Mathematical Institute
Slovak Academy of Sciences

The Structure of Uninorms with Continuous Underlying Triangular Norms and Conorms and Their Generalizations

Doktorská dizertačná práca

Andrea Zemánková

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Declaration

The thesis consists of works written in the years 2013–2020. These eleven papers were published in leading scientific journals. I hereby declare that none of these works were used before to obtain a scientific degree. A full list of the papers follows.

[UNI1] A. Mesiarová-Zemánková (2016). A note on decomposition of idempotent uninorms into an ordinal sum of singleton semigroups. Fuzzy Sets and Systems 299, pp. 140–145.

[UNI2] A. Mesiarová-Zemánková (2016). Ordinal sum construction for uninorms and generalized uninorms. International Journal of Approximate Reasoning 76, pp. 1–17.

[UNI3] A. Mesiarová-Zemánková (2017). Ordinal sums of representable uninorms. Fuzzy Sets and Systems 308, pp. 42–53.

[UNI4] A. Mesiarová-Zemánková (2017). Characterization of uninorms with continuous underlying t-norm and t-conorm by means of the ordinal sum construction. International Journal of Approximate Reasoning 83, pp. 176–192.

[UNI5] A. Mesiarová-Zemánková (2018). Characterization of uninorms with continuous underlying t-norm and t-conorm by their set of discontinuity points. IEEE Transactions on Fuzzy Systems 26(2), pp. 705–714.

[UNI6] A. Mesiarová-Zemánková (2018). Characterizing set-valued functions of uninorms with continuous underlying t-norm and t-conorm. Fuzzy Sets and Systems 334, pp. 83–93.

[UNI7] A. Mesiarová-Zemánková (2017). Uninorms continuous on $[0,e[^2 \cup]e,1]^2$. Information Sciences 393, pp. 130–143.

[NUN1] A. Mesiarová-Zemánková, Characterization of idempotent n-uninorms, Fuzzy Sets and Systems, https://doi.org/10.1016/j.fss.2020.12.019

[NUN2] A. Mesiarová-Zemánková (2021). The n-uninorms with continuous underlying t-norms and t-conorms. International Journal of General Systems 50(1), pp. 92–116.

[NUN3] A. Mesiarová-Zemánková, Characterizing functions of n-uninorms with continuous underlying functions, IEEE Transactions on Fuzzy Systems, https://doi.org/10.1109/TFUZZ.2021.3057231

[NUN4] A. Mesiarová-Zemánková (2021). Characterization of n-uninorms with continuous underlying functions via z-ordinal sum construction. International Journal of Approximate Reasoning 133, pp. 60–79.

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Preface

The probability theory and classical measure theory are based on sigma-additivity, reflecting genuine properties of the related objects such as areas and volumes. Humanities, economics and related sciences focus on interaction which cannot be modelled by (sigma-)additivity and thus require several generalizations of probability (measure) and related mean values (integrals). While the standard mean values deal with the standard multiplication (this is forced by the distributivity with respect to the addition necessary for a sound definition of the Lebesgue-Stieltjes integral), this is no longer the case when the (sigma-)additivity of measures is relaxed or modified. Therefore the study of integrals based on commutative associative functions on the unit interval is indispensable for development of generalized theory of probability.

The introduction of statistical metric spaces (probabilistic metric spaces as they are called today) by Menger in [54] and exploration of related concepts has initiated a deep study of triangular norms and related operations on $[0, 1]$. In particular, commutative and associative binary functions on $[0, 1]$, including t-norms, t-conorms, uninorms, nullnorms and other special functions were considered and applied in many theoretical and applied fields, for example in probability, statistic, many-valued logic, decision theory, artificial intelligence, neural networks, image processing, data fusion, however, also in economics, social sciences, and many others.

Due to the associativity, t-norms can be comprehended as special semigroups on the unit interval (special topological semigroups called I-semigroups in the case of continuity) or ring operations. With 1 as the neutral element, considering the 2-monotonicity (n-monotonicity), we come to binary (n-ary) copulas, which serve as a tool for modeling the stochastic dependence of random vectors, in particular, they model the links between 1-dimensional marginal distributions and join distributions.

Although the majority of applications focused on continuous t-norms and their dual t-conorms because of their easy characterization, the researchers soon realized that relaxation or the replacement of some axioms can enhance their performance in real-life applications. This led to several generalizations as for example non-continuous t-norms, semi t-operators, pseudo-t-norms, quasi-copulas, semicopulas and overlap functions, among others. Relaxation of the monotonicity yielded concepts of directional and weak monotonicity and replacement of the unit interval, which is a bounded chain, by more general structures yielded t-norms on bounded posets and bounded lattices.

Generalization of the position of the neutral element or the annihilator of a t-norm yielded the definition of uninorms and nullnorms [17, 82]. Since these operations behave differently below and above the neutral element (annihilator) it was soon observed that they can be used in bipolar aggregation, or bipolar many-valued logic [84]. In fact, uninorms and nullnorms can be taken as bipolar t-norms and t-conorms [56]. From an algebraic point of view, proper uninorms are the only binary operations $*$ on $[0, 1]$ which make the structures $([0, 1], \max, *)$ and $([0, 1], \min, *)$ distributive commutative semi-rings (see [32]).

Recently, the concept which brings together uninorms and nullnorms – n-uninorms – was introduced by Prabhakar Akella [5]. This concept on one hand generalizes uninorms in such a way that the global neutral element is replaced by n local neutral elements. On the other hand, this concept shares the same idea with the ordinal sum of t-norms (t-conorms), where on distinct subareas of the unit square the t-norm can behave differently, i.e., on each such area a different t-norm can be applied. The same can be observed in the case of n-uninorms, where on distinct subareas of the unit square different uninorms can be applied. Similarly, we can relate n-uninorms to the idea of k-ary capacities introduced by Grabisch and Labreuche, which are based on reference levels of interest for scores [33].

The investigation of functions mentioned above contribute to the development of generalized theory of probability, where for weakening or modifying the properties of probability we need to introduce integrals based exactly on functions studied in this doctoral dissertation. First approaches to non-additive measures and integrals were proposed and studied for example in [61, 77]. Integrals based on uninorms were proposed for example in [40].

A major part of my research in the last decade was focused on a deep study of uninorms and n-uninorms, and my main results are summarized in this thesis.

My work is dedicated to Professor Erich Peter Klement, a precious man who helped me and many others to start their academic carrier.

Objectives

The aim of this work is to offer a complete characterization of uninorms and n-uninorms with continuous underlying functions and, particularly, to study their continuity on the whole unit square and their decomposition into irreducible sets via the ordinal sum (z-ordinal sum) construction. Therefore the objectives of this thesis are the following:

- Define the ordinal sum construction for uninorms.
- Study semigroups that yield a uninorm via the ordinal sum construction.
- Show one-to-one correspondence between idempotent uninorms and special linear orders on the unit interval.
- Define a characterizing set-valued function of a uninorm with continuous underlying functions and show its relation to the set of points of discontinuity of the given uninorm.
- Show that each uninorm with continuous underlying functions can be expressed as an ordinal sum of semigroups related to continuous Archimedean t-norms, t-conorms, representable uninorms and idempotent semigroups.
- Define a z-ordinal sum construction for partially ordered index sets.
- Show one-to-one correspondence between idempotent n-uninorms and special partial orders on the unit interval.
- Define characterizing (set-valued) functions of an n-uninorm with continuous underlying functions and show their relation to the set of points of discontinuity of the given n-uninorm.
- Show that each n-uninorm with continuous underlying functions can be expressed as a z-ordinal sum of semigroups related to continuous Archimedean t-norms, t-conorms, representable uninorms and idempotent semigroups.

The thesis consists of 11 research papers, divided into two chapters: the first dedicated to uninorms with continuous underlying functions and the second dedicated to n-uninorms with continuous underlying functions. In Part I below, we give a brief overview of the content of the corresponding works and possibilities of further research. The papers can be found in Part II.

Contents

Part I

Summary of the results

Chapter 1

Basic notions and results

In this work we want to extend the characterization known for continuous t-norms and t-conorms to uninorms and n-uninorms with continuous underlying functions. Therefore we first recall the most important results on continuous t-norms. The history of t-norms, an overview of their properties, connection to other aggregations functions, basic applications, relevant literature and many other related results can be found in the two monographs [8, 39].

A *triangular norm* is a binary function $T\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and 1 is its neutral element. Due to the associativity, n-ary form of any t-norm is uniquely given and thus it can be extended to an aggregation function working on $\bigcup_{n\in\mathbb{N}}[0,1]^n$ (see [34]). Dual functions to t-norms are t-conorms. A *triangular conorm* is a binary function $S\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and 0 is its neutral element. The duality between t-norms and t-conorms is expressed by the fact that from any t-norm T we can obtain its dual t-conorm S by the equation

$$S(x,y) = 1 - T(1-x, 1-y)$$

and vice-versa. Linear transformation of a t-norm (t-conorm) to a non-trivial interval $[a,b]$, for $a,b \in \mathbb{R}$ is called a t-norm (t-conorm) on $[a,b]^2$. Moreover, a t-norm (t-conorm) T (S) is called *Archimedean* if for every $x,y \in \,]0,1[$ there exists $n \in \mathbb{N}$ such that $x_T^{(n)} < y$, ($x_S^{(n)} > y$), where $x_T^{(0)} = 1$ and $x_T^{(n)} = T(x, x_T^{(n-1)})$ for all $n \in \mathbb{N}$ (and similarly for S).

As we will see later, ordinal sum of t-norms, t-conorms, uninorms and the z-ordinal sum construction are all based on the following fundamental result of Clifford [19]. We introduce this result as formulated in [39].

Theorem 1.1.1
Let $A \neq \emptyset$ be a totally ordered set and $(G_\alpha)_{\alpha\in A}$ with $G_\alpha = (X_\alpha, *_\alpha)$ be a family of semigroups. Assume that for all $\alpha, \beta \in A$ with $\alpha < \beta$ the sets X_α and X_β are either

disjoint or that $X_\alpha \cap X_\beta = \{x_{\alpha,\beta}\}$*, where* $x_{\alpha,\beta}$ *is both the neutral element of* G_α *and the annihilator of* G_β *and where for each* $\gamma \in A$ *with* $\alpha < \gamma < \beta$ *we have* $X_\gamma = \{x_{\alpha,\beta}\}$*. Put* $X = \bigcup\limits_{\alpha \in A} X_\alpha$ *and define the binary operation* $*$ *on* X *by*

$$x * y = \begin{cases} x *_\alpha y & \text{if } (x,y) \in X_\alpha \times X_\alpha, \\ x & \text{if } (x,y) \in X_\alpha \times X_\beta \text{ and } \alpha < \beta, \\ y & \text{if } (x,y) \in X_\alpha \times X_\beta \text{ and } \alpha > \beta. \end{cases}$$

Then $G = (X, *)$ *is a semigroup. The semigroup* G *is commutative if and only if for each* $\alpha \in A$ *the semigroup* G_α *is commutative.*

Remark 1.1.2

As we see in the previous theorem, the ordinal sum construction assumes that the index set A *is totally (linearly) ordered and therefore if we say that semigroups* G_α *for* $\alpha \in A$ *are ordered we are speaking about the order defined on the index set* A*. Then, if for some* $\alpha, \beta \in A$ *we have* $X_\alpha \neq X_\beta$ *and* $x * y = x$ *for all* $x \in X_\alpha$ *and all* $y \in X_\beta$*, necessarily* $\alpha < \beta$*.*

Vice versa, given a commutative, associative function $F\colon [0,1]^2 \longrightarrow [0,1]$ *assume that there exists an index set* A *and semigroups* $(G_\alpha)_{\alpha \in A}$ *with* $G_\alpha = (X_\alpha, F|_{X_\alpha})$*, where* $F|_{X_\alpha}\colon X_\alpha^2 \longrightarrow X_\alpha$ *is the restriction of* F *to* X_α*, such that* $[0,1] = \bigcup_{\alpha \in A} X_\alpha$*, and for* $\alpha, \beta \in A$ *the sets* X_α *and* X_β *are either disjoint, or* $X_\alpha \cap X_\beta = \{x_{\alpha,\beta}\}$*, and* $X_\alpha \neq X_\beta$ *whenever* $\alpha \neq \beta$*. We define a partial order on* A *by* $\alpha \leq_A \beta$ *for* $\alpha, \beta \in A$ *if either* $\alpha = \beta$*, or* $F(x,y) = x$ *for all* $x \in X_\alpha$ *and all* $y \in X_\beta$*. Then* $\leq_A$ *is evidently reflexive, the antisymmetry of* $\leq_A$ *follows from the commutativity of* F *and the fact that* $X_\alpha \neq X_\beta$ *whenever* $\alpha \neq \beta$*, and the transitivity of* $\leq_A$ *follows from the associativity of* F*. In the case when* $\leq_A$ *is a total (linear) order it is easy to check that* $([0,1], F)$ *is an ordinal sum of* $(G_\alpha)_{\alpha \in A}$ *with respect to order* $\leq_A$ *.*

Therefore, in order to show that F *is an ordinal sum of semigroups* $(G_\alpha)_{\alpha \in A}$ *it is enough to show that these semigroups are totally ordered by the order* $\leq_A$ *defined above.*

The ordinal sum construction for t-norms and t-conorms is given as follows [48].

Proposition 1.1.3

Let K *be a finite or countably infinite index set and let* $(]a_k, b_k[)_{k \in K}$ $((]c_k, d_k[)_{k \in K})$ *be a system of open, disjoint subintervals of* $[0,1]$*. Let* $(T_k)_{k \in K}$ $((S_k)_{k \in K})$ *be a system of t-norms (t-conorms). Then the ordinal sum* $T = (\langle a_k, b_k, T_k \rangle \mid k \in K)$ $(S = (\langle c_k, d_k, S_k \rangle \mid k \in K))$ *given by*

$$T(x,y) = \begin{cases} a_k + (b_k - a_k) T_k\left(\frac{x - a_k}{b_k - a_k}, \frac{y - a_k}{b_k - a_k}\right) & \text{if } (x,y) \in [a_k, b_k[^2, \\ \min(x,y) & \text{else} \end{cases}$$

and

$$S(x,y) = \begin{cases} c_k + (d_k - c_k) S_k(\frac{x-c_k}{d_k-c_k}, \frac{y-c_k}{d_k-c_k}) & \textit{if } (x,y) \in \,]c_k, d_k]^2 \,, \\ \max(x,y) & \textit{else} \end{cases}$$

is a t-norm (t-conorm). The t-norm T (t-conorm S) is continuous if and only if all summands T_k (S_k) for $k \in K$ are continuous.

We see that semigroups in the ordinal sum of t-norms are for $k \in K$ given by $G_k = ([a_k, b_k[\,, T_k^*)$, where T_k^* is a linear transformation of the t-norm T_k to the interval $[a_k, b_k]$ (restricted to $[a_k, b_k[$) and the remainder of the unit square is filled by the minimum. Here we can observe two facts. At first, an ordinal sum of t-norms is in fact an ordinal sum of semigroups G_k and the trivial semigroups $G_x = (\{x\}, \mathrm{Id})$, where $x \in [0,1] \setminus \bigcup_{k \in K} [a_k, b_k[$ and $\mathrm{Id}\colon \{x\}^2 \longrightarrow \{x\}$ is the unique operation which can be defined on a trivial semigroup given by $\mathrm{Id}(x,x) = x$. On the other hand, the linear transformation between $[0,1]$ and $[a_k, b_k]$ is an increasing isomorphism and thus it preserves the commutativity, the associativity, the monotonicity as well as the location of the neutral element ($e = 0$, or $e = 1$, or $e \in \,]0,1[$). Moreover, this linear transformation is also a homeomorphism and thus it preserves the continuity of a t-norm (t-conorm).

The characterization of all continuous t-norms (t-conorms) is based on two constructions [48]. The first result shows that each continuous t-norm (t-conorm) is equal to an ordinal sum of continuous Archimedean t-norms (t-conorms). Note that a continuous t-norm (t-conorm) is Archimedean if and only if it has only trivial idempotent points 0 and 1. The second result shows that each continuous Archimedean t-norm (t-conorm) has a continuous additive generator.

Proposition 1.1.4
Let $t\colon [0,1] \longrightarrow [0,\infty]$ ($s\colon [0,1] \longrightarrow [0,\infty]$) be a continuous strictly decreasing (increasing) function such that $t(1) = 0$ ($s(0) = 0$). Then the binary operation $T\colon [0,1]^2 \longrightarrow [0,1]$ ($S\colon [0,1]^2 \longrightarrow [0,1]$) given by

$$T(x,y) = t^{-1}(\min(t(0), t(x) + t(y)))$$

$$S(x,y) = s^{-1}(\min(s(1), s(x) + s(y)))$$

is a continuous Archimedean t-norm (t-conorm). The function t (s) is called an additive generator *of T (S).*

An additive generator of a continuous t-norm T (t-conorm S) is uniquely determined up to a positive multiplicative constant. A continuous Archimedean t-norm T (t-conorm S) is either *strict*, i.e., strictly increasing on $]0,1]^2$ (on $[0,1[^2$), or *nilpotent*, i.e., there exists $(x,y) \in \,]0,1[^2$ such that $T(x,y) = 0$ ($S(x,y) = 1$). For an additive generator t of a strict t-norm there is $t(0) = +\infty$, and for an additive generator f of a nilpotent t-norm there is $f(0) < +\infty$.

If we relax the condition of the neutral element we obtain the notion of a *triangular subnorm* [37]. A t-subnorm is a binary function $M\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and bounded by the minimum from above, i.e., $M(x,y) \leq \min(x,y)$ for all $x, y \in [0,1]$. Evidently, each t-norm is also a t-subnorm. The dual operation to a t-subnorm is a *t-superconorm* which is a binary function $R\colon [0,1]^2 \longrightarrow [0,1]$ that is commutative, associative, non-decreasing in both variables and bounded by the maximum from below, i.e., $R(x,y) \geq \max(x,y)$ for all $x, y \in [0,1]$. More details on t-norms and t-conorms can be found in [8, 39].

The neutral element of a t-norm (t-conorm) is in the point 1 (0). Generalization of the position of the neutral element yields a uninorm introduced in [82]. A *uninorm* is a binary function $U\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and has a neutral element $e \in]0,1[$ (see also [30]). If we take uninorm in a broader sense, i.e., if for a neutral element we have $e \in [0,1]$, then the class of uninorms covers also the class of t-norms and the class of t-conorms. In order the stress that we assume a uninorm with $e \in]0,1[$ we will call such a uninorm *proper*. For each uninorm the value $U(1,0) \in \{0,1\}$ is the annihilator of U. A uninorm is called *conjunctive* (*disjunctive*) if $U(1,0) = 0$ ($U(1,0) = 1$).

For each uninorm U with the neutral element $e \in]0,1[$, the restriction of U to $[0,e]^2$ is a t-norm on $[0,e]^2$, i.e., a linear transformation of some t-norm T_U on $[0,1]^2$ and the restriction of U to $[e,1]^2$ is a t-conorm on $[e,1]^2$, i.e., a linear transformation of some t-conorm S_U. The t-norm T_U and a the t-conorm S_U are called the *underlying functions* of the uninorm U. In the case that for a uninorm U we have $e = 1$ ($e = 0$) then its underlying t-norm (t-conorm) is just U and its underlying t-conorm (t-norm) is degenerated to a trivial binary operation on a single point 1 (0). We will denote the set of uninorms with continuous underlying functions by $\mathcal{U}$. Moreover, $\min(x,y) \leq U(x,y) \leq \max(x,y)$ for all $(x,y) \in [0,e] \times [e,1] \cup [e,1] \times [0,e]$.

From any pair of a t-norm and a t-conorm we can construct the minimal and the maximal uninorm with the given underlying functions (see [47]).

Proposition 1.1.5
Let $T\colon [0,1]^2 \longrightarrow [0,1]$ be a t-norm and $S\colon [0,1]^2 \longrightarrow [0,1]$ a t-conorm and assume $e \in]0,1[$. Then the two functions $U_{\min}, U_{\max}\colon [0,1]^2 \longrightarrow [0,1]$ given by

$$U_{\min}(x,y) = \begin{cases} e \cdot T(\frac{x}{e}, \frac{y}{e}) & \textit{if } (x,y) \in [0,e[^2, \\ e + (1-e) \cdot S(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \textit{if } (x,y) \in [e,1]^2, \\ \min(x,y) & \textit{otherwise} \end{cases}$$

and

$$U_{\max}(x,y) = \begin{cases} e \cdot T(\frac{x}{e}, \frac{y}{e}) & \textit{if } (x,y) \in [0,e]^2\,, \\ e + (1-e) \cdot S(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \textit{if } (x,y) \in \,]e,1]^2\,, \\ \max(x,y) & \textit{otherwise} \end{cases}$$

are proper uninorms. We will denote the set of all uninorms of the first type by $\mathcal{U}_{\min}$ *and of the second type by* $\mathcal{U}_{\max}$.

A uninorm $U\colon [0,1]^2 \longrightarrow [0,1]$ is called

- *internal* if $U(x,y) \in \{x,y\}$ for all $(x,y) \in [0,1]^2$,
- *d-internal* if it is internal and there exists a continuous and strictly decreasing function $g_U\colon [0,1] \longrightarrow [0,1]$ such that $U(x,y) = \min(x,y)$ if $y < g_U(x)$ and $U(x,y) = \max(x,y)$ if $y > g_U(x)$,
- *locally internal* on $A(e)$ if U is internal on $A(e) = [0,e] \times [e,1] \cup [e,1] \times [0,e]$,
- *idempotent* if $U(x,x) = x$ for all $x \in [0,1]$.

Observe that if a uninorm U is internal then it is also idempotent and vice-versa.

For example all uninorms from $\mathcal{U}_{\min} \cup \mathcal{U}_{\max}$ are locally internal on $A(e)$.

The characterization of idempotent uninorms was given in [67].

Theorem 1.1.6
Let $U\colon [0,1]^2 \longrightarrow [0,1]$ *be a binary function. Then* U *is an idempotent uninorm with the neutral element* $e \in \,]0,1[$ *if and only if there exists a non-increasing function* $g\colon [0,1] \longrightarrow [0,1]$, *which is Id-symmetric, with* $g(e) = e$, *such that*

$$U(x,y) = \begin{cases} \min(x,y) & \textit{if } y < g(x) \textit{ or } (y = g(x) \textit{ and } x < g(g(x))), \\ \max(x,y) & \textit{if } y > g(x) \textit{ or } (y = g(x) \textit{ and } x > g(g(x))), \\ x \text{ or } y & \textit{if } y = g(x) \textit{ and } x = g(g(x)), \end{cases}$$

being commutative in the points (x,y) *such that* $y = g(x)$ *with* $x = g(g(x))$.

Similarly as in the case of t-norms and t-conorms we can construct uninorms using additive generators (see [30]).

Proposition 1.1.7
Let $f\colon [0,1] \longrightarrow [-\infty,\infty]$, $f(0) = -\infty$, $f(1) = \infty$ *be a continuous strictly increasing function. Then the binary function* $U\colon [0,1]^2 \longrightarrow [0,1]$ *given by*

$$U(x,y) = f^{-1}(f(x) + f(y)),$$

where $f^{-1}\colon [-\infty,\infty] \longrightarrow [0,1]$ *is an inverse function to* f*, with the convention* $\infty + (-\infty) = \infty$ $(\infty + (-\infty) = -\infty)$ *is a uninorm, which will be called a* representable *uninorm. The unique point* $e \in\,]0,1[$ *such that* $f(e) = 0$ *is then the neutral point of* U*.*

Note that if we relax the monotonicity of the additive generator then the neutral element will be lost and by relaxing the condition $f(0) = -\infty$, $f(1) = \infty$ the associativity will be lost (if $f(0) < 0$ and $f(1) > 0$). In [66] (see also [56]) we can find the following result.

Proposition 1.1.8
Let $U\colon [0,1]^2 \longrightarrow [0,1]$ *be a uninorm. Then* U *is representable if and only if it is continuous on* $[0,1]^2 \setminus \{(0,1),(1,0)\}$*.*

This result completely characterizes the set of representable uninorms.

Further results on uninorms with continuous underlying functions offer the characterizations of uninorms with continuous Archimedean underlying functions (see [31, 50, 44, 64] and [68] for the discrete case), uninorms with underlying functions given as ordinal sums (see [23]) and uninorms with continuous underlying functions that are locally internal in $A(e)$ (see [25]). In this work we will generalize these results for any uninorm with continuous underlying functions.

If we generalize the position of the annihilator of a t-norm (t-conorm) we obtain the following definition of a nullnorm [17]. Note that t-operators were independently defined in [53] and in [52] it was shown that t-operators and nullnorms coincide.

A binary function $V\colon [0,1]^2 \longrightarrow [0,1]$ is called a *nullnorm* if it is commutative, associative, non-decreasing in each variable and there exists a $z \in [0,1]$ such that $V(0,x) = x$ for all $x \le z$ and $V(1,x) = x$ for all $x \ge z$. The monotonicity then implies that z is the annihilator of V.

If $z = 0$ ($z = 1$) then V is a t-norm (t-conorm). Observe that for a commutative, associative and non-decreasing function $F\colon [0,1]^2 \longrightarrow [0,1]$, with $F(0,0) = 0$, $F(1,1) = 1$, the value $F(0,1)$ is always an annihilator of F. Thus for a nullnorm $z = V(0,1)$. In [17] the following result was shown.

Theorem 1.1.9
Let $z \in\,]0,1[$*. Then* $V\colon [0,1]^2 \longrightarrow [0,1]$ *is a nullnorm with the annihilator* z *if and only if there exists a t-norm* T_V *and a t-conorm* S_V *such that*

$$V(x,y) = \begin{cases} z \cdot S_V(\frac{x}{z}, \frac{y}{z}) & \text{if } x,y \in [0,z[^2\,, \\ z + (1-z) \cdot T_V(\frac{x-z}{1-z}, \frac{y-z}{1-z}) & \text{if } x,y \in\,]z,1]^2\,, \\ z & \text{otherwise.} \end{cases}$$

Therefore the characterization of nullnorms with continuous underlying functions is easy – since they are uniquely given by their underlying functions.

Further generalization that covers both uninorms and nullnorms was introduced by Akella [5]. Assume an $n \in \mathbb{N} \setminus \{1\}$. Let $V\colon [0,1]^2 \longrightarrow [0,1]$ be a commutative binary function. Then $\{e_1, \ldots, e_n\}_{z_1,\ldots,z_{n-1}}$ is called an *n-neutral element* of V if for $0 = z_0 < z_1 < \cdots < z_n = 1$ and $e_i \in [z_{i-1}, z_i]$, $i = 1, \ldots, n$ we have $V(e_i, x) = x$ for all $x \in [z_{i-1}, z_i]$.

A binary function $U^n\colon [0,1]^2 \longrightarrow [0,1]$ is an *n-uninorm* if it is commutative, associative, non-decreasing in each variable and has an n-neutral element $\{e_1, \ldots, e_n\}_{z_1,\ldots,z_{n-1}}$. Observe that the notation U^n should not be confused with the n-th power of U, and we keep it to follow the original notation of Akella [5].

The basic structure of n-uninorms was described by Akella in [5, 6] and the characterization of the five main classes of 2-uninorms was given in [85]. Now we will recall these five exhaustive and mutually exclusive classes:

- Class 1: 2-uninorms with $U^2(0,1) = z_1$.
- Class 2a: 2-uninorms with $U^2(0,1) = 0, U^2(1,z_1) = z_1$.
- Class 2b: 2-uninorms with $U^2(0,1) = 1, U^2(0,z_1) = z_1$.
- Class 3a: 2-uninorms with $U^2(0,1) = 0, U^2(1,z_1) = 1$.
- Class 3b: 2-uninorms with $U^2(0,1) = 1, U^2(0,z_1) = 0$.

Each n-uninorm has the following building blocks around the main diagonal.

Proposition 1.1.10
Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be an n-uninorm and let $\{e_1, \ldots, e_n\}_{z_1,\ldots,z_{n-1}}$ be its n-neutral element. Then

(i) *U^n restricted to $[z_{i-1}, e_i]^2$, for $i = 1, \ldots, n$, is a linear transformation of a t-norm. We will denote this t-norm by T_i.*

(ii) *U^n restricted to $[e_i, z_i]^2$ for $i = 1, \ldots, n$, is a linear transformation of a t-conorm. We will denote this t-conorm by S_i.*

(iii) *U^n restricted to $[z_{i-1}, z_i]^2$ for $i = 1, \ldots, n$, is a linear transformation of a uninorm. We will denote this uninorm by U_i.*

(iv) *U^n restricted to $[z_i, z_j]^2$ for $i, j \in \{0, 1, \ldots, n\}$, $i < j$, is a linear transformation of a $(j-i)$-uninorm.*

Moreover, U^n restricted to $[e_i, e_{i+1}]^2$ for $i \in \{1, \ldots, n-1\}$, is a linear transformation of a nullnorm. For generality, 1-uninorm will denote the standard uninorm.

For $n \in \mathbb{N}$ we will denote the set of all n-uninorms such that their underlying t-norms $T_1, \ldots, T_n$ and t-conorms $S_1, \ldots, S_n$ are continuous by $\mathcal{U}_n$. Recall that here the linear transformation from $[0,1]$ to $[a,b]$ as well as the backward transformation are homeomorphisms and therefore an n-uninorm from $\mathcal{U}_n$ is continuous on $[z_{i-1},e_i]^2$ and on $[e_i,z_i]^2$ for $i=1,\dots,n$.

If for a 2-uninorm there is $e_2=1$ we obtain a uni-nullnorm and if $e_1=0$ we obtain a null-uninorm [78]. Our aim is the characterization of n-uninorms from $\mathcal{U}_n$. Note that the first result in this direction was the characterization of uni-nullnorms with continuous Archimedean underlying functions given in [79].

Chapter 2

Uninorms with continuous underlying functions

As we have seen in the previous chapter, there are two main construction methods that are used for the characterization of continuous t-norms (t-conorms). The ordinal sum construction and the construction via an additive generator. While the concept of an additive generator was easily introduced also for uninorms and yields representable uninorms, the ordinal sum construction was not so evident. The results known so far were based merely on the ordinal sum decomposition of underlying functions of uninorms and not uninorms themselves. Therefore the natural question arises – which semigroups can yield a uninorm in the ordinal sum construction? In this chapter we will define ordinal sums of uninorms and study the semigroups that yield uninorms via the ordinal sum construction. Using these results we are able to completely characterize uninorms with continuous underlying functions. We will study characterizing set-valued functions of such uninorms, their relation to the set of points of discontinuity of such uninorms and finally we will provide their decomposition into irreducible semigroups with respect to the ordinal sum construction. This chapter is based on my papers [UNI1,UNI2,UNI3,UNI4,UNI5,UNI6,UNI7].

2.1 Generalized uninorms

Following [41], where it was shown that the most general semigroups that yield a t-norm via the ordinal sum construction are t-subnorms, we have done the same analysis for uninorms in [UNI2]. Since below (above) the neutral element the uninorm corresponds to a t-norm (t-conorm) we immediately see that each such an operation, when restricted to the points smaller (greater) than the neutral element, yields a t-subnorm (t-superconorm).

Thus, beside trivial elements, t-subnorms and t-supreconorms we have identified four kinds of operations that can be used for the construction of a uninorm via the ordinal sum construction. These four operations are generalized sub-uninorms, generalized super-uninorms, generalized composite uninorms and standard uninorms transformed to the corresponding subset of the unit interval (see below).

All kinds of generalized uninorms are commutative, associative and non-decreasing binary functions and have an averaging behaviour when a point below the dividing element and a point above the dividing element are combined (for uninorms this dividing element is the neutral element, however, generalized uninorms do not have to have a neutral element). Moreover, roughly speaking, a generalized sub-uninorm represents an operation whose underlying functions are a t-subnorm and a t-conorm (without one or both boundary points). Similarly, a generalized super-uninorm represents an operation whose underlying functions are a t-norm and a t-superconorm. Finally, a generalized composite uninorm represents an operation whose underlying functions are a t-subnorm and a t-superconorm. As we already know, underlying functions of a uninorm are a t-norm and a t-conorm. Note that since each t-subnorm (t-superconorm) is a t-norm (t-conorm) then uninorms are both generalized sub-uninorms as well as generalized super-uninorms. The intervals from which these generalized uninorms arise can be seen on Figure 2.1 (see [UNI2] for more details).

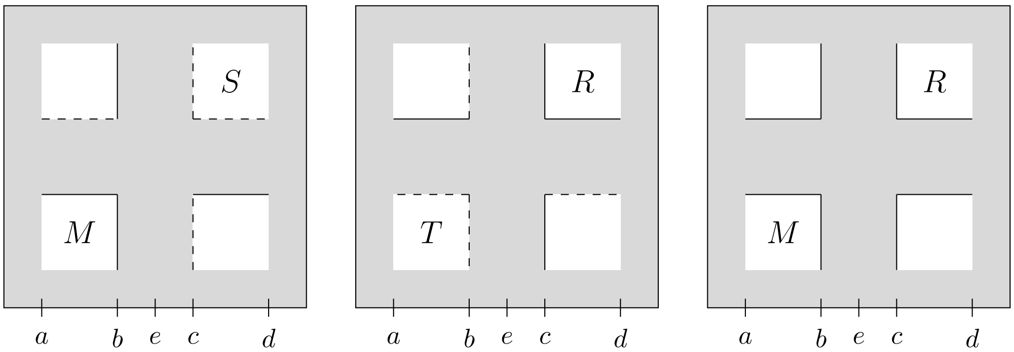

Figure 2.1: Sketch of areas from which generalized uninorms originate: generalized sub-uninorm (left), generalized super-uninorm (center) and generalized composite uninorm (right). Here M denotes a t-subnorm, R a t-superconorm, T a t-norm and S a t-conorm.

Before we introduce a proper definition of the three generalized uninorms let us observe that the class of generalized composite uninorms differ from the other three operations as it cannot be easily transformed (by a monotone transformation) to an operation on $[0,1]^2$. On the other hand, for other three operations it is possible. Obviously, we cannot use a linear transformation, as in the case of t-norms, as the part below the division point and above the division point should be transformed separately. Therefore we will use the following piece-wise linear isomorphism. For any $0 \leq a < b \leq c < d \leq 1$, $v \in [b,c]$ and a point $e \in]0,1[$ assume the function $f\colon [0,1] \longrightarrow [a,b[\cup \{v\} \cup]c,d]$, given by

$$f(x) = \begin{cases} (b-a)\cdot \frac{x}{e} + a & \text{if } x \in [0,e[\,, \\ v & \text{if } x = e, \\ d - \frac{(1-x)(d-c)}{(1-e)} & \text{otherwise.} \end{cases} \tag{2.1}$$

Then f is linear on $[0,e[$ and on $]e,1]$ and thus it is a piece-wise linear isomorphism of $[0,1]$ to $([a,b[\cup \{v\} \cup]c,d])$. Assume a function $GU\colon [0,1]^2 \longrightarrow [0,1]$. Then we can define the binary function $GU_v^{a,b,c,d}\colon ([a,b[\cup \{v\} \cup]c,d])^2 \longrightarrow ([a,b[\cup \{v\} \cup]c,d])$ given by

$$GU_v^{a,b,c,d}(x,y) = f(GU(f^{-1}(x), f^{-1}(y))). \tag{2.2}$$

Similarly, using the backward transformation f^{-1} we can transform a binary function defined on $([a,b[\cup\{v\}\cup]c,d])^2$ to a binary function defined on $[0,1]^2$. Since f is an increasing isomorphism it preserves the commutativity, the associativity and the monotonicity. Further, if e is the neutral element of GU then v is the neutral element of $GU_v^{a,b,c,d}$. Thus if GU is a uninorm on $[0,1]^2$ then $GU_v^{a,b,c,d}$ is a commutative, associative, non-decreasing function on $([a,b[\cup \{v\} \cup]c,d])^2$ with the neutral element v, which we will call simply a *uninorm on* $([a,b[\cup \{v\} \cup]c,d])^2$. Note that in the case when $b = c = v$ then f is a continuous, piece-wise linear transformation from $[0,1]$ to $[a,d]$ such that $f(e) = v$. This transformation can be used for all kinds of generalized uninorms except of generalized composite uninorms and therefore we give the following definitions.

Definition 2.1.1
An associative, commutative, binary operation $GU\colon [0,1]^2 \longrightarrow [0,1]$ *which is non-decreasing in each variable will be called*

(i) generalized sub-uninorm *if there exists an* $e \in [0,1]$ *such that there is* $GU(x,y) \leq \min(x,y)$ *for all* $(x,y) \in [0,e]^2$, $GU(x,y) \geq \max(x,y)$ *for all* $(x,y) \in]e,1]^2$, *and* $GU(x,y) \in [x,y]$ *for all* $(x,y) \in [0,e] \times]e,1] \cup]e,1] \times [0,e]$.

(ii) generalized super-uninorm *if there exists an* $e \in [0,1]$ *such that there is* $GU(x,y) \leq \min(x,y)$ *for all* $(x,y) \in [0,e[^2$, $GU(x,y) \geq \max(x,y)$ *for all* $(x,y) \in [e,1]^2$, *and* $GU(x,y) \in [x,y]$ *for all* $(x,y) \in [0,e[\times [e,1] \cup [e,1] \times [0,e[$.

It can be easily observed that generalized sub-uninorms and generalized super-uninorms differ only on the set $\{e\} \times [0,1] \cup [0,1] \times \{e\}$.

Definition 2.1.2
A binary operation $GU\colon ([a,b] \cup [c,d])^2 \longrightarrow ([a,b] \cup [c,d])$, *where* $a < b < c < d$, $a,b,c,d \in [0,1]$ *will be called* a generalized composite uninorm *if it is associative, commutative, non-decreasing in both coordinates and* GU *restricted to* $[a,b]^2$ *is a t-subnorm on* $[a,b]^2$, GU *restricted to* $[c,d]^2$ *is a t-superconorm on* $[c,d]^2$, *and* $GU(x,y) \in [x,y]$ *for all* $(x,y) \in [a,b] \times [c,d]$ *and all* $(x,y) \in [c,d] \times [a,b]$.

Several results on the structure of generalized uninorms, especially representable generalized uninorms, can be found in [UNI2]. However, in our investigation of uninorms with continuous underlying functions, which necessarily have a continuous diagonal, we can observe that all components which correspond to a t-subnorm (t-superconorm) reduce to (a restriction of) a continuous t-norm (t-conorm) and thus in this case it is enough to assume standard uninorms instead of generalized sub-uninorms and generalized super-uninorms. Moreover, in this case every generalized composite uninorm GU reduces to an operation with underlying continuous t-norm – acting on the interval $[a, b]$, and underlying continuous t-conorm – acting on the interval $[c, d]$, with an averaging behaviour on $[a, b] \times [c, d] \cup [c, d] \times [a, b]$. Due to the monotonicity such an operation possesses the neutral element $e \in \{b, c\} \setminus \{GU(b, c)\}$.

The structure of a generalized composite uninorm GU with continuous underlying t-norm (t-conorm) can be very simple, for example in the case when it is similar to $U_{\min}$ ($U_{\max}$), i.e., when $GU(x, y) = \min(x, y)$ ($GU(x, y) = \max(x, y)$) for all $(x, y) \in [a, b] \times [c, d] \cup [c, d] \times [a, b]$. In this case GU can be expressed as an ordinal sum of the corresponding continuous t-norm on $[a, b]^2$ and the corresponding continuous t-conorm on $[c, d]^2$. However, the structure of a generalized composite uninorm can be also complicated as it can be seen in the following example.

Example 2.1.3
Let $A = \{a_i \mid i \in \mathbb{N} \cup \{0\}\}$, $B = \{b_i \mid i \in \mathbb{N} \cup \{0\}\}$, and let $X_{a_i} = \left]\frac{1}{3^{i+1}}, \frac{1}{3^i}\right]$ for all $i \in \mathbb{N}$, $X_{a_0} = \{0\}$, $X_{b_i} = \left[1 - \frac{1}{3^i}, 1 - \frac{1}{3^{i+1}}\right[$ for all $i \in \mathbb{N}$, $X_{b_0} = \{1\}$. Assume semigroups $G_{a_i} = (X_{a_i}, \min)$ and $G_{b_i} = (X_{b_i}, \max)$ for all $i \in \mathbb{N} \cup \{0\}$ and a linear order on the set $A \cup B$ given by $a_i >^* b_i >^* a_{i+1} >^* a_0 >^* b_0$ for all $i \in \mathbb{N}$. Then the ordinal sum (X, F) of semigroups $(G_\alpha)_{\alpha \in A \cup B}$ with respect to order $\leq^*$ is a commutative, associative function $F\colon X^2 \longrightarrow X$, where $X = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ and F restricted to $\left[0, \frac{1}{3}\right]$ is the minimum, F restricted to $\left[\frac{2}{3}, 1\right]$ is the maximum and $F(x, y) \in [x, y]$ for all $(x, y) \in \left[0, \frac{1}{3}\right] \times \left[\frac{2}{3}, 1\right] \cup \left[\frac{2}{3}, 1\right] \times \left[0, \frac{1}{3}\right]$. Further, we can observe the following facts: $F(1, x) = 1$ for all $x \in X$. $F(0, x) = 0$ for all $x \in X$ with $x < 1$. $F(x, y) = \min(x, y)$ for all $x \in X_{a_i}$ and $y \in X_{b_j}$ for $i > j$ and $F(x, y) = \max(x, y)$ for all $x \in X_{a_i}$ and $y \in X_{b_j}$ for $i \leq j$. Therefore the monotonicity of F is easily verified. Thus F is an idempotent generalized composite uninorm with continuous underlying t-norm and continuous underlying t-conorm (see Figure 2.2). It is interesting to observe that this generalized composite uninorm was constructed as an ordinal sum of a countably many semigroups with operations min *and* max*, however, it cannot be expressed as an ordinal sum of a uninorm (a semigroup isomorphic to a uninorm via (2.1)) and a finite number of semigroups with operations* min *and* max*, i.e., it cannot be reduced to a uninorm in a finite number of steps.*

When we look for building stones for construction of uninorms with continuous un-

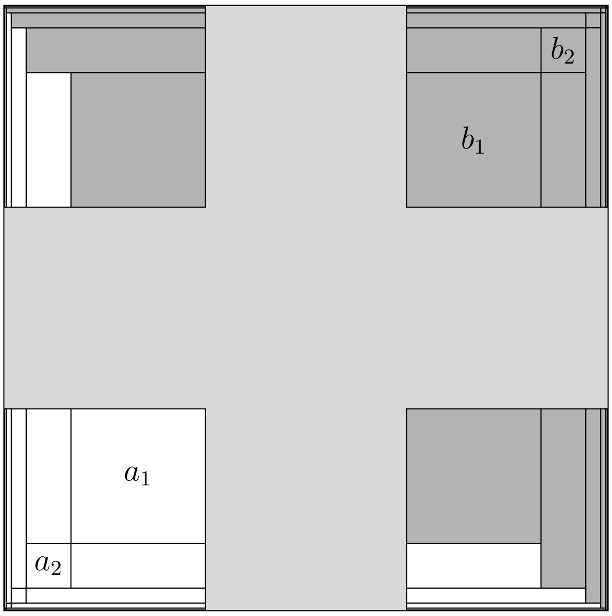

Figure 2.2: The generalized composite uninorm F from Example 2.1.3. On white areas F is equal to the minimum and on black areas F is equal to the maximum.

derlying functions via the ordinal sum then we should focus on generalized composite uninorms with continuous Archimedean underlying functions. For better visualization we add the following result which was not shown in papers from Part II (for proof see Appendix).

Proposition 2.1.4
Let $GU\colon ([a,b]\cup[c,d])^2 \longrightarrow ([a,b]\cup[c,d])$, where $a<b<c<d$, $a,b,c,d \in [0,1]$ be a generalized composite uninorm with underlying functions which are a continuous Archimedean t-norm and a continuous Archimedean t-conorm, respectively. Then GU can be expressed either as an ordinal sum of a uninorm with continuous Archimedean underlying functions and a trivial semigroup, or as an ordinal sum of a continuous Archimedean t-norm (possibly without one or both boundary points), a continuous Archimedean t-conorm (possibly without one or both boundary points) and few trivial semigroups (corresponding to points form $\{a,b,c,d\}$).

Thus instead of generalized composite uninorms it is enough to assume uninorms (including t-norms, t-conorms and trivial semigroups) as building stones in our ordinal sum construction.

Remark 2.1.5
Observe that it is easy to show that generalized composite uninorms with continuous underlying t-norm (t-conorm) have a similar ordinal sum structure as uninorms with continuous underlying functions, i.e., they can be expressed as an ordinal sum of Archimedean t-norms, Archimedean t-conorms, representable uninorms, idempotent uninorms (includ-

ing the min *and the* max *operator) and trivial semigroups (see Section 2.5). This can be simply observed from the fact that an ordinal sum of a generalized composite uninorm on* $([a,b] \cup [c,d])^2$ *and any internal proper uninorm linearly transformed to* $[b,c]^2$ *restricted to* $]b,c[^2$ *(with the respective linear order induced by the monotonicity) is a uninorm (on* $[a,d]^2$*) with continuous underlying functions.*

Note that generalized composite uninorms will play a role also in the characterization of n-uninorms (see discussion under Theorem 3.1.6). Although the corresponding generalized composite uninorm can be further decomposed via the ordinal sum construction, its structure can be quite complicated and thus a notion of a generalized composite uninorm is useful for an easy characterization.

