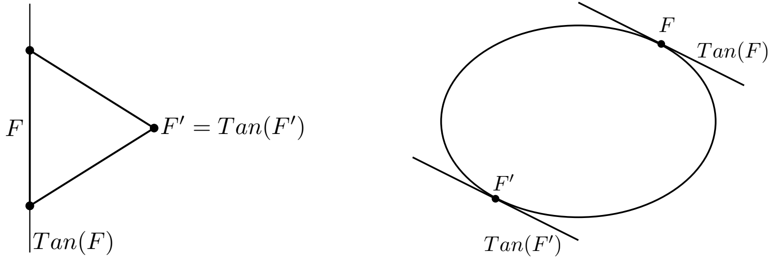

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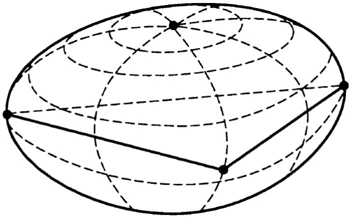

$E(K)$ is spectral, one context having 3 elements and uncountably many contexts having 2 elements.

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Constructions with spectral effect algebras

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Direct product:

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- $(a_1, a_2) \oplus (b_1, b_2) = (a_1 \oplus b_1, a_2 \oplus b_2)$
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Direct sum:

$E_1 \oplus E_2 = \{[(\lambda a_1, (1-\lambda)a_2)]_\sim, a_1 \in E_1, a_2 \in E_2, \lambda \in [0,1]\}$, where

- $\sim$ is determined by $(1,0) \sim (0,1)$
- is a convex effect algebra

but not spectral (unless E_1 or E_2 is isomorphic to $[0,1]$)

A non-example: effects on the square

Let $K = [0,1] \times [0,1] \subset \mathbb{R}^2$. Then

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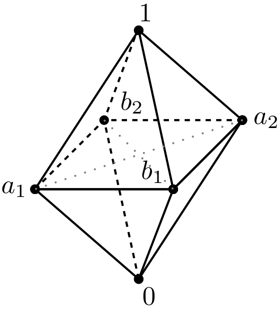

$E(K)$ has exactly 2 contexts $\{a_1, a_2\}$ and $\{b_1, b_2\}$, inherited from the two classical spectral effect algebras

Sharply determining state spaces

- state space of an effect algebra

$$\mathfrak{S}(E) = \{\text{EA morphisms } E \to [0,1]\}$$

- if E is convex, all $s \in \mathfrak{S}(E)$ are affine
- sharply determining $\mathfrak{S}(E)$: for any sharp $e \in E$ and $a \in E$,

$$s(a) = 0 \text{ whenever } s \in \mathfrak{S}(E),\ s(e) = 0 \implies a \leq e$$

Gudder, Pulmannová, Bugajski, Beltrametti, 1999

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- the set of sharp elements is an orthomodular lattice

Sharply determining state spaces

Uniqueness of spectral decomposition:

Theorem

Let E be a spectral effect algebra, $\mathfrak{S}(E)$ sharply determining. Let $a \in E$ have spectral decompositions in contexts $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_m\}$:

$$a = \oplus_{i=1}^{N} \lambda_i \left(\oplus_j a_j^i\right) = \oplus_{k=1}^{M} \mu_k \left(\oplus_l b_l^k\right),$$

$\lambda_1 > \cdots > \lambda_N > 0$, $\mu_1 > \cdots > \mu_M > 0$. *Then* $N = M$, $\lambda_i = \mu_i$ *and*

$$\oplus_j a_j^i = \oplus_l b_l^i.$$

Thank you for your attention.