Since in this work we focus on uninorms with continuous underlying functions, we should investigate the relation between the isomorphism f given by (2.1) and the continuity of underlying functions. We cannot claim that f is a homeomorphism as in the case of linear transformations, since it transforms a connected area to four unconnected areas. On the other hand, observe that f is in fact composed of 3 increasing homeomorphisms: one acting on $[0,e[$, one on $\{e\}$, and one on $]e,1]$. For better understanding we add the following result which was not introduced in papers from Part II (for proof see Appendix).

Proposition 2.1.6
Assume $0 \leq a < b \leq c < d \leq 1$, $v \in [b,c]$, $e \in]0,1[$, *a binary function* $U\colon [0,1]^2 \longrightarrow [0,1]$ *and the function* f *given by* (2.1). *Then* U *is a uninorm with the neutral element* e *and continuous underlying functions if and only if the function* $U^*\colon ([a,b[\cup \{v\} \cup]c,d])^2 \longrightarrow ([a,b[\cup \{v\} \cup]c,d])$ *given by* $U^*(x,y) = f(U(f^{-1}(x), f^{-1}(y)))$ *is a uninorm on* $([a,b[\cup \{v\} \cup]c,d])^2$ *which is continuous on* $[a,b[^2$ *and on* $]c,d]^2$ *and fulfills* $\lim\limits_{t \longrightarrow b^-} U^*(x,t) = x$ *for all* $x \in [a,b[$ *and* $\lim\limits_{t \longrightarrow c^+} U^*(y,t) = y$ *for all* $y \in]c,d]$.

This result shows us that in order to transform a restriction of a uninorm to a uninorm with continuous underlying functions this restriction has to fulfill all conditions of the previous proposition. In other words, U restricted to $[a,b]^2$ has to be a continuous t-norm (on $[a,b]^2$) and U restricted to $[c,d]^2$ has to be a continuous t-conorm (on $[c,d]^2$). This condition is obviously fulfilled whenever U is a uninorm with continuous underlying functions.

2.2 Ordinal sum of uninorms

In the case of an ordinal sum of t-norms (a similarly for t-conorms) the procedure is the following: define a subdivision of the unit interval into non-empty subintervals $[a_k, b_k[$; assume a linear transformation T_k^* of a t-norm T_k, restricted to $[0,1[^2$, to the interval

$[a_k, b_k[$; apply the ordinal sum of semigroups $([a_k, b_k[\,, T_k^*)$ with the order induced by the monotonicity; fill the remainder with the minimum.

In [UNI2] we have defined a similar procedure also for uninorms, however, in the case of uninorms the situation is a bit more complicated. The building stones here are t-norms, t-conorms, uninorms and trivial semigroups. Using (2.1) each proper uninorm is transformed to the set $([a_k, b_k[\cup \{v_k\} \cup]c_k, d_k])^2$. Note however, that if $U_k(x, y) = e_k$, where e_k is the neutral element of U_k, for some $x \neq e_k$, $y \neq e_k$ then $([0, e_k[\cup]e_k, 1])^2$ is not closed under U_k and thus in order to preserve the associativity v_k has to be an annihilator of U restricted to $[b_k, c_k]^2$. Observe that due to the monotonicity the annihilator of U on $[b_k, c_k]^2$ is equal to $U(b_k, c_k)$.

To avoid duplication, in the ordinal sum of uninorms the values of respective summands will be specified on intervals $[a_k, b_k[\cup]c_k, d_k]$ and in the case that the summand corresponds to a proper (representable) uninorm the respective v_k will be chosen is such a way that after applying the ordinal sum construction we will obtain $v_k = U(b_k, c_k)$. In the case that $a_k = b_k$ ($c_k = d_k$) the corresponding summand is isomorphic to a t-conorm (t-norm). In this case the restriction of the t-conorm S_k (t-norm T_k) to $]0, 1]^2$ ($[0, 1[^2$) is linearly transformed to $]c_k, d_k]^2$ ($[a_k, b_k[^2$) and the neutral element 0 (1) is transformed to the neutral element v_k. Similarly as above, it is enough to specify this summand on $]c_k, d_k]^2$ ($[a_k, b_k[^2$).

In order to obtain the monotonicity of the resulting uninorm the order of summands, denoted by $\preceq$, has to be compatible with the standard order $\leq$ on $[0, e]$ and reversed with respect to the standard order on $[e, 1]$. This means that $k_1 \prec k_2$ for $k_1, k_2 \in K$ implies $b_{k_1} \leq a_{k_2}$ and $c_{k_1} \geq d_{k_2}$, i.e.,

$$[a_{k_2}, d_{k_2}]^2 \subseteq [b_{k_1}, c_{k_1}]^2 \subseteq [a_{k_1}, d_{k_1}]^2 .$$

Example 2.2.1

- *Any uninorm from $\mathcal{U}_{\min}$ ($\mathcal{U}_{\max}$) is an ordinal sum of a t-norm on $[0, e]^2$ and a t-conorm on $[e, 1]^2$. The order of these two semigroups in the ordinal sum then determines whether the corresponding uninorm belongs to $\mathcal{U}_{\min}$, or to $\mathcal{U}_{\max}$.*

- *Assume two proper uninorms U_1 and U_2, an $e \in]0, 1[$ and points $a, b \in [0, 1]$ such that $0 < a < e < b < 1$. Then the ordinal sum of semigroups G_1 and G_2, where G_1 corresponds to a uninorm U_1 on $[a, b]^2$, i.e., $G_1 = ([a, b]\,, (U_1)_e^{a,e,e,b})$ and G_2 corresponds to a uninorm U_2 on $([0, a[\cup \{v\} \cup]b, 1])^2$, i.e., $G_2 = ([0, a[\cup \{v\} \cup]b, 1]\,, (U_2)_v^{0,a,b,1})$ is a uninorm with the neutral element e if and only if $2 \prec 1$ and $v = (U_1)_e^{a,e,e,b}(a, b)$ (see Figure 2.3).*

- *Assume a proper uninorm U, a t-norm T and points $a, b \in [0, 1]$ such that $0 < a < b < 1$. Then the ordinal sum of semigroups G_1 and G_2, where G_1 corresponds to a* *t-norm T on $[a,b]^2$, i.e., $G_1 = ([a,b], (T)^{a,b})$, where $(T)^{a,b}$ is a linear transformation of the t-norm T to the interval $[a,b]$ and G_2 corresponds to a uninorm U on $([0,a[\cup \{v\} \cup]b,1])^2$, i.e., $G_2 = ([0,a[\cup \{v\} \cup]b,1], (U)_v^{0,a,b,1})$, is a uninorm with the neutral element b if and only if $2 \prec 1$ and $v = a$.*

- *Assume a proper uninorm U, a t-norm T, an $e \in]0,1[$ and point $a \in]0,e[$. Then the ordinal sum of semigroups G_1 and G_2, where G_1 corresponds to a uninorm U on $[a,1]^2$, i.e., $G_1 = ([a,1], (U)_e^{a,e,e,1})$ and G_2 corresponds to a respective transformation of a t-norm T to $([0,a[\cup \{v\} \cup]1,1])^2$, i.e., $G_2 = ([0,a[\cup \{v\}, (T)_v^{0,a,1,1})$, is a uninorm with the neutral element e if and only if $2 \prec 1$ and $v = (U)_e^{a,e,e,1}(a,1)$ (see Figure 2.3).*

  Observe, however, that v can be deleted from the semigroup G_2 and therefore it is enough to assume $G_2 = ([0,a[, T^*)$, where T^* is a linear transformation of the restriction of T to $[0,1[^2$ to the interval $[0,a[$.

This example illustrates the ordinal sum of two uninorms. Further, we want to extend the ordinal sum of uninorms to a countably many summands.

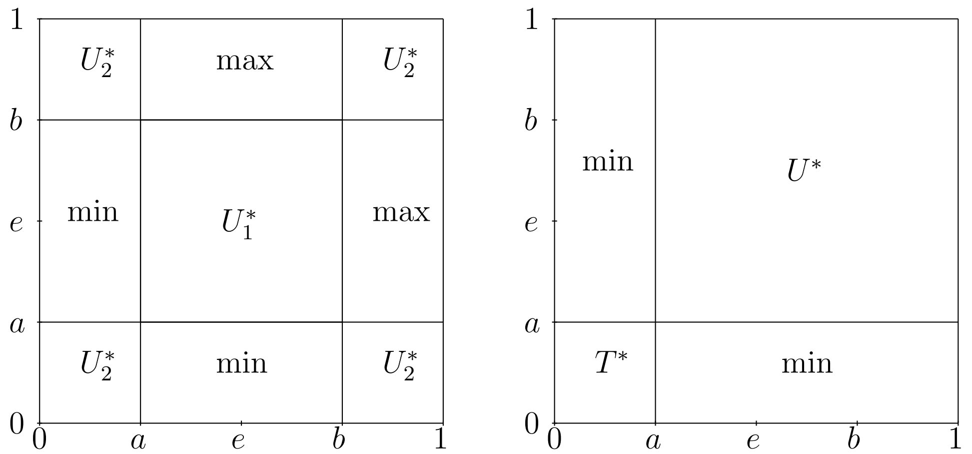

Figure 2.3: Ordinal sum of uninorms with two summands from Example 2.2.1 (i), where $U_2^* = (U_2)^{0,a,b,1}_{U_1^*(a,b)}$, $U_1^* = (U_1)_e^{a,e,e,b}$ (left) and from Example 2.2.1 (iii), where T^* is a linear transformation of $T|_{[0,1[^2}$ to interval $[0,a[$ and $U^* = (U)_e^{a,e,e,1}$ (right).

In the ordinal sum of t-norms, intervals $[a_k, b_k]$ need not cover the whole unit interval as the rest is covered by the minimum. However, in the case of uninorms the remainder should be covered by a mixture of the minimum and the maximum – more precisely by a commutative, associative, non-decreasing and idempotent operation, which is not uniquely determined. Therefore in the case of uninorms we will suppose that $\bigcup\limits_{k\in K} [a_k, b_k] = [0, e]$

and $\bigcup_{k\in K} [c_k, d_k] = [e, 1]$. Thus the ordinal sum uniquely determines the operation on the whole unit interval, except for the sets

$$B_1 = \bigcup_{k\in K} [a_k, b_k] \setminus \bigcup_{k\in K} [a_k, b_k[$$

$$C_1 = \bigcup_{k\in K} [c_k, d_k] \setminus \bigcup_{k\in K}]c_k, d_k] \,.$$

Note that for all $x \in [0, 1]$ there is $[x, x[\; = \;]x, x] = \emptyset$ and therefore if we denote $K_* = \{k \in K \mid]a_k, b_k[\neq \emptyset\}$ and $K^* = \{k \in K \mid]c_k, d_k[\neq \emptyset\}$ then $B_1 = \{b_k \mid k \in K\} \setminus \{a_k \mid k \in K_*\}$ and $C_1 = \{c_k \mid k \in K\} \setminus \{d_k \mid k \in K^*\}$. Since K is assumed to be countable then every $b \in B_1 \setminus \{e\}$ is an accumulation point of $\{a_k \mid k \in K_*\}$ (and similarly for $c \in C_1 \setminus \{e\}$). We denote $B_2 = B_1 \setminus \{e\}$, $C_2 = C_1 \setminus \{e\}$ and define functions $g\colon B_2 \longrightarrow [e, 1]$, $h\colon C_2 \longrightarrow [0, e]$, such that if for $b \in B_2$ we have $b = \lim_{i\longrightarrow\infty} a_{k_i}$ for $k_i \in K_*$, then

$$g(b) = \lim_{i\longrightarrow\infty} d_{k_i}. \tag{2.3}$$

Similarly, if for $c \in C_2$ we have $c = \lim_{i\longrightarrow\infty} d_{k_i}$ for $k_i \in K^*$, then

$$h(c) = \lim_{i\longrightarrow\infty} a_{k_i}. \tag{2.4}$$

If $g(b) \notin C_2$ for some $b \in B_2$ ($h(c) \notin B_2$ for some $c \in C_2$) then the value of $U(b, g(b))$ ($U(c, h(c))$) follows from the monotonicity of U. Therefore we have to separately cover only the case when $g(b) \in C_2$ ($h(c) \in B_2$).

Now we can introduce the ordinal sum of uninorms (Proposition 8 in [UNI2]).

Proposition 2.2.2

Assume $e \in [0, 1]$. Let K be an index set which is finite or countably infinite and let $(]a_k, b_k[)_{k\in K}$ be a disjoint system of open subintervals (which can be also empty) of $[0, e]$, such that $\bigcup_{k\in K} [a_k, b_k] = [0, e]$. Similarly, let $(]c_k, d_k[)_{k\in K}$ be a disjoint system of open subintervals (which can be also empty) of $[e, 1]$, such that $\bigcup_{k\in K} [c_k, d_k] = [e, 1]$. Let further these two systems be anti-comonotone, i.e., $b_k \leq a_i$ if and only if $c_k \geq d_i$ for all $i, k \in K$. We will denote $K_* = \{k \in K \mid]a_k, b_k[\neq \emptyset\}$ and $K^* = \{k \in K \mid]c_k, d_k[\neq \emptyset\}$. Assume a family $(U_k)_{k\in K_*\cap K^*}$ of proper uninorms on $[0, 1]^2$, a family $(U_k)_{k\in K_*\setminus K^*}$ of t-norms on $[0, 1]^2$ and a family $(U_k)_{k\in K^*\setminus K_*}$ of t-conorms on $[0, 1]^2$. Denote $B_1 = \{b_k \mid k \in K\} \setminus \{a_k \mid k \in K_*\}$ and $C_1 = \{c_k \mid k \in K\} \setminus \{d_k \mid k \in K^*\}$ and let $B = \{b \in B_1 \setminus \{e\} \mid g(b) \in C_1\}$, $C = \{c \in C_1 \setminus \{e\} \mid h(c) \in B_1\}$, where the functions g and h are defined by (2.3) and (2.4). Further assume a function $n\colon B \longrightarrow B \cup C$ given for all $b \in B$ by

$$n(b) \in \{b, g(b)\}.$$

Let the ordinal sum $U^e = (\langle a_k, b_k, c_k, d_k, U_k\rangle \mid k \in K)^e$ be given by $U^e(x,y) =$

$$\begin{cases} y & \text{if } x = e, \\ x & \text{if } y = e, \\ (U_k)^{a_k,b_k,c_k,d_k}_{v_k}(x,y) & \text{if } (x,y) \in ([a_k,b_k[\cup]c_k,d_k])^2, k \in K_* \cap K^*, \\ (U_k)^{a_k,b_k}(x,y) & \text{if } (x,y) \in ([a_k,b_k[\cup]c_k,d_k])^2, k \in K_* \setminus K^*, \\ (U_k)^{c_k,d_k}(x,y) & \text{if } (x,y) \in ([a_k,b_k[\cup]c_k,d_k])^2, k \in K^* \setminus K_*, \\ x & \text{if } y \in [b_k,c_k], x \in [a_k,d_k] \setminus [b_k,c_k], k \in K_* \cup K^*, \\ y & \text{if } x \in [b_k,c_k], y \in [a_k,d_k] \setminus [b_k,c_k], k \in K_* \cup K^*, \\ \min(x,y) & \text{if } (x,y) \in [b,c]^2 \setminus (]b,c[^2 \cup \{(b,c),(c,b)\}), \\ & \quad \text{where } b \in B,\ c = g(b),\ x+y < c+b, \\ \max(x,y) & \text{if } (x,y) \in [b,c]^2 \setminus (]b,c[^2 \cup \{(b,c),(c,b)\}), \\ & \quad \text{where } b \in B,\ c = g(b),\ x+y > c+b, \\ n(b) & \text{if } (x,y) = (b,c) \text{ or } (x,y) = (c,b),\ b \in B,\ c = g(b), \\ \min(x,y) & \text{if } (x,y) \in \{b\} \times [b,c] \cup [b,c] \times \{b\} \text{ and} \\ & \qquad b \in B_1 \setminus (B \cup \{e\}), c = g(b), \\ \max(x,y) & \text{if } (x,y) \in \{c\} \times [b,c] \cup [b,c] \times \{c\} \text{ and} \\ & \qquad c \in C_1 \setminus (C \cup \{e\}), b = h(c), \end{cases}$$

where $v_k = c_k$ ($v_k = b_k$) if there exists an $i \in K$ such that $b_k = a_i$, $c_k = d_i$ and U_i is disjunctive (conjunctive); $v_k = e$ if $b_k = c_k$; and otherwise

$$v_k = \begin{cases} n(b_k) & \text{if } b_k \in B, \\ b_k & \text{if } b_k \in B_1 \setminus (B \cup \{e\}), \\ c_k & \text{if } c_k \in C_1 \setminus (C \cup \{e\}). \end{cases}$$

Further, $(U_k)^{a_k,b_k,c_k,d_k}_{v_k}$ is given by the formula (2.2), $(U_k)^{a_k,b_k}$ ($(U_k)^{c_k,d_k}$) is a linear transformation of U_k to $[a_k,b_k]^2$ ($[c_k,d_k]^2$). Then U^e is a uninorm.

This result defines the ordinal sum of uninorms. In the following sections we will see that uninorms with continuous underlying functions can be expressed as an ordinal sum of semigroups related to uninorms (including t-norms, t-conorms and trivial semigroups), however, these semigroups need not be uninorms. This is caused by the fact that the boundary points of a transformed uninorm can behave differently than the remainder of the respective semigroup (for more details see Section 2.5).

2.3 Idempotent uninorms and ordinal sums of representable uninorms

The first step towards characterization of uninorms with continuous underlying functions was done in papers [UNI1] and [UNI3]. The main results on idempotent uninorms with respect to the ordinal sum construction are the following (Propositions 2 and 3 in [UNI1]).

Proposition 2.3.1
Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be an idempotent uninorm. Then $([0,1], U)$ is an ordinal sum of singleton semigroups $(\{x\}, \mathrm{Id})$ for $x \in [0,1]$.

The total order that yields an idempotent uninorm via an ordinal sum of trivial semigroups can be characterized as follows.

Proposition 2.3.2
Let P be an index set isomorphic with $[0,1]$ via the isomorphism φ. For all $p \in P$ we put $X_p = \{x\}$ if $\varphi(p) = x$. Let $e \in [0,1]$ and let $\preceq$ be a linear order on P. If $([0,1], U)$ is the ordinal sum of $\{(X_p, \mathrm{Id})\}_{p \in P}$ with the linear order $\preceq$ then U is an idempotent uninorm with the neutral element e if and only if the following two conditions are fulfilled:

(i) $p_1 \prec p_2$ *for all* $p_1, p_2 \in P$ *such that* $X_{p_1} = \{x_1\}$, $X_{p_2} = \{x_2\}$, $x_1 < x_2$ and $x_1, x_2 \in [0, e]$,

(ii) $p_1 \prec p_2$ *for all* $p_1, p_2 \in P$ *such that* $X_{p_1} = \{y_1\}$, $X_{p_2} = \{y_2\}$, $y_1 > y_2$ and $y_1, y_2 \in [e, 1]$.

These two results completely characterize the construction of idempotent uninorms via the ordinal sum. We see that idempotent uninorms are in one-to-one correspondence with special linear orders on $[0,1]$. In [UNI1] we can further find how are the corresponding special orders related for dual uninorms and for isomorphic uninorms.

Assume an idempotent uninorm U, a non-increasing function g from Theorem 1.1.6 and the corresponding linear order $\preceq$ on $[0,1]$. Then for all $x \in [0,e]$, $y \in [e,1]$ there is $x \prec y$ if and only if either $y < g(x)$, or $y = g(x)$, $x < g(g(x))$, or $y = g(x)$, $x = g(g(x))$ and $U(x,y) = x$.

The paper [UNI3] deals with ordinal sums of representable uninorms. If a uninorm can be expressed as an ordinal sum of representable uninorms then the underlying t-norm (t-conorm) is an ordinal sum of continuous t-norms (t-conorms) and thus the underlying functions of such a uninorm are continuous. Observe that if U is a representable uninorm then for all $x \in\,]0,1[$ there exists a unique $y \in\,]0,1[$ such that $U(x,y) = e$. Thus we can define a strictly decreasing function $h\colon\,]0,1[\, \longrightarrow\,]0,1[$ by $h(x) = y$ if $U(x,y) = e$. Moreover, each representable uninorm is discontinuous in exactly two points $(0,1)$ and $(1,0)$. After transformation of a representable uninorm to the set $([a,b[\, \cup \{v\} \cup\,]c,d])^2$ the set of all points of discontinuity of this transformed uninorm on $[a, b[\times]c, d]$ corresponds exactly to the graph of the respective transformation of the function h (restricted to $]0, e[$) to the function $h^*\colon\ ]a, b[\longrightarrow]c, d[$, with additional point (a, d), which corresponds to the transformation of the point $(0, 1)$ (see Figure 2.4).

For an ordinal sum of representable uninorms we then obtain the following result (Proposition 7 in [UNI3]).

Proposition 2.3.3

Assume a uninorm $U\colon [0,1]^2 \longrightarrow [0,1]$. If U is an ordinal sum of representable uninorms, i.e., $U = (\langle a_k, b_k, c_k, d_k, U_k\rangle \mid k \in K)^e$, for $e \in]0,1[$ and some suitable systems $(]a_k, b_k[)_{k\in K}$ and $(]c_k, d_k[)_{k\in K}$ with $a_k < b_k$ and $c_k < d_k$ for all $k \in K$, and a family of (proper) representable uninorms $(U_k)_{k\in K}$, then there exists a continuous strictly decreasing function $r\colon [0,1] \longrightarrow [0,1]$ with $r(0) = 1$, $r(e) = e$ and $r(1) = 0$ such that U is continuous on $[0,1] \setminus \{(x, r(x)) \mid x \in [0,1]\}$. Note that U need not be non-continuous on the whole set $\{(x, r(x)) \mid x \in [0,1]\}$.

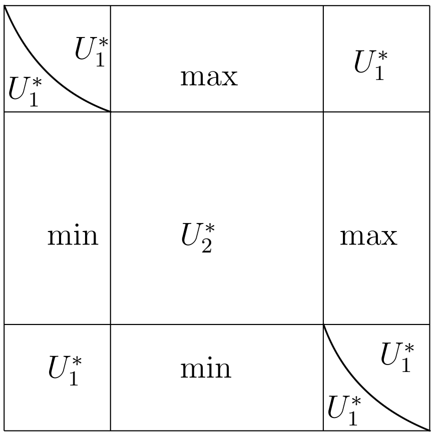

Figure 2.4: The ordinal sum of two representable uninorms. Here U_1^* is a transformation of U_1 to $([0, \frac{1}{4}[\cup[\frac{3}{4}, 1])^2$ given by (2.2), and U_2^* is a linear transformation of U_2 to $\left[\frac{1}{4}, \frac{3}{4}\right]^2$. The oblique lines denote the points of discontinuity of U.

The function r from the previous result is not unique, however, we usually assume the function r such that $U(x, y) = e$ implies $r(x) = y$ for all $x, y \in]0, 1[$. As we will observe in the following papers such a function r divides the unit square (except of the graph of r) into two parts: the set on which the uninorm attains values smaller than the neutral element e and the set on which the uninorm attains values greater than e. For more general uninorms with continuous underlying functions we obtain a similar division, however, here the dividing line needs not be strictly decreasing (see Section 2.4).

If we denote the set of all uninorms U such that $U(x,0) = 0$ for all $x \in [0,1[$ and $U(x,1) = 1$ for all $x \in]0,1]$ by $\mathcal{N}$ then we get the following result (Proposition 8 in [UNI3]).

Proposition 2.3.4
Assume a uninorm $U\colon [0,1]^2 \longrightarrow [0,1]$ such that $U \in \mathcal{U}$ and $U \notin \mathcal{N}$. Then U is an ordinal sum of a uninorm and a non-proper uninorm (i.e., a t-norm or a t-conorm).

This result shows us that an ordinal sum of representable uninorms always belongs to $\mathcal{N}$.

Observe, however, that both conditions from Propositions 2.3.3 and 2.3.4 are fulfilled also in the case when some summands in the ordinal sum are d-internal uninorms. We have the following result (Proposition 12 in [UNI3]).

Proposition 2.3.5
Assume a uninorm $U\colon [0,1]^2 \longrightarrow [0,1]$, $U \in \mathcal{U} \cap \mathcal{N}$ and let there exist a continuous strictly decreasing function $r\colon [0,1] \longrightarrow [0,1]$ with $r(0) = 1$, $r(e) = e$ and $r(1) = 0$ such that U is continuous on $[0,1] \setminus \{(x, r(x)) \mid x \in [0,1]\}$. Then U is an ordinal sum of representable uninorms and d-internal uninorms.

From this result we can conclude that a uninorm $U \in \mathcal{U} \cap \mathcal{N}$ is an ordinal sum of representable and d-internal uninorms if and only if there exists a continuous strictly decreasing function r from Proposition 2.3.5.

Moreover, a uninorm $U \in \mathcal{U} \cap \mathcal{N}$ is an ordinal sum of representable uninorms if and only if there exists a continuous strictly decreasing function r from Proposition 2.3.5 and U has countably many idempotent points. Indeed, if U has uncountably many idempotent points then at least one summand has to be equal to a d-internal uninorm.

Finally note that if for a uninorm U there exists a continuous strictly decreasing function $r\colon [0,1] \longrightarrow [0,1]$ with $r(0) = 1$, $r(e) = e$ and $r(1) = 0$ such that U is continuous on $[0,1] \setminus \{(x, r(x)) \mid x \in [0,1]\}$ then evidently $U \in \mathcal{U}$ and Proposition 2.3.4 implies that $U \in \mathcal{N}$. Indeed, if $U \notin \mathcal{N}$ then U is an ordinal sum of a uninorm and a t-norm (t-conorm). Then we can show that the set of the points of discontinuity cannot be covered by the graph of a strictly decreasing function defined on $[0,1]$ since this set contains either a horizontal, or a vertical segment (see Section 2.4).

2.4 Characterizing set-valued functions and the set of points of discontinuity

In the case of ordinal sums of representable uninorms we have seen that the set of points of discontinuity of the resulting uninorm is covered by the graph of a strictly decreasing function r such that $r(0) = 1$, $r(e) = e$ and $r(1) = 0$. This inspired us to examine the set of points of discontinuity also for general uninorms with continuous underlying functions in [UNI5]. However, in a general case, r is not a real-valued function since it can contain also vertical segments. Therefore we have to work with set-valued functions.

Definition 2.4.1
A mapping $p\colon [0,1] \longrightarrow \mathcal{P}([0,1])$ is called set-valued function *on $[0,1]$. Such a set-valued function assigns to every $x \in [0,1]$ a subset of $[0,1]$, i.e., $p(x) \subseteq [0,1]$. Assuming the standard order on $[0,1]$, a set-valued function p is called*

(i) non-increasing *if for all $x_1, x_2 \in [0,1]$, $x_1 < x_2$, we have $y_1 \geq y_2$ for all $y_1 \in p(x_1)$ and all $y_2 \in p(x_2)$ and thus the cardinality $\mathrm{Card}(p(x_1) \cap p(x_2)) \leq 1$,*

(ii) symmetric *if $y \in p(x)$ if and only if $x \in p(y)$ for all $x, y \in [0,1]$.*

The graph of a set-valued function p will be denoted by $G(p)$, i.e., $(x,y) \in G(p)$ if and only if $y \in p(x)$.

A set-valued function $p\colon [0,1] \longrightarrow \mathcal{P}([0,1])$ is called u-surjective *if for all $y \in [0,1]$ there exists an $x \in [0,1]$ such that $y \in p(x)$. It is easy to show that a symmetric set-valued function p is u-surjective whenever $p(x) \neq \emptyset$ for all $x \in [0,1]$.*

In Theorem 11 from [UNI5] we have shown the following result.

Theorem 2.4.2
Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{U}$. Then there exists a symmetric, u-surjective, non-increasing set-valued function r on $[0,1]$ such that U is continuous on $[0,1]^2 \setminus G(r)$ and $U(x,y) = e$ implies $(x,y) \in G(r)$ for all $(x,y) \in [0,1]^2$.

We have shown that $r(x)$ is a closed interval (including singletons) for all $x \in [0,1]$. Moreover, U need not be non-continuous in all points from $G(r)$. In fact, U is continuous in all points $(x,y) \in [0,1]^2$ such that $U(x,y) = e$. Moreover, U is continuous in all points $(0,x), (x,0)$ such that $x > \inf\{t \in [0,1] \mid U(t,0) > e\}$ and in all points $(1,x), (x,1)$ such that $x < \sup\{t \in [0,1] \mid U(t,1) < e\}$ (see Figure 2.5).

However, the claim opposite to the previous result does not hold in general, i.e., the existence of a set-valued function from Theorem 2.4.2 is not enough to ensure that a uninorm has continuous underlying functions.

Example 2.4.3
Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be given by

$$U(x,y) = \begin{cases} 0 & \textit{if } \max(x,y) < e, \\ x & \textit{if } y = e, \\ y & \textit{if } x = e, \\ \max(x,y) & \textit{otherwise.} \end{cases}$$

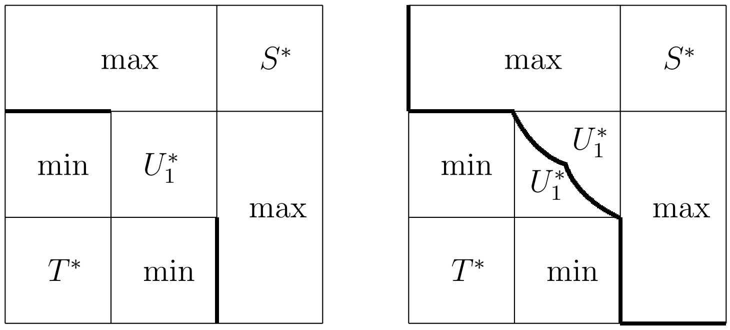

Figure 2.5: The uninorm U which is an ordinal sum of a representable uninorm, a continuous t-norm and a continuous t-conorm. Left: the bold lines denote the points of discontinuity of U. Right: the bold lines denote the characterizing set-valued function of U.

Then $U \in \mathcal{U}_{\max}$ is a uninorm, where the underlying t-norm is the drastic product $T_D\colon [0,1]^2 \longrightarrow [0,1]$ given by $T_D(x,y) = 0$ if $\max(x,y) < 1$ and $T_D(x,y) = \min(x,y)$ otherwise, and the underlying t-conorm is the maximum. This uninorm is non-continuous in points from $\{e\} \times [0,e] \cup [0,e] \times \{e\}$. Thus the corresponding set-valued function is given by

$$r(x) = \begin{cases} [e,1] & \text{if } x = 0, \\ e & \text{if } x \in \left]0,e\right[, \\ [0,e] & \text{if } x = e, \\ 0 & \text{otherwise.} \end{cases}$$

Since $U(x,y) = e$ implies $x = y = e$ we see that U is continuous on $[0,1]^2 \setminus G(r)$ and r is a symmetric, u-surjective, non-increasing set-valued function such that $U(x,y) = e$ implies $(x,y) \in G(r)$. However, the drastic product t-norm is not continuous and thus $U \notin \mathcal{U}$.

As we see from the previous example, the problem arises whenever $G(r)$ contains points $(x,e),(e,x)$ for some $x \in [0,1]$, $x \neq e$. If the underlying t-norm (t-conorm) is continuous then U is left-(right-)continuous in all points $(x,e),(e,x)$ for $x \in [0,e]$ ($x \in [e,1]$). However, in the previous example the uninorm U is neither left- nor right continuous in all points $(x,e),(e,x)$ for $x \in \left]0,e\right[$. Generally, we were able to show that if $U \in \mathcal{U}$ then in each point $(x,y) \in [0,1]^2$ the uninorm U is either left-continuous or right-continuous (see Proposition 13 in [UNI5]). Therefore the following result completely characterizes uninorms with continuous underlying functions (see Theorem 14 in [UNI5]).

Theorem 2.4.4
Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm. Then $U \in \mathcal{U}$ if and only if U is continuous on $[0,1]^2 \setminus G(r)$, where r is a symmetric, u-surjective, non-increasing set-valued function

such that $U(x,y)=e$ implies $(x,y)\in G(r)$, and in each point $(x,y)\in[0,1]^2$ the uninorm U is either left-continuous or right-continuous (or continuous).

The paper [UNI6] studies characterizing functions of uninorms with continuous underlying functions further. Although the paper is quite technical, these properties are necessary for further results on the characterization of uninorms from $\mathcal{U}$ via the ordinal sum construction.

As we mentioned above, the graph of the characterizing set-valued function of a uninorm $U\in\mathcal{U}$ divides the unit square (except of the graph itself) into two parts: above the graph of the characterizing set-valued function the uninorm U attains values greater than e and below this graph the uninorm U attains values smaller than e. Note that this means that in the case of idempotent uninorms the graph of the characterizing set-valued function contains the graph of the non-increasing function g from Theorem 1.1.6.

The characterizing set-valued function can be decomposed into maximal horizontal, maximal vertical and maximal strictly decreasing segments. Moreover, the border points of these segments are always idempotent points of the corresponding uninorm. Here horizontal segments correspond to t-norm summands, vertical segments correspond to t-conorm summands and strictly decreasing segments, as we have seen above (see Section 2.3), correspond to summands composed of representable and d-internal uninorms.

Observe that if a uninorm U is continuous on $[0,1]^2\setminus G(r)$, where r is a symmetric, u-surjective, non-increasing set-valued function such that $U(x,y)=e$ implies $(x,y)\in G(r)$ then U is evidently continuous on $[0,e[^2\cup\,]e,1]^2$. This inspired us to study uninorms continuous on $[0,e[^2\cup\,]e,1]^2$, see Section 2.6.

2.5 Decomposition of uninorms with continuous underlying functions via the ordinal sum

In paper [UNI4] we have studied the decomposition of a uninorm with continuous underlying functions into Archimedean, representable and idempotent semigroups with respect to the ordinal sum construction (in the sense of Clifford). Observe that an idempotent semigroup can be further decomposed via the ordinal sum (see Section 2.3), however, an Archimedean (representable) semigroup cannot be expressed as a non-trivial ordinal sum of two (or more) semigroups. Note that if we speak about non-trivial ordinal sum then each of the summands is a proper subsemigroup of the resulting semigroup. In this case, i.e., if $(X,*)$ cannot be expressed as a non-trivial ordinal sum of two (or more) semigroups we will say that $(X,*)$ (or simply X) is irreducible with respect to the ordinal sum construction. For the simplicity, in the following we will write (X,U) instead of

$(X, U|_{X^2})$.

For better understanding, in [UNI4], we have first described the decomposition for uninorms with continuous Archimedean underlying functions. The characterizing formulas for such uninorms can be found in [44, 45].

Let us start with the case when both underling functions are nilpotent. Then $U \in \mathcal{U}_{\min} \cup \mathcal{U}_{\max}$. For a nilpotent t-norm (t-conorm) the interval $[0,1[$ $(]0,1])$ is an irreducible set and therefore in this case $[0,e[$ and $]e,1]$ are irreducible sets. Thus each uninorm with nilpotent underlying functions can be decomposed into three semigroups, $G_1 = ([0,e[\,,U)$, $G_2 = (\{e\},U)$, $G_3 = (]e,1]\,,U)$. Since e is the neutral element for the corresponding linear order in the ordinal sum we have $1 < 2$ and $3 < 2$. Then $1 < 3$ implies $U \in \mathcal{U}_{\min}$ and $3 < 1$ implies $U \in \mathcal{U}_{\max}$.

In the case that the underlying functions are strict there are two possibilities. If U is representable then irreducible sets are $\{0\}$, $]0,1[$ and $\{1\}$. In the opposite case the irreducible sets are $\{0\}$, $]0,e[\,$, $\{e\}$, $]e,1[\,$, $\{1\}$. Similarly as above we can obtain the following:

- If U is representable then it can be decomposed into three semigroups, $G_1 = (\{0\},U)$, $G_2 = (]0,1[\,,U)$, $G_3 = (\{1\},U)$. Here again $1 < 2$ and $3 < 2$. Then $1 < 3$ implies that U is conjunctive and $3 < 1$ implies that U is disjunctive.

- If U is not representable and T_U is strict and S_U is strict then U is an ordinal sum of five semigroups $G_1 = (\{0\},U)$, $G_2 = (]0,e[\,,U)$, $G_3 = (\{e\},U)$, $G_4 = (]e,1[\,,U)$ and $G_5 = (\{1\},U)$. Due to the monotonicity we get $1 < 2 < 3$ and $5 < 4 < 3$. Thus there are six possible orderings, each corresponding to one form of a uninorm with strict underlying functions from [44] (see Example 2 in [UNI4]).

- If T_U is nilpotent and S_U is strict then U is an ordinal sum of four semigroups $G_1 = ([0,e[\,,U)$, $G_2 = (\{e\},U)$, $G_3 = (]e,1[\,,U)$ and $G_4 = (\{1\},U)$. Due to the monotonicity we get $1 < 2$ and $4 < 3 < 2$. Then $1 < 4 < 3 < 2$ implies $U \in \mathcal{U}_{\min}$, $4 < 3 < 1 < 2$ implies $U \in \mathcal{U}_{\max}$ and $4 < 1 < 3 < 2$ implies that $U(1,x) = 1$ for all $x \in [0,1]$ and $U(x,y) = \min(x,y)$ for $x < e \leq y < 1$.

- If T_U is strict and S_U is nilpotent then U is an ordinal sum of four semigroups $G_1 = (\{0\},U)$, $G_2 = (]0,e[\,,U)$, $G_3 = (\{e\},U)$ and $G_4 = (]e,1]\,,U)$ and we get similar results as in the previous case.

From the previous discussion we can observe that even though we have defined summands in the ordinal sum of uninorms on sets $[a_k,b_k[\, \cup \{v_k\} \cup\,]c_k,d_k]$ $([a_k,b_k[\, \cup\,]c_k,d_k])$, in the case when the underlying t-norm (t-conorm) is strict the semigroup $([a_k,b_k[\, \cup$

$\{v_k\} \cup]c_k, d_k], U)$ can be further decomposed to $(]a_k, b_k[\cup \{v_k\} \cup]c_k, d_k], U)$ and $(\{a_k\}, U)$ $\big(([a_k, b_k[\cup \{v_k\} \cup]c_k, d_k[, U)$ and $(\{d_k\}, U)\big)$.

In the following definition we describe irreducible subsemigroups of a uninorm with continuous underlying functions and its internal subsemigroups which we will not decompose further. This is motivated by the definition of the ordinal sum of t-norms in Proposition 1.1.3 (and by the representation of continuous t-norms) where the minimum is not decomposed further into trivial semigroups.

Definition 2.5.1

Let $a, b, c, d \in [0,1]$ with $a < b < c < d$, $v \in [b, c]$. Then

(i) *a semigroup $(]a, b[\cup \{v\} \cup]c, d[, *)$ will be called a* representable *semigroup if $*$ is isomorphic via (2.2) to a restriction of a representable uninorm to $]0,1[^2$,*

(ii) *a semigroup $(]a, b[, *)$ will be called a* t-strict *semigroup if $*$ is linearly isomorphic to a restriction of a strict t-norm to $]0,1[^2$,*

(iii) *a semigroup $(]c, d[, *)$ will be called an* s-strict *semigroup if $*$ is linearly isomorphic to a restriction of a strict t-conorm to $]0,1[^2$,*

(iv) *a semigroup $([a, b[, *)$ will be called a* t-nilpotent *semigroup if $*$ is linearly isomorphic to a restriction of a nilpotent t-norm to $[0,1[^2$,*

(v) *a semigroup $(]c, d], *)$ will be called an* s-nilpotent *semigroup if $*$ is linearly isomorphic to a restriction of a nilpotent t-conorm to $]0,1]^2$,*

(vi) *a semigroup $(]a, b[\cup]c, d[, *)$ will be called a* d-internal *semigroup if $*$ is isomorphic via (2.2) to a restriction of an d-internal uninorm to $(]0,1[\setminus \{e\})^2$,*

(vii) *a semigroup $(]a, b[, *)$ will be called a* t-internal *semigroup if $* = \min$,*

(viii) *a semigroup $(]c, d[, *)$ will be called an* s-internal *semigroup if $* = \max$.*

We will denote the set of semigroups from the previous definition and trivial semigroups by $\mathcal{H}$. The decomposition of a uninorm U with continuous underlying functions into semigroups from $\mathcal{H}$ is rather technical. Not going much into details, using the ordering induced by the characterizing set-valued function of U and the partition of the unit interval induced by the characterizing set-valued function and the set of idempotent points of U (see Definition 7 and Lemma 12 in [UNI4]) we were able to show the following result (see Proposition 11 in [UNI4]).

Theorem 2.5.2

Let $U \in \mathcal{U}$. Then U can be expressed as an ordinal sum of a countable number of semigroups from $\mathcal{H}$.

An opposite result was shown as well. Using the semigroups from $\mathcal{H}$, with a suitable linear order on the corresponding index set which preserves the monotonicity, we can always construct a uninorm with continuous underlying functions via the ordinal sum construction (see Proposition 12 in [UNI4]).

Remark 2.5.3
After publishing the paper [UNI4], I have realized that there is a mistake in Remark 1 *and in discussion above Proposition* 9*. The problem is that the closure of a countable set need not be countable. As an example we can take rational numbers from the unit interval. I was misled by the ordinal sum of t-norms which is defined for a countable index set of summands. However, the ordinal sum of t-norms specifies only non-trivial summands and the rest is filled by the minimum. Therefore a possible uncountability is hidden in the area which is not covered by non-trivial summands.*

On the other hand, in the ordinal sum of uninorms, as well as for t-norms, each non-trivial summand contains an interval, which contains at least one rational number. Thus the number of non-trivial summands is always countable. The correction which contains an example of a uninorm (t-norm) which cannot be expressed as an ordinal sum of a countable number of Archimedean, representable and idempotent summands, based on the Cantor set, will be published in the paper [58] which discusses the ordinal sum construction for aggregation functions on the real unit interval. Therefore the correct wording of the main result of [UNI4] is that each uninorm with continuous underlying functions can be expressed as an ordinal sum of a countable number of eight types of semigroups from Definition 2.5.1 and a possibly uncountable number of trivial semigroups defined on singletons. The proof is exactly the same as the proof of Proposition 11 in [UNI4], the only correction has to be done in the partition of the unit interval which was given in Definition 7 in [UNI4], where the sets A and D are possibly uncountable and then also the index sets M_3 and O_3 are possibly uncountable.

2.6 Uninorms continuous on $[0, e[^2 \cup]e, 1]^2$

In [UNI7] we extended the results from the previous sections also for uninorms which are continuous on $[0, e[^2 \cup]e, 1]^2$. In such a case the underlying t-norm (t-conorm) need not be continuous. Moreover, the following example from [46] shows that a uninorm continuous on $[0, e[^2 \cup]e, 1]^2$ cannot be expressed as an ordinal sum of representable, Archimedean and idempotent semigroups, in general.

Example 2.6.1
For $e \in \left]0,1\right[$, $x_0 \in [0,e[$ and a t-conorm S, the operation $U\colon [0,1]^2 \longrightarrow [0,1]$ given by

$$U(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,e[^2, \\ e + (1-e)\cdot S(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ 1 & \text{if } \max(x,y) = 1, \\ x_0 & \text{if } (x,y) \in \left]0,x_0\right[\times \left]e,1\right[\cup \\ & \qquad \left]e,1\right[\times \left]0,x_0\right[, \\ \min(x,y) & \text{otherwise} \end{cases}$$

is a uninorm if and only if $S(x,y) < 1$ for all $(x,y) \in [0,1[^2$. We see that for $x_0 > 0$ and a continuous t-conorm S with no divisors of 1 the uninorm U can be expressed as an ordinal sum of a semigroup acting on $[0,1[$ and a trivial semigroup defined on $\{1\}$, however, it cannot be decomposed further. Observe that T_U is the drastic product t-norm (see Example 2.4.3 for the definition of the drastic product t-norm).

The underlying functions of uninorms continuous on $[0,e[^2 \cup \left]e,1\right]^2$ are related to continuous t-subnorms and t-superconorms as follows. For each t-norm (and similarly for t-conorms) we can define its border continuous projection (see [36] and [UNI7]) $M_T\colon [0,1]^2 \longrightarrow [0,1]$ by

$$M_T(x,y) = \begin{cases} T(x,y) & \text{if } (x,y) \in [0,1[^2, \\ \lim\limits_{u\longrightarrow 1^-} T(u,y) & \text{if } x = 1, y < 1, \\ \lim\limits_{u\longrightarrow 1^-} T(x,u) & \text{if } x < 1, y = 1, \\ \lim\limits_{\substack{u\longrightarrow 1^- \\ v\longrightarrow 1^-}} T(u,v) & \text{if } x = y = 1. \end{cases}$$

However, such a border continuous projection need not be associative. Therefore we have shown the following results (see Proposition 5 and Corollary 1 in [UNI7]).

Proposition 2.6.2
For a t-norm $T\colon [0,1]^2 \longrightarrow [0,1]$ its border-continuous projection M_T is a t-subnorm if and only if the following two conditions are satisfied:

(i) *for all $x,y \in [0,1[$ either $T(u_0,x) = \lim\limits_{u\longrightarrow 1^-} T(u,x)$ for some $u_0 \in [0,1[$, or $T(a,y) = \lim\limits_{v\longrightarrow a^-} T(v,y)$, where $a = \lim\limits_{u\longrightarrow 1^-} T(u,x)$,*

(ii) *either $\lim\limits_{u\longrightarrow 1^-} T(u,u) = 1$, or $T(u_0,v_0) = \lim\limits_{u\longrightarrow 1^-} T(u,u)$ for some $u_0, v_0 \in [0,1[$, or for all $x \in [0,1[$ there is $T(b,x) = \lim\limits_{v\longrightarrow b^-} T(v,x)$, where $b = \lim\limits_{u\longrightarrow 1^-} T(u,u)$.*

Corollary 2.6.3
Let $T\colon [0,1]^2 \longrightarrow [0,1]$ be a t-norm left-continuous on $[0,1[^2$. Then its border-continuous projection $M_T\colon [0,1]^2 \longrightarrow [0,1]$ is a t-subnorm.

Since we focus on uninorms which are continuous on $[0,e[^2 \cup]e,1]^2$ then M_{T_U} is always a t-subnorm and similar result can be obtained also for S_U. This means that T_U (and similarly for S_U) coincides on $[0,1[^2$ with a continuous t-subnorm.

If T_U is not continuous its border-continuous projection is a continuous proper t-subnorm, which can be decomposed into an ordinal sum of continuous t-subnorms (see [38, 55]).

Theorem 2.6.4
A mapping $M\colon [0,1]^2 \longrightarrow [0,1]$ is a continuous proper t-subnorm if and only if it is an ordinal sum of continuous Archimedean t-norms and a continuous Archimedean proper t-subnorm, $M = (\langle a_k, b_k, M_k\rangle \mid k \in K)$, where $(]a_k, b_k[)_{k\in K}$ is a disjoint system of open subintervals of $[0,1]$ with $b_{k_0} = 1$ for some $k_0 \in K$, M_{k_0} is a continuous Archimedean proper t-subnorm and M_k is a continuous Archimedean t-norm for all $k \neq k_0$, i.e.,

$$M(x,y) = \begin{cases} a_k + (b_k - a_k) M_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right) & \text{if } (x,y) \in]a_k, b_k]^2, \\ \min(x,y) & \text{else.} \end{cases}$$

Observe that in the ordinal sum of t-subnorms left-open intervals are used, while for the ordinal sum of t-norms right-open intervals are used, compare Proposition 1.1.3 (see [38, 55] for more details). Thus we see that if T_U is not continuous then it can be expressed as an ordinal sum of continuous Archimedean t-norms, a continuous Archimedean proper t-subnorm restricted to $[0,1[^2$ and a trivial semigroup acting on $\{1\}$. We will now distinguish two cases. First we will assume that border-continuous projections of T_U and S_U are Archimedean, and later we will assume the case when one or both of them is not Archimedean.

As we have seen in Example 2.6.1, generally we cannot extend the results from the previous sections also for uninorms continuous on $[0,e[^2 \cup]e,1]^2$, however, under assumption of cancellativity on some subareas of the unit square we can characterize also these uninorms. For continuous, cancellative border-continuous projections of T_U and S_U we obtain Propositions 8, 9 and 10 from [UNI7].

Proposition 2.6.5
Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm such that M_{T_U} is a proper, continuous, cancellative t-subnorm and M_{S_U} is a proper, continuous, cancellative t-superconorm. Then there exists an increasing isomorphism $\varphi\colon [0,1] \longrightarrow [0,1]$ such that $U(x,y) = \varphi^{-1}(UP(\varphi(x), \varphi(y)))$ for all $(x,y) \in [0,1]^2$, where UP is a uninorm such that $M_{T_{UP}} = \frac{x\cdot y}{2}$ and $M_{S_{UP}} = \frac{1+x+y-x\cdot y}{2}$.

Proposition 2.6.6
Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm such that M_{T_U} is a proper, continuous, cancellative t-subnorm and M_{S_U} is a continuous, cancellative t-conorm. Then there exists an increasing isomorphism $\varphi\colon [0,1] \longrightarrow [0,1]$ such that $U(x,y) = \varphi^{-1}(UPT(\varphi(x), \varphi(y)))$ for all $(x,y) \in [0,1]^2$, where UPT is a uninorm such that $M_{T_{UPT}} = \frac{x \cdot y}{2}$ and $M_{S_{UPT}} = x + y - x \cdot y$.

Proposition 2.6.7
Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm such that M_{T_U} is a continuous, cancellative t-norm and M_{S_U} is a proper, continuous, cancellative t-superconorm. Then there exists an increasing isomorphism $\varphi\colon [0,1] \longrightarrow [0,1]$ such that $U(x,y) = \varphi^{-1}(UPS(\varphi(x), \varphi(y)))$ for all $(x,y) \in [0,1]^2$, where UPS is a uninorm such that $M_{T_{UPS}} = x \cdot y$ and $M_{S_{UPS}} = \frac{1+x+y-x \cdot y}{2}$.

By these results we have characterized uninorms continuous on $[0,e[^2 \cup]e,1]^2$, which have Archimedean and cancellative underlying functions, on $[0,e]^2$ and on $[e,1]^2$. For the remainder of the unit square we obtain the following result, which is similar to the corresponding result on uninorms with strict underlying functions from [44] (see Lemma 7 in [UNI7]).

Proposition 2.6.8
Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm with neutral element $e \in]0,1[$, such that M_{T_U} and M_{S_U} are continuous and cancellative. Then exactly one of the following seven statements holds:

(i) $U \in \mathcal{U}_{\min}$,

(ii)
$$U(x,y) = \begin{cases} e \cdot T_U(\frac{x}{e}, \frac{y}{e}) & \textit{if } (x,y) \in [0,e]^2, \\ e + (1-e) \cdot S_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \textit{if } (x,y) \in [e,1]^2, \\ 1 & \textit{if } x = 1 \textit{ or } y = 1, \\ \min(x,y) & \textit{otherwise,} \end{cases}$$

(iii)
$$U(x,y) = \begin{cases} e \cdot T_U(\frac{x}{e}, \frac{y}{e}) & \textit{if } (x,y) \in [0,e]^2, \\ e + (1-e) \cdot S_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \textit{if } (x,y) \in [e,1]^2, \\ 1 & \textit{if } x = 1, y > 0 \textit{ or } y = 1, x > 0, \\ \min(x,y) & \textit{otherwise,} \end{cases}$$

(iv) $U \in \mathcal{U}_{\max}$,

(v)

$$U(x,y)=\begin{cases} e\cdot T_U(\frac{x}{e},\frac{y}{e}) & \textit{if } (x,y)\in[0,e]^2\,,\\ e+(1-e)\cdot S_U(\frac{x-e}{1-e},\frac{y-e}{1-e}) & \textit{if } (x,y)\in[e,1]^2\,,\\ 0 & \textit{if } x=0 \textit{ or } y=0,\\ \max(x,y) & \textit{otherwise,}\end{cases}$$

(vi)

$$U(x,y)=\begin{cases} e\cdot T_U(\frac{x}{e},\frac{y}{e}) & \textit{if } (x,y)\in[0,e]^2\,,\\ e+(1-e)\cdot S_U(\frac{x-e}{1-e},\frac{y-e}{1-e}) & \textit{if } (x,y)\in[e,1]^2\,,\\ 0 & \textit{if } x=0,y<1 \textit{ or } y=0,x<1,\\ \max(x,y) & \textit{otherwise,}\end{cases}$$

(vii) *U is representable.*

This completely characterizes the case when border-continuous projections of both underlying functions are cancellative. For a combination of a (continuous) nilpotent underlying function and a non-continuous cancellative underlying function we get Propositions 15 and 16 from [UNI7].

Proposition 2.6.9
Let $U\colon [0,1] \longrightarrow [0,1]^2$ be a uninorm with the neutral element $e\in\,]0,1[$ such that M_{T_U} is continuous and cancellative and S_U is a nilpotent t-conorm. Then exactly one of the following three statements holds:

(i) $U\in\mathcal{U}_{\min}$,

(ii) $U\in\mathcal{U}_{\max}$,

(iii)

$$U(x,y)=\begin{cases} e\cdot T_U(\frac{x}{e},\frac{y}{e}) & \textit{if } (x,y)\in[0,e]^2\,,\\ e+(1-e)\cdot S_U(\frac{x-e}{1-e},\frac{y-e}{1-e}) & \textit{if } (x,y)\in[e,1]^2\,,\\ 0 & \textit{if } x=0 \textit{ or } y=0,\\ \max(x,y) & \textit{otherwise.}\end{cases}$$

Proposition 2.6.10
Let $U\colon [0,1] \longrightarrow [0,1]^2$ be a uninorm with the neutral element $e\in\,]0,1[$ such that M_{S_U} is continuous and cancellative and T_U is a nilpotent t-norm. Then exactly one of the following three statements holds:

(i) $U\in\mathcal{U}_{\min}$,

(ii) $U\in\mathcal{U}_{\max}$,

(iii)

$$U(x,y)=\begin{cases} e\cdot T_U(\frac{x}{e},\frac{y}{e}) & \textit{if } (x,y)\in[0,e]^2\,,\\ e+(1-e)\cdot S_U(\frac{x-e}{1-e},\frac{y-e}{1-e}) & \textit{if } (x,y)\in[e,1]^2\,,\\ 1 & \textit{if } x=1 \textit{ or } y=1,\\ \min(x,y) & \textit{otherwise.}\end{cases}$$

Thus in our investigation of uninorms continuous on $[0,e[^2\cup]e,1]^2$ with Archimedean underlying functions we did not cover only the cases when the border-continuous projection of the underlying t-norm (t-conorm) is a proper t-subnorm (t-superconorm) which is not cancellative.

If M_{T_U} is not Archimedean (and similarly for M_{S_U}) we know that T_U can be expressed as an ordinal sum of continuous Archimedean t-norms, a continuous Archimedean proper t-subnorm restricted to $[0,1[^2$ and a trivial semigroup acting on $\{1\}$. We can now distinguish three cases:

1. If S_U and T_U are not continuous and the respective proper t-subnorm (t-superconorm) from the ordinal sum decomposition of M_{T_U} (M_{S_U}) is cancellative.

2. If T_U is a continuous t-norm, S_U is not continuous and the respective proper t-superconorm from the ordinal sum decomposition of M_{S_U} is cancellative.

3. If S_U is a continuous t-conorm, T_U is not continuous and the respective proper t-subnorm from the ordinal sum decomposition of M_{T_U} is cancellative.

Since all continuous t-norms (t-conorms) can be expressed as an ordinal sum of continuous Archimedean t-norms (t-conorms) in all three cases we can find idempotent points $a\in[0,e]$, $b\in[e,1]$ such that U on $[a,b]^2$ is a uninorm (or a t-norm, or a t-conorm) with Archimedean underlying functions (which was characterized above) and U on $[0,a]^2$ is a continuous t-norm (on $[0,a]^2$) and U on $[b,1]^2$ is a continuous t-conorm (on $[b,1]^2$). Moreover, Proposition 13 in [UNI7] shows that $([0,a[\cup\{U(a,b)\}\cup]b,1])^2$ is closed under U. If $U(a,b)$ is the neutral element of U restricted to $([0,a[\cup\{U(a,b)\}\cup]b,1])^2$ then $([0,1],U)$ can be expressed as an ordinal sum of $G_1=([0,a[\cup\{U(a,b)\}\cup]b,1]\,,U)$ and $G_2=([a,b]\,,U)$, where $1<2$ and G_1 is isomorphic to a uninorm with continuous underlying functions.

If $U(a,b)$ isn't the neutral element of U restricted to $([0,a[\cup\{U(a,b)\}\cup]b,1])^2$ the situation is a bit more complicated and it was not covered in [UNI7]. However, similarly as in the previous section we can show that in this case U can be expressed as an ordinal sum of semigroups from $\mathcal{H}$ and one or two additional semigroups, one corresponding to a restriction of a continuous t-subnorm and second corresponding to a restriction of a continuous t-superconorm.

If we summarize these results we see that uninorms continuous on $[0,e[^2 \cup]e,1]^2$ have a similar structure as uninorms with continuous underlying functions, except for the case when the ordinal sum decomposition of M_{T_U} (M_{S_U}) contains a proper t-subnorm (t-superconorm) which is not cancellative.

Chapter 3

n-Uninorms with continuous underlying functions

n-Uninorms generalize uninorms and are in fact composed of uninorms of lower orders glued together partially by the local annihilator and partially by the ordinal sum construction. This fact inspired us to introduce the z-ordinal sum construction which extends Clifford's ordinal sum also to partially ordered families of semigroups. Using this construction we were then able to provide a similar characterization as in the previous chapter also for n-uninorms with continuous underlying functions. As in the case of uninorms with continuous underlying functions also here we study characterizing set-valued functions of such n-uninorms and provide their decomposition into irreducible subsemigroups with respect to the z-ordinal sum construction. This chapter is based on papers [NUN1,NUN2,NUN3,NUN4].

3.1 Z-ordinal sum construction and n-uninorms with continuous underlying functions

We have described the basic properties of n-uninorms with continuous underlying functions in two papers [NUN1] and [NUN2].

Before we start to describe the properties of n-uninorms, let us turn our attention to the main result on which the characterization of n-uninorms with continuous underlying functions is based – the z-ordinal sum construction, which was introduced in Theorem 4.2 from [NUN1].

Theorem 3.1.1
Let A and B be two index sets such that $A \cap B = \emptyset$ and $C = A \cup B \neq \emptyset$. Let $(G_\alpha)_{\alpha\in C}$ with $G_\alpha = (X_\alpha, *_\alpha)$ be a family of semigroups and let the set C be partially ordered by

the binary relation $\preceq$ such that $(C, \preceq)$ is a meet semi-lattice. Further suppose that each semigroup G_α for $\alpha \in A$ possesses an annihilator z_α, and for all $\alpha, \beta \in C$ such that α and β are incomparable there is $\alpha \wedge \beta \in A$. Assume that for all $\alpha, \beta \in C$, $\alpha \neq \beta$, the sets X_α and X_β are either disjoint or that $X_\alpha \cap X_\beta = \{x_{\alpha,\beta}\}$. In the second case suppose that for all $\gamma \in C$ which is incomparable with $\alpha \wedge \beta$ there is $\alpha \wedge \gamma = \beta \wedge \gamma$ and for each $\gamma \in C$ with $\alpha \wedge \beta \prec \gamma \prec \alpha$ or $\alpha \wedge \beta \prec \gamma \prec \beta$ we have $X_\gamma = \{x_{\alpha,\beta}\}$. Further,

(i) *in the case that $\alpha \wedge \beta \in A$ then $x_{\alpha,\beta} = z_{\alpha\wedge\beta}$ is the annihilator of both G_β and G_α;*

(ii) *in the case that $\alpha \wedge \beta = \alpha \in B$ then $x_{\alpha,\beta}$ is both the annihilator of G_β and the neutral element of G_α.*

Put $X = \bigcup_{\alpha\in C} X_\alpha$ and define the binary operation $*$ on X by

$$x * y = \begin{cases} x *_\alpha y & \text{if } (x,y) \in X_\alpha \times X_\alpha, \\ x & \text{if } (x,y) \in X_\alpha \times X_\beta,\ \alpha \neq \beta, \text{ and } \alpha \wedge \beta = \alpha \in B, \\ y & \text{if } (x,y) \in X_\alpha \times X_\beta,\ \alpha \neq \beta, \text{ and } \alpha \wedge \beta = \beta \in B, \\ z_\gamma & \text{if } (x,y) \in X_\alpha \times X_\beta,\ \alpha \neq \beta, \text{ and } \alpha \wedge \beta = \gamma \in A. \end{cases}$$

Then $G = (X, *)$ is a semigroup. The semigroup G is commutative if and only if for each $\alpha \in C$ the semigroup G_α is commutative.

The set A from the previous theorem will be called the *branching set*. If the branching set is empty, i.e., if $A = \emptyset$ then the set $C = B$ is linearly ordered and the z-ordinal sum reduces to the standard ordinal sum construction of Clifford. Further, if each semigroup G_α for $\alpha \in C$ is trivial and $A = C$ then the z-ordinal sum $(X, *)$ of G_α is given by $x * y = x \wedge^* y$, where the order $\leq^*$ is given for $x \in X_\alpha$ and $y \in X_\beta$ by $x \leq^* y$ if $\alpha \preceq \beta$.

If $x \in X_\alpha \cap X_\beta$ for some $\alpha, \beta \in C$ then the condition $\alpha \wedge \gamma = \beta \wedge \gamma$ for all $\gamma \in C$ incomparable with $\alpha \wedge \beta$ is necessary, otherwise the associativity could be violated. However, alternatively we can require that if $x \in X_\alpha \cap X_\beta$ and for some $\gamma \in C$, which is incomparable with $\alpha \wedge \beta$, there is $\alpha \wedge \gamma \neq \beta \wedge \gamma$ then $X_\gamma = \{z_{\alpha\wedge\beta\wedge\gamma}\}$.

Observe that the z-ordinal sum construction enables us to construct non-decreasing functions $F\colon [0,1]^2 \longrightarrow [0,1]$, $F(0,0) = 0$, $F(1,1) = 1$, with an annihilator inside the unit interval, while in the case of the ordinal sum the annihilator of such functions was always on its boundary, i.e., at 0 or at 1. For example, we can construct nullnorms and n-uninorms via the z-ordinal sum construction.

Similarly as in the case of idempotent uninorms also for idempotent n-uninorms we have the following result (see Proposition 4.16 in [NUN1]).

Proposition 3.1.2
Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be an idempotent n-uninorm. Then $([0,1], U^n)$ is a z-ordinal sum of singleton semigroups $(\{x\}, \mathrm{Id})$ for $x \in [0,1]$.

If we ask which partial orders on $[0,1]$ yield idempotent n-uninorms we can use the following result (see Proposition 4.17 in [NUN1]).

Proposition 3.1.3
Let P be an index set isomorphic with $[0,1]$ via the isomorphism φ. For all $p \in P$ we put $X_p = \{x\}$ if $\varphi(p) = x$. Let $e_1, \dots, e_n, z_1, \dots, z_{n-1} \in [0,1]$, $0 = z_0 < z_1 < \dots < z_n = 1$, $e_i \in [z_{i-1}, z_i]$ for $i = 1, \dots, n$. Denote $A = \{q_1, \dots, q_{n-1}\}$, where $X_{q_i} = \{z_i\}$ for $i = 1, \dots, n-1$ and $B = P \setminus A$. Let $\preceq$ be a partial order on P such that all requirements of Theorem 3.1.1 are fulfilled. If $([0,1], U^n)$ is the z-ordinal sum of $\{(X_p, \mathrm{Id})\}_{p \in P}$ with the partial order $\preceq$ then U^n is an idempotent n-uninorm with the n-neutral element $\{e_1, \dots, e_n\}_{z_1, \dots, z_{n-1}}$ if and only if the following conditions are fulfilled:

(i) *$a_1 \prec a_2$ for all $a_1, a_2 \in P$ such that $X_{a_1} = \{x_1\}$, $X_{a_2} = \{x_2\}$, $x_1 < x_2$ and $x_1, x_2 \in [z_{i-1}, e_i]$, for $i = 1, \dots, n$.*

(ii) *$b_1 \prec b_2$ for all $b_1, b_2 \in P$ such that $X_{b_1} = \{y_1\}$, $X_{b_2} = \{y_2\}$, $y_1 > y_2$ and $y_1, y_2 \in [e_i, z_i]$ for $i = 1, \dots, n$.*

(iii) *For $a, b \in P$, $X_a = \{x\}$, $X_b = \{y\}$, are a and b incomparable if and only if there exists an $i \in \{1, \dots, n-1\}$ such that $q_i \preceq a$, $q_i \preceq b$ and $z_i \in]x, y[$.*

(iv) *a_1 and a_2 are comparable for all $a_1, a_2 \in P$ such that $X_{a_1} = \{x_1\}$, $X_{a_2} = \{x_2\}$, where $(x_1, x_2) \in [z_{i-1}, z_i]^2$ for $i = 1, \dots, n$.*

From [NUN1] we know that an idempotent n-uninorm induce a partial order which resembles a binary tree, where nodes of this tree correspond to division points $z_1, \dots, z_{n-1}$ (see Figure 3.1).

In [6, Theorem 2] it was shown that for an idempotent n-uninorm U^n and $x, y \in [0,1]$, $x \le y$ there is $U^n(x,y) \in \{x, y\} \cup \{z_i \mid z_i \in]x, y[\}$. We have shown a similar result for all n-uninorms with continuous underlying functions in Lemma 5.1 from [NUN2].

Lemma 3.1.4
Let $U^n \colon [0,1]^2 \longrightarrow [0,1]$ be an n-uninorm, $U^n \in \mathcal{U}_n$. If $a \in [0,1]$ is an idempotent point of U^n then $U^n(a,x) \in \{x, a\} \cup \{z_i \mid z_i \in]\min(a,x), \max(a,x)[\}$ for all $x \in [0,1]$.

We stress this result since it is very useful in decomposition of an n-uninorm $U^n \in \mathcal{U}_n$. We immediately see that $U^n(0,1) \in \{z_0, \dots, z_n\}$. Moreover, if $U^n(0,1) = z_k$ for some $k \in \{1, \dots, n-1\}$ then U^n has a very simple structure: it is a linear transformation of a k-uninorm from $\mathcal{U}_k$ on $[0, z_k]^2$, a linear transformation of an $(n-k)$-uninorm from $\mathcal{U}_{n-k}$ on $[z_k, 1]^2$, and otherwise it attains the value z_k. In this case, i.e., when $0 < k < n$ we say that U^n belongs to the Class 1.

For each $U^n \in \mathcal{U}_n$ there is $U^n(e_1, e_n) = z_k$ for some $k \in \{1, \dots, n-1\}$. The point z_k is important since it is the annihilator of U^n on $[e_1, e_n]$ and for all $x \in [0,1]$ there is $U^n(x, z_k) \in \{x, z_k\}$ (see Lemmas 5.2 and 5.3 in [NUN2]). The monotonicity of U^n then

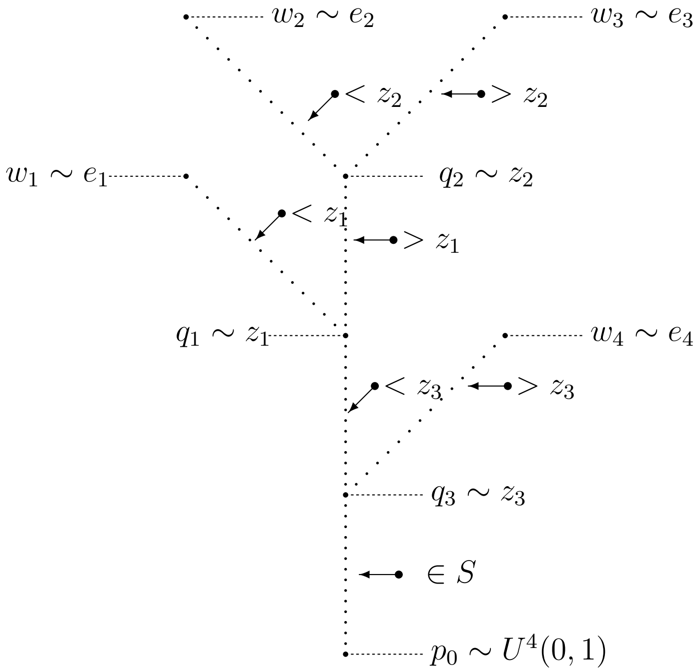

Figure 3.1: A lower semi-lattice corresponding to an idempotent 4-uninorm, where $k = 3$. For more details see [NUN1].

implies that there exists an $x_0 \in [0, e_1]$ and a $y_0 \in [e_n, 1]$ such that $U^n(x, z_k) = x$ for all $x < x_0$ and $U^n(x, z_k) = z_k$ for all $x_0 < x \leq z_k$, and $U^n(y, z_k) = y$ for all $y > y_0$ and $U^n(y, z_k) = z_k$ for all $z_k \leq y < y_0$. The points x_0 and y_0 are idempotent points of U^n and if $x \in]x_0, z_k]$ and $y \in [z_k, y_0[$ then $U^n(x, z_k) = z_k$, $U^n(y, z_k) = z_k$ and the monotonicity of U^n implies $U^n(x, y) = z_k$ (see Figure 3.2).

Therefore

- If $U^n(x_0, z_k) = z_k = U^n(y_0, z_k)$ then U^n on $[x_0, y_0]^2$ is a linear transformation of an n-uninorm from Class 1,

- If $U^n(x_0, z_k) = z_k$, $U^n(y_0, z_k) = y_0$ then U^n on $[x_0, y_0[^2$ is a linear transformation of a restriction of an n-uninorm from Class 1 to $[0, 1[^2$,

- If $U^n(x_0, z_k) = x_0$, $U^n(y_0, z_k) = z_k$ then U^n on $]x_0, y_0]^2$ is a linear transformation of a restriction of an n-uninorm from Class 1 to $]0, 1]^2$,

- If $U^n(x_0, z_k) = x_0$, $U^n(y_0, z_k) = y_0$ then U^n on $]x_0, y_0[^2$ is a linear transformation of a restriction of an n-uninorm from Class 1 to $]0, 1[^2$.

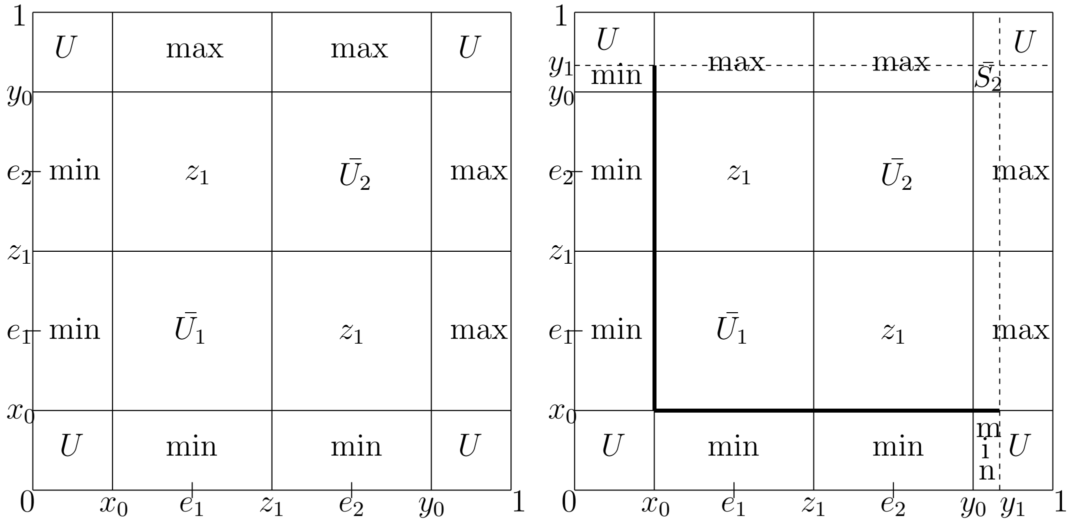

Figure 3.2: A 2-uninorm with $U^2(x_0, y_0) = z_1$ (left) and with $U^2(x_0, y_0) = x_0$, $y_1 > y_0$, $U^2(x_0, y_1) = x_0$ (right). $\bar{U}_1$ ($\bar{U}_2, \bar{S}_2$) indicates that U^2 is on the given area isomorphic to a restriction of U_1 (U_2, S_2) to a subinterval of $[0,1]$. Further, $([0, x_0[\cup \{z_1\} \cup]y_0, 1], U)$ $(([0, x_0] \cup]y_1, 1], U))$ is isomorphic to a uninorm with continuous underlying functions.

We see that in the core of each n-uninorm from $\mathcal{U}_n$ there is an n-uninorm from Class 1.

The main result of the paper [NUN1] for idempotent n-uninorms and of [NUN2] for uninorms with continuous underlying functions shows how is this core composed with the remainder of the unit square. Since $U^n(x_0, y_0) \in \{x_0, y_0, z_k\}$ we have to discuss several cases. If $U^n(x_0, y_0) = z_k$ we get the following result (see Theorem 5.10 in [NUN2]).

Theorem 3.1.5
Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be an n-uninorm and let $U^n \in \mathcal{U}_n$. If $U^n(x_0, y_0) = z_k$ then U^n is an ordinal sum of two semigroups $G_1 = ([0, x_0[\cup \{z_k\} \cup]y_0, 1], U^n)$ and $G_2 = ([x_0, y_0], U^n)$, where G_2 is isomorphic to an n-uninorm from Class 1 and G_1 is isomorphic to a uninorm with continuous underlying functions. Moreover, the order of semigroups in the ordinal sum construction is $1 < 2$.

For the case when $U^n(x_0, y_0) \in \{x_0, y_0\}$ we define

- $y_1 = \sup\{y \in [y_0, 1] \mid U^n(x_0, y) = x_0\}$ if $U^n(x_0, y_0) = x_0$,
- $x_1 = \inf\{x \in [0, x_0] \mid U^n(y_0, x) = y_0\}$ if $U^n(x_0, y_0) = y_0$.

The following is a summary of Theorems 5.11–5.14 from [NUN2].

Theorem 3.1.6
Let $U^n \in \mathcal{U}_n$, $U^n(x_0, y_0) = x_0$ and $U^n(y_1, x_0) = x_0$ ($U^n(x_0, y_0) = y_0$ and $U^n(x_1, y_0) = y_0$). Then U^n can be expressed as an ordinal sum of a semigroup which is a linear transforma-

tion of (a restriction of) an n-uninorm from Class 1 to interval $[x_0, y_0]^2$ ($[x_0, y_0[^2$, $]x_0, y_0]^2$, $]x_0, y_0[^2$) and at most 2 other semigroups.

- *The smallest of these semigroups in the corresponding linear order is always a semigroup which is isomorphic to a uninorm with continuous underlying functions.*
- *If $U^n(x_0, y_0) = x_0$, $U^n(z_k, y_0) = y_0$, $y_1 = y_0$ then the semigroup $(\{y_0\}, \mathrm{Id})$ is included.*
- *If $U^n(x_0, y_0) = y_0$, $U^n(z_k, x_0) = x_0$, $x_1 = x_0$ then the semigroup $(\{x_0\}, \mathrm{Id})$ is included.*
- *If $U^n(x_0, y_0) = x_0$ and $y_1 > y_0$ ($U^n(x_0, y_0) = y_0$ and $x_1 < x_0$) then a semigroup which is isomorphic to (a restriction of) a continuous t-conorm (t-norm) is included.*

In the case when $U^n(y_1, x_0) = y_1$ ($U^n(x_1, y_0) = x_1$) we get a similar result, however, here the smallest semigroup is not isomorphic to a uninorm, but to a generalized composite uninorm with continuous underling functions (see Definition 2.1.2). From Remark 2.1.5 we know that such a semigroup can be expressed as an ordinal sum of semigroups from $\mathcal{H}$ (see Theorems 5.15 and 5.16 in [NUN2]).

From these results we can observe that n-uninorms from Class 1 play a major role in our investigation. Observe that these n-uninorms have a similar structure as nullnorms, i.e., they have uninorms of lower orders glued together by the global annihilator. Such a structure can be easily expressed as a z-ordinal sum, where the respective annihilator corresponds to the bottom element of our partial order. Above the bottom element we have two branches each corresponding to a respective uninorm of lower order. This observation yields a question whether each n-uninorm form $\mathcal{U}_n$ can be expressed as a z-ordinal sum of Archimedean, representable and idempotent semigroups. This question will be positively answered in Section 3.3.

3.2 Characterizing functions of n-uninorms with continuous underlying functions

The characterizing functions of n-uninorms from $\mathcal{U}_n$ were discussed in paper [NUN3]. Before we start to discuss characterizing functions of n-uninorms we need to clarify several anomalous situations. At first it can happen that for an n-uninorm there is $e_i = e_j$ for some $i, j \in \{1, \ldots, n\}$, $i < j$. However, in such a situation $e_i = e_k = z_m$ for all $k \in \{i, \ldots, j\}$ and $m \in \{i, \ldots, j-1\}$. Then e_i is the neutral element of U^n on $[z_{i-1}, z_j]$ and thus U^n is in fact a $(n-j+i)$-uninorm and neutral elements $e_{i+1}, \ldots, e_j$ and division points $z_i, \ldots, z_{j-1}$ can be omitted. Therefore when investigating characterizing functions of n-uninorms we will assume only n-uninorms where $e_1 < e_2 < \cdots < e_n$. Further, in the case of uninorms for each $x \in [0,1]$ there exists at most one point $y \in [0,1]$ such that $U(x,y) = e$. This is no longer true for n-uninorms (see Example III.1 in [NUN3]). However, we have shown the following result (see Proposition III.3 in [NUN3]).

Proposition 3.2.1
Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be an n-uninorm and let $U^n \in \mathcal{U}_n$. If $U^n(x,y_1) = U^n(x,y_2) = e_i$ for some $x, y_1, y_2 \in [0,1]$, $y_1 < y_2$, and $i \in \{1,\ldots,n\}$ then $e_i \in \{z_{i-1}, z_i\}$.

In this case, however, we can reduce the order of the given n-uninorm (see Theorem III.4 in [NUN3]).

Theorem 3.2.2
Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be an n-uninorm, $U^n \in \mathcal{U}_n$. If for some $i \in \{1,\ldots,n\}$ there is $e_i = z_j$ for $j \in \{1,\ldots,n-1\}$ then U^n is an $(n-1)$-uninorm from $\mathcal{U}_{(n-1)}$ with the $(n-1)$-neutral element $\{e_1,\ldots,e_{i-1},e_{i+1},\ldots,e_n\}_{z_1,\ldots,z_{j-1},z_{j+1},\ldots,z_{n-1}}$.

The previous theorem shows that in this case we can reduce the order of the n-uninorm by one. Using this procedure repeatedly we see that each n-uninorm U^n from $\mathcal{U}_n$ can be seen as an m-uninorm U^m from $\mathcal{U}_m$ such that if e_i^* is the i-th local neutral element of U^m then $e_i^* \in \{z_{i-1}^*, z_i^*\}$ implies $e_i^* \in \{0,1\}$. Then the m-uninorm U^m will be called the reduced form of the n-uninorm U^n (reduced m-uninorm for short). Therefore in the following section it is enough to focus just on reduced n-uninorms. Observe that for 2-uninorms $e_1 = 0$ yields null-uninorms and $e_2 = 1$ yields uni-nullnorms [78].

For a reduced n-uninorm U^n and $x, y_1, y_2 \in [0,1]$, $y_1 < y_2$, the equality $U^n(x,y_1) = U^n(x,y_2) = e_i$ for some $i \in \{1,\ldots,n\}$ implies $e_i \in \{0,1\}$, i.e., $i \in \{1,n\}$. However, since $e_1 = 0$ ($e_n = 1$) is the neutral element of U^n on $[0,z_1]$ ($[z_{n-1},1]$) there is $U^n(x,0) > 0$ for all $x > 0$ ($U^n(x,1) < 1$ for all $x < 1$). Therefore in the case of reduced n-uninorms there for any $i \in \{1,\ldots,n\}$ and any $x \in [0,1]$ exists at most one $y \in [0,1]$ such that $U^n(x,y) = e_i$.

Now we can define characterizing functions and characterizing set-valued functions for n-uninorm from $\mathcal{U}_n$.

Definition 3.2.3
Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be a reduced n-uninorm, let $U^n \in \mathcal{U}_n$ and assume an $i \in \{1,\ldots,n\}$. Define a function $g_i\colon [0,1] \longrightarrow [0,1]$ by

$$g_i(x) = \sup\{t \in [0,1] \mid U^n(x,t) < e_i\},$$

where $\sup\emptyset = 0$. The function g_i will be called the i-th characterizing function of the n-uninorm U^n.

Note that evidently $g_i(e_i) = e_i$ for all $i \in \{1, \ldots, n\}$. Further, if $e_1 = 0$ then $g_1(x) = 0$ for all $x \in [0,1]$. Similarly, if $e_n = 1$ then $g_n(x) = 1$ for all $x \in [0,1]$.

The characterizing function g_i is non-increasing for all $i = 1, \ldots, n$ (see Proposition III.10 in [NUN3]) and from the definition we see that $U^n(x,t) < e_i$ for all $t < g_i(x)$ and $U^n(x,t) > e_i$ for all $t > g_i(x)$.

Further, if $e_i \notin \mathrm{Ran}(U^n(x,\cdot))$ then evidently $U^n(x,\cdot)$ is non-continuous in $g_i(x)$ (or $g_i(x) \in \{0,1\}$). If $e_i \in \mathrm{Ran}(U^n(x,\cdot))$ then $U^n(x,\cdot)$ is continuous in $g_i(x)$ and then $U^n(x, g_i(x)) = e_i$. Summarizing, either $U^n(x, g_i(x)) = e_i$, or $U^n(x,\cdot)$ is non-continuous in $g_i(x)$, or $g_i(x) \in \{0,1\}$. Observe that if $U^n(x,\cdot)$ is continuous in $g_i(x)$ and $g_i(x) = 0$ ($g_i(x) = 1$) then $U(x,t) \geq e_i$ ($U(x,t) \leq e_i$) for all $t \in [0,1]$.

Definition 3.2.4

Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be a reduced n-uninorm, $U^n \in \mathcal{U}_n$, and assume an $i \in \{1, \ldots, n\}$. We define the characterizing set-valued function $r_i\colon [0,1] \longrightarrow \mathcal{P}([0,1])$ by

$$r_i(x) = \begin{cases} \left[\lim\limits_{t\longrightarrow 0^+} g_i(t), 1\right] & \text{if } x = 0, \\ \left[0, \lim\limits_{t\longrightarrow 1^-} g_i(t)\right] & \text{if } x = 1, \\ \left[\lim\limits_{t\longrightarrow x^+} g_i(t), \lim\limits_{t\longrightarrow x^-} g_i(t)\right] & \text{otherwise.} \end{cases}$$

Observe that $g_i(x) \in r_i(x)$ for all $x \in [0,1]$, $i \in \{1, \ldots, n\}$ and if g_i is continuous in $x \in]0,1[$ for some $i \in \{1, \ldots, n\}$ then $r_i(x) = \{g_i(x)\}$. For an example of characterizing set-valued functions see Figure 3.3.

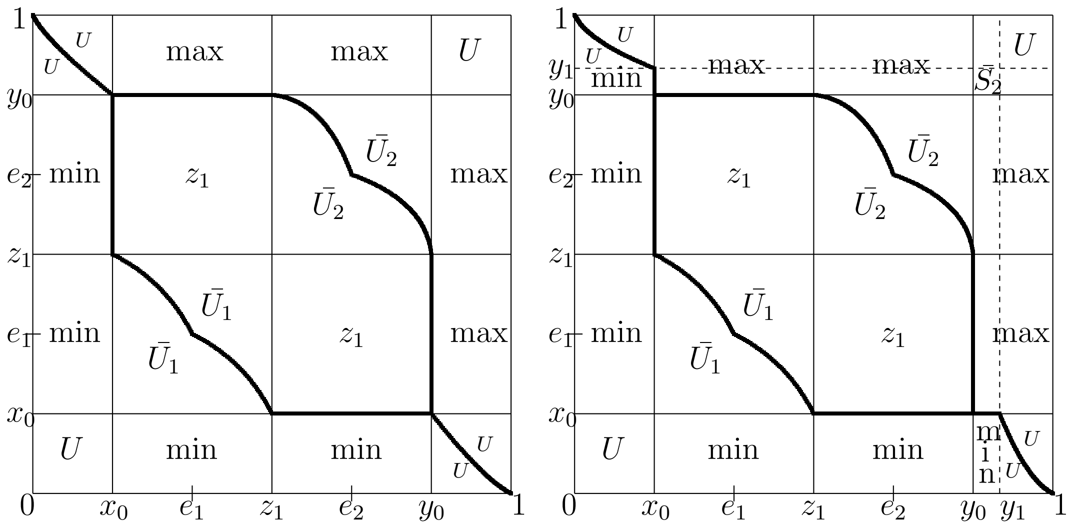

Figure 3.3: The two 2-uninorms from Figure 3.2 (Figure 1 in [NUN2]). The bold lines denote characterizing set-valued functions r_1 and r_2.

In the following theorem we summarize Lemmas III.13, III.14 and III.15 from [NUN3].

Theorem 3.2.5
The characterizing set-valued function r_i of a reduced n-uninorm $U^n \in \mathcal{U}_n$ is non-increasing, symmetric and u-surjective (see Definition 2.4.1) for all $i = 1, \ldots, n$. Further, above (below) the graph of the characterizing set-valued function r_i the n-uninorm U^n attains values greater (smaller) than e_i.

Each point of discontinuity of a reduced n-uninorm $U^n \in \mathcal{U}_n$ can be associated with at least one local neutral element e_i for $i \in \{1, \ldots, n\}$ (see Lemma IV.1 in [NUN3]). Moreover, if $U^n(x, y) = e_i \in]0, 1[$ for some $x, y \in [0, 1]$, $i \in \{1, \ldots, n\}$ then $x, y \in]z_{i-1}, z_i[$. It is easy to show that $U^n(x, y) = e_i$ implies $(x, y) \in G(r_i)$ (see Lemmas IV.2 and IV.6 in [NUN3]).

Vice versa, from the definition we easily see that if $(x_0, y_0) \in G(r_i) \cap]0, 1[^2$ then either $U^n(x_0, y_0) = e_i$, or $U^n(x_0, y_0)$ is a point of discontinuity of U^n. On the lower boundary of the unit square (and similarly on the upper boundary of the unit square) we know that U^n is continuous in point $(0, 0)$ and U^n is non-continuous in each point $(0, t), (t, 0)$ such that $t \in \left] \lim_{t \longrightarrow 0^+} g_i(t), g_i(0) \right]$ for some $i \in \{1, \ldots, n\}$, where $1 > e_i > 0$.

The main result of [NUN3] shows that each point of discontinuity of $U^n \in \mathcal{U}_n$ is covered by the union of the graphs of its characterizing set-valued functions (see Theorem IV.8 in [NUN3]).

Theorem 3.2.6
Let $U^n \colon [0, 1]^2 \longrightarrow [0, 1]$ be a reduced n-uninorm and let $U^n \in \mathcal{U}_n$. If $(x_0, y_0) \in [0, 1]^2$ is a point of discontinuity of U^n then $(x_0, y_0) \in \bigcup_{i=1}^{n} G(r_i)$.

For a uninorm with continuous underlying functions we have shown that U is in each point from the unit square either left-continuous, or right-continuous (or continuous). This is no loner true in the case of n-uninorms from $\mathcal{U}_n$. However, this situation can occur only in the case when the corresponding point of discontinuity belongs to graphs of at least two characterizing set-valued functions.

Example 3.2.7
Assume $0 < e_1 < z_1 < e_2 < 1$ an let a binary function $U^2 \colon [0, 1]^2 \longrightarrow [0, 1]$ be given by:

$$U^2(x,y) = \begin{cases} \min(x,y) & \text{if } \min(x,y) < e_1, \\ \max(x,y) & \text{if } \min(x,y) > e_1, \max(x,y) > e_2, \\ \min(x,y) & \text{if } x, y \in [z_1, e_2], \\ \max(x,y) & \text{if } x, y \in [e_1, z_1], \\ z_1 & \text{otherwise.} \end{cases}$$

Then U^2 is a 2-uninorm with continuous underlying functions and $U^n(x,y) = e_i$ implies $x = y = e_i$. Evidently, U^2 is in the reduced form. However, $U^2(e_1, e_2) = z_1$ and $U^n(s,t) = s < e_1$ for all $s < e_1$, $e_1 < t < e_2$ and $U^n(s,t) = t > e_2$ for all $e_2 > s > e_1$, $t > e_2$. Therefore in point (e_1, e_2) the 2-uninorm U^2 is neither left-continuous, nor right-continuous.

In the previous example point (e_1, e_2) belongs to graphs of both characterizing set-valued functions r_1 and r_2. This holds for all points in which an n-uninorm from $\mathcal{U}_n$ is neither left-continuous nor right continuous (see Propostion IV.10 in [NUN3]).

Proposition 3.2.8
Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be a reduced n-uninorm, $U^n \in \mathcal{U}_n$, and assume a point $(x_0, y_0) \in [0,1]^2$. If there exists exactly one $i \in \{1, \ldots, n\}$ such that $(x_0, y_0) \in G(r_i)$ then U^n is left-continuous or right-continuous (or continuous) at point (x_0, y_0).

Using Lemma IV.11 from [NUN3] which shows that $[z_{i-1}, z_i]^2 \cap \bigcup\limits_{j=1}^{n} G(r_j) = G(r_i)$ for all $i \in \{1, \ldots, n\}$ we can show Theorem IV.12 in [NUN3].

Theorem 3.2.9
Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be a reduced n-uninorm. Suppose that U^n is continuous on $[0,1]^2 \setminus \bigcup\limits_{i=1}^{n} G(r_i)$, where r_i is a symmetric, u-surjective, non-increasing set-valued function on $[0,1]$, such that $U^n(x,y) = e_i$ implies $(x,y) \in G(r_i)$ for $i = 1, \ldots, n$. Further assume that U^n is either left-continuous, or right-continuous (or continuous) in each point $(x_0, y_0) \in [0,1]^2$ such that there is exactly one $i \in \{1, \ldots, n\}$ for which $(x_0, y_0) \in G(r_i)$. Then $U^n \in \mathcal{U}_n$.

In the proof of this theorem we have also shown the connection between the i-th characterizing set-valued function and the characterizing set-valued function of the i-th underlying uninorm U_i.

Remark 3.2.10
Observe that if $e_1 = 0$ $(e_n = 1)$ we claimed that U^n is continuous on $[0, z_1]^2$ $([z_{n-1}, 1]^2)$ since each t-norm (t-conorm) is continuous on the lower (the upper) boundary of the unit square. However, in such cases we obtain a t-conorm on $[0, z_1]^2$ (a t-norm on $[z_{n-1}, 1]^2$) and thus it is an exact opposite. For a t-conorm on $[0, z_1]^2$ the graph of the characterizing set-valued function r_1 coincides with the lower boundary and for a t-norm on $[z_{n-1}, 1]^2$ the graph of the characterizing set-valued function r_n coincides with the upper boundary. For $e_1 = 0$ we have $r_1(0) = [0,1]$ and $r_1(x) = 0$ for all $x \in \,]0,1]$. However, in all points $(x,0), (0,x)$ for $x \in [0,1]$ the left-sided limit does not exist. Similarly, for $e_n = 1$ we have $r_n(x) = 1$ for all $x \in [0,1[$ and $r(1) = [0,1]$ and in all points $(x,1), (1,x)$ for $x \in [0,1]$ the right-sided limit does not exist. Since U^n is either left-continuous, or right-continuous (or continuous) in each point $(x_0, y_0) \in [0,1]^2$ such that there is exactly one $i \in \{1, \ldots, n\}$

for which $(x_0, y_0) \in G(r_i)$*, we see that for* $e_1 = 0$ *the underlying uninorm* U_1 *(which is a t-conorm) is (right-)continuous in all points* $(x, 0), (0, x)$ *for* $x \in [0, z_1]$*, i.e.,* U_1 *is a continuous t-conorm. Observe that* $(0, z_1) \in G(r_i)$ *implies* $i = 1$ *since* U^n *is in the reduced form and* $i > 1$ *would imply* $z_1 = e_i > e_1 = 0$*, i.e.,* $z_1 = e_i = 1$*, which means that* U^n *is a standard uninorm and* $i \in \{1\}$*. Similarly we can show that if* $e_n = 1$ *then the underlying uninorm* U_n *is a continuous t-norm.*

3.3 Decomposition of n-uninorms with continuous underlying functions via the z-ordinal sum construction

Our work on characterization of uninorms and n-uninorms with continuous underlying functions was concluded in [NUN4], where all n-uninorms from $\mathcal{U}_n$ were characterized. For better understanding we have started with decomposition of nullnorms with continuous underlying functions via the z-ordinal sum construction. We have shown that each such a nullnorm with an annihilator z can be decomposed into three semigroups: $G_1 = ([0, z], S^*)$, $G_2 = ([z, 1], T^*)$ and $G_3 = (\{z\}, \mathrm{Id})$, where S^* (T^*) is a linear transformation of some t-conorm (t-norm) to the interval $[0, z]$ ($[z, 1]$), with the branching set $A = \{3\}$ and the respective partial order given by $1 \wedge 2 = 3$ (see Lemma 1 in [NUN4]). Each continuous t-norm (t-conorm) can be expressed as an ordinal sum of continuous Archimedean t-norms (t-conorms). Moreover, each continuous Archimedean t-norm (t-conorm) can be decomposed via ordinal (z-ordinal) sum only to one non-trivial and one or two trivial semigroups, which correspond to boundary points 0 and 1. Therefore we say that a semigroup is related to a continuous Archimedean t-norm (t-conorm) if it can be obtained from a continuous Archimedean t-norm (t-conorm) by exclusion of one or both boundary points. Then we obtain Theorem 5 in [NUN4].

Theorem 3.3.1
Let $V\colon [0,1]^2 \longrightarrow [0,1]$ *be a nullnorm with annihilator* $z \in \,]0,1[$ *and let* T_V *and* S_V *be continuous. Then* V *is a* z*-ordinal sum of a countable number of semigroups related to continuous Archimedean t-norms, continuous Archimedean t-conorms and idempotent t-norms and t-conorms (including trivial semigroups).*

Observe that all semigroups from the previous theorem, irreducible with respect to the ordinal sum (the z-ordinal sum), belong to $\mathcal{H}$.

For n-uninorms with continuous underlying functions we have first discussed the interaction of points from different intervals, i.e., for $x \in \,]z_{i-1}, z_i[$ and $y \in \,]z_{j-1}, z_j[$, where

$x < y$. Generally we can show that $U^n(x,y) \in [x, e_i[\cup \{U^n(e_1, e_j)\} \cup]e_j, y]$. Then depending on the position of x (y) with respect the local neutral element e_i (e_j) we can describe the values of U^n on the respective regions as depicted on Figure 3.4 (see Proposition 6 in [NUN4]).

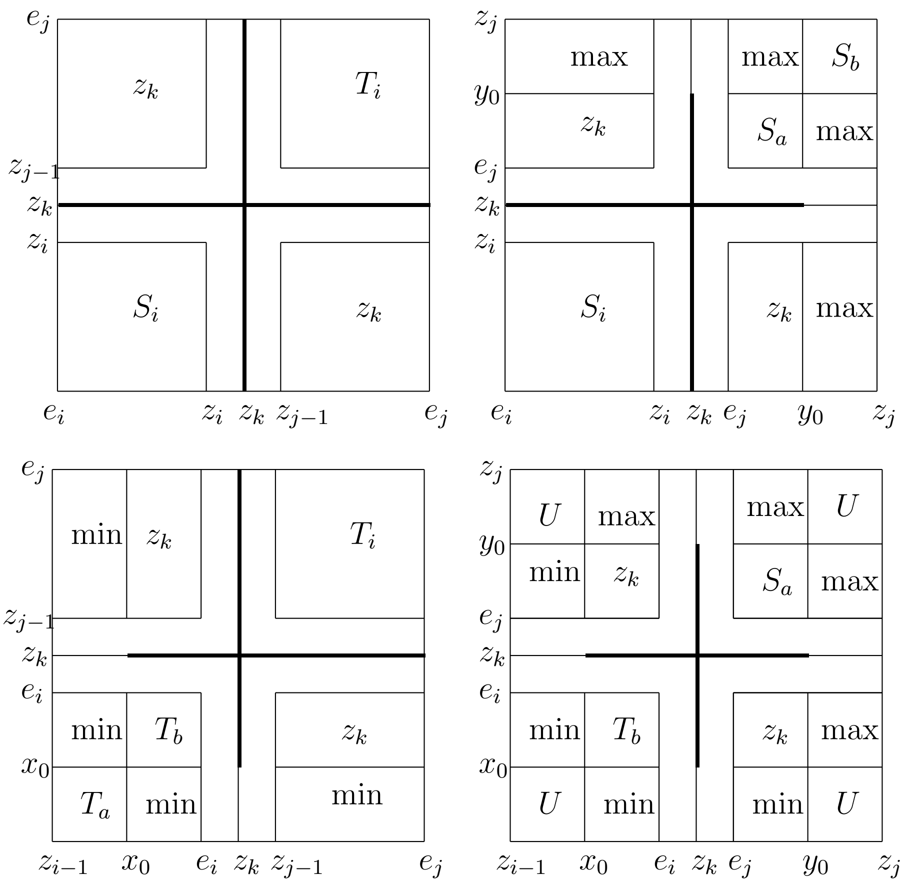

Figure 3.4: Sketch of n-uninorm U^n on selected regions. Bold lines denote the set where the functions $U^n(z_k, \cdot)$ and $U^n(\cdot, z_k)$ attain value $z_k = U^n(e_i, e_j)$.

In papers [NUN1] and [NUN2] we have shown that in the core of each n-uninorm $U^n \in \mathcal{U}_n$ there is an n-uninorm U^n_* from Class 1, i.e., such that $U^n_*(0,1) = z_k$ for some $k \in \{1, \dots, n-1\}$ (see Theorems 3.1.5, 3.1.6). Observe that an n-uninorm from Class 1 has a similar structure as a nullnorm, i.e., it can be expressed as a z-ordinal sum of a semigroup acting on $[0, z_k]$ (which is a linear transformation of a k-uninorm), a semigroup acting on $[z_k, 1]$ (which is a linear transformation of an $(n-k)$-uninorm) and a trivial semigroup $(\{z_k\}, \mathrm{Id})$.

Our aim is to decompose U^n into semigroups from $\mathcal{H}$, i.e., semigroups related to continuous Archimedean t-norms and t-conorms and representable and idempotent uninorms, via the z-ordinal sum. In the first step we decompose U^n into an ordinal sum of (a restriction of) an n-uninorm from Class 1 acting on $[x_0, y_0]$ ($[x_0, y_0[$, $]x_0, y_0]$, $]x_0, y_0[$) and semigroups from $\mathcal{H}$, according to results from Section 3.1. In the second step we decompose this n-uninorm from Class 1, via z-ordinal sum, into uninorms of lower orders. For $n = 2$ we obtain standard uninorms which, according to Chapter 2, can be expressed as an ordinal sum of semigroups from $\mathcal{H}$. For $n > 2$ we will use induction.

Since we will compose the global z-ordinal sum from several local parts we have shown the following useful results that helped us to shorten proofs considerably (see Lemmas 3, 4 and 5 in [NUN4]).

Definition 3.3.2
Let $(C, \preceq)$ be a partially ordered set. We say that $(C, \preceq)$ has a tree structure *if for each $p_1, p_2 \in C$ such that p_1 and p_2 are incomparable there is no upper bound for p_1 and p_2.*

Lemma 3.3.3
Let $(C, \preceq)$ be a meet semi-lattice which has a tree structure. For $\alpha, \beta, \gamma \in C$, if γ is incomparable with $\alpha \wedge \beta$ then $\alpha \wedge \gamma = \beta \wedge \gamma$.

In the following result we assume a z-ordinal sum of semigroups G_α for $\alpha \in A \cup B$, in which each semigroup G_α for $\alpha \in B$ can be expressed as an ordinal sum of semigroups.

Lemma 3.3.4
Let $(X, *)$ be a z-ordinal sum of semigroups $(G_\alpha)_{\alpha \in C}$ with respect to sets A and B and a partial order $\preceq$. Assume that for each $\alpha \in B$ the semigroup G_α is an ordinal sum of semigroups $(H_\beta)_{\beta \in B_\alpha}$ for some linearly ordered index set $(B_\alpha, \leq_\alpha)$ and $H_\beta = G_\beta$ for all $\beta \in A$. Then $(X, *)$ is a z-ordinal sum of semigroups $(H_\beta)_{\beta \in A' \cup B'}$ with respect to sets $A' = A$, $B' = \bigcup\limits_{\alpha \in B} B_\alpha$ and a partial order $\preceq'$ given by:

(i) *If $p_1, p_2 \in B_\alpha$ for some $\alpha \in B$ then $p_1 \preceq' p_2$ if $p_1 \leq_\alpha p_2$.*
(ii) *If $p_1 \in B_\alpha$ and $p_1 \in B_\beta$ for some $\alpha, \beta \in B$ then $p_1 \preceq' p_2$ if $\alpha \preceq \beta$ and $p_2 \preceq' p_1$ if $\beta \preceq \alpha$.*
(iii) *If $p_1 \in B_\alpha$ for some $\alpha \in B$ and $p_2 \in A$. Then $p_1 \preceq' p_2$ if $\alpha \preceq p_2$ and $p_2 \preceq' p_1$ if $p_2 \preceq \alpha$.*
(iv) *If $p_1, p_2 \in A$ then $p_1 \preceq' p_2$ if $p_1 \preceq p_2$.*

Moreover, if $(C, \preceq)$ for $C = A \cup B$ has a tree structure then also $(C', \preceq')$ for $C' = A' \cup B'$ has a tree structure.

In the following result we assume a z-ordinal sum of semigroups which has structure of a tree with two branches, where each branch can be expressed as a z-ordinal sum of semigroups.

Lemma 3.3.5
Let $(X, *)$ be a z-ordinal sum of semigroups G_1, G_2, G_3 and G_4, where $A = \{3\}$ and $\preceq$ is given by $1 \wedge 2 = 3$ and $4 \prec 3$. Assume that G_1 is a z-ordinal sum of semigroups H_α with respect to A_1, B_1 and $\preceq_1$, where $(C_1, \preceq_1)$ for $C_1 = A_1 \cup B_1$ has a tree structure. Similarly,

assume that G_2 is a z-ordinal sum of semigroups H_α with respect to A_2, B_2 and $\preceq_2$, where $(C_2, \preceq_2)$ for $C_2 = A_2 \cup B_2$ has a tree structure; and $H_3 = G_3$, $H_4 = G_4$. Then $(X, *)$ is a z-ordinal sum of semigroups $(H_\alpha)_{\alpha \in C_1 \cup C_2 \cup \{3,4\}}$ with respect to $A' = A_1 \cup A_2 \cup \{3\}$, $B' = B_1 \cup B_2 \cup \{4\}$ and $\preceq'$ given by

(i) $\alpha \preceq' \beta$ *if* $\alpha, \beta \in C_1$ and $\alpha \preceq_1 \beta$.

(ii) $\alpha \preceq' \beta$ *if* $\alpha, \beta \in C_2$ and $\alpha \preceq_2 \beta$.

(iii) $4 \prec' 3 \prec' \alpha$ *for all* $\alpha \in C_1 \cup C_2$.

(iv) *If* $\alpha \in C_1$ and $\beta \in C_2$ *then* α and β *are incomparable.*

Moreover, $(C', \preceq')$ has a tree structure.

The previous results imply that z-ordinal sum decomposition of a 2-uninorm from $\mathcal{U}_2$ resembles a tree with two branches and only one node – which corresponds to semigroup $(\{z_1\}, \mathrm{Id})$. Semigroups that are smaller than $(\{z_1\}, \mathrm{Id})$ are those from the ordinal sum decomposition of semigroups which remain when the 2-uninorm U^2_* from Class 1 is removed from the core. Further, one branch corresponds to the first underlying uninorm of U^2_* and the second branch corresponds to the second underlying uninorm of U^2_* (see Figure 3.5). If we summarize Theorems 6, 7 and 8 from [NUN4] we get the following. Recall that the relation $A \sim S$ expresses that the set A consists of indices that correspond to trivial semigroups defined on points from S.

Theorem 3.3.6
Let $U^2\colon [0,1] \longrightarrow [0,1]$ be a 2-uninorm, $U^2 \in \mathcal{U}_2$. Then U^2 can be expressed as a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$, where $A \sim \{z_1\}$ and $(C, \preceq)$ has a tree structure.

Note that a countable number of semigroups in this result was incorrectly transferred from [UNI4] (see Remark 2.5.3). The corrected result, with exactly the same proof, is that U^2 can be expressed as a z-ordinal sum of a countable number of semigroups from Definition 2.5.1 and a possibly uncountable number of trivial semigroups, where $A \sim \{z_1\}$ and $(C, \preceq)$ has a tree structure.

For an n-uninorm from $\mathcal{U}_n$ the smallest node from the branching set A in the respective partial order corresponds to $z_k = U^n(e_1, e_n)$. Similar to 2-uninorms, for n-uninorms semigroups that are smaller than $(\{z_k\}, \mathrm{Id})$ are those from the ordinal sum decomposition of semigroups which remain when the n-uninorm U^n_* from Class 1 is removed from the core of U^n. Above $(\{z_k\}, \mathrm{Id})$ there are two branches, one corresponds to the linear transformation of (a restriction of) a k-uninorm to the interval $[x_0, z_k]$ ($]x_0, z_k]$) and the second corresponds to the linear transformation of (a restriction of) an $(n-k)$-uninorm to the interval $[z_k, y_0]$ ($[z_k, y_0[$). By induction we can then express each branch further as a z-ordinal sum of semigroups from $\mathcal{H}$. Then we obtain Theorem 10 in [NUN4].

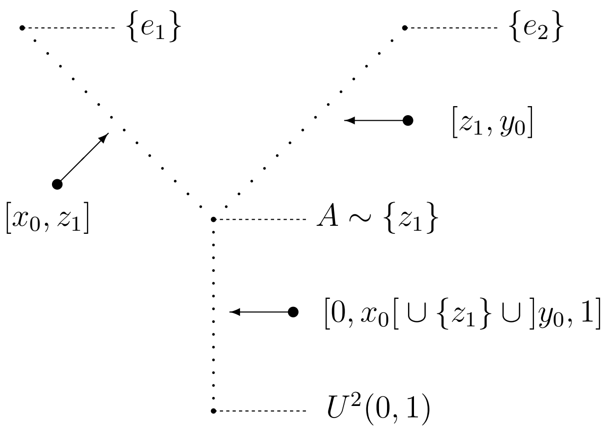

Figure 3.5: A partial order related to z-ordinal sum decomposition of a 2-uninorm with $U^2(x_0, y_0) = z_1$. Labeled areas consist of semigroups which contain points from the given set. Note that e_1 and e_2 can belong to semigroups which are not trivial.

Theorem 3.3.7
Let $U^n \colon [0,1] \longrightarrow [0,1]$ be an n-uninorm, $U^n \in \mathcal{U}_n$. Then U^n can be expressed as a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$, where $A \sim \{z_1, \ldots, z_{n-1}\}$ and $(C, \preceq)$ has a tree structure.

Similarly as before, the correct result, with exactly the same proof, is that U^n can be expressed as a z-ordinal sum of a countable number of semigroups from Definition 2.5.1 and a possibly uncountable number of trivial semigroups, where $A \sim \{z_1, \ldots, z_{n-1}\}$ and $(C, \preceq)$ has a tree structure.

This result completely characterizes n-uninorms with continuous underlying functions.

3.4 Conclusions

If we summarize our results we see that each n-uninorm from $\mathcal{U}_n$ can be reduced to an m-uninorm from $\mathcal{U}_m$, where $m \leq n$ and for the m-neutral element $\{e_1, \ldots, e_m\}_{z_1,\ldots,z_{m-1}}$ of U^m there $e_i = z_j$ for some $i \in \{1, \ldots, n\}$, $j \in \{0, \ldots, n\}$ implies $e_i = z_j \in \{0, 1\}$. Moreover, if $U^m(x, y_1) = U^m(x, y_2) = e_i$ for some $x, y_1, y_2 \in [0,1]$ and $i \in \{1, \ldots, n\}$ then $y_1 = y_2$.

Further, each reduced n-uninorm (including 1-uninorms which are just standard uninorms) with continuous underlying functions can be expressed as a z-ordinal sum of semigroups related to continuous Archimedean t-norms, t-conorms, representable uninorms and idempotent semigroups, where the meet semi-lattice $(C, \preceq)$ resembles a binary tree and it has an n-top element, i.e., it possesses n top branches. Here we say that a partially ordered set has an n-top element if there exist $k_1, \ldots, k_n \in C$ such that

- k_i is incomparable with k_j for $i \neq j$, $i, j \in \{1, \ldots, n\}$,
- for each $i \in \{1, \ldots, n\}$ there is no $k \in C$ such that $k_i \prec k$,
- for all $k \in C$ there is $k \preceq k_i$ for some $i \in \{1, \ldots, n\}$.

For an n-uninorm U^n and all $i \in \{1, \ldots, n\}$ there is $e_i \in X_{k_j}$ for some $j \in \{1, \ldots, n\}$ and vice versa for all $j \in \{1, \ldots, n\}$ there exists an $i \in \{1, \ldots, n\}$ such that $e_i \in X_{k_j}$. Therefore we can assume that $e_i \in X_{k_i}$ for all $i \in \{1, \ldots, n\}$. Then $k_i \wedge k_{i+1} \sim \{z_i\}$ and for all $i, j \in \{1, \ldots, n\}$, $i < j$, there is $k_i \wedge k_j \sim \{z_k\}$ for some $k \in \{1, \ldots, n-1\}$ such that z_k is the annihilator of U^n on $[e_i, e_j]^2$.

The set of points of discontinuity of each reduced n-uninorm can be covered by graphs of n characterizing set-valued functions, each related to one of the local neutral elements e_i for $i \in \{1, \ldots, n\}$. Thus the structure of n-uninorms with continuous underlying functions is completely characterized in this work.

A complete characterization of (continuous) t-norms is an important result that facilitates insight into the structure of the inference apparatus used in many applications, including probabilistic metric spaces and non-additive measures and integrals, which are used in generalized theory of probability to model the interaction when calculating the mean value. In the case of non-additive integrals t-norms and related operations replace the standard multiplication which is used in the additive case. One class of such operations are uninorms, which are studied in this doctoral dissertation. The main advantage of uninorms is that they can be used when working on a bipolar scale.

From the application point of view is probably the most interesting the class of uninorms with continuous underlying functions since it is big enough and still has quite nice properties. That is the reason why this class was studied by many authors, however, the achieved results covered only a number of special cases. A complete characterization of this class of uninorms was given solely in the papers that are contained in this doctoral dissertation.

One of the further generalizations of the bipolar scale yields an approach where the corresponding binary function in the respective non-additive integral has different properties depending on the specific subarea of the unit square, i.e., input values are divided into so-called reference levels. This approach brings us to n-uninorms which generalize uninorms. The class of n-uninorms with continuous underlying functions is also completely characterized in this work. Similarly as in the case of uninorms, up to now only some special cases of n-uninorms were characterized. Therefore this doctoral dissertation contributes to the development of generalized theory of probability, specifically to extension of knowledge of non-additive measures and integrals that model the mean value in the case when interaction is assumed.

3.5 Future and related work

In our recent work, we have applied results on uninorms with continuous underlying functions in [57] which studies convex combinations of uninorms. Further, we have shown a close connection between z-ordinal sum, introduced in this thesis, and natural partial ordering introduced in [59] in the case of commutative, associative and idempotent functions. Based on these results we have developed a method for an easy verification of associativity of a commutative and idempotent function (or a special commutative non-idempotent function).

In the future work we will study the z-ordinal sum construction further, especially in connection with associative functions on the unit interval. We plan to show that each semigroup that can be expressed as a z-ordinal sum of semigroups can be also expressed as a z-ordinal sum of semigroups such that $\alpha \in A$ implies that G_α is a trivial semigroup. For simplicity, we will define the basic form of a z-ordinal sum for which $\alpha \in A$ implies that G_α is a trivial semigroup and $X_p = X_k$ for some $p, k \in A \cup B$ implies $p = k$. Our aim is to show that each z-ordinal sum can be reduced to its basic form.

We will also investigate binary functions on the unit interval which can be obtained as a z-ordinal sum of semigroups from $\mathcal{H}$. It is easy to observe that all such functions are commutative, associative, have a continuous diagonal and have continuous Archimedean components. On the other hand, to ensure the monotonicity of a z-ordinal sum is not so easy. Therefore another important work will be the investigation of the compatibility of the standard order on the unit interval and the partial order $\preceq$ from the z-ordinal sum construction. In other words, we will investigate the conditions under which a z-ordinal sum yields a monotone function, i.e., a function non-decreasing with respect to the standard order on the unit interval.

Vice versa, we plan to show that each commutative, associative and non-decreasing function on the unit interval which is continuous on the diagonal and has continuous Archimedean components can be expressed as a z-ordinal sum of semigroups from $\mathcal{H}$.

As another streaming of our further research, we aim to focus on ordinal sum (z-ordinal sum) constructions of general aggregation functions [12, 34] or of some particular aggregation functions, such as overlap and grouping functions [14, 15], particular integrals [13, 62, 80], and of some related functions, such as fuzzy implications [9].

Appendix

Here we introduce the proof of Proposition 2.1.4 from Section 2.1.

Proposition 2.1.4
Let $GU\colon ([a,b]\cup[c,d])^2 \longrightarrow ([a,b]\cup[c,d])$, where $a<b<c<d$, $a,b,c,d\in[0,1]$ be a generalized composite uninorm with underlying functions which are a continuous Archimedean t-norm and a continuous Archimedean t-conorm, respectively. Then GU can be expressed either as an ordinal sum of a uninorm with continuous Archimedean underlying functions and a trivial semigroup, or as an ordinal sum of a continuous Archimedean t-norm (possibly without one or both boundary points), a continuous Archimedean t-conorm (possibly without one or both boundary points) and few trivial semigroups (corresponding to points form $\{a,b,c,d\}$).

PROOF: Since $GU(b,c)\in[b,c]$ there is $GU(b,c)\in\{b,c\}$. Further we will assume that $GU(b,c)=b$ as the case when $GU(b,c)=c$ is analogous. Then also $GU(x,c)=GU(GU(x,b),c)=GU(x,GU(b,c))=GU(x,b)=x$ for all $x\in[a,b]$, i.e., c is the neutral element of GU. Then we can distinguish the following cases:

(i) If $GU(b,y)=y$ for all $y\in\,]c,d[$. In this case the monotonicity implies also $GU(b,d)=d$, i.e., b is the neutral element of GU restricted to $([a,b]\cup\,]c,d])^2$. Therefore GU restricted to the set $([a,b]\cup\,]c,d])^2$ is isomorphic to a uninorm with continuous underlying functions. In this case $([a,b]\cup[c,d], GU)$ can be expressed as an ordinal sum of $G_1=(\{c\},\mathrm{Id})$ and $G_2=([a,b]\cup\,]c,d], GU|_{([a,b]\cup]c,d])^2})$, which is isomorphic to a uninorm with continuous underlying functions and the order on $\{1,2\}$ in this ordinal sum is $2<1$.

(ii) If $GU(b,y_0)\neq y_0$ for some $y_0\in\,]c,d[$. We have two possibilities: either $GU(b,y_0)=b$, or $GU(b,y_0)\geq c$. First we will assume that $GU(b,y_0)=y_1\geq c$. Then $y_1\in[b,y_0[$, i.e., $y_1<y_0$ and

$$y_1=GU(GU(b,b),y_0)=GU(b,GU(b,y_0))=GU(b,y_1).$$

However, since GU on $[c,d]^2$ is a continuous t-conorm there exists a $p\in[c,d]$ such that $GU(y_1,p)=y_0$. Then we get

$$GU(b,y_0)=GU(b,GU(y_1,p))=GU(GU(b,y_1),p)=GU(y_1,p)=y_0,$$

which is a contradiction.

Thus $GU(b, y_0) = b$. Then the monotonicity gives $GU(b, y) = b$ for all $y \in [c, y_0]$ and the associativity gives

$$b = GU(b, y_0) = GU(b, GU(y_0, y_0)) = \cdots = GU(b, GU(\underbrace{y_0, \ldots, y_0}_{n-times}))$$

for all $n \in \mathbb{N}$. Since GU is on $[c, d]^2$ Archimedean we get $GU(b, y) = b$ for all $y \in [c, d[$. Then also

$$GU(x, y) = GU(GU(x, b), y) = GU(x, GU(b, y)) = GU(x, b) = x$$

for all $x \in [a, b]$ and all $y \in [c, d[$.

Now we will show that $GU(x, d) \in \{x, d\}$ for all $x \in [a, b]$. Indeed, if $GU(x, d) = x_1 > x$, $x_1 \neq d$ for some $x \in [a, b]$ then the associativity implies $GU(x_1, d) = x_1$ and thus the monotonicity and $GU(c, d) = d$ imply $x_1 \leq b$. Since GU is on $[a, b]^2$ a continuous t-norm there exists a $p \in [a, b]$ such that $GU(x_1, p) = x$ and then

$$GU(x, d) = GU(GU(p, x_1), d) = GU(p, GU(x_1, d)) = GU(p, x_1) = x,$$

which is a contradiction. Therefore $GU(x, d) \in \{x, d\}$ for all $x \in [a, b]$.

Further, if $GU(x_2, d) = x_2$ for some $x_2 \in \,]a, b[$ then the monotonicity gives $GU(x, d) = x$ for all $x \in [a, x_2]$ and the Archimedean property on $[a, b]^2$ gives $GU(x, d) = x$ for all $x \in [a, b[$.

Therefore we have the following possibilities:

1. If $GU(b, d) = b$ then also $GU(x, d) = x$ for all $x \in [a, b]$ and we see that $([a, b] \cup [c, d], GU)$ is an ordinal sum of $G_1 = ([a, b], GU|_{[a,b]^2})$ and $G_2 = ([c, d], GU|_{[c,d]^2})$, i.e., of a continuous t-norm and a continuous t-conorm, and the order on $\{1, 2\}$ in this ordinal sum is given by $1 < 2$.

2. If $GU(b, d) = d$ and $GU(x, d) = x$ for all $x \in [a, b[$ then $([a, b] \cup [c, d], GU)$ is an ordinal sum of semigroups $G_1 = ([a, b[, GU|_{[a,b[^2})$, $G_2 = (\{b\}, \mathrm{Id})$, $G_3 = ([c, d[, GU|_{[c,d[^2})$ and $G_4 = (\{d\}, \mathrm{Id})$. The order on $\{1, 2, 3, 4\}$ in this ordinal sum is given by $1 < 4 < 2 < 3$.

3. If $GU(a, d) = a$ and $GU(x, d) = d$ for all $x \in \,]a, b]$ then $([a, b] \cup [c, d], GU)$ is an ordinal sum of semigroups $G_1 = (]a, b], GU|_{]a,b]^2})$, $G_2 = (\{a\}, \mathrm{Id})$, $G_3 = ([c, d[, GU|_{[c,d[^2})$ and $G_4 = (\{d\}, \mathrm{Id})$. The order on $\{1, 2, 3, 4\}$ in this ordinal sum is given by $2 < 4 < 1 < 3$.

4. If $GU(a,d) = d$ then $([a,b] \cup [c,d], GU)$ is an ordinal sum of semigroups $G_1 = ([a,b], GU|_{[a,b]^2})$, $G_2 = ([c,d[, GU|_{[c,d[^2})$ and $G_3 = (\{d\}, \mathrm{Id})$. The order on the set $\{1,2,3\}$ in this ordinal sum is given by $3 < 1 < 2$.

$\square$

Here we introduce the proof of Proposition 2.1.6 from Section 2.1.

Proposition 2.1.6
Assume $0 \le a < b \le c < d \le 1$, $v \in [b,c]$, $e \in]0,1[$, a binary function $U\colon [0,1]^2 \longrightarrow [0,1]$ and the function f given by (2.1). Then U is a uninorm with the neutral element e and continuous underlying functions if and only if the function $U^*\colon ([a,b[\cup \{v\} \cup]c,d])^2 \longrightarrow ([a,b[\cup \{v\} \cup]c,d])$ given by $U^*(x,y) = f(U(f^{-1}(x), f^{-1}(y)))$ is a uninorm on $([a,b[\cup \{v\} \cup]c,d])^2$ which is continuous on $[a,b[^2$ and on $]c,d]^2$ and fulfills $\lim\limits_{t \longrightarrow b^-} U^*(x,t) = x$ for all $x \in [a,b[$ and $\lim\limits_{t \longrightarrow c^+} U^*(y,t) = y$ for all $y \in]c,d]$.

PROOF: Assume that U is a uninorm with continuous underlying functions. Then U^* is a uninorm on $([a,b[\cup \{v\} \cup]c,d])^2$. Since f^{-1} restricted to $[a,b[$ ($]c,d]$) is an increasing homeomorphism we know that U^* is continuous on $[a,b[^2$ and on $]c,d]^2$. Assume $x \in [a,b[$ with $f^{-1}(x) = s \in [0,e[$ and any $t \in [a,x[$ with $f^{-1}(t) = s_1 \le s$, $s_1 \in [0,e[$. Since U is a uninorm with continuous underlying functions we know that $U(e,w) = w$ and $U(0,w) = 0$ for all $w \in [0,e]$ and the continuity ensures the existence of $q \in [0,e[$ such that $U(s,q) = s_1$. Then $U^*(x, f(q)) = f(U(s,q)) = f(s_1) = t$. Furthermore, for all $p \in [0,e[$ there is $U(s,p) \le s$ and therefore $U^*(x,y) = f(U(s, f^{-1}(y))) \le f(s) = x$ for all $y \in [a,b[$. These two facts together with the monotonicity give us

$$\lim_{t \longrightarrow b^-} U^*(x,t) = x.$$

Similarly we can show that

$$\lim_{t \longrightarrow c^+} U^*(y,t) = y$$

for all $y \in]c,d]$.

Vice versa, assume that U^* is a uninorm on $([a,b[\cup \{v\} \cup]c,d])^2$ which is continuous on $[a,b[^2$ and on $]c,d]^2$ and fulfills $\lim\limits_{t \longrightarrow b^-} U^*(x,t) = x$ for all $x \in [a,b[$ and $\lim\limits_{t \longrightarrow c^+} U^*(y,t) = y$ for all $y \in]c,d]$. Then similarly as above we can show that U is a uninorm with the neutral element e which is continuous on $[0,e[^2$ and on $]e,1]^2$. Further,

$$\lim_{t \longrightarrow e^-} U(x,t) = x$$

for all $x \in [0,e[$ and

$$\lim_{t \longrightarrow e^+} U(y,t) = y$$

for all $y \in \,]e, 1]$. Since e is the neutral element of U we easily see that U is a uninorm with continuous underlying functions.

$\square$

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Part II

The papers

Chapter 4

Uninorms with continuous underlying functions

[UNI1] A. Mesiarová-Zemánková (2016). A note on decomposition of idempotent uninorms into an ordinal sum of singleton semigroups. Fuzzy Sets and Systems 299, pp. 140–145.

[UNI2] A. Mesiarová-Zemánková (2016). Ordinal sum construction for uninorms and generalized uninorms. International Journal of Approximate Reasoning 76, pp. 1–17.

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Short Communication

A note on decomposition of idempotent uninorms into an ordinal sum of singleton semigroups

Andrea Mesiarová-Zemánková

Mathematical Institute, Slovak Academy of Sciences, Bratislava, Slovakia

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Abstract

The idempotent uninorms are characterized by means of the ordinal sum of Clifford. It is shown that idempotent uninorms are in one-to-one correspondence with special linear orders on [0, 1]. A connection between respective linear order on [0, 1] and the characterizing multi-function of the uninorm is also investigated.

Keywords: Uninorm; Idempotent element; Internal uninorm; Ordinal sum

1. Introduction, basic notions and results

The uninorms (see [10,12,13,17,22]) generalize both t-norms and t-conorms (see [1,14]). A uninorm is a binary operation $U\colon [0,1]^2 \longrightarrow [0,1]$ that is commutative, associative, non-decreasing in both coordinates and has a neutral element $e \in]0,1[$. Due to the associativity, n-ary form of any uninorm is uniquely given and thus it can be extended into an aggregation function working on $\bigcup_{n\in\mathbb{N}}[0,1]^n$.

If we take uninorm in a broader sense, i.e., if for a neutral element we have $e \in [0,1]$, then the class of uninorms covers also the class of t-norms (here $e = 1$) and the class of t-conorms (here $e = 0$). For each uninorm the value $U(1,0) \in \{0,1\}$ is the annihilator of U. A uninorm is said to be *conjunctive* (*disjunctive*) if $U(1,0) = 0$ ($U(1,0) = 1$).

For each uninorm U with the neutral element $e \in]0,1[$ the restriction of U to $[0,e]^2$ is a t-norm on $[0,e]^2$, i.e., a linear transformation of some t-norm T_U on $[0,1]^2$ and the restriction of U to $[e,1]^2$ is a t-conorm on $[e,1]^2$, i.e., a linear transformation of some t-conorm S_U. Moreover, $\min(x,y) \le U(x,y) \le \max(x,y)$ for all $(x,y) \in [0,e] \times [e,1] \cup [e,1] \times [0,e]$.

On the other hand, from any pair of a t-norm and a t-conorm we can construct the minimal and the maximal uninorm with the given underlying functions.

E-mail address: zemankova@mat.savba.sk.

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Proposition 1. *(See [15].) Let* $T: [0,1]^2 \longrightarrow [0,1]$ *be a t-norm and* $S: [0,1]^2 \longrightarrow [0,1]$ *a t-conorm and assume* $e \in [0,1]$. *Then the two functions* $U_{\min}, U_{\max}: [0,1]^2 \longrightarrow [0,1]$ *given by*

$$U_{\min}(x,y) = \begin{cases} e \cdot T(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e) \cdot S(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ \min(x,y) & \text{otherwise} \end{cases}$$

and

$$U_{\max}(x,y) = \begin{cases} e \cdot T(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e) \cdot S(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ \max(x,y) & \text{otherwise} \end{cases}$$

are uninorms. We will denote the set of all uninorms of the first type by $\mathcal{U}_{\min}$ *and of the second type by* $\mathcal{U}_{\max}$.

One important subclass of uninorms are idempotent uninorms, i.e., uninorms where $U(x,x) = x$ for all $x \in [0,1]$. In the case of t-norms and t-conorms there is only one idempotent t-norm – the minimum, and only one idempotent t-conorm – the maximum. Therefore idempotent uninorms are uniquely given and continuous on $[0,e]^2 \cup [e,1]^2$. Idempotent uninorms were studied in several papers (see Refs., [7,9,16,21] and references therein).

Lemma 1. *(See [9].) Let* $U: [0,1]^2 \longrightarrow [0,1]$ *be an idempotent uninorm. Then* U *is internal, i.e.,* $U(x,y) \in \{x,y\}$ *holds for all* $(x,y) \in [0,1]^2$.

Further, idempotent uninorms that are left-continuous, or right-continuous were characterized in [7]. Idempotent uninorms on finite ordinal scales were studied in [5]. The complete characterization of idempotent uninorms from [16] was later corrected in [21]. In the following a non-increasing function $g: [0,1] \longrightarrow [0,1]$ is called Id-symmetrical if its completed graph F_g is Id-symmetrical, i.e., $(x,y) \in F_g$ if and only if $(y,x) \in F_g$. Note that a completed graph was defined in [21] as follows: let $g: [0,1] \longrightarrow [0,1]$ be any decreasing function and let G be the graph of g, that is

$$G = \{(x, g(x)) \mid x \in [0,1]\};$$

for any point of discontinuity s of g, let s^- and s^+ be the corresponding lateral limits. Then, we define the completed graph of g, denoted by F_g, as the set obtained from G by adding the vertical segments in any discontinuity point s, from s^- to s^+.

Theorem 1. *(See [21].) Consider* $e \in]0,1[$. *The following items are equivalent:*

(i) U *is an idempotent uninorm with neutral element* e.
(ii) There exists a decreasing, Id-symmetrical function $g: [0,1] \longrightarrow [0,1]$ *with fixed point* e *such that* U *is for all* $(x,y) \in [0,1]^2$ *given by*

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } y < g(x) \text{ or } y = g(x), x < g(g(x)), \\ \max(x,y) & \text{if } y > g(x) \text{ or } y = g(x), x > g(g(x)), \\ x \text{ or } y & \text{if } y = g(x), x = g(g(x)), \end{cases}$$

being commutative on the set of points $(x, g(x))$ *such that* $x = g(g(x))$.

Note that the function g coincides with the characterizing multi-function of U which we now recall.

Definition 1. (See [18].) A mapping $p: [0,1] \longrightarrow \mathcal{P}([0,1])$ is called a multi-function if to every $x \in [0,1]$ it assigns a subset of $[0,1]$, i.e., $p(x) \subseteq [0,1]$. A multi-function p is called

(i) *non-increasing* if for all $x_1, x_2 \in [0,1]$, $x_1 < x_2$ there is $p(x_1) \geq p(x_2)$, i.e, for all $y_1 \in p(x_1)$ and all $y_2 \in p(x_2)$ we have $y_1 \geq y_2$ and thus $\mathrm{Card}(p(x_1) \cap p(x_2)) \leq 1$,
(ii) *symmetric* if $y \in p(x)$ if and only if $x \in p(y)$.

The graph of a multi-function p will be denoted by $G(p)$, i.e., $(x, y) \in G(p)$ if and only if $y \in p(x)$.

Lemma 2. *(See [18].) A symmetric multi-function $p\colon [0,1] \longrightarrow \mathcal{P}([0,1])$ is surjective, i.e., for all $y \in [0,1]$ there exists an $x \in [0,1]$ such that $y \in p(x)$, if and only if we have $p(x) \neq \emptyset$ for all $x \in [0,1]$. The graph of a symmetric, surjective, non-increasing multi-function $p\colon [0,1] \longrightarrow \mathcal{P}([0,1])$ is a connected line.*

We will denote the set of all uninorms $U\colon [0,1]^2 \longrightarrow [0,1]$ such that U is continuous on $[0,1]^2 \setminus R$, where $R = G(r)$ and r is a symmetric, surjective, non-increasing multi-function such that $U(x, y) = e$ implies $(x, y) \in R$, by $\mathcal{UR}$. Further, the corresponding multi-function r will be called the characterizing multi-function of U.

Theorem 2. *(See [18].) Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm. Then U is continuous on $[0,e]^2$ and on $[e,1]^2$ if and only if $U \in \mathcal{UR}$ and in each point $(x, y) \in [0,1]^2$ the uninorm U is either left-continuous or right-continuous.*

The previous theorem characterizes uninorms with continuous underlying functions via their characterizing multi-function. In the case of idempotent uninorms the graph of their characterizing multi-function coincides with the completed graph of the function g from Theorem 1.

Since uninorms are special semigroups we can use here the result of Clifford.

Theorem 3. *(See [8].) Let $A \neq \emptyset$ be a totally ordered set and $(G_\alpha)_{\alpha \in A}$ with $G_\alpha = (X_\alpha, *_\alpha)$ be a family of semigroups. Assume that for all $\alpha, \beta \in A$ with $\alpha < \beta$ the sets X_α and X_β are either disjoint or that $X_\alpha \cap X_\beta = \{x_{\alpha,\beta}\}$, where $x_{\alpha,\beta}$ is both the neutral element of G_α and the annihilator of G_β and where for each $\gamma \in A$ with $\alpha < \gamma < \beta$ we have $X_\gamma = \{x_{\alpha,\beta}\}$. Put $X = \bigcup_{\alpha \in A} X_\alpha$ and define the binary operation $*$ on X by*

$$x * y = \begin{cases} x *_\alpha y & \text{if } (x, y) \in X_\alpha \times X_\alpha, \\ x & \text{if } (x, y) \in X_\alpha \times X_\beta \text{ and } \alpha < \beta, \\ y & \text{if } (x, y) \in X_\alpha \times X_\beta \text{ and } \alpha > \beta. \end{cases}$$

Then $G = (X, *)$ is a semigroup. The semigroup G is commutative if and only if for each $\alpha \in A$ the semigroup G_α is commutative.

It is immediate that both uninorms $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ discussed in Proposition 1 can be seen as ordinal sums. This is (in our best knowledge), up to minor generalizations including the ordinal sum construction on $[0,e]^2$ and/or on $[e,1]^2$ subdomains (see [11]), the only application of Clifford's ordinal sums in construction/representation of uninorms. Our results presented in the next section show a novel and surprising fact concerning the construction/representation of idempotent uninorms, showing their link through the Clifford ordinal sum construction with particular linear orders on the set $[0,1]$.

2. Main result

In this section we would like to show that each idempotent uninorm can be decomposed to an ordinal sum of singleton semigroups. For these semigroups, i.e., semigroups that are defined on singletons, the only possible operation is

$$\mathrm{Id}\colon \{x\}^2 \longrightarrow \{x\} \text{ given by } \mathrm{Id}(x, x) = x.$$

Proposition 2. *Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be an idempotent uninorm. Then $([0,1], U)$ is an ordinal sum of singleton semigroups $(\{x\}, \mathrm{Id})$ for $x \in [0,1]$.*

Proof. Let P be an index set isomorphic with $[0, 1]$ via the isomorphism i. For all $p \in P$ we put $X_p = \{x\}$ if $i(p) = x$. On the set P we define a relation $\preceq$ by $p_1 \preceq p_2$ if $U(x, y) = x$, where $X_{p_1} = \{x\}$ and $X_{p_2} = \{y\}$. Now we will show that $\preceq$ is a linear order. Since U is idempotent we have $U(x, x) = x$ for all $x \in [0, 1]$ and thus $\preceq$ is reflexive. If $U(x, y) = x$ and $U(y, x) = y$ then the commutativity of U implies that $x = y$, i.e., $\preceq$ is anti-symmetric. If $U(x, y) = x$ and $U(y, z) = y$ then the associativity of U implies

$$U(x, z) = U(U(x, y), z) = U(x, U(y, z)) = U(x, y) = x,$$

i.e., $U(x, z) = x$ and thus $\preceq$ is transitive. Finally, by Lemma 1 we have $U(x, y) \in \{x, y\}$ for all $x, y \in [0, 1]$ and thus $\preceq$ is a linear order. Now let $([0, 1], U^*)$ be the ordinal sum of $\{(X_p, \mathrm{Id})\}_{p\in P}$ with the linear order $\preceq$. Then it is easy to see that $U(x, y) = U^*(x, y)$ for all $x, y \in [0, 1]$. □

Proposition 3. *Let P be an index set isomorphic with $[0, 1]$ via the isomorphism i. For all $p \in P$ we put $X_p = \{x\}$ if $i(p) = x$. Let $e \in [0, 1]$ and let $\preceq$ be a linear order on P. If $([0, 1], U)$ is the ordinal sum of $\{(X_p, \mathrm{Id})\}_{p\in P}$ with the linear order $\preceq$ then U is an idempotent uninorm with the neutral element e if and only if the following two conditions are fulfilled:*

(i) $p_1 \prec p_2$ for all $p_1, p_2 \in P$ if $X_{p_1} = \{x_1\}$, $X_{p_2} = \{x_2\}$, $x_1 < x_2$ and $x_1, x_2 \in [0, e]$,
(ii) $p_1 \prec p_2$ for all $p_1, p_2 \in P$ if $X_{p_1} = \{y_1\}$, $X_{p_2} = \{y_2\}$, $y_1 > y_2$ and $y_1, y_2 \in [e, 1]$.

Proof. Let $([0, 1], U)$ be the ordinal sum of $\{(X_p, \mathrm{Id})\}_{p\in P}$ with the linear order $\preceq$. Then U is associative and commutative. If the two conditions are satisfied then $p_1 \prec p$ and $p_2 \prec p$, where $X_{p_1} = \{x\}$, $X_{p_2} = \{y\}$ and $X_p = \{e\}$ for all $x \in [0, e]$, $y \in [e, 1]$. Therefore $U(x, e) = x$ for all $x \in [0, 1]$. Finally, let us show that U is non-decreasing. The two conditions imply that $U|_{[0,e]^2} = \min$ and $U|_{[e,1]^2} = \max$. Thus we only have to show the monotonicity on $[0, e] \times [e, 1] \cup [e, 1] \times [0, e]$. We will focus on $[0, e] \times [e, 1]$ as the other case is analogical. Due to the commutativity it is enough to show that for $x \in [0, e]$ and $y_1, y_2 \in [e, 1]$, $y_1 < y_2$ there is $U(x, y_1) \le U(x, y_2)$. Denote $X_p = \{y\}$, $X_{p_1} = \{y_1\}$ and $X_{p_2} = \{y_2\}$. Then the second condition implies that $p_2 \prec p_1$. Now there are three possibilities:

(i) $p_2 \prec p_1 \prec p$,
(ii) $p_2 \prec p \prec p_1$,
(iii) $p \prec p_2 \prec p_1$.

In the first case we have $U(x, y_1) = y_1 < y_2 = U(x, y_2)$. In the second case there is $U(x, y_1) = x < y_2 = U(x, y_2)$. Finally, in the third case we get $U(x, y_1) = x = U(x, y_2)$. Thus in all cases $U(x, y_1) \le U(x, y_2)$ and therefore U is non-decreasing in both coordinates. Summarizing, U is an internal uninorm.

Vice versa, let U be an idempotent uninorm. From Proposition 2 it then follows that $([0, 1], U)$ is an ordinal sum of $\{(X_p, \mathrm{Id})\}_{p\in P}$ with the linear order $\preceq_2$ given by $p_1 \preceq_2 p_2$ if $U(x, y) = x$ for $X_{p_1} = \{x\}$, $X_{p_2} = \{y\}$. Since for $x_1, x_2 \in [0, e]$ we have $U(x, y) = \min(x, y)$ we get $p_1 \prec p_2$ for all $p_1, p_2 \in P$ such that $X_{p_1} = \{x_1\}$, $X_{p_2} = \{x_2\}$, $x_1 < x_2$ and $x_1, x_2 \in [0, e]$. Similarly, for $y_1, y_2 \in [e, 1]$ we have $U(x, y) = \max(x, y)$ and thus we get $p_1 \prec p_2$ for all $p_1, p_2 \in P$ such that $X_{p_1} = \{y_1\}$, $X_{p_2} = \{y_2\}$, $y_1 > y_2$ and $y_1, y_2 \in [e, 1]$. □

The previous result shows that idempotent uninorms are in one-to-one correspondence with linear orders $\preceq$ on $[0, 1]$, such that $x_1 < x_2$ implies $x_1 \prec x_2$ for all $x_1, x_2 \in [0, e]$ and $y_1 > y_2$ implies $y_1 \prec y_2$ for all $y_1, y_2 \in [e, 1]$. We will denote the set of all linear orders on $[0, 1]$ that fulfill this condition by $\mathcal{R}_e$.

Example 1.

(i) Let $U_1 \colon [0, 1]^2 \longrightarrow [0, 1]$ be an idempotent uninorm, $U_1 \in \mathcal{U}_{\min}$. Then for the linear order $\preceq_1 \in \mathcal{R}_e$ related to the decomposition of U_1 into singleton semigroups we have $0 \prec_1 x \prec_1 1 \prec_1 y \prec_1 e$ for all $x \in]0, e[$, $y \in]e, 1[$.
(ii) Let $U_2 \colon [0, 1]^2 \longrightarrow [0, 1]$ be an idempotent uninorm, $U_2 \in \mathcal{U}_{\max}$. Then for the linear order $\preceq_2 \in \mathcal{R}_e$ related to the decomposition of U_2 into singleton semigroups we have $1 \prec_2 y \prec_2 0 \prec_2 x \prec_2 e$ for all $x \in]0, e[$, $y \in]e, 1[$.

(iii) Let $U_3\colon [0,1]^2 \longrightarrow [0,1]$ be an idempotent uninorm given by

$$U_3(x,y) = \begin{cases} \min(x,y) & \text{if } x+y<1, \\ \max(x,y) & \text{otherwise.} \end{cases}$$

Then for the linear order $\preceq_3 \in \mathcal{R}_e$ related to the decomposition of U_3 into singleton semigroups we have for all $x \in]0,e[$, $y \in]e,1[$ that $x \prec_3 y$ if and only if $x+y<1$.

Corollary 1. *Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be an idempotent uninorm and let $\preceq \in \mathcal{R}_e$ be the linear order related to the decomposition of U into singleton semigroups. If $g\colon [0,1] \longrightarrow [0,1]$ is the decreasing, Id-symmetrical function with fixed point e from Theorem 1 then for all $x \in [0,e]$, $y \in [e,1]$ there is $x \prec y$ if and only if either $y < g(x)$, or $y = g(x)$, $x < g(g(x))$, or $y = g(x)$, $x = g(g(x))$ and $U(x,y) = x$.*

Proposition 4. *Let $U_1\colon [0,1]^2 \longrightarrow [0,1]$ be an idempotent uninorm and let $U_2\colon [0,1]^2 \longrightarrow [0,1]$ be the dual uninorm of U_1, i.e., there is $U_1(x,y) = 1 - U_2(1-x, 1-y)$ for all $x, y \in [0,1]$. Then for the linear order $\preceq_1$ related to the decomposition of U_1 into singleton semigroups and the linear order $\preceq_2$ related to the decomposition of U_2 into singleton semigroups we have $x \preceq_1 y$ if and only if $1-x \preceq_2 1-y$ for all $x, y \in [0,1]$.*

Proof. We have $x \preceq_1 y$ if and only if $x = U_1(x,y) = 1 - U_2(1-x, 1-y)$, i.e., $U_2(1-x,1-y) = 1-x$ and thus $1-x \preceq_2 1-y$ for all $x, y \in [0,1]$. □

Using any increasing isomorphism $\varphi\colon [0,1] \longrightarrow [0,1]$ we can construct from a uninorm U a new uninorm U_φ by

$$U_\varphi(x,y) = \varphi^{-1}(U(\varphi(x), \varphi(y)))$$

for all $(x,y) \in [0,1]^2$. Then we have the following result.

Proposition 5. *Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be an idempotent uninorm and let $\varphi\colon [0,1] \longrightarrow [0,1]$ be an increasing isomorphism on $[0,1]$. Then $U_\varphi\colon [0,1]^2 \longrightarrow [0,1]$ given for all $(x,y) \in [0,1]^2$ by $U_\varphi(x,y) = \varphi^{-1}(U(\varphi(x), \varphi(y)))$ is an idempotent uninorm. Further, for the linear order $\preceq$ related to the decomposition of U into singleton semigroups and the linear order $\preceq_\varphi$ related to the decomposition of U_φ into singleton semigroups we have $x \preceq_\varphi y$ if and only if $\varphi(x) \preceq \varphi(y)$ for all $x, y \in [0,1]$.*

Proof. It is easy to see that U_φ is an idempotent uninorm. Further, for all $x, y \in [0,1]$ we have $x \preceq_\varphi y$ if and only if

$$x = U_\varphi(x,y) = \varphi^{-1}(U(\varphi(x), \varphi(y))),$$

i.e., $U(\varphi(x), \varphi(y)) = \varphi(x)$ which holds if and only if $\varphi(x) \preceq \varphi(y)$. □

Remark 1. Proposition 3 is valid also for discrete uninorms, i.e., for commutative, associative, non-decreasing operations $U\colon \{0,\dots,n-1\}^2 \longrightarrow \{0,\dots,n-1\}$ with neutral element $e \in \{0,\dots,n-1\}$, where $n \in \mathbb{N}$. If we fix $i \in \{0,\dots,n-1\}$ then there are exactly i elements smaller than i and $n-1-i$ elements bigger than i. Therefore the number of idempotent discrete uninorms on $\{0,\dots,n-1\}$ with the neutral element $e = i$ is equal to the number of ways how to divide $n-1-i$ elements into $i+1$ groups which is the combinatorial number $C(n-1,i)$. Then we immediately see that there are exactly $\sum_{i=0}^{n-1} C(n-1,i) = 2^{n-1}$ idempotent uninorms on $\{0,\dots,n-1\}$.

3. Conclusions

In this short contribution we have shown the characterization of idempotent uninorms via the ordinal sum of singleton semigroups. Respective linear orders on $[0,1]$ were also studied. Up to the theoretical importance of our results, we expect also their application in several fields, where the uninorms were already successfully applied, such as in expert systems [6], approximate reasoning [20], data mining [26], image processing [3,4], fuzzy systems modeling [23–25], neural networks [2], etc.

Our results cover and generalize results known from the literature. For example, Corollary 1 shows transparently what is the link of representation of idempotent uninorms by means of Id-symmetrical functions (see Theorem 1) and the representation by means of linear orders on [0, 1] (see Proposition 3).

In [19] we will continue to characterize uninorms with continuous underlying functions via the ordinal sum of Clifford.

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Ordinal sum construction for uninorms and generalized uninorms

Andrea Mesiarová-Zemánková

Mathematical Institute, Slovak Academy of Sciences, Bratislava, Slovakia

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ABSTRACT

The ordinal sum construction yielding uninorms is studied. A special case when all summands in the ordinal sum are isomorphic to uninorms is discussed and the most general semigroups that yield a uninorm via the ordinal sum construction, called generalized uninorms, are studied.

1. Introduction

The (left-continuous) t-norms and their dual t-conorms play an indispensable role in many domains such as probabilistic metric spaces [21], fuzzy logic [4], fuzzy control [22], non-additive measures and integrals [19], multi-criteria-decision making [26] and others. Each continuous t-norm is an ordinal sum of continuous Archimedean t-norms, and each continuous Archimedean t-norm possesses a continuous additive generator. However, in [9] (see also [10]) it was shown that the most general operations that yield a t-norm via the ordinal sum construction are t-subnorms.

In order to model bipolar behavior, uninorms were introduced in [24] (see also [3]). A uninorm U restricted to $[0, e]^2$, where e is the neutral element of U is a t-norm on $[0, e]^2$, and U restricted to $[e, 1]^2$ is a t-conorm on $[e, 1]^2$. Each uninorm is isomorphic to a bipolar t-conorm on $[-1, 1]$ (see [15]), i.e., a bipolar operation that is disjunctive with respect to the neutral point 0 (i.e., aggregated values diverge from the neutral point).

T-norms, t-conorms as well as uninorms are Abelian semigroups and therefore it is possible to apply the ordinal sum of Clifford for their construction. As uninorms are closely related to t-norms and t-conorms, it is clear that an ordinal sum that yields a uninorm will be closely connected with the ordinal sum that yields the corresponding underlying t-norm and t-conorm. In the case of t-norms (t-conorms) the basic stones in the ordinal sum construction are t-subnorms (t-superconorms). In this paper we investigate which operations can be used in the construction of uninorms via the ordinal sum and we call them generalized uninorms.

As we mentioned above, each continuous Archimedean t-norm possesses a continuous additive generator which has a range from $[0, \infty]$. Moreover, also t-subnorms can be additively generated. In the case of t-subnorms the strict monotonicity

E-mail address: zemankova@mat.savba.sk.

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of additive generators of t-norms can be relaxed. If we consider a strictly monotone, continuous additive generator with the range $[-\infty, \infty]$ the generated operation will be a uninorm. Therefore our other interest is whether by relaxing the strict monotonicity of the additive generator of a uninorm we can generate a generalized uninorm.

The paper is structured as follows. In Section 2, some basic notions and results are recalled. The ordinal sum construction of Clifford is used to construct uninorms (Section 3) and a special case when all summands in this ordinal sum are isomorphic to uninorms is discussed in Section 4. In Section 5 we show the basic facts on generalized uninorms and in Section 6 we then study generated generalized uninorms. We give our conclusions in Section 7.

2. Basic notions and results

We will start with several important definitions (see [8,14]).

Definition 1.

(i) A triangular norm is a binary function $T\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and 1 is its neutral element.
(ii) A triangular conorm is a binary function $C\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and 0 is its neutral element.
(iii) A triangular subnorm is a binary function $M\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and there is $M(x,y) \le \min(x,y)$ for all $(x,y) \in [0,1]^2$.
(iv) A triangular superconorm is a binary function $R\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and there is $R(x,y) \ge \max(x,y)$ for all $(x,y) \in [0,1]^2$.

Due to the associativity n-ary form of any t-norm (t-conorm) is uniquely given and thus it can be extended to an aggregation function working on $\bigcup_{n\in\mathbb{N}}[0,1]^n$.

The duality between t-norms and t-conorms is expressed by the fact that from any t-norm T we can obtain its dual t-conorm C by the equation

$$C(x,y) = 1 - T(1-x, 1-y)$$

and vice-versa. The same duality holds between t-subnorms and t-superconorms.

Now let us recall an ordinal sum construction for t-norms and t-conorms [8].

Proposition 1. *Let K be a finite or countably infinite index set and let $(]a_k, b_k[)_{k\in K}$ $((]c_k, d_k[)_{k\in K})$ be a system of open disjoint subintervals of $[0,1]$. Let $(T_k)_{k\in K}$ $((C_k)_{k\in K})$ be a system of t-norms (t-conorms). Then the ordinal sum $T = (\langle a_k, b_k, T_k\rangle \mid k \in K)$ $(C = (\langle c_k, d_k, C_k\rangle \mid k \in K))$ given by*

$$T(x,y) = \begin{cases} a_k + (b_k - a_k)T_k(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}) & \text{if } (x,y) \in [a_k, b_k[^2\,, \\ \min(x,y) & \text{else} \end{cases}$$

and

$$C(x,y) = \begin{cases} c_k + (d_k - c_k)C_k(\frac{x-c_k}{d_k-c_k}, \frac{y-c_k}{d_k-c_k}) & \text{if } (x,y) \in]c_k, d_k]^2\,, \\ \max(x,y) & \text{else} \end{cases}$$

is a t-norm (t-conorm). The t-norm T (t-conorm C) is continuous if and only if all summands T_k (C_k) for $k \in K$ are continuous.

Proposition 2 *([8]). Let $t\colon [0,1] \longrightarrow [0,\infty]$ $(c\colon [0,1] \longrightarrow [0,\infty])$ be a continuous strictly decreasing (increasing) function such that $t(1) = 0$ $(c(0) = 0)$. Then the binary operation $T\colon [0,1]^2 \longrightarrow [0,1]$ $(C\colon [0,1]^2 \longrightarrow [0,1])$ given by*

$$T(x,y) = t^{-1}(\min(t(0), t(x) + t(y)))$$
$$C(x,y) = c^{-1}(\min(c(1), c(x) + c(y)))$$

is a continuous t-norm (t-conorm). The function t (c) is called an additive generator *of T (C).*

An additive generator of a continuous t-norm T (t-conorm C) is uniquely determined up to a positive multiplicative constant. Each continuous t-norm (t-conorm) is equal to an ordinal sum of continuous Archimedean t-norms (t-conorms). Note that a continuous t-norm (t-conorm) is Archimedean if and only if it has only trivial idempotent points 0 and 1. A continuous Archimedean t-norm T (t-conorm C) is either strict, i.e., strictly increasing on $]0,1]^2$ (on $[0,1[^2$), or nilpotent, i.e., there exists $(x, y) \in]0, 1[^2$ such that $T(x, y) = 0$ ($C(x, y) = 1$). Moreover, each continuous Archimedean t-norm (t-conorm) has a continuous additive generator. More details on t-norms and t-conorms can be found in [1,8].

Definition 2 *([24]).* A uninorm is a binary function $U\colon [0, 1]^2 \longrightarrow [0, 1]$ which is commutative, associative, non-decreasing in both variables and has a neutral element $e \in]0, 1[$.

If the class of uninorms is taken in a broader sense, i.e., if for the neutral element we have $e \in [0, 1]$ then the class of uninorms covers also t-norms and t-conorms. In the case that we will assume a uninorm with $e \in]0, 1[$ we will call such a uninorm *proper*. For each uninorm the value $U(1, 0) \in \{0, 1\}$ is the annihilator of U. A uninorm is called *conjunctive* (*disjunctive*) if $U(1, 0) = 0$ ($U(1, 0) = 1$) (see [3,24]).

For each uninorm U with the neutral element $e \in [0, 1]$, the restriction of U to $[0, e]^2$ is a t-norm T^*_U on $[0, e]^2$, i.e., a linear transformation of some t-norm T_U on $[0, 1]^2$ and the restriction of U to $[e, 1]^2$ is a t-conorm C^*_U on $[e, 1]^2$, i.e., a linear transformation of some t-conorm C_U on $[0, 1]^2$. Moreover, $\min(x, y) \le U(x, y) \le \max(x, y)$ for all $(x, y) \in [0, e] \times [e, 1] \cup [e, 1] \times [0, e]$.

On the other hand, from [11] we have the following result.

Proposition 3. *Let $T\colon [0, 1]^2 \longrightarrow [0, 1]$ be a t-norm and $C\colon [0, 1]^2 \longrightarrow [0, 1]$ a t-conorm and assume $e \in [0, 1]$. Then the two functions $U_{\min}, U_{\max}\colon [0, 1]^2 \longrightarrow [0, 1]$ given by*

$$U_{\min}(x, y) = \begin{cases} e \cdot T(\frac{x}{e}, \frac{y}{e}) & \text{if } (x, y) \in [0, e]^2, \\ e + (1 - e) \cdot C(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x, y) \in [e, 1]^2, \\ \min(x, y) & \text{otherwise} \end{cases}$$

and

$$U_{\max}(x, y) = \begin{cases} e \cdot T(\frac{x}{e}, \frac{y}{e}) & \text{if } (x, y) \in [0, e]^2, \\ e + (1 - e) \cdot C(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x, y) \in [e, 1]^2, \\ \max(x, y) & \text{otherwise} \end{cases}$$

are uninorms. We will denote the set of all uninorms of the first type by $\mathcal{U}_{\min}$ and of the second type by $\mathcal{U}_{\max}$.

3. Ordinal sum construction for uninorms

At first we recall the basic result of Clifford [2] on which the ordinal sum construction for t-norms, and generally for semigroups is based.

Theorem 1. *Let $A \neq \emptyset$ be a totally ordered set and $(G_\alpha)_{\alpha \in A}$ with $G_\alpha = (X_\alpha, *_\alpha)$ be a family of semigroups. Assume that for all $\alpha, \beta \in A$, $\alpha \neq \beta$, the sets X_α and X_β are disjoint. Put $X = \bigcup_{\alpha \in A} X_\alpha$ and define the binary operation $*$ on X by*

$$x * y = \begin{cases} x *_\alpha y & \text{if } (x, y) \in X_\alpha \times X_\alpha, \\ x & \text{if } (x, y) \in X_\alpha \times X_\beta \text{ and } \alpha < \beta, \\ y & \text{if } (x, y) \in X_\alpha \times X_\beta \text{ and } \alpha > \beta. \end{cases} \tag{1}$$

Then $G = (X, *)$ is a semigroup. The semigroup G is commutative if and only if for each $\alpha \in A$ the semigroup G_α is commutative.

Remark 1. Note that the condition that for $\alpha, \beta \in A$, $\alpha \neq \beta$, the sets X_α and X_β are disjoint can be replaced by the condition that for all $\alpha, \beta \in A$ with $\alpha < \beta$ the sets X_α and X_β are either disjoint or $X_\alpha \cap X_\beta = \{x_{\alpha,\beta}\}$, where $x_{\alpha,\beta}$ is both the neutral element of G_α and the annihilator of G_β and for each $\gamma \in A$ with $\alpha < \gamma < \beta$ we have $X_\gamma = \{x_{\alpha,\beta}\}$.

In Proposition 1 we see that the ordinal sum construction for t-norms is an ordinal sum construction in the sense of Clifford (see [9]), where we have a family $(X_k, *_k)_{k \in K}$, with $X_k = [a_k, b_k[$ and

$$x *_k y = a_k + (b_k - a_k) \cdot T_k(\frac{x - a_k}{b_k - a_k}, \frac{y - a_k}{b_k - a_k}).$$

Further we take

$$[0,1] \setminus \bigcup_{k\in K} [a_k, b_k[= \bigcup_{l\in L} Y_l,$$

where the sets Y_l are components of $[0,1] \setminus \bigcup_{k\in K} [a_k, b_k[$ with respect to conectedness, with $x *_l y = \min(x, y)$, i.e., we obtain a family $(Y_l, *_l)_{l\in L}$ of semigroups. Equipping the set $K \cup L$ with the linear order $\preceq$, which is compatible with the usual order $\leq$ on $[0,1]$, then the semigroup $([0,1], T)$ from Proposition 1 can be rewritten as an ordinal sum of semigroups $(X_\alpha, *_\alpha)_{\alpha\in K\cup L}$ in the sense of Clifford. Note that the sets Y_l are either subintervals of $[0,1]$ or singletons.

In the case of t-conorms we can proceed similarly, however, here the linear order of semigroups will be reversed with respect to the usual order on $[0,1]$, and the operation on the set Y_l will be given by $x *_l y = \max(x, y)$.

Let us now focus on the class of uninorms. If a uninorm is an ordinal sum of semigroups $(X_\alpha, *_\alpha)$ for $\alpha \in A$ then $U|_{[0,e]^2}$ is an ordinal sum of semigroups $(X_\alpha \cap [0,e], *_\alpha)$ and $U|_{[e,1]^2}$ is an ordinal sum of semigroups $(X_\alpha \cap [e,1], *_\alpha)$ for $\alpha \in A$. Since $U|_{[0,e]^2}$ ($U|_{[e,1]^2}$) is linearly isomorphic to the underlying t-norm T_U (t-conorm C_U) the structure of X_α for $\alpha \in A$ can be derived from the respective results on the ordinal sum of semigroups yielding a t-norm (t-conorm) which we will now recall.

In [5] Jenei introduced t-subnorms which not only generalize t-norms but are the basic stones in the construction and characterization of t-norms (see Definition 1). Continuous t-subnorms were studied in [14]. In [6] the following construction of t-norms using t-subnorms was shown.

Proposition 4. *Let K be a finite or countably infinite index set and let $(]a_k, b_k[)_{k\in K}$ be a system of open disjoint subintervals of $[0,1]$. Let $(M_k)_{k\in K}$ be a family of t-subnorms such that if $b_{k_0} = 1$ for some $k_0 \in K$ then M_{k_0} is a t-norm, and if $b_{k_1} = a_{k_2}$ for some $k_1, k_2 \in K$ then either M_{k_1} is a t-norm or M_{k_2} has no zero divisors. Then the ordinal sum $T = (\langle a_k, b_k, M_k\rangle \mid k \in K)$ given by*

$$T(x,y) = \begin{cases} a_k + (b_k - a_k) M_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right) & \text{if } (x,y) \in \,]a_k, b_k]^2, \\ \min(x,y) & \text{else} \end{cases}$$

is a t-norm.

Thus t-norms are not the most general semigroups that yield a t-norm via the ordinal sum construction. In [9] the most general semigroups that can be used for construction of t-norms via the ordinal sum construction (in the sense of Clifford) were studied. Let us recall several results from this paper.

Proposition 5. *Let $(A, \preceq)$ be a linearly ordered set, $A \neq \emptyset$ and $((X_\alpha, *_\alpha))_{\alpha\in A}$ be a family of semigroups such that $(X_\alpha)_{\alpha\in A}$ is a partition of the closed unit interval $[0,1]$. If operation $* \colon [0,1]^2 \longrightarrow [0,1]$ given by (1) is a triangular norm then we have:*

(i) Each X_α is a subinterval on $[0,1]$.
(ii) Each semigroup $(X_\alpha, *_\alpha)$ is a totally ordered Abelian semigroup where the operation $*_\alpha$ is bounded from above by the minimum, i.e., we have $x *_\alpha y \leq \min(x,y)$ for all $x, y \in X_\alpha$.
(iii) The order $\preceq$ on A is compatible with the usual order $\leq$ on $[0,1]$, i.e., for $\alpha, \beta \in A$ we have $\alpha \prec \beta$ if and only if $x < y$ for all $x \in X_\alpha$ and $y \in X_\beta$.
(iv) For all $(x,y) \in [0,1]^2$ we have

$$x * y = \begin{cases} x *_\alpha y & \text{if } (x,y) \in X_\alpha^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

Proposition 6. *Let T be a t-norm. Then the following are equivalent:*

(i) $([0,1], T)$ is a non-trivial ordinal sum of semigroups.
(ii) T is a non-trivial ordinal sum of t-subnorms.

The previous result shows that t-subnorms are the most general semigroups that can be used in the construction via the ordinal sum. This result is based on Proposition 5, which implies that sets X_α in the ordinal sum construction are of five kinds: singletons, and then subintervals of the form $[a_\alpha, b_\alpha]$, or $[a_\alpha, b_\alpha[$, or $]a_\alpha, b_\alpha]$, or $]a_\alpha, b_\alpha[$. It is easy to see that on singletons the corresponding operation $*_\alpha$ is given by $x *_\alpha y = \min(x,y)$. Further, the interval $[a_\alpha, b_\alpha]$ corresponds to a t-subnorm M_α given by

$$M_\alpha(x,y) = \frac{(a_\alpha + (b_\alpha - a_\alpha)\cdot x) *_\alpha (a_\alpha + (b_\alpha - a_\alpha)\cdot y) - a_\alpha}{b_\alpha - a_\alpha} \tag{2}$$

for all $x, y \in [0,1]^2$. The t-subnorm that corresponds to the interval $]a_\alpha, b_\alpha]$ is given by (2) for all $x, y \in]0,1]^2$ and $M_\alpha(x,y) = 0$ for all $x, y \in [0,1]$ with $\min(x,y) = 0$. The t-subnorm that corresponds to the interval $[a_\alpha, b_\alpha[$ is given by (2) for all $x, y \in [0,1[$ and $M_\alpha(x,y) = \min(x,y)$ for all $x, y \in [0,1]$ such that $\max(x,y) = 1$. Finally, the t-subnorm that corresponds to the interval $]a_\alpha, b_\alpha[$ is given by (2) for all $x, y \in]0,1[$ and $M_\alpha(x,y) = \min(x,y)$ for all $x, y \in [0,1]$ such that $\max(x,y) = 1$, $M_\alpha(x,y) = 0$ for all $x, y \in [0,1]$ with $\min(x,y) = 0$.

As we mentioned above, if for a uninorm U the semigroup $([0,1], U)$ is an ordinal sum of semigroups $((X_\alpha, *_\alpha))_{\alpha\in A}$ then $U|_{[0,e]^2}$, which is linearly isomorphic to T_U, is an ordinal sum of semigroups $((X_\alpha \cap [0,e], *_\alpha))_{\alpha\in A}$ and $U|_{[e,1]^2}$, which is linearly isomorphic to C_U, is an ordinal sum of semigroups $((X_\alpha \cap [e,1], *_\alpha))_{\alpha\in A}$. Therefore Proposition 5 easily yields the following result.

Proposition 7. *Let $(A, \preceq)$ be a linearly ordered set, $A \neq \emptyset$ and $((X_\alpha, *_\alpha))_{\alpha\in A}$ be a family of semigroups such that $(X_\alpha)_{\alpha\in A}$ is a partition of the closed unit interval $[0,1]$. If operation $* \colon [0,1]^2 \longrightarrow [0,1]$ given by (1) is a uninorm with the neutral element $e \in [0,1]$ then we have:*

(i) Each X_α has a form $X_\alpha = I_\alpha \cup J_\alpha$, where I_α is a subinterval of $[0,e]$ and J_α is a subinterval of $[e,1]$.
(ii) Each semigroup $(X_\alpha, *_\alpha)$ is a totally ordered Abelian semigroup, where for the operation $*_\alpha$ we have $x *_\alpha y \le \min(x,y)$ for all $x, y \in I_\alpha$, $x *_\alpha y \ge \max(x,y)$ for all $x, y \in J_\alpha$, and $x *_\alpha y \in [x,y]$ for all $x \in I_\alpha$, $y \in J_\alpha$ and all $x \in J_\alpha$, $y \in I_\alpha$.
(iii) The order $\preceq$ on A is compatible with the usual order $\le$ on $[0,e]$ and reversed to the usual order $\le$ on $[e,1]$, i.e., for $\alpha, \beta \in A$ we have $\alpha \prec \beta$ if and only if $x < y$ for all $x \in I_\alpha$ and $y \in I_\beta$ and $u > v$ for all $u \in J_\alpha$ and $v \in J_\beta$.

Remark 2. Proposition 7 remains valid also in the case that for the family of semigroups $((X_\alpha, *_\alpha))_{\alpha\in A}$ the sets X_α are not disjoint but fulfill the conditions from Remark 1.

Since I_α is a subinterval of $[0,e]$ and J_α is a subinterval of $[e,1]$, where both I_α and J_α can have one of the five forms (singleton, closed interval, open interval, left-open interval, right-open interval), the set X_α can have one of the 25 possible forms.

Similarly as in Proposition 6 where we have related corresponding semigroups to t-subnorms which generalize t-norms, also here we can relate corresponding semigroups to operations that generalize uninorms and therefore we will call them generalized uninorms. We can distinguish here three kinds of generalized uninorms.

Definition 3. An associative, commutative, binary operation $V \colon [0,1]^2 \longrightarrow [0,1]$ which is non-decreasing in each variable will be called

(i) generalized sub-uninorm if there exists an $e \in [0,1]$ such that $V(x,y) \le \min(x,y)$ for all $(x,y) \in [0,e]^2$, $V(x,y) \ge \max(x,y)$ for all $(x,y) \in]e,1]^2$, $V(x,y) \in [x,y]$ for all $(x,y) \in [0,e] \times]e,1] \cup]e,1] \times [0,e]$.
(ii) generalized super-uninorm if there exists an $e \in [0,1]$ such that $V(x,y) \le \min(x,y)$ for all $(x,y) \in [0,e[^2$, $V(x,y) \ge \max(x,y)$ for all $(x,y) \in [e,1]^2$, $V(x,y) \in [x,y]$ for all $(x,y) \in [0,e[\times [e,1] \cup [e,1] \times [0,e[$.

Definition 4. A binary operation $V \colon ([a,b] \cup [c,d])^2 \longrightarrow ([a,b] \cup [c,d])$, where $a < b < c < d$, $a, b, c, d \in [0,1]$ will be called a generalized composite uninorm if it is associative, commutative, non-decreasing in both coordinates and V restricted to $[a,b]^2$ is a t-subnorm on $[a,b]^2$, V restricted to $[c,d]^2$ is a t-superconorm on $[c,d]^2$, and $V(x,y) \in [x,y]$ for all $x \in [a,b]$, $y \in [c,d]$ and all $x \in [c,d]$, $y \in [a,b]$.

Thus the class of generalized uninorms consists of generalized sub-uninorms, generalized super-uninorms, and generalized composite uninorms. The last class, however, differs from the others as it cannot (in general) be transformed to an operation on $[0,1]^2$. Note that uninorms are both generalized sub-uninorms as well as generalized super-uninorms.

The generalized sub-uninorms arise from semigroups defined on sets $[a_\alpha, b_\alpha] \cup]c_\alpha, d_\alpha]$, $]a_\alpha, b_\alpha] \cup]c_\alpha, d_\alpha]$, $[a_\alpha, b_\alpha] \cup]c_\alpha, d_\alpha[$, and $]a_\alpha, b_\alpha] \cup]c_\alpha, d_\alpha[$. Further, generalized super-uninorms arise from semigroups defined on sets $[a_\alpha, b_\alpha[\cup [c_\alpha, d_\alpha]$, $[a_\alpha, b_\alpha[\cup [c_\alpha, d_\alpha[$, $]a_\alpha, b_\alpha[\cup [c_\alpha, d_\alpha]$, and $]a_\alpha, b_\alpha[\cup [c_\alpha, d_\alpha[$. The generalized composite uninorms arise from semigroups defined on sets $[a_\alpha, b_\alpha] \cup [c_\alpha, d_\alpha]$, $]a_\alpha, b_\alpha] \cup [c_\alpha, d_\alpha]$, $[a_\alpha, b_\alpha] \cup [c_\alpha, d_\alpha[$, and $]a_\alpha, b_\alpha] \cup [c_\alpha, d_\alpha[$. Finally, standard uninorms correspond to semigroups defined on sets $[a_\alpha, b_\alpha[\cup]c_\alpha, d_\alpha]$, $[a_\alpha, b_\alpha[\cup]c_\alpha, d_\alpha[$, $]a_\alpha, b_\alpha[\cup]c_\alpha, d_\alpha]$, and $]a_\alpha, b_\alpha[\cup]c_\alpha, d_\alpha[$.

If I_α and J_α are singletons, $I_\alpha = \{a_\alpha\}$, $J_\alpha = \{d_\alpha\}$, then the respective summand is a semigroup $(\{a_\alpha, d_\alpha\}, *_\alpha)$, where for $*_\alpha$ we have either

$$x *_\alpha y = \begin{cases} x & \text{if } x = y, \\ a_\alpha & \text{if } x \neq y, \end{cases}$$

or

$$x *_\alpha y = \begin{cases} x & \text{if } x = y, \\ d_\alpha & \text{if } x \neq y. \end{cases}$$

The last possibility is that I_α is a non-singleton subinterval of $[0, e]$ and $J_\alpha = \{d_\alpha\}$ (or analogically, $I_\alpha = \{a_\alpha\}$ and J_α is a non-singleton subinterval of $[e, 1]$). In such a case we obtain a semigroup $(I_\alpha \cup \{d_\alpha\}, *_\alpha)$, where $d_\alpha *_\alpha d_\alpha = d_\alpha$ and $*_\alpha$ on $(I_\alpha)^2$ is isomorphic to a t-subnorm (possibly without one or both border points of the unit interval).

Example 1. Let $I_\alpha = \left[0, \frac{1}{4}\right]$ and $J_\alpha = \{\frac{3}{4}\}$. Let $*_\alpha : (I_\alpha \cup J_\alpha)^2 \longrightarrow I_\alpha \cup J_\alpha$ be a commutative operation which is for $x \le y$ given by

$$x *_\alpha y = \begin{cases} \frac{3}{4} & \text{if } x = y = \frac{3}{4}, \\ x & \text{if } x \in \left[0, \frac{1}{8}\right], y = \frac{3}{4}, \\ \frac{1}{4} & \text{if } x \in \left]\frac{1}{8}, \frac{1}{4}\right], y = \frac{3}{4}, \\ \min(x, \frac{1}{8}) \cdot \min(y, \frac{1}{8}) & \text{if } (x, y) \in \left[0, \frac{1}{4}\right]^2. \end{cases}$$

Then it is easily checked that $G_1 = (I_\alpha \cup J_\alpha, *_\alpha)$ is a commutative semigroup that fulfills all properties from Proposition 7 and therefore it can be used as a summand in the ordinal sum construction. If we assume additional semigroups $G_2 = (\frac{1}{2}, \min)$, $G_3 = (\left]\frac{1}{4}, \frac{1}{2}\right[, \min)$, $G_4 = (\left]\frac{1}{2}, \frac{3}{4}\right[, \max)$ and $G_5 = (\left]\frac{3}{4}, 1\right], \max)$, with the linear order $5 \prec 1 \prec 3 \prec 4 \prec 2$ then the ordinal sum yields a semigroup $([0, 1], U)$, where $U : [0, 1]^2 \longrightarrow [0, 1]$ is a uninorm given for $x \le y$ by

$$U(x, y) = \begin{cases} \min(x, \frac{1}{8}) \cdot \min(y, \frac{1}{8}) & \text{if } (x, y) \in \left[0, \frac{1}{4}\right]^2, \\ \min(x, y) & \text{if } (x, y) \in \left[0, \frac{1}{2}\right]^2 \setminus \left[0, \frac{1}{4}\right]^2, \\ \max(x, y) & \text{if } (x, y) \in \left[\frac{1}{2}, 1\right]^2, \\ \min(x, y) & \text{if } x \in \left[0, \frac{1}{2}\right], y \in \left[\frac{1}{2}, \frac{3}{4}\right[, \\ \max(x, y) & \text{if } x \in \left[0, \frac{1}{2}\right], y \in \left]\frac{3}{4}, 1\right], \\ x & \text{if } x \in \left[0, \frac{1}{8}\right], y = \frac{3}{4}, \\ \frac{1}{4} & \text{if } x \in \left]\frac{1}{8}, \frac{1}{4}\right], y = \frac{3}{4}, \\ y & \text{if } x \in \left]\frac{1}{4}, \frac{1}{2}\right], y = \frac{3}{4}. \end{cases}$$

In the rest of the paper we will first focus on a special case when all summands in the ordinal sum construction are standard uninorms and then we will closer study generalized uninorms.

4. Ordinal sum of uninorms

In this section we will focus on a special case when all summands in the ordinal sum construction are either isomorphic to uninorms (including t-norms and t-conorms), or they are singletons. The aim is to obtain a construction similar to that in Proposition 1, which will be called the ordinal sum of uninorms. In this case the non-singleton summands are defined on sets $[a_k, b_k[\cup]c_k, d_k]$ and then for the resulting uninorm U and any $v \in [b_k, c_k]$ the ordinal sum construction and monotonicity ensures that the restriction of U to $[a_k, b_k[\cup \{v\} \cup]c_k, d_k]$ is isomorphic to some uninorm U_k on $[0, 1]^2$ with the neutral element e_k which corresponds to the transformation of v. Note however, that if $U_k(x, y) = e_k$ for some $x \ne e_k$, $y \ne e_k$ then $[0, e_k[\cup]e_k, 1]$ is not closed under U_k and thus in order to preserve associativity v has to be an annihilator of U restricted to $[b_k, c_k]^2$. Thus we will obtain the ordinal sum construction in the sense of Remark 1.

In the case that $a_k = b_k$ ($c_k = d_k$) then the corresponding summand is isomorphic to a t-conorm (t-norm).

The isomorphism between U_k and the respective semigroup on $[a_k, b_k[\cup \{v\} \cup]c_k, d_k]$ will be given by the following transformation. For any $0 \le a < b \le c < d \le 1$, $v \in [b, c]$, and a uninorm U with the neutral element $e \in]0, 1[$ let $f : [0, 1] \longrightarrow [a, b[\cup \{v\} \cup]c, d]$ be given by

$$f(x) = \begin{cases} (b - a) \cdot \frac{x}{e} + a & \text{if } x \in [0, e[, \\ v & \text{if } x = e, \\ d - \frac{(1-x)(d-c)}{(1-e)} & \text{otherwise}. \end{cases} \tag{3}$$

Then f is linear on $[0, e[$ and on $]e, 1]$ and thus it is a piece-wise linear isomorphism of $[0, 1]$ to $([a, b[\cup \{v\} \cup]c, d])$ and a binary function $U_v^{a,b,c,d}\colon ([a, b[\cup \{v\} \cup]c, d])^2 \longrightarrow ([a, b[\cup \{v\} \cup]c, d])$ given by

$$U_v^{a,b,c,d}(x, y) = f(U(f^{-1}(x), f^{-1}(y))) \tag{4}$$

is a uninorm on $([a, b[\cup\{v\}\cup]c, d])^2$. The function f is piece-wise linear, however, more generally, we can use any increasing isomorphic transformation.

In the case that U_k is a t-norm (t-conorm) we will take the standard linear isomorphism between $[0, 1]$ and $[a_k, b_k]$ ($[c_k, d_k]$).

In order to obtain the monotonicity of the resulting uninorm the order of summands have to be compatible with the standard order on $[0, e]$ and reversed with respect to the standard order on $[e, 1]$. This means that $k_1 \prec k_2$ for $k_1, k_2 \in K$ implies $b_{k_1} \le a_{k_2}$ and $c_{k_1} \ge d_{k_2}$, i.e.,

$$\left[a_{k_2}, d_{k_2}\right]^2 \subseteq \left[b_{k_1}, c_{k_1}\right]^2 \subseteq \left[a_{k_1}, d_{k_1}\right]^2 .$$

Note that $\left[a_{k_2}, d_{k_2}\right]^2 = \left[a_{k_1}, d_{k_1}\right]^2$ implies $a_{k_1} = b_{k_1}$ and $c_{k_1} = d_{k_1}$ which means that the summand corresponding to k_1 acts on an empty set.

In the ordinal sum of t-norms, intervals $[a_k, b_k[$ need not to cover the whole unit interval as the rest is covered by the minimum. However, as in the case of uninorms we should have on the remaining semigroups corresponding to Y_l a mixture of min and max (which corresponds to an internal uninorm) which is not closer determined, in the case of uninorms we will suppose that $\bigcup_{k\in K} [a_k, b_k] = [0, e]$ and $\bigcup_{k\in K} [c_k, d_k] = [e, 1]$. In this way there will be no sets Y_l which are subintervals of $[0, 1]$, i.e., the sets Y_l will be only singletons such that

$$B_1 = \bigcup_{k\in K} [a_k, b_k] \setminus \bigcup_{k\in K} [a_k, b_k[= \bigcup_{l\in L_1} Y_l,$$

$$C_1 = \bigcup_{k\in K} [c_k, d_k] \setminus \bigcup_{k\in K}]c_k, d_k] = \bigcup_{l\in L_2} Y_l.$$

Note that for all $x \in [0, 1]$ there is $[x, x[=]x, x] = \emptyset$ and therefore if we denote $K_* = \{k \in K \mid]a_k, b_k[\ne \emptyset\}$ and $K^* = \{k \in K \mid]c_k, d_k[\ne \emptyset\}$ then $B_1 = \{b_k \mid k \in K\} \setminus \{a_k \mid k \in K_*\}$ and $C_1 = \{c_k \mid k \in K\} \setminus \{d_k \mid k \in K^*\}$. Since K is assumed to be countable then every $b \in B_1 \setminus \{e\}$ is an accumulation point of $\{a_k \mid k \in K_*\}$ (and similarly for $c \in C_1 \setminus \{e\}$). We denote $B_2 = B_1 \setminus \{e\}$, $C_2 = C_1 \setminus \{e\}$ and define functions $g\colon B_2 \longrightarrow [e, 1]$, $h\colon C_2 \longrightarrow [0, e]$, such that if for $b \in B_2$ we have $b = \lim_{i\longrightarrow\infty} a_{k_i}$ for $k_i \in K_*$, then

$$g(b) = \lim_{i\longrightarrow\infty} d_{k_i}. \tag{5}$$

Similarly, if for $c \in C_2$ we have $c = \lim_{i\longrightarrow\infty} d_{k_i}$ for $k_i \in K^*$, then

$$h(c) = \lim_{i\longrightarrow\infty} a_{k_i}. \tag{6}$$

Now if $g(b) \notin C_2$ for some $b \in B_2$ ($h(c) \notin B_2$ for some $c \in C_2$) then the value of $U(b, g(b))$ ($U(c, h(c))$) follows from the monotonicity of U. Therefore we have only to solve the case when $g(b) \in C_2$ ($h(c) \in B_2$).

We summarize the observations made above in the following proposition.

Proposition 8. *Assume $e \in [0, 1]$. Let K be an index set which is finite or countably infinite and let $(]a_k, b_k[)_{k\in K}$ be a system of open disjoint subintervals (which can be also empty) of $[0, e]$, such that $\bigcup_{k\in K} [a_k, b_k] = [0, e]$. Similarly, let $(]c_k, d_k[)_{k\in K}$ be a system of open disjoint subintervals (which can be also empty) of $[e, 1]$, such that $\bigcup_{k\in K} [c_k, d_k] = [e, 1]$. Let further these two systems be anti-comonotone, i.e., $b_k \le a_i$ if and only if $c_k \ge d_i$ for all $i, k \in K$. We will denote $K_* = \{k \in K \mid]a_k, b_k[\ne \emptyset\}$ and $K^* = \{k \in K \mid]c_k, d_k[\ne \emptyset\}$. Assume a family of proper uninorms $(U_k)_{k\in K_*\cap K^*}$ on $[0, 1]^2$, a family of t-norms $(U_k)_{k\in K_*\setminus K^*}$ on $[0, 1]^2$ and a family of t-conorms $(U_k)_{k\in K^*\setminus K_*}$ on $[0, 1]^2$. Denote $B_1 = \{b_k \mid k \in K\} \setminus \{a_k \mid k \in K_*\}$ and $C_1 = \{c_k \mid k \in K\} \setminus \{d_k \mid k \in K^*\}$ and let $B = \{b \in B_1 \setminus \{e\} \mid g(b) \in C_1\}$, $C = \{c \in C_1 \setminus \{e\} \mid h(c) \in B_1\}$, where the functions g and h are defined by (5) and (6). Further assume a function $n\colon B \longrightarrow B \cup C$ given for all $b \in B$ by*

$$n(b) \in \{b, g(b)\}.$$

Let the ordinal sum $U^e = (\langle a_k, b_k, c_k, d_k, U_k\rangle \mid k \in K)^e$ be given by

$$U^e(x,y)=\begin{cases} y & \text{if } x=e,\\ x & \text{if } y=e,\\ (U_k)^{a_k,b_k,c_k,d_k}_{v_k}(x,y) & \text{if } (x,y)\in([a_k,b_k[\,\cup\,]c_k,d_k])^2, k\in K_*\cap K^*,\\ (U_k)^{a_k,b_k}(x,y) & \text{if } (x,y)\in([a_k,b_k[\,\cup\,]c_k,d_k])^2, k\in K_*\setminus K^*,\\ (U_k)^{c_k,d_k}(x,y) & \text{if } (x,y)\in([a_k,b_k[\,\cup\,]c_k,d_k])^2, k\in K^*\setminus K_*,\\ x & \text{if } y\in[b_k,c_k]\,, x\in[a_k,d_k]\setminus[b_k,c_k]\,, k\in K_*\cup K^*,\\ y & \text{if } x\in[b_k,c_k]\,, y\in[a_k,d_k]\setminus[b_k,c_k]\,, k\in K_*\cup K^*,\\ \min(x,y) & \text{if } (x,y)\in[b,c]^2\setminus(]b,c[^2\cup\{(b,c),(c,b)\}),\\ & \quad \text{where } b\in B, c=g(b), x+y<c+b,\\ \max(x,y) & \text{if } (x,y)\in[b,c]^2\setminus(]b,c[^2\cup\{(b,c),(c,b)\}),\\ & \quad \text{where } b\in B, c=g(b), x+y>c+b,\\ n(b) & \text{if } (x,y)=(b,c) \text{ or } (x,y)=(c,b), b\in B, c=g(b),\\ \min(x,y) & \text{if } (x,y)\in\{b\}\times[b,c]\cup[b,c]\times\{b\} \text{ and}\\ & \qquad b\in B_1\setminus(B\cup\{e\}), c=g(b),\\ \max(x,y) & \text{if } (x,y)\in\{c\}\times[b,c]\cup[b,c]\times\{c\} \text{ and}\\ & \qquad c\in C_1\setminus(C\cup\{e\}), b=h(c), \end{cases}$$

where $v_k=c_k$ ($v_k=b_k$) if there exists an $i\in K$ such that $b_k=a_i$, $c_k=d_i$ and U_i is disjunctive (conjunctive) and $v_k=n(b_k)$ if $b_k\in B$, $v_k=b_k$ if $b_k\in B_1\setminus B$, $v_k=c_k$ if $c_k\in C_1\setminus C$, and $(U_k)^{a_k,b_k,c_k,d_k}_{v_k}$ is given by the formula (4), $(U_k)^{a_k,b_k}$ ($(U_k)^{c_k,d_k}$) is a linear transformation of U_k (U_k) to $[a_k,b_k]^2$ ($[c_k,d_k]^2$). Then U^e is a uninorm.

Proof. The commutativity and the neutral element of U^e are obvious. Due to the commutativity, to show the monotonicity of U^e it is enough to show the monotonicity in the second coordinate. If $x=e$ then monotonicity of $U(e,\cdot)$ is clear. Now assume that $x\in[a_k,b_k[$ for some $k\in K$ (similarly we can show the case when $x\in]c_k,d_k]$). Then $U(x,y)=\min(x,y)$ if $b_k\le y\le c_k$ and $U(x,y)=\max(x,y)$ if $y>d_k$. Further, $U(x,y)\in[x,y]$ if $y\in]c_k,d_k]\neq\emptyset$ and monotonicity in this case follows from the monotonicity of U_k. If $y\in[a_k,b_k[$ then $U(x,y)\le\min(x,y)$ and monotonicity on this interval follows from the monotonicity of U_k and $U(x,a_k)=a_k$. Finally, if $y<a_k$ then $U(x,y)=\min(x,y)$. Thus, summarizing, if $x\in[a_k,b_k[$ then $U(x,\cdot)$ is non-decreasing. Now suppose that $x=b\in B_1$ (similarly we can show the case when $x=c\in C_1$). Then $U(x,y)=\min(x,y)$ if $y<g(b)$, $U(x,y)=\max(x,y)$ if $y>g(b)$ and $U(x,y)\in\{x,y\}$ if $y=g(b)$, i.e., the monotonicity holds.

For the associativity it is enough to observe that the above ordinal sum of uninorms is an ordinal sum in the sense of Clifford. This ordinal sum consists of six kinds of semigroups: $([a_k,b_k[\,\cup\{v_k\}\cup\,]c_k,d_k]\,,(U_k)^{a_k,b_k,c_k,d_k}_{v_k})$ for $k\in K_*\cap K^*$, $([a_k,b_k[\,,(U_k)^{a_k,b_k})$ if $k\in K_*\setminus K^*$, $(]c_k,d_k]\,,(U_k)^{c_k,d_k})$ if $k\in K^*\setminus K_*$, $(\{b\},\min)$ for $b\in B_1\setminus\{e\}$, $(\{c\},\min)$ for $c\in C_1\setminus\{e\}$, and the last semigroup is $(\{e\},\min)$. These semigroups are equipped with the linear order, where the semigroup $(\{e\},\min)$ is the biggest in this order and $k_1\prec k_2$ if and only if $b_{k_1}\le a_{k_2}$ and $c_{k_1}\ge d_{k_2}$. It is easy to observe that if X_{k_1} and X_{k_2} for some $k_1,k_2\in K$, $k_1\neq k_2$, are not disjoint then they have just one element in common, which in one semigroup act as a neutral element and in the other as the annihilator. Thus according to Remark 1 $([0,1],U^e)$ is the ordinal sum of semigroups in the sense of Clifford, i.e., U^e is associative (see Fig. 1). □

Example 2. Assume $U_1\in\mathcal{U}_{\min}$ and $U_2\in\mathcal{U}_{\max}$ with respective neutral elements e_1,e_2. Then U_1 and U_2 are ordinal sums of uninorms, $U_1=(\langle e_1,e_1,e_1,1,C_{U_1}\rangle,\langle 0,e_1,1,1,T_{U_1}\rangle)^{e_1}$ and $U_2=(\langle 0,e_2,e_2,e_2,T_{U_2}\rangle,\langle 0,0,e_2,1,C_{U_2}\rangle)^{e_2}$.

Example 3. Assume the triple Π operator from [25] which is a uninorm given by

$$UP(x,y)=\frac{x\cdot y}{x\cdot y+(1-x)\cdot(1-y)}$$

and we will assume $UP(0,1)=1$. Then the ordinal sum

$$U=(\langle 0,\frac{1}{3},\frac{2}{3},1,UP\rangle,\langle\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{2}{3},UP\rangle)^{\frac{1}{2}}$$

is a uninorm given for $x\le y$ by

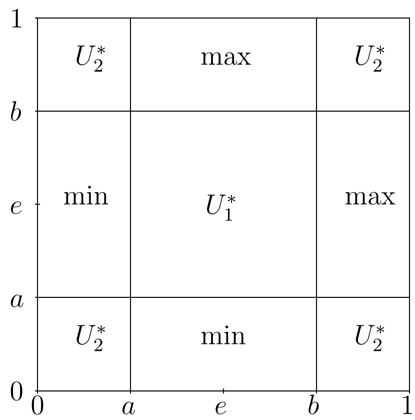

Fig. 1. Ordinal sum of uninorms from Proposition 8 with two summands, where $U_2^* = (U_2)^{0,a,b,1}_{U_1^*(a,b)}$, $U_1^* = (U_1)_e^{a,e,e,b}$.

$$U(x,y)=\begin{cases}\frac{2\cdot(3x-1)(3y-1)+(2-3x)(2-3y)}{3\cdot[(3x-1)(3y-1)+(2-3x)(2-3y)]} & \text{if } (x,y)\in\left[\frac{1}{3},\frac{2}{3}\right]^2,\\ \frac{6\cdot x\cdot y}{9\cdot x\cdot y+(2-3x)\cdot(2-3y)} & \text{if } (x,y)\in\left[0,\frac{1}{3}\right]^2,\\ \frac{3\cdot(3x-1)\cdot(3y-1)+(3-3x)(3-3y)}{3\cdot[(3x-1)\cdot(3y-1)+(3-3x)(3-3y)]} & \text{if } (x,y)\in\left[\frac{2}{3},1\right]^2,\\ \frac{6\cdot x\cdot(3y-1)}{3\cdot[3\cdot x\cdot(3y-1)+(2-3x)(3-3y)]} & \text{if } x\in\left[0,\frac{1}{3}\right[,\ y\in\left]\frac{2}{3},1\right],\ x+y<1,\\ \frac{9\cdot x\cdot(3y-1)+(2-3x)(3-3y)}{3\cdot[3\cdot x\cdot(3y-1)+(2-3x)(3-3y)]} & \text{if } x\in\left[0,\frac{1}{3}\right[,\ y\in\left]\frac{2}{3},1\right],\ x+y>1,\\ \frac{1}{3} & \text{if } x\in\left[0,\frac{1}{3}\right[,\ y\in\left]\frac{2}{3},1\right],\ x+y=1,\\ \frac{1}{3} & \text{if } x=\frac{1}{3},\ y=\frac{2}{3},\\ 0 & \text{if } x=0,\ y=1,\\ x & \text{if } x\in\left[0,\frac{1}{3}\right[,\ y\in\left]\frac{1}{3},\frac{2}{3}\right[,\\ y & \text{if } x\in\left]\frac{1}{3},\frac{2}{3}\right[,\ y\in\left]\frac{2}{3},1\right],\end{cases}$$

compare Fig. 1.

Remark 3. For the uninorm U^e from Proposition 8 which is an ordinal sum of semigroups $((X_k,*_k))_{k\in K}$ we see that $U^e|_{[0,e]^2}$, which is linearly isomorphic with T_{U^e}, is an ordinal sum of semigroups $((X_k\cap[0,e],*_k))_{k\in K}$. Then for $k\in K$ we have the following possibilities:

(i) $X_K\cap[0,e]=[a_k,b_k]$ for $k\in K_*\cap K^*$ if $U^e(b_k,c_k)=b_k$. Then U^e on $[a_k,b_k[^2$ is linearly isomorphic with T_{U_k} on $[0,1[^2$.
(ii) $X_K\cap[0,e]=[a_k,b_k[$ for $k\in K_*\cap K^*$ if $U^e(b_k,c_k)=c_k$. Then again U^e on $[a_k,b_k[^2$ is linearly isomorphic with T_{U_k} on $[0,1[^2$.
(iii) $X_K\cap[0,e]=[a_k,b_k[$ for $k\in K_*\setminus K^*$. Then U^e on $[a_k,b_k[^2$ is linearly isomorphic with U_k on $[0,1[^2$.
(iv) $X_K\cap[0,e]=\{b\}$ for $b\in B_1$.
(v) $X_K\cap[0,e]=\emptyset$.

Therefore, since $U^e|_{[0,e]^2}$ is linearly isomorphic with T_{U^e}, the composition of linear isomorphisms is again a linear isomorphism, and since between two semigroups on right-open intervals there can exist only one linear isomorphism, we see that T_{U^e} is an ordinal sum of t-norms from Proposition 1,

$$T_{U^e}=((\langle a_k,b_k,T_{U_k}\rangle)_{k\in K_*\cap K^*},(\langle a_k,b_k,T_k\rangle)_{k\in K_*\setminus K^*}).$$

Thus generally we can say that the underlying t-norm of U^e is an ordinal sum of the underlying t-norms of the corresponding summands. Similar result can be shown also for the underlying t-conorm C_{U^e}.

In [12] several classes of uninorms were summarized. Let us now see how does the class of summands in the ordinal sum influence the class of the U^e.

(i) A uninorm U is called idempotent if $T_U = \min$ and $C_U = \max$. It is easy to show that then U is internal, i.e., $U(x, y) \in \{x, y\}$ for all $x, y \in [0, 1]$. If all summands in the ordinal sum are idempotent then evidently also U^e is idempotent. More on idempotent uninorms and ordinal sum construction can be found in [16].
(ii) As the transformation (3) suggests, the continuity of a uninorm in the open unit square is not preserved by the ordinal sum construction.
(iii) If all summands are locally internal in $A(e) = [0, e] \times [e, 1] \cup [e, 1] \times [0, e]$ then again it is easy to see that also U^e is locally internal in $A(e)$.
(iv) Since ordinal sum of continuous t-norms (t-conorms) is continuous (see Proposition 1), if all summands in the ordinal sum have continuous underlying functions then also U^e will have continuous underlying functions. More on uninorms with continuous underlying functions and ordinal sum construction can be found in [18].
(v) Ordinal sums of representable uninorms were studied in [17].
(vi) Finally, let us assume that $U_k \in \mathcal{U}_{\min}$ for all $k \in K$. Note that all t-norms and t-conorms belong to the class $\mathcal{U}_{\min}$. This class is not closed under ordinal sum construction in general. In order to ensure $U^e \in \mathcal{U}_{\min}$ the condition $a_k = b_k = e$ for all $k \in K^*$ has to be fulfilled, which means that U^e is in fact an ordinal sum of a t-conorm and a t-norm which is confirmed in Example 2. A similar observation can be made also for the class $\mathcal{U}_{\max}$.

5. Generalized uninorms

As in the case of uninorms, also for a generalized uninorm V (either on $[0, 1]^2$ or on $([0, b] \cup [c, 1])^2$), the associativity and monotonicity together with $V(0, 0) = 0$ and $V(1, 1) = 1$ imply that the value $V(0, 1) \in \{0, 1\}$ is the annihilator of V.

Remark 4. Each generalized sub-uninorm (super-uninorm) V on $[0, 1]^2$ with $e \in]0, 1[$ can be isomorphically transformed to $([a, b] \cup]c, d])^2$ $(([a, b[\cup [c, d])^2)$ for any $a, b, c, d \in [0, 1]$, $a < b < c < d$. Then the ordinal sum of $([a, b] \cup]c, d], V)$ and $(\{c\}, \min)$ $(([a, b[\cup [c, d], V)$ and $(\{b\}, \min))$ is a generalized composite uninorm.

Vice versa, if for a generalized uninorm G on $([a, b] \cup [c, d])^2$ we have $G(x, y) = b$ implies $b \in \{x, y\}$ ($G(x, y) = c$ implies $c \in \{x, y\}$) for all $x, y \in [a, b] \cup [c, d]$ then restriction of G to $([a, b[\cup [c, d])^2$ $(([a, b] \cup]c, d])^2)$ is isomorphic with some generalized super-uninorm (sub-uninorm).

At first we will take a closer look on generalized composite uninorms. If a uninorm U is on some set $([a, b] \cup [c, d])^2$ where, $a < b \le e \le c < d$, a generalized composite uninorm then since $U(b, c) \in [b, c] \cap ([a, b] \cup [c, d])$ we get $U(b, c) \in \{b, c\}$. Further, we have the following easy result.

Lemma 1. *Assume $a, b, c, d \in [0, 1]$, $a < b < c < d$ and let M be a t-subnorm on $[a, b]^2$ and R a t-superconorm on $[c, d]^2$. Then the binary function $V : ([a, b] \cup [c, d])^2 \longrightarrow ([a, b] \cup [c, d])$ given by*

$$V(x, y) = \begin{cases} M(x, y) & \text{if } (x, y) \in [a, b]^2, \\ R(x, y) & \text{if } (x, y) \in [c, d]^2, \\ A(x, y) & \text{otherwise,} \end{cases}$$

where $A(x, y) = \min(x, y)$ for all $(x, y) \in [0, 1]^2$ or $A(x, y) = \max(x, y)$ for all $(x, y) \in [0, 1]^2$ is a generalized composite uninorm on $([a, b] \cup [c, d])^2$.

We have also the following partial result.

Proposition 9. *Let $V : ([a, b] \cup [c, d])^2 \longrightarrow ([a, b] \cup [c, d])$, where $a < b < c < d$, $a, b, c, d \in [0, 1]$, be a generalized composite uninorm, such that $V(b, c) = b$ and V restricted to $[a, b]^2$ is a continuous cancellative t-subnorm on $[a, b]^2$, and V restricted to $[c, d]^2$ is a nilpotent, Archimedean t-superconorm on $[c, d]^2$. Then $V(x, y) = \min(x, y)$ for all $x \in [a, b]$, $y \in [c, d]$ and all $x \in [c, d]$, $y \in [a, b]$.*

Proof. Since $V(b, c) = b$ then also $V(b, \underbrace{c, \dots, c}_{n\text{-times}}) = b$ for all $n \in \mathbb{N}$ and since V restricted to $[c, d]^2$ is nilpotent and Archimedean we have $V(\underbrace{c, \dots, c}_{N\text{-times}}) = d$ for some $N \in \mathbb{N}$. Thus $V(b, d) = b$ and monotonicity implies $V(b, y) = b$ for all $y \in [c, d]$. Now let $V(b, b) = p \in [a, b]$ then $V(p, c) = V(b, V(b, c)) = V(b, b) = p$ and similarly as before we get $V(p, y) = p$ for all $y \in [c, d]$. Since V restricted to $[a, b]^2$ is continuous for all $x \in [a, p]$ there exists a $q \in [a, b]$ such that $V(b, q) = x$. Then $V(x, c) = V(q, V(b, c)) = V(q, b) = x$ and similarly as above we get $V(x, y) = x$ for all $y \in [c, d]$. Finally, assume $x \in]p, b[$. Then $V(x, b) \le p$ and thus $V(V(x, c), b) = V(x, V(b, c)) = V(x, b)$. Since $a \le V(x, c) \le V(b, c) = b$ the cancellativity of V restricted to $[a, b]^2$ gives us $V(x, c) = x$ and similarly as above we get $V(x, y) = x$ for all $y \in [c, d]$. Summarizing, $V(x, y) = \min(x, y)$ for all $x \in [a, b]$, $y \in [c, d]$ and by commutativity $V(x, y) = \min(x, y)$ for all $x \in [c, d]$, $y \in [a, b]$. □

Analogously we can show the following result.

Proposition 10. *Let $V\colon ([a,b]\cup[c,d])^2 \longrightarrow ([a,b]\cup[c,d])$, where $a<b<c<d$, $a,b,c,d\in[0,1]$, be a generalized composite uninorm, such that $V(b,c)=c$ and V restricted to $[a,b]^2$ is a nilpotent, Archimedean t-subnorm on $[a,b]^2$, and V restricted to $[c,d]^2$ is a continuous cancellative t-superconorm on $[c,d]^2$. Then $V(x,y)=\max(x,y)$ for all $x\in[a,b]$, $y\in[c,d]$ and all $x\in[c,d]$, $y\in[a,b]$.*

Example 4. Let $V\colon (\left[0,\frac{1}{4}\right]\cup\left[\frac{3}{4},1\right])^2 \longrightarrow (\left[0,\frac{1}{4}\right]\cup\left[\frac{3}{4},1\right])$ be partially given by

$$V(x,y)=\begin{cases}0 & \text{if } (x,y)\in\left[0,\frac{1}{4}\right]^2,\\ 1 & \text{if } (x,y)\in\left[\frac{3}{4},1\right]^2.\end{cases}$$

Then $V(\frac{1}{4},\frac{3}{4})\in\{\frac{1}{4},\frac{3}{4}\}$. Let us assume $V(\frac{1}{4},\frac{3}{4})=\frac{1}{4}$. Then $V(\frac{1}{4},1)=V(V(\frac{1}{4},\frac{3}{4}),\frac{3}{4})=\frac{1}{4}$ and monotonicity implies $V(\frac{1}{4},y)=\frac{1}{4}$ for all $y\in\left[\frac{3}{4},1\right]$. Further, $V(0,1)=V(\frac{1}{4},V(\frac{1}{4},1))=V(\frac{1}{4},\frac{1}{4})=0$ and monotonicity implies $V(0,y)=0$ for all $y\in\left[\frac{3}{4},1\right]$. Further, if $x\in\left]0,\frac{1}{4}\right[$ is such that $V(x,1)=y>x$ then $V(y,1)=1$ and for all $z\in\left[\frac{3}{4},1\right]$ there is $V(x,z)\in[x,y]$. As an example of such a generalized composite uninorm we can take the one which is on $\left[0,\frac{1}{4}\right]\times\left[\frac{3}{4},1\right]$ given by

$$V(x,y)=\begin{cases}0 & \text{if } x=0,\, y\in\left[\frac{3}{4},1\right],\\ \frac{1}{2^i} & \text{if } x\in\left]\frac{1}{2^{i+1}},\frac{1}{2^i}\right],\, i\in\mathbb{N}\setminus\{1\},\, y\in\left[\frac{3}{4},1\right].\end{cases}$$

In the following we will focus on the remaining two classes, i.e., generalized sub-uninorms and generalized super-uninorms. In some cases these operations are convertible uninorms.

Definition 5. Let $O\colon [0,1]^2 \longrightarrow [0,1]$ be a binary function and let there exists an $e\in[0,1]$ such that the function $U\colon [0,1]^2 \longrightarrow [0,1]$ given by

$$U(x,y)=\begin{cases}x & \text{if } y=e,\\ y & \text{if } x=e,\\ O(x,y) & \text{otherwise,}\end{cases}$$

is a uninorm. Then O will be called a *convertible* uninorm.

Lemma 2. *If for a generalized sub-uninorm (super-uninorm) $V\colon [0,1]^2 \longrightarrow [0,1]$ there $V(x,y)=e$ implies $e\in\{x,y\}$ then V is a convertible uninorm.*

Proof. If $V(x,y)=e$ implies $e\in\{x,y\}$ then V can be restricted to $([0,1]\setminus\{e\})^2$ and it is easy to see that then the ordinal sum of $([0,1]\setminus\{e\},V)$ and $(\{e\},\min)$ is a uninorm. $\square$

Similar results as in Lemma 1 can be shown also for generalized sub-uninorms and generalized super-uninorms.

Lemma 3. *Assume $e\in[0,1]$ and let M be a t-subnorm on $[0,e]^2$ and R a t-superconorm on $[e,1]^2$. Then the binary function $V_1\colon [0,1]^2 \longrightarrow [0,1]$ given by*

$$V_1(x,y)=\begin{cases}M(x,y) & \text{if } (x,y)\in[0,e]^2,\\ R(x,y) & \text{if } (x,y)\in\,]e,1]^2,\\ A(x,y) & \text{otherwise}\end{cases}$$

is a generalized sub-uninorm and the binary function $V_2\colon [0,1]^2 \longrightarrow [0,1]$ given by

$$V_2(x,y)=\begin{cases}M(x,y) & \text{if } (x,y)\in[0,e[^2,\\ R(x,y) & \text{if } (x,y)\in[e,1]^2,\\ A(x,y) & \text{otherwise}\end{cases}$$

is a generalized super-uninorm, where $A(x,y)=\min(x,y)$ for all $(x,y)\in[0,1]^2$ or $A(x,y)=\max(x,y)$ for all $(x,y)\in[0,1]^2$.

In the following section we will focus on additively generated generalized uninorms.

6. Generalization of representable uninorms

In this section we will discuss generated generalized uninorms. We will start with representable uninorms (see [3]).

Proposition 11. *Let $f\colon [0,1] \longrightarrow [-\infty, \infty]$, $f(0) = -\infty$, $f(1) = \infty$ be a continuous strictly increasing function. Then a binary function $U\colon [0,1]^2 \longrightarrow [0,1]$ given by*

$$U(x, y) = f^{-1}(f(x) + f(y)),$$

where $f^{-1}\colon [-\infty, \infty] \longrightarrow [0,1]$ is an inverse function to f, is a uninorm, which will be called a representable *uninorm.*

Since the function f from the previous proposition is continuous and $f(0) = -\infty$, $f(1) = \infty$, there exists an $e \in \,]0,1[$ such that $f(e) = 0$. Then e is the neutral element of U. The function f in the previous proposition is called an additive generator of the uninorm U and it defines an isomorphism between the addition on $[-\infty, \infty]$ and a uninorm U on $[0,1]$. An additive generator of the uninorm U is uniquely given up to the positive multiplicative constant.

For a representable uninorm U generated by some function f we then see that f restricted to $[0, e]$ is an additive generator of the t-norm T_U^* on $[0,e]^2$ and f restricted to $[e, 1]$ is an additive generator of the t-conorm C_U^* on $[e,1]^2$. Due to the boundary conditions and continuity of f both T_U and C_U are then continuous and strict.

In [13] a generator of a representable uninorm was called a h-generator and authors use such generators for construction of implications. The h-generator was further generalized to the generalized h-generator which is a continuous strictly increasing function f with $f(e) = 0$ for some $e \in \,]0,1[$. Thus the condition $f(0) = -\infty$, $f(1) = \infty$ is relaxed. Here the inverse function is replaced by the pseudo-inverse function of f (see [7]).

Definition 6. Let $f\colon [a,b] \longrightarrow [c,d]$ be a non-decreasing function. The function $f^{(-1)}\colon \mathbb{R} \longrightarrow [a,b]$ given by

$$f^{(-1)}(x) = \sup\{y \mid f(y) < x\},$$

with the convention $\sup \emptyset = a$ will be called the pseudo-inverse of f. Additionally, the function $f^{(-1)u}\colon \mathbb{R} \longrightarrow [a,b]$ given by

$$f^{(-1)u}(x) = \inf\{y \mid f(y) > x\},$$

with the convention $\inf \emptyset = b$ will be called the upper pseudo-inverse of f.

For pseudo-inverse we have:

$$f \circ f^{(-1)}|_{\mathrm{Ran}(f)} = \mathrm{id}_{\mathrm{Ran}(f)}. \tag{7}$$

However, the following result was shown in [3].

Proposition 12. *Let $f\colon [0,1] \longrightarrow [-\infty, \infty]$ be a continuous strictly increasing function with $f(x) < 0$, $f(y) = 0$ and $f(z) > 0$ for some $x, y, z \in [0,1]$. Let the binary function $O\colon [0,1]^2 \longrightarrow [0,1]$ be given by $O(x,y) = f^{(-1)}(f(x) + f(y))$. Then if $f(1) = d < \infty$ or if $f(0) = c > -\infty$ the function O is not associative.*

The previous result implies that in such a case the function O cannot be a generalized uninorm.

The generated continuous t-subnorms (see [14]) generalize continuous generated t-norms. Although in the case of continuous generated t-norms the existence of the neutral element force that the respective additive generators are strictly monotone, in the case of continuous t-subnorms also generators that are not strictly monotone can be assumed. That is why we would like to do the same generalization also in the case of representable uninorms, i.e., examine the binary functions generated by a continuous non-decreasing function f with $f(e) = 0$ for some $e \in \,]0,1[$, and study under which conditions such a generated function is a generalized uninorm. Since f need not to be strictly monotone, it need not have an inverse and therefore pseudo-inverse should be used. However, in order to be more general, we will investigate not only construction based on pseudo-inverses, but more generally on quasi-inverses (see [7]).

Definition 7. Let $f\colon [a,b] \longrightarrow [c,d]$ be a non-decreasing function. Then each function $f^*\colon \mathbb{R} \longrightarrow [a,b]$ satisfying

$$f \circ f^*|_{\mathrm{Ran}(f)} = \mathrm{id}_{\mathrm{Ran}(f)} \tag{8}$$

and

$$f^{(-1)} \le f^* \le f^{(-1)u} \tag{9}$$

will be called a quasi-inverse of f. The family of all quasi-inverses of f will be denoted by $Q(f)$.

For every non-decreasing function there exists at least one quasi-inverse. The pseudo-inverse of a non-decreasing function is the weakest quasi-inverse and it is left-continuous. Similarly, the upper pseudo-inverse of a non-decreasing function is the strongest quasi-inverse of the given function and it is right-continuous.

Definition 8. Let $f: [0,1] \longrightarrow [-\infty, \infty]$ be a continuous non-decreasing function with $f(e) = 0$ for some $e \in]0,1[$. Let $g \in Q(f)$, i.e., g is a quasi-inverse of f. Then the binary function $O: [0,1]^2 \longrightarrow [0,1]$ given by $O(x,y) = g(f(x) + f(y))$ will be called *generated* and (f, g) will be called a *quasi-generating pair* of O, and more generally, f will be called a *generator* of O.

In the following we will investigate the properties of generated binary functions, and we will focus mainly on generated generalized uninorms.

For a generator f, let us define the set

$$K_f = \{x \in [0,1] \mid \text{ there exists } y \in [0,1], x \neq y, f(x) = f(y)\}$$

and $E_f = [0,1] \setminus K_f$. Then for $e \in]0,1[$ such that $f(e) = 0$ we have $O(e,x) = x$ for all $x \in E_f$. We further denote $V_f = f(K_f)$, i.e., the set of values of the function f on the intervals of constantness. From the properties of quasi-inverses we see that a generated binary function O generated by a generator f is uniquely given if we fix the values of the generating quasi-inverse $g \in Q(f)$ on the set V_f.

We will categorize generated binary functions based on the properties of their (quasi)-generators.

Class I will be a class of generated binary functions which are generated by generators f, which are strictly increasing.
Class II will be a class of generated binary functions which are generated by generators f such that $0 \notin V_f$ and $K_f \neq \emptyset$.
Class IIIa will be a class of generated binary functions which are generated by generators f such that $V_f = \{0\}$.
Class IIIb will be a class of generated binary functions which are generated by generators f such that $0 \in V_f$ and $\text{Card}(V_f) > 1$.

The generated binary functions from the Class I are just representable uninorms which were covered in the previous text and therefore we will study only the generated binary functions from the Class II and III, i.e., such that are generated by a (quasi-)generator f that have at least one interval of constantness. As f is continuous, each interval of constantness is closed and monotonicity implies that there are at most countably many intervals of constantness of f.

First let us note that if for $a, b \in [0,1]$, $a < b$, there is $f(a) = f(b)$ then the generated binary function O with generator f does not have a neutral element since $O(a,x) = O(b,x)$ for all $x \in [0,1]$. Thus behavior of points between a and b is indistinguishable in the aggregation by O.

Assume that $f: [0,1] \longrightarrow [-\infty, \infty]$ is a continuous non-decreasing function with $f(1) = 0$. Then the function $m: [0,1] \longrightarrow [0, \infty]$ given by $m(x) = -f(x)$ for $x \in [0,1]$ defines a generator of a left-continuous t-subnorm. This is due to the following result shown in [14].

Proposition 13. *Let $m: [0,1] \longrightarrow [0,\infty]$ be a continuous non-increasing mapping. Then the binary function $M: [0,1]^2 \longrightarrow [0,1]$ given by $M(x,y) = m^{(-1)}(m(x) + m(y))$ is a left-continuous t-subnorm.*

In the case that instead of the pseudo-inverse $m^{(-1)}$ we would use a quasi-inverse, the condition $M(x,y) \leq \min(x,y)$ could be violated.

Example 5. Let $m: [0,1] \longrightarrow [0, \infty]$ be a continuous non-increasing mapping given by

$$m(x) = \begin{cases} 3 - 2x & \text{if } x \in \left[0, \frac{1}{2}\right], \\ 2 & \text{if } x \in \left]\frac{1}{2}, \frac{3}{4}\right], \\ 8 - 8x & \text{otherwise.} \end{cases}$$

Let us assume a quasi-inverse $g \in Q(m)$ of m given by

$$g(x) = \begin{cases} 0 & \text{if } x > 3, \\ \frac{3-x}{2} & \text{if } x \in]2,3], \\ \frac{3}{5} & \text{if } x = 2, \\ \frac{8-x}{8} & \text{if } x \in [0,2[, \\ 1 & \text{otherwise.} \end{cases}$$

Then for the binary function $M: [0,1]^2 \longrightarrow [0,1]$ given by $M(x,y) = g(m(x) + m(y))$ we have $M(1, \frac{1}{2}) = g(2) = \frac{3}{5} > \min(1, \frac{1}{2})$.

Similar result can be shown for the case when $f(0) = 0$. Then the generated binary function generated by f is a t-superconorm if and only if the quasi-inverse is equal to the upper pseudo-inverse. We summarize these observations in the following proposition.

Proposition 14.

(i) *Let $m\colon [0,1] \longrightarrow [0,\infty]$ be a continuous non-increasing mapping, $m(1) = 0$, and let $g \in Q(m)$. Then the binary function $M\colon [0,1]^2 \longrightarrow [0,1]$ given by $M(x,y) = g(m(x) + m(y))$ is a t-subnorm if and only if g is the pseudo-inverse of m, i.e., $g = m^{(-1)}$.*

(ii) *Let $r\colon [0,1] \longrightarrow [0,\infty]$ be a continuous non-decreasing mapping, $r(0) = 0$, and let $q \in Q(r)$. Then the binary function $R\colon [0,1]^2 \longrightarrow [0,1]$ given by $R(x,y) = q(r(x) + r(y))$ is a t-superconorm if and only if q is the upper pseudo-inverse of r, i.e., $q = r^{(-1)u}$.*

Thus we see that if $f(1) = 0$ ($f(0) = 0$) then the generated binary function generated by a quasi-generating pair (f, g) is a generalized uninorm if and only if g is a pseudo-inverse (upper pseudo-inverse) of f.

From now on we will assume a general case where we have $x, y, z \in [0,1]$ such that $f(x) < 0$, $f(y) = 0$ and $f(z) > 0$. We have the following easy result which we introduce without proof.

Proposition 15. *Let $f\colon [0,1] \longrightarrow [-\infty,\infty]$ be a continuous non-decreasing function with $x, y, z \in [0,1]$ such that $f(x) < 0$, $f(y) = 0$ and $f(z) > 0$ and let $g \in Q(f)$. Then the binary function $O\colon [0,1]^2 \longrightarrow [0,1]$ given by $O(x,y) = g(f(x) + f(y))$ is non-decreasing and commutative. Moreover, O is associative if and only if $f(0) = -\infty$ and $f(1) = \infty$.*

For a generated binary function O we will now study when is $O|_{[e,1]^2}$ ($O|_{]e,1]^2}$, $O|_{[0,e]^2}$, $O|_{[0,e[^2}$) equal to a t-superconorm on $[e,1]^2$ (on $]e,1]^2$, t-subnorm on $[0,e]^2$, on $[0,e[^2$). Proposition 14 implies that restriction of a generated binary function O, generated by a function f to $[e,1]^2$ ($]e,1]^2$, $[0,e]^2$, $[0,e[^2$) is a t-superconorm (t-subnorm) if and only if g is equal to the upper pseudo-inverse of f on $[e,1]$ ($]e,1]$) (pseudo-inverse of f on $[0,e]$, on $[0,e[$). For the generated binary functions of the Class II we have $0 \notin V_f$ and thus there is only one $e \in [0,1]$ such that $f(e) = 0$. Thus for any quasi-inverse $g \in Q(f)$, i.e., also for both the pseudo-inverse and the upper pseudo-inverse, we have $g(0) = e$. Then we get the following result.

Proposition 16. *Let $O\colon [0,1]^2 \longrightarrow [0,1]$ be a generated binary function of the Class II generated by a generator f and its quasi-inverse g. Then O is a generalized sub-uninorm if and only if $f(0) = -\infty$, $f(1) = \infty$, and g on $[-\infty, 0]$ is equal to the pseudo-inverse of f on $[0,e]$, $-\infty \notin V_f$ and f is increasing on $]e,1]$.*

Proof. Necessity: If O is a generalized sub-uninorm then Proposition 15 implies that $f(0) = -\infty$, $f(1) = \infty$, and Proposition 14 further implies that g on $[-\infty, 0]$ is equal to the pseudo-inverse of f on $[0,e]$, and g on $]0,\infty]$ is equal to the upper pseudo-inverse of f on $[e,1[$, and since e is the only point with the functional value of f equal to 0 then g on $[0,\infty]$ is equal to the upper pseudo-inverse of f on $[e,1]$. If $-\infty \in V_f$, where $f(0) = f(a) = -\infty$ for some $0 < a \le e$ then $O(a,y) = 0 < \min(a,y)$ for all $y \in [e,1]$ such that $f(y) < \infty$, i.e., O is not a generalized sub-uninorm. Finally, since $O(e,y) \in [e,y]$ for all $y \in]e,1]$ and $f(e) = 0$ we get $g(f(y)) \in [e,y]$. However, g on $[0,\infty]$ is equal to the upper pseudo-inverse of f on $[e,1]$, i.e., $g(f(y)) \ge y$. Thus $O(e,y) = y = g(f(y))$ for all $y \in]e,1]$, which means that f is increasing on $]e,1]$.

Sufficiency: Proposition 15 implies that O is monotone, commutative and associative. Further, Proposition 14 implies that O on $[0,e]^2$ is a t-subnorm. Since f is continuous on $[0,1]$ and increasing on $]e,1]$ we see that O on $]e,1]^2$ is a t-conorm. Finally, we have to check the averaging behavior on $[0,e] \times]e,1]$ (and similarly on $]e,1] \times [0,e]$). Note that since f is increasing on $]e,1]$ then $\infty \notin V_f$. If $x = e$ then $O(e,y) = g(f(y)) = y$ for all $y \in]e,1]$. If $x < e$ and $y > e$ then $f(x) < f(x) + f(y) < f(y)$ which means that $O(x,y) \in [x,y]$ for all $x, y \in [0,1]$, $x < e$ and $y > e$. Summarizing, O is a generalized sub-uninorm. □

Note that in the previous result since $f(e) = 0$ and $g(0) = e$ we get $O(e,e) = e$, i.e., O on $[0,e]^2$ is a boundary weak t-norm [23]. O on $[e,1]^2$ is then a t-conorm. However, O is not a convertible uninorm. Indeed, if O is of the Class II then corresponding f has at least one interval of constantness and since $\infty, -\infty \notin V_f$ we are able to find $x_1, x_2, y \in [0,1]$, $x_1 \ne x_2$, such that $-\infty < f(x_1) = f(x_2) = -f(y) < 0$. Then $O(x_1, O(y, x_2)) = O(O(x_1, y), x_2) = g(f(x_1))$, however, if U is a uninorm converted from O then $U(x_1, U(y, x_2)) = U(x_1, e) = x_1$ and $U(U(x_1, y), x_2) = U(e, x_2) = x_2$, i.e., associativity is violated.

Analogously we can show the following result.

Proposition 17. *Let $O\colon [0,1]^2 \longrightarrow [0,1]$ be a generated binary function of the Class II generated by a generator f and its quasi-inverse g. Then O is a generalized super-uninorm if and only if $f(0) = -\infty$, $f(1) = \infty$, and g on $[0,\infty]$ is equal to the upper pseudo-inverse of f on $[e,1]$, $\infty \notin V_f$ and f is increasing on $[0,e[$.*

Now we will focus on the Class III.

For a generated binary function O of the Class III we have $0 \in V_f$. Let the preimage of $\{0\}$ in the mapping f be the interval $[a,b]$, $0 \le a < b \le 1$. If $g(0) = c \in]a,b[$ we would get $O(a,a) = c > a$ $O(b,b) = c < b$ which means that O is not a generalized uninorm. Thus we have the following lemma.

Lemma 4. *Let $O\colon [0,1]^2 \longrightarrow [0,1]$ be a generated binary function of the Class III generated by a generator f and its quasi-inverse g. Let the interval $[a,b]$, $0 \le a < b \le 1$, be the preimage of $\{0\}$ in the mapping f. Then if O is a generalized uninorm we have $g(0) \in \{a,b\}$.*

If $g(0) = a$ ($g(0) = b$) then $O(b,b) = a$ ($O(a,a) = b$) and thus the t-subnorm part should act on $[0,b]^2$, i.e., O is a generalized sub-uninorm with $e = b$ (t-superconorm part should act on $[a,1]^2$, i.e., O is a generalized super-uninorm with $e = a$). We get the following result.

Proposition 18. *Let $O\colon [0,1]^2 \longrightarrow [0,1]$ be a generated binary function of the Class III generated by a generator f and its quasi-inverse g. Let the interval $[a,b]$, $0 \le a < b \le 1$, be the preimage of $\{0\}$ in the mapping f. Then O is a generalized sub-uninorm if and only if $f(0) = -\infty$, $f(1) = \infty$, $g(0) = a$ and g on $[-\infty, 0]$ is equal to the pseudo-inverse of f on $[0,b]$, $-\infty \notin V_f$ and f is increasing on $]b,1]$.*

Proof. Necessity: If O is a generalized sub-uninorm then Proposition 15 implies that $f(0) = -\infty$, $f(1) = \infty$ and similarly as in Proposition 16 we can show that $-\infty \notin V_f$. Further, from above it follows that $g(0) = a$ and $O(b,b) = a$, i.e., O should be a t-subnorm on $[0,b]^2$ and a t-superconorm on $]b,1]^2$. Proposition 14 then implies that g on $[-\infty, 0]$ is equal to the pseudo-inverse of f on $[0,b]$, and g on $]0,\infty]$ is equal to the upper pseudo-inverse of f on $]b,1]$. Similarly as in Proposition 16 the averaging behavior of O on $\{b\} \times]b,1]$ then implies that f is increasing on $]b,1]$.

Sufficiency can be shown similarly as in Proposition 16. Here just instead of averaging behavior on $\{e\} \times]e,1]$ we have to check averaging behavior on $[a,b] \times]b,1]$. However, for $x \in [a,b]$ and $y \in]b,1]$ we have $O(x,y) = g(f(y)) = y$, i.e., $O(x,y) \in [x,y]$. $\square$

In this case, i.e., if O is a generalized sub-uninorm of the Class III we have $g(f(x) + f(y)) \neq b$ for all $(x,y) \in [0,1]^2$ and therefore O is a convertible uninorm.

Similar result holds also for generalized super-uninorms.

Proposition 19. *Let $O\colon [0,1]^2 \longrightarrow [0,1]$ be a generated binary function of the Class III generated by a generator f and its quasi-inverse g. Let the interval $[a,b]$, $0 \le a < b \le 1$, be the preimage of $\{0\}$ in the mapping f. Then O is a generalized super-uninorm if and only if $f(0) = -\infty$, $f(1) = \infty$, $g(0) = b$ and g on $[0,\infty]$ is equal to the upper pseudo-inverse of f on $[a,1]$, $\infty \notin V_f$ and f is increasing on $[0,a[$.*

Triangular subnorms (t-superconorms) generated by a continuous additive generator need not to be continuous. In [14] we can find the following result.

Proposition 20. *A continuous non-increasing mapping $m\colon [0,1] \longrightarrow [0,\infty]$ is an additive generator of some continuous t-subnorm M if and only if $m|_{[0,m^{(-1)}(2m(1))]}$ is strictly monotone.*

From this we can conclude the following.

Proposition 21. *Let $O\colon [0,1]^2 \longrightarrow [0,1]$ be a generalized sub-uninorm generated by a quasi-generating pair (f,g) and let the preimage of $\{0\}$ in the mapping f be the interval $[a,b]$, $0 \le a \le b \le 1$. If O on $[0,e]^2$ is a continuous t-subnorm then f is strictly increasing on $[0,a]$.*

Proof. From Proposition 19 we know that $g(0) = a$ and Proposition 20 then implies that O on $[0,b]^2$ is a continuous t-subnorm if and only if f is strictly increasing on $[0, O(b,b)] = [0,a]$. $\square$

If we put Propositions 16, 18 and 21 together we obtain the following.

Proposition 22. *Let $O\colon [0,1]^2 \longrightarrow [0,1]$ be a generalized sub-uninorm generated by a quasi-generating pair (f,g) and let the preimage of $\{0\}$ in the mapping f be the interval $[a,b]$, $0 \le a \le b \le 1$. Then O on $[0,e]^2$ is a continuous t-subnorm if and only if f is strictly increasing on $[0,a]$ and on $]b,1]$.*

In this case O is continuous on $[0,e]^2$ and also on $]e,1]^2$. Similarly we can show the result for generalized super-uninorms.

Proposition 23. *Let $O: [0,1]^2 \longrightarrow [0,1]$ be a generalized super-uninorm generated by quasi-generating pair (f, g) and let the preimage of $\{0\}$ in the mapping f be the interval $[a,b]$, $0 \le a \le b \le 1$. Then O on $[e,1]^2$ is a continuous t-superconorm if and only if f is strictly increasing on $[0,a[$ and on $[b,1]$.*

Thus in both cases we obtain a generator from the class IIIa.

Example 6. Let $f: [0,1] \longrightarrow [-\infty, \infty]$ be given by

$$f(x) = \begin{cases} \ln 4x & \text{if } x \in \left[0, \frac{1}{4}\right], \\ 0 & \text{if } x \in \left]\frac{1}{4}, \frac{3}{4}\right[, \\ -\ln(4-4x) & \text{otherwise.} \end{cases}$$

Then if $g \in Q(f)$, $g(0) = \frac{1}{4}$, with the convention $\infty + (-\infty) = -\infty$, the binary operation $O: [0,1]^2 \longrightarrow [0,1]$ generated by (f, g) is a generalized sub-uninorm given for $x \le y$ by

$$O(x, y) = \begin{cases} 4xy & \text{if } (x, y) \in \left[0, \frac{1}{4}\right]^2, \\ \frac{1}{4} & \text{if } (x, y) \in \left[\frac{1}{4}, \frac{3}{4}\right]^2, \\ 1 - 4(1-x)(1-y) & \text{if } (x, y) \in \left[\frac{3}{4}, 1\right]^2, \\ x & \text{if } x \in \left]0, \frac{1}{4}\right[, \ y \in \left]\frac{1}{4}, \frac{3}{4}\right], \\ y & \text{if } y \in \left]\frac{3}{4}, 1\right[, \ x \in \left[\frac{1}{4}, \frac{3}{4}\right[, \\ \frac{1}{4} & \text{if } (x, y) \in \left]0, \frac{1}{4}\right[\times \left]\frac{3}{4}, 1\right[, \ x = 1 - y, \\ \frac{x}{4-4y} & \text{if } (x, y) \in \left]0, \frac{1}{4}\right[\times \left]\frac{3}{4}, 1\right[, \ x < 1 - y, \\ \frac{4x+y-1}{4x} & \text{if } (x, y) \in \left]0, \frac{1}{4}\right[\times \left]\frac{3}{4}, 1\right[, \ x > 1 - y, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } y = 1, x > 0. \end{cases}$$

If $q \in Q(f)$, $q(0) = \frac{3}{4}$, with the convention $\infty + (-\infty) = -\infty$, the binary operation $O_2: [0,1]^2 \longrightarrow [0,1]$ generated by (f, q) is a generalized super-uninorm given by

$$O_2(x, y) = \begin{cases} \frac{3}{4} & \text{if } (x, y) \in \left[\frac{1}{4}, \frac{3}{4}\right]^2, \\ \frac{3}{4} & \text{if } x = 1 - y, x > 0, \\ O(x, y) & \text{otherwise.} \end{cases}$$

7. Conclusions

In our work we have studied the ordinal sum construction yielding uninorms. We have introduced the basic formula for ordinal sum of uninorms and then we have examined the most general operations that can be used in the construction of uninorms via the ordinal sum construction. We have distinguished four kinds of operations: uninorms, generalized sub-uninorms, generalized super-uninorms and generalized composite uninorms. The first three types of operations can be isomorphically transformed to the unit interval and therefore we have investigated additive generators that yield these operations. We have generalized generators of uninorms by relaxing the strict monotonicity which means the lost of the neutral element. Another possibility of generalization of the generator of a representable uninorm is to keep strict monotonicity and relax the continuity condition. In the case when the range of such a generator is contained in the set $[0, \infty]$, several interesting results can be found in [20].

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Ordinal sums of representable uninorms

Andrea Mesiarová-Zemánková

Mathematical Institute, Slovak Academy of Sciences, Bratislava, Slovakia

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Abstract

We investigate properties of an ordinal sum of uninorms in the case that the summands are proper representable uninorms. We show sufficient and necessary conditions for a uninorm to be an ordinal sum of representable uninorms. An example is also included.

Keywords: Uninorm; Representable uninorm; t-norm; Ordinal sum

1. Introduction

Triangular norms, t-conorms and uninorms [1,7] are applied in many domains and therefore several construction methods for such aggregation functions were developed. Among others, let us recall the construction using the additive generators and the ordinal sum construction. A triangular norm is a binary function $T\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and 1 is its neutral element. Due to the associativity the n-ary form of any t-norm is uniquely given and thus it can be extended to an aggregation function working on $\bigcup_{n\in\mathbb{N}}[0,1]^n$. Dual functions to t-norms are t-conorms. A triangular conorm is a binary function $S\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and 0 is its neutral element. The duality between t-norms and t-conorms is expressed by the fact that from any t-norm T we can obtain its dual t-conorm S by the equation

$$S(x,y) = 1 - T(1-x, 1-y)$$

and vice versa.

Proposition 1. *Let $t\colon [0,1] \longrightarrow [0,\infty]$ be a continuous strictly decreasing function such that $t(1)=0$. Then the binary operation $T\colon [0,1]^2 \longrightarrow [0,1]$ given by*

$$T(x,y) = t^{-1}(\min(t(0), t(x)+t(y)))$$

is a continuous t-norm. The function t is called an additive generator *of T.*

E-mail address: zemankova@mat.savba.sk.

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An additive generator of a continuous t-norm T is uniquely determined up to a positive multiplicative constant. Similarly, an additive generator of a continuous t-conorm S is a continuous strictly increasing function $c\colon [0,1] \longrightarrow [0,\infty]$ such that $c(0) = 0$.

Now let us recall an ordinal sum construction for t-norms and t-conorms [2,7].

Proposition 2. *Let K be a finite or countably infinite index set and let $(]a_k, b_k[)_{k\in K}$ $((]c_k, d_k[)_{k\in K})$ be a disjoint system of open subintervals of $[0,1]$. Let $(T_k)_{k\in K}$ $((S_k)_{k\in K})$ be a system of t-norms (t-conorms). Then the ordinal sum $T = (\langle a_k, b_k, T_k\rangle \mid k \in K)$ $(S = (\langle c_k, d_k, S_k\rangle \mid k \in K))$ given by*

$$T(x,y) = \begin{cases} a_k + (b_k - a_k)T_k(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}) & \text{if } (x,y) \in [a_k, b_k[^2, \\ \min(x,y) & \text{else} \end{cases}$$

and

$$S(x,y) = \begin{cases} c_k + (d_k - c_k)S_k(\frac{x-c_k}{d_k-c_k}, \frac{y-c_k}{d_k-c_k}) & \text{if } (x,y) \in]c_k, d_k]^2, \\ \max(x,y) & \text{else} \end{cases}$$

is a t-norm (t-conorm). The t-norm T (t-conorm S) is continuous if and only if all summands T_k (S_k) for $k \in K$ are continuous.

More details on t-norms and t-conorms can be found in [1,7]. In order to model bipolar behaviour, uninorms were introduced in [16] as binary functions on $[0,1]$ which are commutative, associative, non-decreasing in both variables and have a neutral element $e \in]0,1[$ (see also [5]). A uninorm can be also taken as a bipolar t-conorm on $[-1,1]$ (see [13]), i.e., a bipolar operation that is disjunctive with respect to the neutral point 0 (i.e., aggregated values diverge from the neutral point). If we take a uninorm in a broader sense, i.e., if for a neutral element we have $e \in [0,1]$, then the class of uninorms covers also the class of t-norms and the class of t-conorms. In order to stress that we assume a uninorm with $e \in]0,1[$ we will call such a uninorm *proper*. For each uninorm the value $U(1,0) \in \{0,1\}$ is the annihilator of U. A uninorm is called *conjunctive* (*disjunctive*) if $U(1,0) = 0$ $(U(1,0) = 1)$. Due to the associativity we can uniquely define the n-ary form of any uninorm for any $n \in \mathbb{N}$ and therefore in some proofs we will use its ternary form instead of binary, where suitable.

For each uninorm U with neutral element $e \in [0,1]$, the restriction of U to $[0,e]^2$ is a t-norm on $[0,e]^2$ (i.e., a linear transformation of some t-norm T_U) and the restriction of U to $[e,1]^2$ is a t-conorm on $[e,1]^2$ (i.e., a linear transformation of some t-conorm S_U). Moreover, $\min(x,y) \le U(x,y) \le \max(x,y)$ for all $(x,y) \in [0,e] \times [e,1] \cup [e,1] \times [0,e]$.

Definition 1. A uninorm $U\colon [0,1]^2 \longrightarrow [0,1]$ is called *internal* if $U(x,y) \in \{x,y\}$ for all $(x,y) \in [0,1]^2$.

Lemma 1 *([3]). Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm such that $T_U = \min$ and $S_U = \max$. Then U is internal.*

For more details and a survey on other classes of uninorms we recommend [12].

In the following result we see that from any pair of a t-norm and a t-conorm we can construct the minimal and the maximal uninorm with the given underlying functions.

Proposition 3 *([9]). Let $T\colon [0,1]^2 \longrightarrow [0,1]$ be a t-norm and $S\colon [0,1]^2 \longrightarrow [0,1]$ a t-conorm and assume $e \in [0,1]$. Then the two functions $U_{\min}, U_{\max}\colon [0,1]^2 \longrightarrow [0,1]$ given by*

$$U_{\min}(x,y) = \begin{cases} e \cdot T(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)\cdot S(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ \min(x,y) & \text{otherwise} \end{cases}$$

and

$$U_{\max}(x,y)=\begin{cases} e\cdot T(\frac{x}{e},\frac{y}{e}) & \textit{if } (x,y)\in[0,e]^2\,,\\ e+(1-e)\cdot S(\frac{x-e}{1-e},\frac{y-e}{1-e}) & \textit{if } (x,y)\in[e,1]^2\,,\\ \max(x,y) & \textit{otherwise}\end{cases}$$

are uninorms. We will denote the set of all uninorms of the first type by $\mathcal{U}_{\min}$ *and of the second type by* $\mathcal{U}_{\max}$.

Similarly as in the case of t-norms and t-conorms we can construct uninorms using additive generators (see [5]).

Proposition 4. *Let* $f\colon [0,1]\longrightarrow[-\infty,\infty]$, $f(0)=-\infty$, $f(1)=\infty$ *be a continuous strictly increasing function. Then a binary function* $U\colon [0,1]^2\longrightarrow[0,1]$ *given by*

$$U(x,y)=f^{-1}(f(x)+f(y)),$$

where $f^{-1}\colon [-\infty,\infty]\longrightarrow[0,1]$ *is an inverse function to* f*, is a uninorm, which will be called a* representable *uninorm. This uninorm is conjunctive if we take the convention* $\infty+(-\infty)=-\infty$ *and it is disjunctive if we take the convention* $\infty+(-\infty)=\infty$.

Note that if we relax the monotonicity of the additive generator then the neutral element will be lost and by relaxing the condition $f(0)=-\infty$, $f(1)=\infty$ the associativity will be lost. In [15] (see also [13]) we can find the following result.

Proposition 5. *Let* $U\colon [0,1]^2\longrightarrow[0,1]$ *be a uninorm continuous everywhere on the unit square except of the two points* $(0,1)$ *and* $(1,0)$*. Then* U *is representable, i.e., there exists such a function* $u\colon [0,1]\longrightarrow[-\infty,\infty]$ *with* $u(e)=0$, $u(0)=-\infty$, $u(1)=\infty$ *that* $U(x,y)=u^{-1}(u(x)+u(y))$.

Thus a uninorm U is representable if and only if it is continuous on $[0,1]^2\setminus\{(0,1),(1,0)\}$, which completely characterises the set of representable uninorms.

Definition 2. We will denote the set of all uninorms U such that T_U and S_U are continuous by $\mathcal{U}$, and the set of all uninorms V such that $V(x,0)=0$ for all $x\in[0,1[$ and $V(x,1)=1$ for all $x\in]0,1]$ by $\mathcal{N}$. Further, we will denote by $\mathcal{N}_{\max}$ ($\mathcal{N}_{\min}$) the set of all uninorms $U\in\mathcal{N}$ such that there exists a uninorm $U_1\in\mathcal{U}_{\max}$ ($U_1\in\mathcal{U}_{\min}$) such that $U=U_1$ on $]0,1[^2$.

Note that the class of representable uninorms belongs to the intersection $\mathcal{U}\cap\mathcal{N}$.

In the case of t-norms (t-conorms), each continuous t-norm (t-conorm) is an ordinal sum of continuous generated t-norms (t-conorms). The aim of this paper is the characterisation of the uninorms that are ordinal sums of proper representable uninorms. In the following section we will investigate properties of ordinal sums of proper representable uninorms and in Section 3 we will completely characterise uninorms which are ordinal sums of proper representable uninorms. We give our conclusions in Section 4.

2. Ordinal sum of representable uninorms

An ordinal sum of uninorms was introduced in [14] (see also [2]). We will use the following transformation. For any $0\le a<b\le c<d\le 1$, $v\in[b,c]$, and a uninorm U with neutral element $e\in[0,1]$ let $f\colon [0,1]\longrightarrow[a,b[\cup\{v\}\cup]c,d]$ be given by

$$f(x)=\begin{cases}(b-a)\cdot\frac{x}{e}+a & \text{if } x\in[0,e[,\\ v & \text{if } x=e,\\ d-\frac{(1-x)(d-c)}{(1-e)} & \text{otherwise.}\end{cases}\tag{1}$$

Then f is linear on $[0,e[$ and on $]e,1]$ and thus it is a piece-wise linear isomorphism of $[0,1]$ to $([a,b[\cup\{v\}\cup]c,d])$ and a function $U_v^{a,b,c,d}\colon ([a,b[\cup\{v\}\cup]c,d])^2\longrightarrow([a,b[\cup\{v\}\cup]c,d])$ given by

$$U_v^{a,b,c,d}(x,y)=f(U(f^{-1}(x),f^{-1}(y)))\tag{2}$$

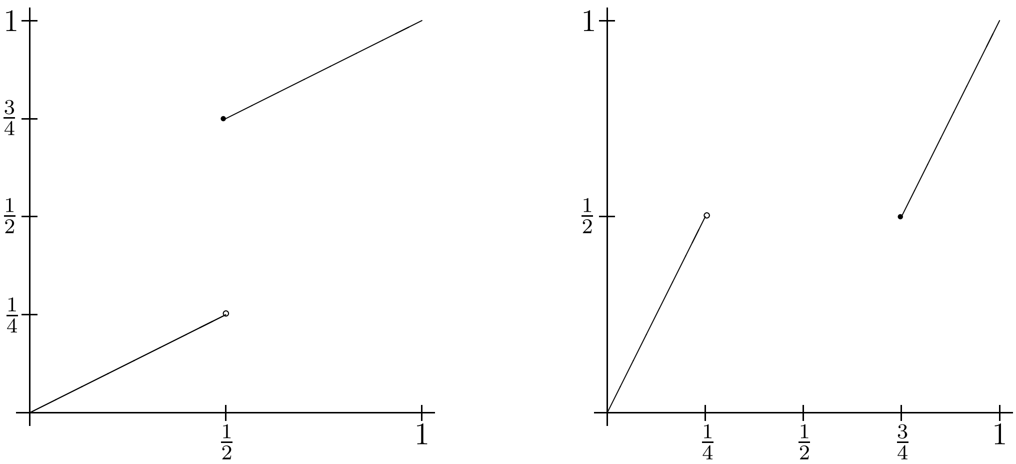

Fig. 1. The function f (left) and its inverse f^{-1} (right) from Example 1.

is an operation on $([a, b[\cup \{v\} \cup]c, d])^2$ which is commutative, associative, non-decreasing in both variables (with respect to the standard order) and v is its neutral element. The function f is piece-wise linear, however, more generally we can assume any increasing isomorphic transformation.

Example 1. Let $a = 0$, $b = \frac{1}{4}$, $v = c = \frac{3}{4}$, $d = 1$, $e = \frac{1}{2}$. Then the function $f\colon [0, 1] \longrightarrow \left[0, \frac{1}{4}\right[\cup \left[\frac{3}{4}, 1\right]$ from (1) is given by

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \in \left[0, \frac{1}{2}\right[, \\ \frac{1+x}{2} & \text{otherwise,} \end{cases}$$

see Fig. 1. If we assume uninorm U_1 from Example 3 the operation $(U_1)_{\frac{3}{4}}^{0,\frac{1}{4},\frac{3}{4},1}(x, y)$ can be then found in Example 3 as that part of the uninorm U^e which is defined on $(\left[0, \frac{1}{4}\right[\cup \left[\frac{3}{4}, 1\right])^2$.

Proposition 6 *([14]). Assume $e \in [0, 1]$. Let K be an index set which is finite or countably infinite and let $(]a_k, b_k[)_{k\in K}$ be a system of open disjoint subintervals (including empty subintervals) of $[0, e]$, such that $\bigcup_{k\in K} [a_k, b_k] = [0, e]$. Similarly, let $(]c_k, d_k[)_{k\in K}$ be a system of open disjoint subintervals (including empty subintervals) of $[e, 1]$, such that $\bigcup_{k\in K} [c_k, d_k] = [e, 1]$. Further, let these two systems be anti-comonotone, i.e., $b_k \le a_i$ if and only if $c_k \ge d_i$ for all $i, k \in K$. We will denote by $K_* = \{k \in K \mid]a_k, b_k[\neq \emptyset\}$ and by $K^* = \{k \in K \mid]c_k, d_k[\neq \emptyset\}$. Assume a family of proper uninorms $(U_k)_{k\in K_*\cap K^*}$ on $[0, 1]^2$, a family of t-norms $(T_k)_{k\in K_*\setminus K^*}$ on $[0, 1]^2$ and a family of t-conorms $(S_k)_{k\in K^*\setminus K_*}$ on $[0, 1]^2$. Denote $B_1 = \{b_k \mid k \in K\} \setminus \{a_k \mid k \in K_*\}$ and $C_1 = \{c_k \mid k \in K\} \setminus \{d_k \mid k \in K^*\}$ and let $g\colon B_1 \setminus \{e\} \longrightarrow [e, 1]$, $h\colon C_1 \setminus \{e\} \longrightarrow [0, e]$ be two functions given for $b \in B_1 \setminus \{e\}$ and $c \in C_1 \setminus \{e\}$ by*

$$g(b) = \lim_{i\longrightarrow\infty} d_{k_i},$$

where $b = \lim_{i\longrightarrow\infty} a_{k_i}$ for $k_i \in K_*$ and

$$h(c) = \lim_{i\longrightarrow\infty} a_{k_i},$$

where $c = \lim_{i\longrightarrow\infty} d_{k_i}$ for $k_i \in K^*$. We further denote $B = \{b \in B_1 \setminus \{e\} \mid g(b) \in C_1\}$, $C = \{c \in C_1 \setminus \{e\} \mid h(c) \in B_1\}$ and we assume a function $n\colon B \longrightarrow B \cup C$ given for all $b \in B$ by

$$n(b) \in \{b, g(b)\}.$$

Let the ordinal sum $U^e = (\langle a_k, b_k, c_k, d_k, U_k\rangle \mid k \in K)$ be given by

$$U^e(x,y)=\begin{cases} y & \text{if } x=e,\\ x & \text{if } y=e,\\ (U_k)^{a_k,b_k,c_k,d_k}_{v_k} & \text{if } (x,y)\in([a_k,b_k[\cup]c_k,d_k])^2, k\in K_*\cap K^*,\\ (T_k)^{a_k,b_k} & \text{if } (x,y)\in([a_k,b_k[\cup]c_k,d_k])^2, k\in K_*\setminus K^*,\\ (S_k)^{c_k,d_k} & \text{if } (x,y)\in([a_k,b_k[\cup]c_k,d_k])^2, k\in K^*\setminus K_*,\\ x & \text{if } y\in[b_k,c_k], x\in[a_k,d_k]\setminus[b_k,c_k], k\in K_*\cup K^*,\\ y & \text{if } x\in[b_k,c_k], y\in[a_k,d_k]\setminus[b_k,c_k], k\in K_*\cup K^*,\\ \min(x,y) & \text{if } (x,y)\in[b,c]^2\setminus(]b,c[^2\cup\{(b,c),(c,b)\}),\\ & \quad \text{where } b\in B, c=g(b), x+y<c+b,\\ \max(x,y) & \text{if } (x,y)\in[b,c]^2\setminus(]b,c[^2\cup\{(b,c),(c,b)\}),\\ & \quad \text{where } b\in B, c=g(b)\ x+y>c+b,\\ n(b) & \text{if } (x,y)=(b,c) \text{ or } (x,y)=(c,b), b\in B, c=g(b),\\ \min(x,y) & \text{if } (x,y)\in\{b\}\times[b,c]\cup[b,c]\times\{b\}, b\in B_1\setminus(B\cup\{e\}), c=g(b),\\ \max(x,y) & \text{if } (x,y)\in\{c\}\times[b,c]\cup[b,c]\times\{c\}, c\in C_1\setminus(C\cup\{e\}), b=h(c), \end{cases}$$

where $v_k=c_k$ ($v_k=b_k$) if there exists an $i\in K$ such that $b_k=a_i$, $c_k=d_i$ and U_i is disjunctive (conjunctive) and $v_k=n(b_k)$ if $b_k\in B$, $v_k=b_k$ if $b_k\in B_1\setminus B$, $v_k=c_k$ if $c_k\in C_1\setminus C$ and $(U_k)^{a_k,b_k,c_k,d_k}_{v_k}$ is given by the formula (2), $(T_k)^{a_k,b_k}$ ($(S_k)^{c_k,d_k}$) is a linear transformation of T_k (S_k) to $[a_k,b_k]^2$ ($[c_k,d_k]^2$). Then U^e is a uninorm.

An example of an ordinal sum of two uninorms can be found in Example 3.

Remark 1. It is evident that ordinal sum of uninorms $U^e=(\langle a_k,b_k,c_k,d_k,U_k\rangle\mid k\in K)$ is on $[0,e]^2$ equal to an ordinal sum of t-norms, i.e., $T_U=(\langle a_k,b_k,T_{U_k}\rangle\mid k\in K)$ and on $[e,1]^2$ to an ordinal sum of t-conorms $S_U=(\langle c_k,d_k,S_{U_k}\rangle\mid k\in K)$. In the case that we assume an ordinal sum of uninorms such that $\bigcup_{k\in K}[a_k,b_k]\neq[0,e]$ ($\bigcup_{k\in K}[c_k,d_k]\neq[e,1]$) this can be given by the above ordinal sum, where the missing summands are covered by internal uninorms. Later we will see that this holds also vice versa, i.e., if U is an ordinal sum of uninorms and $T_U=(\langle a_k,b_k,T_k\rangle\mid k\in K)$ and $S_U=(\langle c_k,d_k,S_k\rangle\mid k\in K)$ and $[a,b[\cup]c,d]$ is a missing summand support, i.e., such that is not covered by $\bigcup_{k\in K}[a_k,b_k]\cup\bigcup_{k\in K}[c_k,d_k]$, then $U(x,y)=\min(x,y)$ on $[a,b]^2$ and $U(x,y)=\max(x,y)$ on $[c,d]^2$. Moreover, similarly as in Lemma 1 we get that U is internal on $([a,b]\cup[c,d])^2$.

Example 2. Assume $U_1\in\mathcal{U}_{\min}$ and $U_2\in\mathcal{U}_{\max}$ then U_1 and U_2 are ordinal sums of uninorms, $U_1=(\langle e,e,e,1,S_{U_1}\rangle, \langle 0,e,1,1,T_{U_1}\rangle)$ and $U_2=(\langle 0,e,e,e,T_{U_2}\rangle,\langle 0,0,e,1,S_{U_2}\rangle)$.

Assume a uninorm $U\colon[0,1]^2\longrightarrow[0,1]$ such that $U=(\langle a_k,b_k,c_k,d_k,U_k\rangle\mid k\in K)$, where all conditions from Proposition 6 are satisfied and both $]a_k,b_k[$ and $]c_k,d_k[$ are non-empty for all $k\in K$. We will call such an ordinal sum *complete*. Let each U_k for $k\in K$ be a representable uninorm. Then by Proposition 5 the uninorm U_k is continuous on $[0,1]\setminus\{(0,1),(1,0)\}$. Since the summand corresponding to U_k acts on $[a_k,b_k[\cup]c_k,d_k]$ the uninorm U is continuous on $([a_k,b_k[\cup]c_k,d_k])^2$ except the set $\{(f_k(x),f_k(y))\mid U_k(x,y)=e_k\}\cup\{(a_k,d_k),(d_k,a_k)\}$, where e_k is the neutral element of U_k and f_k is the transformation given by (2) respective to the summand corresponding to U_k. Here let us note that for a representable uninorm U_k there exists a strictly decreasing function $r_k\colon\,]0,1[\longrightarrow]0,1[$ with $r_k(e_k)=e_k$ such that $U_k(x,y)=e_k$ if and only if $r_k(x)=y$.

The ordinal sum construction further implies that for $x\in\{a_k,b_k,c_k,d_k\}$ and $y\in[0,1]$ we have $U(x,y)\in\{x,y\}$. Moreover, $U(a_k,y)=a_k$ and $U(d_k,y)=d_k$ for $y\in\,]a_k,d_k[$ and $U(b_k,y)=b_k$ and $U(c_k,y)=c_k$ for $y\in\,]b_k,c_k[$. Since also $U_k(z,e)=\max(z,e)$ for $z>e$ and $U_k(z,e)=\min(z,e)$ for $z<e$ we see that U is continuous on $\{b_k\}\times[a_k,b_k]$, $[a_k,b_k]\times\{b_k\}$, $\{c_k\}\times[c_k,d_k]$, $[c_k,d_k]\times\{c_k\}$, $\{b_k\}\times[c_k,d_k]\setminus\{(b_k,c_k)\}$, $[c_k,d_k]\times\{b_k\}\setminus\{(c_k,b_k)\}$ and on $\{c_k\}\times[a_k,b_k]\setminus\{(c_k,b_k)\}$, $[a_k,b_k]\times\{c_k\}\setminus\{(b_k,c_k)\}$. If we summarise this over all summands for $k\in K$ we obtain the following result.

Proposition 7. *Assume a uninorm $U\colon[0,1]^2\longrightarrow[0,1]$. If U is a complete ordinal sum of representable uninorms, i.e., $U=(\langle a_k,b_k,c_k,d_k,U_k\rangle\mid k\in K)$, for some suitable systems $(]a_k,b_k[)_{k\in K}$ and $(]c_k,d_k[)_{k\in K}$ and a family of (proper)*

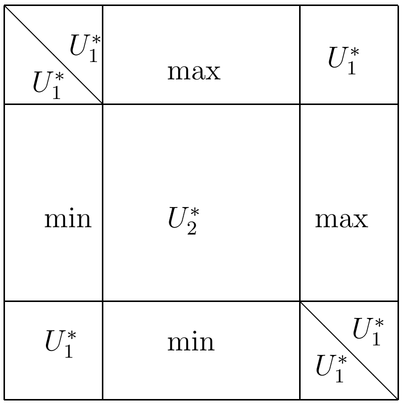

Fig. 2. The uninorm U^e from Example 3. Here U_1^* is a transformation of U_1 to $(\left[0, \frac{1}{4}\right[\cup \left[\frac{3}{4}, 1\right])^2$ given by (2), and U_2^* is a linear transformation of U_2 to $\left[\frac{1}{4}, \frac{3}{4}\right]^2$. The oblique lines denote the points of discontinuity of U.

representable uninorms $(U_k)_{k\in K}$ *then there exists a continuous strictly decreasing function* $r\colon [0,1] \longrightarrow [0,1]$ *with* $r(0) = 1$, $r(e) = e$ *and* $r(1) = 0$ *such that* U *is continuous on* $[0,1] \setminus \{(x, r(x)) \mid x \in [0,1]\}$. *Note that* U *need not be non-continuous on the whole set* $\{(x, r(x)) \mid x \in [0,1]\}$.

Example 3. Let $U_1, U_2\colon [0,1]^2 \longrightarrow [0,1]$ with $U_1 = U_2$ be representable uninorms generated by

$$f(x) = \begin{cases} \ln(2x) & \text{if } x \le \frac{1}{2} \\ -\ln(2-2x) & \text{otherwise,} \end{cases}$$

with $U_1(1,0) = 1$. Then the ordinal sum $U^e = (\langle 0, \frac{1}{4}, \frac{3}{4}, 1, U_1\rangle, \langle \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, U_2\rangle)$, with $e = \frac{1}{2}$ is given in the following table

$x\backslash y$	$\left[0, \frac{1}{4}\right[$	$\left[\frac{1}{4}, \frac{1}{2}\right]$	$\left]\frac{1}{2}, \frac{3}{4}\right]$	$\left]\frac{3}{4}, 1\right]$
$\left[0, \frac{1}{4}\right[$	$\frac{x}{4-4y}$ if $x+y<1$ $\frac{4x-1+y}{4x}$ if $x+y>1$	$\max(x,y)$	$\max(x,y)$	$1-4(1-x)(1-y)$
$\left[\frac{1}{4}, \frac{1}{2}\right]$	$\min(x,y)$	$\frac{x-y+\frac{1}{2}}{3-4y}$ if $x+y<1$ $\frac{3x+y-\frac{3}{2}}{4x-1}$ if $x+y>1$	$\frac{3}{4}-4(\frac{3}{4}-x)(\frac{3}{4}-y)$	$\max(x,y)$
$\left]\frac{1}{2}, \frac{3}{4}\right]$	$\min(x,y)$	$4(x-\frac{1}{4})(y-\frac{1}{4})+\frac{1}{4}$	$\frac{y-x+\frac{1}{2}}{3-4x}$ if $x+y<1$ $\frac{3y+x-\frac{3}{2}}{4y-1}$ if $x+y>1$	$\max(x,y)$
$\left]\frac{3}{4}, 1\right]$	$4xy$	$\min(x,y)$	$\min(x,y)$	$\frac{y}{4-4x}$ if $x+y<1$ $\frac{4y-1+x}{4y}$ if $x+y>1$

where if $x+y=1$ then $U(x,y) = \frac{1}{2}$ for $(x,y) \in \left]\frac{1}{4}, \frac{3}{4}\right[^2$ and otherwise $U(x,y) = \frac{3}{4}$. Thus here U^e is non-continuous only in the points from the set $\{(x, 1-x) \mid x \in \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]\}$. Moreover, evidently $U^e \in \mathcal{U} \cap \mathcal{N}$. The uninorm U^e can be seen in Fig. 2.

Further we will show some general properties of uninorms that we will use later.

Proposition 8. *Assume a uninorm* $U\colon [0,1]^2 \longrightarrow [0,1]$ *such that* $U \in \mathcal{U}$ *and* $U \notin \mathcal{N}$. *Then* U *is an ordinal sum of a uninorm and a non-proper uninorm (i.e., a t-norm or a t-conorm).*

Proof. Assume $U(1,0)=1$, the case when $U(1,0)=0$ can be shown analogically. Then $U(x,1)=1$ for all $x\in[0,1]$ and $U(x,0)=0$ for all $x\in[0,e]$. If $U\notin\mathcal{N}$ then there exists a $y\in\,]e,1[$ such that $U(0,y)=z>0$, i.e., $z\in\,]0,y]$. If $z\le e$ we get $0=U(0,e)\ge U(0,z)=U(0,0,y)=z$ what is a contradiction, i.e., $z\in\,]e,y]$. Moreover, $U(0,z)=z$. Since S_U is continuous then for all $x\in[z,1]$ there exists a $u\in[e,1]$ such that $U(z,u)=x$. Then

$$U(0,x)=U(0,z,u)=U(z,u)=x.$$

Thus if $U(0,y)>0$ for some $y\in\,]e,1[$ then $U(0,y)=y$. Let $b=\inf\{y\in[e,1]\mid U(0,y)>0\}$. Then b is an idempotent point of U, since otherwise the continuity of S_U implies existence of $x_1\in\,]e,b[$ such that $U(x_1,x_1)=v>b$ which means

$$b<v=U(0,v)=U(0,x_1,x_1)=U(0,x_1)=0$$

what is a contradiction. Further, if $U(0,y)=y$ for some $y\in[e,1]$ then $U(x,y)=U(x,0,y)=U(0,y)=y$ for all $x\in[0,e]$ and thus $U(x,y)=\max(x,y)$ if $(x,y)\in[0,e]\times\,]b,1]\cup\,]b,1]\times[0,e]$.

Since b is an idempotent point, S_U (its transformation onto $[e,1]^2$) is an ordinal sum $S_U=(\langle e,b,S_1\rangle,\langle b,1,S_2\rangle)$ and thus U on $[b,1]^2$ corresponds to S_2 and $U(x,y)=\max(x,y)$ for all $(x,y)\in[0,b]\times\,]b,1]\cup\,]b,1]\times[0,b]$. Also, $[0,b]^2$ is closed under U. Thus

$$U=(\langle 0,e,e,b,U^*\rangle,\langle 0,0,b,1,S_2\rangle),$$

where U^* is uninorm which is a linear transformation of U on $[0,b]^2$. $\square$

Definition 3. Let p be a relation on $X\times Y$ and denote $p(x)=\{y\in Y\mid (x,y)\in p\}$. Then p will be called a *continuous non-increasing pseudo-function* if

(i) for all $x_1,x_2\in X$, $x_1<x_2$ there is $p(x_1)\ge p(x_2)$, i.e., for all $y_1\in p(x_1)$ and all $y_2\in p(x_2)$ we have $y_1\ge y_2$ and thus $\mathrm{Card}(p(x_1)\cap p(x_2))\le 1$,
(ii) for all $x\in X$ and all $y\in Y$ there exist $y_1\in Y$ and $x_1\in X$ such that $(x,y_1)\in p$ and $(x_1,y)\in p$,
(iii) if $y_1,y_2\in p(x)$ for some $x\in X$ then $y\in p(x)$ for all $y\in[y_1,y_2]$.

A relation p is called symmetric if $(x,y)\in p$ if and only if $(y,x)\in p$.

Remark 2.

(i) In the case that U is an ordinal sum of representable uninorms which is not complete, then the function that determine the non-continuity points need not be strictly decreasing, just non-increasing. If there is a non-proper summand such that $a_k=b_k=0$ ($c_k=d_k=1$) then for r_k we have $r_k(0)=c_k$ ($r_k(1)=b_k$). Further if there is a non-proper summand, i.e., $a_k=b_k>0$ ($c_k=d_k<1$) for some $k\in K$ then U is non-continuous on $\{a_k\}\times[c_k,d_k]$ ($\{c_k\}\times[a_k,b_k]$). Thus in such a case r_k is no longer a function since it contains also some vertical segments, however, r_K is a symmetric continuous non-increasing pseudo-function.
(ii) If we assume an ordinal sum, where some summands are representable uninorms and some summands are internal uninorms then again we can obtain a symmetric continuous non-increasing pseudo-function r such that U is continuous on $[0,1]\setminus\{(x,r(x))\mid x\in[0,1]\}$. This follows from the fact that for an internal uninorm V the monotonicity implies existence of a symmetric continuous non-increasing pseudo-function p_V such that $V(x,y)=\max(x,y)$ if $y>p_V(x)$ and $V(x,y)=\min(x,y)$ if $y<p_V(x)$.

In the following section we will discuss the characterisation of uninorms that are ordinal sums of proper representable uninorms.

3. Characterisation of uninorms that are equal to an ordinal sum of proper representable uninorms

For a given uninorm $U\colon [0,1]^2\longrightarrow[0,1]$ and each $x\in[0,1]$ we define a function $u_x(z)=U(x,z)$ for $z\in[0,1]$. We will start with the following useful result.

Lemma 2. *Let $U: [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{N} \cap \mathcal{U}$. If for some $a \in]0,e[\cup]e,1[$, a is an idempotent element of U, then u_a is non-continuous.*

The above result follows from the fact that if a is an idempotent element of U different from $0, e$, resp. 1, then $e \notin \text{Ran}(u_a)$. Indeed, if $U(a,b) = e$ for some $b \in [0,1]$ then $e = U(a,a,b) = a$ what is a contradiction.

Proposition 9. *Let $U: [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{N} \cap \mathcal{U}$. Then if $U(a,b) = e$ for some $a, b \in [0,1]$, $a < e$ then U is continuous on $[0,1]^2 \setminus ([0,a[\cup]b,1])^2$.*

Proof. If $U(a,b) = e$ then $a \notin \{0,1\}$. Also if $U(a,b) = U(a,c) = e$ then $b = U(b,a,c) = c$, i.e., $b = c$. First we show that u_a is continuous on $[0,1]$. Since U is monotone and $u_a(0) = 0$, $u_a(1) = 1$ the continuity of u_a is equivalent with the equality $\text{Ran}(u_a) = [0,1]$. Assume that $\text{Ran}(u_a) \neq [0,1]$, i.e., there exists a $c \in [0,1]$ such that $c \notin \text{Ran}(u_a)$. However, we have $U(a,b,c) = c$, i.e., for $z = U(b,c)$ we have $u_a(z) = c$ what is a contradiction.

Thus u_a and similarly u_b are continuous functions. Next we will show that for all $x \in]a,b[$ there exists a $v^x \in [0,1]$ such that $U(x, v^x) = e$. Assume $f \in]a,e]$ (for $f \in [e,b[$ the proof is analogous). Since T_U is continuous and $U(a,f) \leq a$, $U(f,e) = f$ there exists a $a^f \in [0,e]$ such that $U(f, a^f) = a$. Then

$$e = U(a,b) = U(f, a^f, b)$$

and if $w^f = U(a^f, b)$ then $U(f, w^f) = e$. Summarising we get that all sections u_x with $x \in [a,b]$ are continuous. From the previous lemma we see that a and b are not idempotent elements, i.e., there exist g, h with $g < a < b < h$ such that $U(g,h) = e$, i.e., all u_x for $x \in \left[g,h\right]$ are continuous. Now the monotonicity of U implies the continuity on $[0,1]^2 \setminus ([0,a[\cup]b,1])^2$ (see [8]). □

From the previous proposition we see that if $U \in \mathcal{N} \cap \mathcal{U}$ and U is non-continuous in some point $(c,d) \in [0,1]^2$ then u_x is non-continuous for all $x \in [0,c] \cup [d,1]$.

Lemma 3. *Assume a uninorm $U: [0,1]^2 \longrightarrow [0,1]$, $U \in \mathcal{U} \cap \mathcal{N}$. If $a \in [0,1]$ is an idempotent element of U then U is internal on $\{a\} \times [0,1]$.*

Proof. If $a \in \{0,1,e\}$ the result is evident. Otherwise we will assume $a < e$ (the proof for $a > e$ is analogous). Since a is an idempotent point we have $U(a,x) = \min(x,y)$ for all $x \in [0,e]$. From Lemma 2 it follows that u_a is non-continuous and $e \notin \text{Ran}(u_a)$. Assume $y > e$ and let $U(a,y) = v \leq y$. Then if $v \leq e$ we have $v = U(a,a,y) = U(a,v) \leq a$, i.e., $v = a$. Thus if $v > a$ also $v > e$. Denote

$$b = \inf\{y \in [0,1] \mid U(a,y) > a\}.$$

Then $U(a,y) > a$ for $y > b$ and $U(a,y) = \min(a,y)$ for $y < b$. For $v = U(a,y) > a$ we further have $v = U(a,a,y) = U(a,v)$. Since $U \in \mathcal{U}$ the continuity of S_U ensures for any $y_2 > v$ existence of y_1 such that $S(v,y_1) = y_2$. Then

$$U(a,y_2) = U(a,v,y_1) = U(v,y_1) = y_2.$$

Summarising, for all $y > b$ we have $U(a,y) = y$ and for all $x < b$ we have $U(x,a) = \min(a,x)$. To conclude the proof we have only to check the value $U(a,b)$. Assume $U(a,b) = c \in]a,b[$. Then $c \geq e$ and $U(a,c) = c < b$, what is a contradiction since $U(a,x) = \min(a,x)$ for all $x < b$. □

The above lemma shows, that if we denote the set of all idempotent points of U by I_U, then U restricted to I_U^2 is an internal uninorm.

Lemma 4. *Each uninorm $U: [0,1]^2 \longrightarrow [0,1]$ with $U \in \mathcal{U}$ is continuous in (e,e).*

Proof. If T_U and S_U are continuous, since U is commutative, we only have to check that for two monotone sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ with $\lim\limits_{n\longrightarrow\infty} a_n = e = \lim\limits_{n\longrightarrow\infty} b_n$ and $a_n < e$, $b_n > e$ for $n \in \mathbb{N}$ there is $\lim\limits_{n\longrightarrow\infty} U(a_n,b_n) = e$. However, monotonicity gives us $a_n \leq U(a_n,b_n) \leq b_n$ and thus $e \leq \lim\limits_{n\longrightarrow\infty} U(a_n,b_n) \leq e$. □

From now on we will investigate uninorms such that there exists a continuous strictly decreasing function $r: [0,1] \longrightarrow [0,1]$ with $r(0) = 1$, $r(e) = e$ and $r(1) = 0$ such that U is continuous on $[0,1] \setminus \{(x, r(x)) \mid x \in [0,1]\}$. Then if U is non-continuous only in $(0,1), (1,0)$ the uninorm U is representable. Thus if U is not representable then due to Proposition 9 there exist $a, b \in [0,1]$ such that U is non-continuous in all points of $\{(x, r(x)) \mid x \in [0,a] \cup [b,1]\}$, where $a > 0$ and $b < 1$.

Each continuous t-norm (t-conorm) is equal to an ordinal sum of continuous Archimedean t-norms (t-conorms). Note that a continuous t-norm (t-conorm) is Archimedean if and only if it has only trivial idempotent points 0 and 1. A continuous Archimedean t-norm T (t-conorm S) is either strict, i.e., strictly increasing on $]0,1]^2$, (on $[0,1[^2$) or nilpotent, i.e., there exist $(x,y) \in]0,1[^2$ such that $T(x,y) = 0$ ($S(x,y) = 1$).

For Archimedean underlying functions we can show the following result (compare [6,10,11]).

Proposition 10. *Assume a uninorm $U: [0,1]^2 \longrightarrow [0,1]$, $U \in \mathcal{U} \cap \mathcal{N}$. If T_U and S_U are Archimedean then either U is a representable uninorm or $U \in \mathcal{N}_{\min} \cup \mathcal{N}_{\max}$.*

Proof. If for all $x \in]0,1[$ there exists a $y \in [0,1]$ such that $U(x,y) = e$ then since T_U and S_U are continuous and Archimedean U is continuous on $[0,1]^2$ except points $(0,1)$ and $(1,0)$ and thus U is a representable uninorm. If for any $x \in]0,1[$, $x \neq e$ there exists a $y \in [0,1]$ such that $U(x,y) = e$ then also $U(x,x,y,y) = e$ and since T_U and S_U are continuous and Archimedean Proposition 9 implies that for all $x \in]0,1[$ there exists a $y \in [0,1]$ such that $U(x,y) = e$ and thus U is representable. Therefore we will suppose that $U(x,y) = e$ for $(x,y) \in [0,1]^2$ if and only if $x = y = e$.

If $U(x,y) = x = U(x,e)$ (or similarly if $U(x,y) = y$) for some $x, y \in]0,1[$, $x < e$ and $y > e$ then $U(x, \underbrace{y, \ldots, y}_{n\text{-times}}) = x$ for all $n \in \mathbb{N}$ and since S_U is continuous and Archimedean we have $U(x,z) = x$ for all $z \in [e,1[$. Further, since T_U is continuous and Archimedean for all $0 < q \leq x$ there exists $x_1 \in [0,e]$ such that $U(x,x_1) = q$ and thus $U(q,z) = U(x_1,x,z) = U(x_1,x) = q$ for all $z \in [e,1[$. Let

$$b = \inf\{x \in [0,e] \mid U(x, \frac{1+e}{2}) > x\}.$$

Then $U(x,y) = x$ for all $x < b$ and $y \in [e,1[$ and $U(x,y) > x$ for all $x > b$ and $y \in [e,1[$. If b is not an idempotent point then since T_U is continuous and Archimedean there exists b_1, $b < b_1 < e$ such that $U(b_1,b_1) = v < b$. Then for a $y \in [e,1[$ and $U(b_1,y) = w > b_1$ we have $U(b_1,w) = U(b_1,b_1,y) = U(v,y) = v = U(b_1,b_1)$ what is possible only if there is an idempotent point in $[b_1,w]$. However, then

$$b > v = U(b_1,b_1) = U(b_1,w) = b_1 > b$$

what is a contradiction. Thus b is an idempotent point. Since T_U and S_U are Archimedean we have $b \in \{0,e\}$. Thus we get that either $U(x,y) = x$ for all $(x,y) \in]0,e[\times]e,1[$, i.e., $U \in \mathcal{N}_{\min}$, or $U(x,y) = y$ for all $(x,y) \in]0,e[\times]e,1[$, i.e., $U \in \mathcal{N}_{\max}$, or there is $U(x,y) \in \big]x,y\big[$ for all $(x,y) \in]0,e[\times]e,1[$. From now on we will suppose that $U(x,y) \in \big]x,y\big[$ for all $(x,y) \in]0,e[\times]e,1[$.

Take any $(x,y) \in]0,e[\times]e,1[$ and then $U(x,y) = c \in \big]x,y\big[$. Without loss of generality assume $c < e$ (the case when $c > e$ is analogous). Then since $x < c < e$ and T_U is continuous and Archimedean there exists a $x_1 \in]0,e[$ such that $U(c,x_1) = x$. Then

$$c = U(x,y) = U(c,x_1,y)$$

which is a contradiction if $U(x_1,y) \in]e,1[$. Since $U(x_1,y) \leq y$ and $U(x_1,y) \neq e$ we have $x_1 \leq U(x_1,y) = z < e$. Then since $z \in]0,e[$ and T_U is Archimedean the equality $c = U(c,z)$ implies that z is an idempotent point what is a contradiction. Summarising, U is either a representable uninorm or $U \in \mathcal{N}_{\min} \cup \mathcal{N}_{\max}$. $\square$

Corollary 1. *Assume a uninorm $U: [0,1]^2 \longrightarrow [0,1]$, $U \in \mathcal{U} \cap \mathcal{N}$ and let there exist a continuous strictly decreasing function $r: [0,1] \longrightarrow [0,1]$ with $r(0) = 1$, $r(e) = e$ and $r(1) = 0$ such that U is continuous on $[0,1] \setminus \{(x, r(x)) \mid x \in [0,1]\}$. Then if there exist $a \in [0,e[$ and $b \in]e,1]$ such that U is an Archimedean (i.e., nilpotent or strict) t-norm on $[a,e]^2$ and U is an Archimedean (i.e., nilpotent or strict) t-conorm on $[e,b]^2$ then U on $[a,b]^2$ is a representable uninorm.*

Proof. Since U is a t-norm on $[a, e]^2$ and a t-conorm on $[e, b]^2$ then a and b are idempotent points of U and U is closed on $[a, b]^2$, i.e., U on $[a, b]^2$ is isomorphic to a uninorm which we denote by U^*. The previous proposition and Proposition 8 imply that either U^* is a representable uninorm or $U^* \in \mathcal{N}_{\min} \cup \mathcal{N}_{\max}$. However, if $U^* \in \mathcal{N}_{\min}$ then U^* is non-continuous in all points from the set $]0, e[\times \{1\}$, i.e., r is not strictly decreasing what is a contradiction. Similarly, if $U^* \in \mathcal{N}_{\max}$ then U^* is non-continuous in all points from the set $]e, 1[\times \{0\}$. Thus U^* is representable. □

Before we introduce another result we recall the claim of [4, Theorem 5.1]. Here $\mathcal{U}(e) = \{U : [0, 1]^2 \longrightarrow [0, 1] \mid U$ is associative, non-decreasing, with the neutral element $e \in [0, 1]\}$. Thus $U \in \mathcal{U}(e)$ is a uninorm if it is commutative.

Theorem 1. *Let $U \in \mathcal{U}(e)$ and $a, b, c, d \in [0, 1]$, $a \le b \le e \le c \le d$ be such that $U|_{[a,b]^2}$ is associative, non-decreasing, with the neutral element b and $U|_{[c,d]^2}$ is associative, non-decreasing, with the neutral element c. Then the set $([a, b] \cup [c, d])^2$ is closed under U.*

Now we can show the following.

Proposition 11. *Assume a uninorm $U : [0, 1]^2 \longrightarrow [0, 1]$, $U \in \mathcal{U} \cap \mathcal{N}$ and let there exist a continuous strictly decreasing function $r : [0, 1] \longrightarrow [0, 1]$ with $r(0) = 1$, $r(e) = e$ and $r(1) = 0$ such that U is continuous on $[0, 1] \setminus \{(x, r(x)) \mid x \in [0, 1]\}$. Then*

(i) if $a, b \in [0, e]$ are idempotent elements such that $U(x, x) < x$ for all $x \in]a, b[$ then also $c = r(b)$ and $d = r(a)$ are idempotent elements and $U(y, y) > y$ for all $y \in]c, d[$;
(ii) if $c, d \in [e, 1]$ are idempotent elements such that $U(y, y) > y$ for all $y \in]c, d[$ then also $b = r(c)$ and $a = r(d)$ are idempotent elements and $U(x, x) < x$ for all $x \in]a, b[$.

Proof. We will only show the first part, the second part is analogous. Let $a, b \in [0, e]$ be idempotent elements of U such that $U(x, x) < x$ for all $x \in]a, b[$ and let $c = r(b)$ and $d = r(a)$. Let g be the smallest idempotent element of U such that $g \ge d$. Then according to Theorem 1 interval $\left[a, g\right]^2$ is closed under U, i.e., it is a linear transformation of some uninorm U^*, $U^* \in \mathcal{U}$. If $U^* \in \mathcal{N}$ then U is non-continuous in (a, g) which means that $g = d$. If $U^* \notin \mathcal{N}$ then Proposition 8 implies that U^* is an ordinal sum of a uninorm and a non-proper uninorm and since U is non-continuous in (a, d) where $d \le g$ we have $U(a, z) < e$ for $z < d$ and $U(a, z) > e$ for $z > d$, i.e., U^* is an ordinal sum of a uninorm and a t-conorm and d is an idempotent point, i.e., $d = g$. Thus in all cases d is an idempotent element of U.

Further, u_b is non-continuous exactly in the point $x = c$ and since b is idempotent Lemma 3 implies $U(b, x) = \min(x, b)$ for $x < c$ and $U(b, x) = x$ for $x > c$, $U(b, c) \in \{b, c\}$. If $U(b, c) = b$ then also $U(b, c, c) = b$ which implies $U(c, c) \le c$, i.e., c is an idempotent point of U. Assume $U(b, c) = c$. Then for $x \in]e, c[$ we have

$$c = U(b, c) = U(b, x, c) = U(x, c)$$

which means that there is an idempotent point in $[x, c]$. Since S_U is continuous, i.e., the set of all idempotent points is closed we see that c is an idempotent point of U. Thus both c and d are idempotent points.

Assume that $h \in]c, d[$ is an idempotent point. Then similarly as above we can show that $r(h)$ is also an idempotent point of U and $a = r(d) < r(h) < r(c) = b$, i.e., there is an idempotent point between a and b what is a contradiction. □

Definition 4. An internal uninorm $U : [0, 1]^2 \longrightarrow [0, 1]$ will be called *s-internal* if there exists a continuous and strictly decreasing function $v_U : [0, 1] \longrightarrow [0, 1]$ such that $U(x, y) = \min(x, y)$ if $y < v_U(x)$ and $U(x, y) = \max(x, y)$ if $y > v_U(x)$.

Proposition 12. *Assume a uninorm $U : [0, 1]^2 \longrightarrow [0, 1]$, $U \in \mathcal{U} \cap \mathcal{N}$ and let there exist a continuous strictly decreasing function $r : [0, 1] \longrightarrow [0, 1]$ with $r(0) = 1$, $r(e) = e$ and $r(1) = 0$ such that U is continuous on $[0, 1] \setminus \{(x, r(x)) \mid x \in [0, 1]\}$. Then U is an ordinal sum of representable uninorms and s-internal uninorms, i.e., $U = (\langle a_m, b_m, c_m, d_m, U_m\rangle \mid m \in M)$, where $(]a_m, b_m[)_{m\in M}$ and $(]c_m, d_m[)_{m\in M}$ are two anti-comonotone systems of* *disjoint non-empty open intervals such that* $\bigcup_{m\in M}[a_m, b_m] = [0, e]$ *and* $\bigcup_{m\in M}[c_m, d_m] = [e, 1]$*, and* $(U_m)_{m\in m}$ *is a family of (proper) representable uninorms and s-internal uninorms on* $[0, 1]^2$.

Proof. Since T_U and S_U are continuous the set of idempotent elements I_U of U is closed and thus $[0, e] \setminus I_U = \bigcup_{m\in M}]a_m, b_m[$ and $[e, 1] \setminus I_U = \bigcup_{l\in L}]c_l, d_l[$ for some countable index sets M, L and two systems of open non-empty disjoint intervals $(]a_m, b_m[)_{m\in M}$ and $(]c_l, d_l[)_{l\in L}$. From Proposition 11 it follows that each interval $]a_m, b_m[$ can be paired with the interval $]c_l, d_l[$ for some $l \in L$ such that $r(a_m) = d_l$ and $r(b_m) = c_l$ and vice versa, i.e., we can set $L = M$ and obtain two anti-comonotone systems of open non-empty disjoint intervals $(]a_m, b_m[)_{m\in M}$ and $(]c_m, d_m[)_{m\in M}$, where $]a_m, b_m[\subset [0, e]$ and $]c_m, d_m[\subset [e, 1]$ for all $m \in M$. Since a_m, b_m, c_m, d_m are idempotent points, Lemma 3 and monotonicity of U implies that $U(x, y) = y$ if $x \in [b_m, c_m]$ and $y \in [a_m, d_m] \setminus [b_m, c_m]$.

Further, $([a_m, b_m] \cup [c_m, d_m])^2$ is closed under U and U on $[a_m, b_m]^2$ is a continuous Archimedean t-norm and U on $[c_m, d_m]^2$ is a continuous Archimedean t-conorm. In order to use backward transformation inverse to (2) we have only to show that

$$\mathrm{Card}(\mathrm{Ran}(U|_{[a_m,b_m[\cup]c_m,d_m]}) \cap [b_m, c_m]) < 2.$$

Assume $U(x_1, y_1) = q$ for some $x_1 \in [a_m, b_m[$, $y_1 \in]c_m, d_m]$ and $q \in [b_m, c_m]$. Then for any $z \in [b_m, c_m]$ we have $U(q, z) = U(x_1, y_1, z) = U(x_1, y_1) = q$. Thus q is the annihilator of U on $[b_m, c_m]$, i.e., $q = U(b_m, c_m)$. Now if we transform U on $([a_m, b_m[\cup \{U(b_m, c_m)\} \cup]c_m, d_m])^2$ using f^{-1}, where f is given in (2), where $c = a_m$, $a = b_m$, $v = U(b_m, c_m)$, $b = c_m$ and $d = d_m$ and $e \in]0, 1[$ we obtain a uninorm U_m on $[0, 1]^2$ with the neutral element e such that T_{U_m} and S_{U_m} are Archimedean and $U_m \in \mathcal{N} \cap \mathcal{U}$. Then by Proposition 1 the uninorm U_m is representable.

If $\bigcup_{m\in M}[a_m, b_m] = [0, e]$ and $\bigcup_{m\in M}[c_m, d_m] = [e, 1]$ the proof is finished. In the opposite case we have $[0, e] \setminus \bigcup_{m\in M}[a_m, b_m] = \bigcup_{o\in O}]g_o, h_o[$, where $(]g_o, h_o[)_{o\in O}$ is a system of non-empty open intervals, i.e., O is a countable index set. Then we have $[e, 1] \setminus \bigcup_{m\in M}[c_m, d_m] = \bigcup_{o\in O}]r(h_o), r(g_o)[$. The set $(]g_o, h_o[\cup \{U(h_o, r(h_o))\} \cup]r(h_o), r(g_o)[)^2$ is closed under U and thus it is isomorphic to some uninorm U_o such that $T_{U_o} = \min$ and $S_{U_o} = \max$. Thus by Lemma 1 U_o is internal. Moreover, since r is continuous and strictly decreasing there exists a continuous and strictly decreasing function $v_{U_o} : [0, 1] \longrightarrow [0, 1]$ such that $U_o(x, y) = \min(x, y)$ if $y < v_{U_o}(x)$ and $U_o(x, y) = \max(x, y)$ if $y > v_{U_o}(x)$, i.e., U_o is an s-internal uninorm. □

Corollary 2. *A uninorm* $U : [0, 1]^2 \longrightarrow [0, 1]$, $U \in \mathcal{U} \cap \mathcal{N}$ *is a complete ordinal sum of representable and s-internal uninorms if and only if there exists a continuous strictly decreasing function* $r : [0, 1] \longrightarrow [0, 1]$ *with* $r(0) = 1$, $r(e) = e$ *and* $r(1) = 0$ *such that* U *is continuous on* $[0, 1] \setminus \{(x, r(x)) \mid x \in [0, 1]\}$.

Corollary 3. *A uninorm* $U : [0, 1]^2 \longrightarrow [0, 1]$, $U \in \mathcal{U} \cap \mathcal{N}$ *is a complete ordinal sum of representable uninorms if and only if there exists a continuous strictly decreasing function* $r : [0, 1] \longrightarrow [0, 1]$ *with* $r(0) = 1$, $r(e) = e$ *and* $r(1) = 0$ *such that* U *is continuous on* $[0, 1] \setminus \{(x, r(x)) \mid x \in [0, 1]\}$ *and* U *has countably many idempotent points.*

This result follows from the fact that if there are countably many idempotent points then there is no interval of idempotent points, i.e., $\bigcup_{m\in M}[a_m, b_m] = [0, e]$ and $\bigcup_{m\in M}[c_m, d_m] = [e, 1]$. On the other hand, if $\bigcup_{m\in M}[a_m, b_m] = [0, e]$ and $\bigcup_{m\in M}[c_m, d_m] = [e, 1]$ then idempotent points are only a_m, b_m, c_m, d_m for $m \in M$ and since M is countable also the set of idempotent points is countable.

Finally, let us note that if we have a uninorm $U : [0, 1]^2 \longrightarrow [0, 1]$, $U \in \mathcal{U}$ and $U \notin \mathcal{N}$ then according to Proposition 8 the uninorm U is an ordinal sum of a uninorm and a t-norm (t-conorm). This means that u_1 (u_0) is non-continuous in some point $x > 0$ ($x < 1$) which means that there cannot exist a continuous strictly decreasing function $r : [0, 1] \longrightarrow [0, 1]$ with $r(0) = 1$, $r(e) = e$ and $r(1) = 0$ such that U is continuous on $[0, 1] \setminus \{(x, r(x)) \mid x \in [0, 1]\}$.

4. Conclusions

In this paper we have shown that a uninorm is equal to a complete ordinal sum of representable uninorms and s-internal uninorms if and only if there exists a continuous strictly decreasing function $r : [0, 1] \longrightarrow [0, 1]$ with $r(0) = 1$,

$r(e) = e$ and $r(1) = 0$ such that U is continuous on $[0, 1] \setminus \{(x, r(x)) \mid x \in [0, 1]\}$. Moreover, such a uninorm U is a complete ordinal sum of representable uninorms if the set of all idempotent elements of U is countable. We conjecture that a similar result can be shown for all uninorms, where T_U and S_U are continuous. In such a case we conjecture that the set of all points of non-continuity is characterised by a symmetric continuous non-decreasing pseudo-function and each such a uninorm can be decomposed into an ordinal sum of representable uninorms, continuous Archimedean t-norms, t-conorms and internal uninorms. However, any uninorm $U \in \mathcal{N}_{\min}$ such that S_U has no non-trivial elements is irreducible with respect to the ordinal sum construction, i.e., can be expressed only as a trivial ordinal sum with summand on $([0, e[\cup]e, 1])^2$, and thus modification of the ordinal sum construction, such where summands will be defined on $(]a_m, b_m[\cup]c_m, d_m[)^2$ should be assumed in this case. However, we leave this research for the future work.

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Characterization of uninorms with continuous underlying t-norm and t-conorm by means of the ordinal sum construction

Andrea Mesiarová-Zemánková

Mathematical Institute, Slovak Academy of Sciences, Bratislava, Slovakia

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ABSTRACT

The uninorms with continuous underlying t-norm and t-conorm are characterized via the ordinal sum construction of Clifford. Using the previous results of the author, where each uninorm with continuous underlying operations was characterized by properties of its set of discontinuity points, it is shown that each such a uninorm can be decomposed into an ordinal sum of a countable number of semigroups related to representable uninorms, continuous Archimedean t-norms, continuous Archimedean t-conorms and internal uninorms (including the min and the max operator).

1. Introduction

The (left-continuous) t-norms and their dual t-conorms play an indispensable role in many domains [10,32,33]. Each continuous t-norm (t-conorm) can be expressed as an ordinal sum of continuous Archimedean t-norms (t-conorms), while each Archimedean t-norm (t-conorm) is generated by an additive generator (see [1,12]). Generalizations of t-norms and t-conorms that can model bipolar behavior are uninorms (see [9,22,34]). The class of uninorms is widely used both in theory [5,7,19,29] and in applications [28,35]. The complete characterization of uninorms with continuous underlying t-norm and t-conorm has been in the center of the interest for a long time, however, only partial results were achieved (see [6,8,11,18, 15,20,31]).

Ordinal sum of uninorms was introduced in [23], where also the most general operations yielding a uninorm via the ordinal sum construction were studied (see also [25]). This paper is a continuation of the paper [26], where we have characterized uninorms with continuous underlying operations by properties of their set of discontinuity points. Our aim is to completely characterize all uninorms with continuous underlying functions and obtain a similar representation as in the case of t-norms and t-conorms. In this paper we will therefore show that each uninorm with continuous underlying t-norm and t-conorm can be decomposed into an ordinal sum of semigroups related to representable uninorms, continuous Archimedean t-norms, continuous Archimedean t-conorms and internal uninorms.

The paper is structured as follows. In Section 2 we will recall all necessary notions and results. We will recall the ordinal sum construction of Clifford (Section 3) and show several examples of basic uninorms that are constructed using this construction. In Section 4 we will recall some results on the characterizing set-valued function of a uninorm with

E-mail address: zemankova@mat.savba.sk.

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continuous underlying functions and add several new. Section 5 then contains the main result of the paper. We give our conclusions in Section 6.

2. Basic notions and results

Let us now recall all necessary basic notions.

A triangular norm is a binary function $T\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and 1 is its neutral element. Due to the associativity, n-ary form of any t-norm is uniquely given and thus it can be extended to an aggregation function working on $\bigcup_{n\in\mathbb{N}}[0,1]^n$. Dual functions to t-norms are t-conorms. A triangular conorm is a binary function $S\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and 0 is its neutral element. The duality between t-norms and t-conorms is expressed by the fact that from any t-norm T we can obtain its dual t-conorm S by the equation

$$S(x,y) = 1 - T(1-x, 1-y)$$

and vice-versa.

Now let us recall an ordinal sum construction for t-norms and t-conorms [12].

Proposition 1. *Let K be a finite or countably infinite index set and let $(]a_k, b_k[)_{k\in K}$ ($(]c_k, d_k[)_{k\in K}$) be a system of open, disjoint subintervals of $[0,1]$. Let $(T_k)_{k\in K}$ ($(S_k)_{k\in K}$) be a system of t-norms (t-conorms). Then the ordinal sum $T = (\langle a_k, b_k, T_k\rangle \mid k\in K)$ ($S = (\langle a_k, b_k, S_k\rangle \mid k \in K)$) given by*

$$T(x,y) = \begin{cases} a_k + (b_k - a_k)T_k(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}) & \text{if } (x,y) \in [a_k, b_k[^2, \\ \min(x,y) & \text{else} \end{cases}$$

and

$$S(x,y) = \begin{cases} c_k + (d_k - c_k)S_k(\frac{x-c_k}{d_k-c_k}, \frac{y-c_k}{d_k-c_k}) & \text{if } (x,y) \in]c_k, d_k]^2, \\ \max(x,y) & \text{else} \end{cases}$$

is a t-norm (t-conorm). The t-norm T (t-conorm S) is continuous if and only if all summands T_k (S_k) for $k \in K$ are continuous.

Each continuous t-norm (t-conorm) is equal to an ordinal sum of continuous Archimedean t-norms (t-conorms). Note that a continuous t-norm (t-conorm) is Archimedean if and only if it has only trivial idempotent points 0 and 1. A continuous Archimedean t-norm T (t-conorm S) is either strict, i.e., strictly increasing on $]0,1]^2$ (on $[0,1[^2$), or nilpotent, i.e., there exists $(x,y) \in]0,1[^2$ such that $T(x,y) = 0$ ($S(x,y) = 1$). Moreover, each continuous Archimedean t-norm (t-conorm) has a continuous additive generator.

Proposition 2. *Let $t\colon [0,1] \longrightarrow [0,\infty]$ ($s\colon [0,1] \longrightarrow [0,\infty]$) be a continuous strictly decreasing (increasing) function such that $t(1) = 0$ ($s(0) = 0$). Then the binary operation $T\colon [0,1]^2 \longrightarrow [0,1]$ ($S\colon [0,1]^2 \longrightarrow [0,1]$) given by*

$$T(x,y) = t^{-1}(\min(t(0), t(x) + t(y)))$$
$$S(x,y) = s^{-1}(\min(s(1), s(x) + s(y)))$$

is a continuous Archimedean t-norm (t-conorm). The function t (s) is called an additive generator *of T (S).*

An additive generator of a continuous t-norm T (t-conorm S) is uniquely determined up to a positive multiplicative constant. More details on t-norms and t-conorms can be found in [1,12].

A uninorm (introduced in [34]) is a binary function $U\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and have a neutral element $e \in]0,1[$ (see also [9]). If we take uninorm in a broader sense, i.e., if for a neutral element we have $e \in [0,1]$, then the class of uninorms covers also the class of t-norms and the class of t-conorms. In order the stress that we assume a uninorm with $e \in]0,1[$ we will call such a uninorm *proper*. For each uninorm the value $U(1,0) \in \{0,1\}$ is the annihilator of U. A uninorm is called *conjunctive* (*disjunctive*) if $U(1,0) = 0$ ($U(1,0) = 1$). Due to the associativity we can uniquely define n-ary form of any uninorm for any $n \in \mathbb{N}$ and therefore in some proofs we will use ternary form instead of binary, where suitable.

For each uninorm U with the neutral element $e \in]0,1[$, the restriction of U to $[0,e]^2$ is a t-norm on $[0,e]^2$, i.e., a linear transformation of some t-norm T_U on $[0,1]^2$ and the restriction of U to $[e,1]^2$ is a t-conorm on $[e,1]^2$, i.e., a linear transformation of some t-conorm S_U. Moreover, $\min(x,y) \le U(x,y) \le \max(x,y)$ for all $(x,y) \in [0,e] \times [e,1] \cup [e,1] \times [0,e]$.

From any pair of a t-norm and a t-conorm we can construct the minimal and the maximal uninorm with the given underlying functions (see [17]).

Proposition 3. *Let $T\colon [0,1]^2 \longrightarrow [0,1]$ be a t-norm and $S\colon [0,1]^2 \longrightarrow [0,1]$ a t-conorm and assume $e \in [0,1]$. Then the two functions $U_{\min}, U_{\max}\colon [0,1]^2 \longrightarrow [0,1]$ given by*

$$U_{\min}(x,y) = \begin{cases} e \cdot T(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)\cdot S(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ \min(x,y) & \text{otherwise} \end{cases}$$

and

$$U_{\max}(x,y) = \begin{cases} e \cdot T(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)\cdot S(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ \max(x,y) & \text{otherwise} \end{cases}$$

are uninorms. We will denote the set of all uninorms of the first type by $\mathcal{U}_{\min}$ and of the second type by $\mathcal{U}_{\max}$.

Definition 1. A uninorm $U\colon [0,1]^2 \longrightarrow [0,1]$ is called *internal* if $U(x,y) \in \{x,y\}$ for all $(x,y) \in [0,1]^2$. Moreover, U is called d-internal if it is internal and there exists a continuous and strictly decreasing function $g_U\colon [0,1] \longrightarrow [0,1]$ such that $U(x,y) = \min(x,y)$ if $y < g_U(x)$ and $U(x,y) = \max(x,y)$ if $y > g_U(x)$. Finally, U is called locally internal on $A(e)$ if U is internal on $A(e) = [0,e] \times [e,1] \cup [e,1] \times [0,e]$.

For example all uninorms from $\mathcal{U}_{\min} \cup \mathcal{U}_{\max}$ are locally internal on $A(e)$. More results on internal and locally internal uninorms can be found in [2,4,8,21,30].

Similarly as in the case of t-norms and t-conorms we can construct uninorms using additive generators (see [9]).

Proposition 4. *Let $f\colon [0,1] \longrightarrow [-\infty,\infty]$, $f(0) = -\infty$, $f(1) = \infty$ be a continuous strictly increasing function. Then the binary function $U\colon [0,1]^2 \longrightarrow [0,1]$ given by*

$$U(x,y) = f^{-1}(f(x) + f(y)),$$

where $f^{-1}\colon [-\infty,\infty] \longrightarrow [0,1]$ is an inverse function to f, is a uninorm, which will be called a representable *uninorm. The unique point $e \in\,]0,1[$ such that $f(e) = 0$ is then the neutral point of U.*

Note that if we relax the monotonicity of the additive generator then the neutral element will be lost and by relaxing the condition $f(0) = -\infty$, $f(1) = \infty$ the associativity will be lost (if $f(0) < 0$ and $f(1) > 0$). In [29] (see also [22]) we can find the following result.

Proposition 5. *Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm. Then U is representable if and only if it is continuous on $[0,1]^2 \setminus \{(0,1),(1,0)\}$.*

This result completely characterizes the set of representable uninorms.

Definition 2. We will denote the set of all uninorms U such that T_U and S_U are continuous by $\mathcal{U}$. Further, for a given uninorm $U\colon [0,1]^2 \longrightarrow [0,1]$ and each $x \in [0,1]$ we define a function $u_x\colon [0,1] \longrightarrow [0,1]$ by $u_x(z) = U(x,z)$ for $z \in [0,1]$.

3. Ordinal sum construction of Clifford

Our aim in this paper is to decompose each uninorm $U \in \mathcal{U}$ using the ordinal sum construction. Therefore we have to first recall the fundamental result of Clifford [3].

Theorem 1. *Let $A \neq \emptyset$ be a totally ordered set and $(G_\alpha)_{\alpha\in A}$ with $G_\alpha = (X_\alpha, *_\alpha)$ be a family of semigroups. Assume that for all $\alpha, \beta \in A$ with $\alpha < \beta$ the sets X_α and X_β are either disjoint or that $X_\alpha \cap X_\beta = \{x_{\alpha,\beta}\}$, where $x_{\alpha,\beta}$ is both the neutral element of G_α and the annihilator of G_β and where for each $\gamma \in A$ with $\alpha < \gamma < \beta$ we have $X_\gamma = \{x_{\alpha,\beta}\}$. Put $X = \bigcup\limits_{\alpha\in A} X_\alpha$ and define the binary operation $*$ on X by*

$$x * y = \begin{cases} x *_\alpha y & \text{if } (x,y) \in X_\alpha \times X_\alpha, \\ x & \text{if } (x,y) \in X_\alpha \times X_\beta \text{ and } \alpha < \beta, \\ y & \text{if } (x,y) \in X_\alpha \times X_\beta \text{ and } \alpha > \beta. \end{cases}$$

Then $G = (X,*)$ is a semigroup. The semigroup G is commutative if and only if for each $\alpha \in A$ the semigroup G_α is commutative.

Note that ordinal sum construction of t-norms, t-conorms and uninorms are all based on this result (see [13,14,23]). Next we will recall several results on uninorms with continuous Archimedean underlying operations and show how are these related to the ordinal sum construction.

For uninorms with continuous nilpotent underlying operations the following result was shown in [15].

Theorem 2 *([15]). Let $U\colon [0,1] \longrightarrow [0,1]^2$ be a uninorm with the neutral element $e \in\,]0,1[$ such that both T_U and S_U are nilpotent. Then either one of the following two statements holds:*

(i) $U \in \mathcal{U}_{\min}$,
(ii) $U \in \mathcal{U}_{\max}$.

Example 1. If $([0,1],U)$ is an ordinal sum of semigroups $\{G_\alpha\}_{\alpha\in A}$ with $G_\alpha = (X_\alpha, *_\alpha)$ for $\alpha \in A$ then T_U is an ordinal sum of semigroups $\{G^1_\alpha\}_{\alpha\in A}$ with $G^1_\alpha = (X_\alpha \cap [0,e], *_\alpha)$ for $\alpha \in A$ and S_U is an ordinal sum of semigroups $\{G^2_\alpha\}_{\alpha\in A}$ with $G^2_\alpha = (X_\alpha \cap [e,1], *_\alpha)$ for $\alpha \in A$. From [13] we know that then each $X_\alpha \cap [0,e]$ ($X_\alpha \cap [e,1]$) is a subinterval of $[0,e]$ ($[e,1]$). If T_U (S_U) are nilpotent then each support of a subsemigroup of $([0,1],U)$ contains at least one point from the set $\{0,e,1\}$. Therefore as the respective sets in the ordinal sum should cover the whole interval $[0,1]$, the finest partition which we can make is to divide $[0,1]$ into $[0,e[$, $\{e\}$ and $]e,1]$. Thus we have three semigroups $G_{a_1} = ([0,e[\,,U)$, $G_{a_2} = (\{e\},U)$ and $G_{a_3} = (]e,1]\,,U)$. Let $\preceq$ be an order on the set $A = \{a_1,a_2,a_3\}$. Since e is the neutral element it is obvious that $a_1 \prec a_2$ and $a_3 \prec a_2$. Further, if $U \in \mathcal{U}_{\min}$ then $a_1 \prec a_3$ and if $U \in \mathcal{U}_{\max}$ then $a_3 \prec a_1$. It is easy to verify that $U \in \mathcal{U}_{\min}$ is an ordinal sum of G_{a_1}, G_{a_2} and G_{a_3} with the order on the set A given by $a_1 \prec a_3 \prec a_2$ and $U \in \mathcal{U}_{\max}$ is an ordinal sum of G_{a_1}, G_{a_2} and G_{a_3} with the order on the set A given by $a_3 \prec a_1 \prec a_2$.

For uninorms with continuous strict underlying operations the following result was shown in [15] (see also [11]).

Theorem 3 *([15]). Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm with the neutral element $e \in\,]0,1[$ such that both T_U and S_U are strict. Then one of the following seven statements holds:*

(i) $U \in \mathcal{U}_{\min}$,
(ii)

$$U(x,y) = \begin{cases} e\cdot T_U(\frac{x}{e},\frac{y}{e}) & \text{if } (x,y)\in[0,e]^2,\\ e+(1-e)\cdot S_U(\frac{x-e}{1-e},\frac{y-e}{1-e}) & \text{if } (x,y)\in[e,1]^2,\\ 1 & \text{if } x=1 \text{ or } y=1,\\ \min(x,y) & \text{otherwise,}\end{cases}$$

(iii)

$$U(x,y) = \begin{cases} e\cdot T_U(\frac{x}{e},\frac{y}{e}) & \text{if } (x,y)\in[0,e]^2,\\ e+(1-e)\cdot S_U(\frac{x-e}{1-e},\frac{y-e}{1-e}) & \text{if } (x,y)\in[e,1]^2,\\ 1 & \text{if } x=1,\ y>0 \text{ or } y=1,\ x>0,\\ \min(x,y) & \text{otherwise,}\end{cases}$$

(iv) $U \in \mathcal{U}_{\max}$,
(v)

$$U(x,y) = \begin{cases} e\cdot T_U(\frac{x}{e},\frac{y}{e}) & \text{if } (x,y)\in[0,e]^2,\\ e+(1-e)\cdot S_U(\frac{x-e}{1-e},\frac{y-e}{1-e}) & \text{if } (x,y)\in[e,1]^2,\\ 0 & \text{if } x=0 \text{ or } y=0,\\ \max(x,y) & \text{otherwise,}\end{cases}$$

(vi)

$$U(x,y) = \begin{cases} e\cdot T_U(\frac{x}{e},\frac{y}{e}) & \text{if } (x,y)\in[0,e]^2,\\ e+(1-e)\cdot S_U(\frac{x-e}{1-e},\frac{y-e}{1-e}) & \text{if } (x,y)\in[e,1]^2,\\ 0 & \text{if } x=0,\ y<1 \text{ or } y=0,\ x<1,\\ \max(x,y) & \text{otherwise,}\end{cases}$$

(vii) *U is representable.*

Example 2. In the case when for a uninorm U operations T_U and S_U are strict we have two possibilities:

Case 1. There exist $x, y \in [0,1]$, $x < e < y$, such that $U(x,y) = e$. From [11] we know that then U is representable. In such a case for all $x \in]0,1[$ there exists a $y \in]0,1[$ such that $U(x,y) = e$. Then the finest possible partition that we can get is to divide $[0,1]$ into $\{0\}$, $]0,1[$, and $\{1\}$. Thus we have three semigroups $G_{a_1} = (\{0\}, U)$, $G_{a_2} = (]0,1[\,, U)$, and $G_{a_3} = (\{1\}, U)$. Let $\preceq$ be an order on the set $A = \{a_1, a_2, a_3\}$. Then the monotonicity implies $a_1 \prec a_2$ and $a_3 \prec a_2$. Therefore we have two possible orders on the set A: either $a_1 \prec a_3 \prec a_2$, which corresponds to a conjunctive representable uninorm, or $a_3 \prec a_1 \prec a_2$, which corresponds to a disjunctive representable uninorm. Thus in both cases, i.e., whether U is conjunctive or disjunctive, it is easy to see that U is equal to an ordinal sum of G_{a_1}, G_{a_2} and G_{a_3}.

Case 2. For all $(x,y) \in [0,1]^2$ the equality $U(x,y) = e$ implies $x = y = e$. Similarly as above we can show that then the finest partition which we can make is to divide $[0,1]$ into $\{0\}$, $]0,e[$, $\{e\}$, $]e,1[$ and $\{1\}$. Thus we have five semigroups $G_{a_1} = (\{0\}, U)$, $G_{a_2} = (]0,e[\,, U)$, $G_{a_3} = (\{e\}, U)$, $G_{a_4} = (]e,1[\,, U)$ and $G_{a_5} = (\{1\}, U)$. Let $\preceq$ be an order on the set $A = \{a_1, a_2, a_3, a_4, a_5\}$. Since e is the neutral element we have $a_i \prec a_3$ for $i = 1, 2, 4, 5$. Further, the monotonicity implies $a_1 \prec a_2$ and $a_5 \prec a_4$. Then we have the following six possible orders on the set A:

(i) $a_1 \prec a_2 \prec a_5 \prec a_4 \prec a_3$,
(ii) $a_1 \prec a_5 \prec a_2 \prec a_4 \prec a_3$,
(iii) $a_1 \prec a_5 \prec a_4 \prec a_2 \prec a_3$,
(iv) $a_5 \prec a_1 \prec a_2 \prec a_4 \prec a_3$,
(v) $a_5 \prec a_1 \prec a_4 \prec a_2 \prec a_3$,
(vi) $a_5 \prec a_4 \prec a_1 \prec a_2 \prec a_3$.

Again it is easy to see that an ordinal sum of G_{a_1}, G_{a_2}, G_{a_3}, G_{a_4} and G_{a_5} with the first order corresponds to the form (i) from Theorem 3, the second to the form (iii), the third to the form (v), the fourth to the form (ii), the fifth to the form (vi) and the last to the form (iv).

Similarly as Theorems 2 and 3 we have the following.

Theorem 4 *([16]). Let $U\colon [0,1] \longrightarrow [0,1]^2$ be a uninorm with the neutral element $e \in]0,1[$ such that T_U is strict and S_U is nilpotent. Then either one of the following three statements holds:*

(i) $U \in \mathcal{U}_{\min}$,
(i) $U \in \mathcal{U}_{\max}$,
(iii)

$$U(x,y) = \begin{cases} e \cdot T_U(\frac{x}{e}, \frac{y}{e}) & \textit{if } (x,y) \in [0,e]^2, \\ e + (1-e) \cdot S_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \textit{if } (x,y) \in [e,1]^2, \\ 0 & \textit{if } x = 0 \textit{ or } y = 0, \\ \max(x,y) & \textit{otherwise.} \end{cases}$$

In this case the corresponding semigroups act on $\{0\}$, $]0,e[$, $\{e\}$, $]e,1]$.

Theorem 5 *([16]). Let $U\colon [0,1] \longrightarrow [0,1]^2$ be a uninorm with the neutral element $e \in]0,1[$ such that T_U is nilpotent and S_U is strict. Then either one of the following three statements holds:*

(i) $U \in \mathcal{U}_{\min}$,
(ii) $U \in \mathcal{U}_{\max}$,
(iii)

$$U(x,y) = \begin{cases} e \cdot T_U(\frac{x}{e}, \frac{y}{e}) & \textit{if } (x,y) \in [0,e]^2, \\ e + (1-e) \cdot S_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \textit{if } (x,y) \in [e,1]^2, \\ 1 & \textit{if } x = 1 \textit{ or } y = 1, \\ \min(x,y) & \textit{otherwise.} \end{cases}$$

In this case the corresponding semigroups act on $[0,e[$, $\{e\}$, $]e,1[$, $\{1\}$.

In the previous we can see several examples of semigroups that can be used for construction of uninorms via the ordinal sum. In order to characterize them we will use the following transformation.

For any $0 \le a \le b < c \le d \le 1$, $v \in [b, c]$, with $[a, b[\cup]c, d] \neq \emptyset$ and a uninorm U with the neutral element $e \in [0, 1]$ we will use a transformation $f : [0, 1] \longrightarrow [a, b[\cup \{v\} \cup]c, d]$, where if $a = b$ then $e = 0$ and $v = c$, if $c = d$ then $e = 1$ and $v = b$, given by

$$f(x) = \begin{cases} (b-a) \cdot \frac{x}{e} + a & \text{if } x \in [0, e[, \\ v & \text{if } x = e, \\ d - \frac{(1-x)(d-c)}{(1-e)} & \text{otherwise.} \end{cases} \tag{1}$$

Then f is linear on $[0, e[$ and on $]e, 1]$ and thus it is a piece-wise linear isomorphism of $[0, 1]$ to $([a, b[\cup \{v\} \cup]c, d])$ and the binary function $U_v^{a,b,c,d} : ([a, b[\cup \{v\} \cup]c, d])^2 \longrightarrow ([a, b[\cup \{v\} \cup]c, d])$ given by

$$U_v^{a,b,c,d}(x, y) = f(U(f^{-1}(x), f^{-1}(y))) \tag{2}$$

is a uninorm on $([a, b[\cup \{v\} \cup]c, d])^2$. Backward transformation f^{-1} can then transform a uninorm on $([a, b[\cup \{v\} \cup]c, d])^2$ to a uninorm on $[0, 1]^2$.

Now we are able to give the following definition.

Definition 3. Let $a, b, c, d \in [0, 1]$ with $a < b < c < d$. Then

(i) a semigroup $(]a, b[\cup \{v\} \cup]c, d[, *)$ will be called a *representable* semigroup if $*$ is isomorphic via (2) to a restriction of a representable uninorm on $[0, 1]^2$ to $]0, 1[^2$,
(ii) a semigroup $(]a, b[, *)$ will be called a *t-strict* semigroup if $*$ is linearly isomorphic to a restriction of a strict t-norm on $[0, 1]^2$ to $]0, 1[^2$,
(iii) a semigroup $(]c, d[, *)$ will be called an *s-strict* semigroup if $*$ is linearly isomorphic to a restriction of a strict t-conorm on $[0, 1]^2$ to $]0, 1[^2$,
(iv) a semigroup $([a, b[, *)$ will be called a *t-nilpotent* semigroup if $*$ is linearly isomorphic to a restriction of a nilpotent t-norm on $[0, 1]^2$ to $[0, 1[^2$,
(v) a semigroup $(]c, d], *)$ will be called an *s-nilpotent* semigroup if $*$ is linearly isomorphic to a restriction of a nilpotent t-conorm on $[0, 1]^2$ to $]0, 1]^2$,
(vi) a semigroup $(]a, b[\cup]c, d[, *)$ will be called a *d-internal* semigroup if $*$ is isomorphic via (2) to a restriction of an d-internal uninorm on $[0, 1]^2$ to $(]0, 1[\setminus \{e\})^2$,
(vii) a semigroup $(]a, b[, *)$ will be called a *t-internal* semigroup if $*$ is linearly isomorphic to the min on $]0, 1[^2$,
(viii) a semigroup $(]c, d[, *)$ will be called an *s-internal* semigroup if $*$ is linearly isomorphic to the max on $]0, 1[^2$.

Results on related operations can be found in the following literature: for strict and nilpotent t-norms see [12], for representable uninorms see [9,11], for internal uninorms see [2,8,21,30].

For semigroups that are defined on singletons we will further use an operation

$$\mathrm{Id} : \{x\}^2 \longrightarrow \{x\} \text{ given by } \mathrm{Id}(x, x) = x.$$

Proposition 6 *([24]). Let $U : [0, 1]^2 \longrightarrow [0, 1]$ be an internal uninorm. Then $([0, 1], U)$ is an ordinal sum of singleton semigroups $(\{x\}, \mathrm{Id})$ for all $x \in [0, 1]$.*

Proposition 7 *([24]). Let P be an index set isomorphic with $[0, 1]$ via the isomorphism i. For all $p \in P$ we put $X_p = \{x\}$ if $i(p) = x$. Let $e \in [0, 1]$ and let $\preceq$ be a linear order on P. Then the ordinal sum of $\{(X_p, \mathrm{Id})\}_{p \in P}$ with the linear order $\preceq$ is an internal uninorm with the neutral element e if and only if the following two conditions are fulfilled:*

(i) $p_1 \prec p_2$ for all $p_1, p_2 \in P$ such that $X_{p_1} = \{x_1\}$, $X_{p_2} = \{x_2\}$, $x_1 < x_2$ and $x_1, x_2 \in [0, e]$,
(ii) $p_1 \prec p_2$ for all $p_1, p_2 \in P$ such that $X_{p_1} = \{y_1\}$, $X_{p_2} = \{y_2\}$, $y_1 > y_2$ and $y_1, y_2 \in [e, 1]$.

Remark 1. In the rest of the paper we will show that each uninorm $U \in \mathcal{U}$ can be decomposed into ordinal sum of the nine types of semigroups, eight from Definition 3 plus semigroups defined on singletons. Due to Proposition 6 we see that internal semigroups can be decomposed further to singletons, however, our aim is to perform such a decomposition where the number of summands is countable. Therefore we include also internal semigroups. It is clear that if $([0, 1], U)$ is an ordinal sum of the above mentioned semigroups then from each of the eight types of semigroups from Definition 3 we can have only a countable number (since each subinterval of $[0, 1]$ contains some rational number). Thus in our decomposition a singleton semigroup will be always between two non-singleton semigroups (or it will be an accumulation point of the set of their end points) which will ensure that the number of summands is countable.

Further we recall several useful results. The first is the result of [6, Theorem 5.1]. Here $\mathcal{U}(e) = \{U : [0, 1]^2 \longrightarrow [0, 1] \mid U$ is associative, non-decreasing, with the neutral element $e \in [0, 1]\}$. Thus $U \in \mathcal{U}(e)$ is a uninorm if it is commutative.

Theorem 6. *Let $U \in \mathcal{U}(e)$ and $a, b, c, d \in [0, 1]$, $a \leq b \leq e \leq c \leq d$ be such that $U|_{[a,b]^2}$ is associative, non-decreasing, with the neutral element b and $U|_{[c,d]^2}$ is associative, non-decreasing, with the neutral element c. Then the set $([a, b] \cup [c, d])^2$ is closed under U.*

Moreover, we have the following result.

Lemma 1 *([26]). Let $U \in \mathcal{U}$. Then if $a \in [0, 1]$ is an idempotent element of U then $U(a, x) \in \{a, x\}$ for all $x \in [0, 1]$, i.e., U is internal on $\{a\} \times [0, 1]$.*

Using these results we can show the following.

Proposition 8. *Let $U \in \mathcal{U}$. If $a, b \in [0, e]$ and $c, d \in [e, 1]$ are idempotent elements, $a \leq b$ and $c \leq d$, then $([a, b[\cup \{U(b, c)\} \cup]c, d])^2$ is closed under U.*

Proof. Since $U \in \mathcal{U}$ we know that b is the neutral element of U on $[a, b]^2$ and c is the neutral element of U on $[c, d]^2$. Thus by Theorem 6 we know that the set $([a, b] \cup [c, d])^2$ is closed under U. Since b and c are idempotent points by Lemma 1 we have $U(b, c) \in \{b, c\}$. If $b = c = e$ then the claim evidently holds. Suppose $b \neq c$ and $U(b, c) = b$ (the case when $U(b, c) = c$ is analogous). Assume that there are $x, y \in ([a, b] \cup]c, d])^2$ such that $U(x, y) = c$. If $x \in [a, b]$ (similarly if $y \in [a, b]$) then

$$b = U(c, b) = U(U(y, x), b) = U(y, U(x, b)) = U(y, x) = c$$

what is a contradiction. Thus both $x, y \in]c, d]$. Then, however,

$$c = U(x, y) \geq \max(x, y) > c$$

what is again a contradiction. Thus $([a, b[\cup \{U(b, c)\} \cup]c, d])^2$ is closed under U. □

Lemma 2. *Let $U \in \mathcal{U}$ and let $([a, b[\cup \{v\} \cup]c, d])^2$ for $a < b \leq v \leq c < d$, with $a, b, c, d, v \in [0, 1]$ be closed under U. Then U restricted to $([a, b[\cup \{v\} \cup]c, d])^2$ is isomorphic with a uninorm U^* on $[0, 1]^2$, via the backward transformation f^{-1} to the transformation f given in (1), if and only if v is the neutral element of U restricted to $([a, b[\cup \{v\} \cup]c, d])^2$.*

Proof. Since f^{-1} is an isomorphism U^* is commutative, associative and non-decreasing in each variable. Further, e is the neutral element of U^* if and only if v is the neutral element of U restricted to $([a, b[\cup \{v\} \cup]c, d])^2$. □

Lemma 3. *Let $U \in \mathcal{U}$ and let $a, b \in [0, e]$ and $c, d \in [e, 1]$ be idempotent elements, $a < b$ and $c < d$, such that there is no idempotent in $]a, b[$ neither in $]c, d[$. If there exist $(x_1, y_1), (x_2, y_2) \in]a, b[\times]c, d[$ such that $U(x_1, y_1) < e$ and $U(x_2, y_2) > e$ then $U(b, c)$ is the neutral element of U restricted to $([a, b[\cup \{U(b, c)\} \cup]c, d])^2$.*

Proof. Since b, c are idempotents $U(b, c) \in \{b, c\}$. Assume $U(b, c) = b$ (the proof for $U(b, c) = c$ is analogous). Then $U(b, x) = x$ for all $x \in [a, b]$. Further, since b is idempotent $U(b, y) \in \{b, y\}$ for all $y \in]c, d]$. If $U(b, y) = b$ for some $y \in]c, d]$ then monotonicity implies $U(b, z) = b$ for all $z < y$ and $b = U(b, \underbrace{y, \ldots, y}_{n\text{-times}})$, i.e., since U is Archimedean on $[c, d]$ we get $U(b, z) = b$ for all $z \in]c, d[$. Now since $U(x_2, y_2) > e$ we have $b = U(b, y_2) \geq U(x_2, y_2) > e \geq b$ what is a contradiction. Thus $U(b, y) = y$ for all $y \in]c, d]$. □

Lemma 4. *Let $U \in \mathcal{U}$ and let $a, b \in [0, e]$ and $c, d \in [e, 1]$ be idempotent elements, $a < b$ and $c < d$, such that there is no idempotent in $]a, b[$ neither in $]c, d[$. If $([a, b[\cup]c, d])^2$ is not closed under U then $U(b, c)$ is the neutral element of U restricted to $([a, b[\cup \{U(b, c)\} \cup]c, d])^2$.*

Proof. Similarly as in the previous proof we have $U(b, c) \in \{b, c\}$ since b, c are idempotents. Assume $U(b, c) = b$ (the proof for $U(b, c) = c$ is analogous). Then $U(b, x) = x$ for all $x \in [a, b]$. Further, since b is idempotent $U(b, y) \in \{b, y\}$ for all $y \in]c, d]$. If $([a, b[\cup]c, d])^2$ is not closed under U then there exist $x_1 \in [a, b[$, $y_1 \in]c, d]$ such that $U(x_1, y_1) = b$. If $U(b, y) = b$ for some $y \in]c, d]$ then $b = U(b, \underbrace{y, \ldots, y}_{n\text{-times}})$, i.e., $U(b, q) = b$ for all $q \in]c, d]$. Then, however,

$$b = U(b, b) = U(x_1, U(y_1, b)) = U(x_1, b) = x_1,$$

what is a contradiction. Therefore $U(b, q) = q$ for all $q \in]c, d]$. Summarizing, $U(b, c)$ is the neutral element of U restricted to $([a, b[\cup \{U(b, c)\} \cup]c, d])^2$. □

Remark 2. A uninorm U from Lemma 4 (as well as from Lemma 3) is on $([a, b[\cup \{U(b, c)\} \cup]c, d])^2$ isomorphic to a representable uninorm.

Before we will show the main result we should recall the characterizing set-valued function of a uninorm $U \in \mathcal{U}$ which will help us to divide U into respective semigroups.

4. Characterizing set-valued function

In this section we will recall several results from [26,27].

Definition 4. A mapping $p\colon [0,1] \longrightarrow \mathcal{P}([0,1])$ is called a set-valued function on $[0,1]$ if to every $x \in [0,1]$ it assigns a subset of $[0,1]$, i.e., $p(x) \subseteq [0,1]$. Assuming the standard order on $[0,1]$, a set-valued function p is called

(i) *non-increasing* if for all $x_1, x_2 \in [0,1]$, $x_1 < x_2$, we have $y_1 \geq y_2$ for all $y_1 \in p(x_1)$ and all $y_2 \in p(x_2)$ and thus the cardinality $\mathrm{Card}(p(x_1) \cap p(x_2)) \leq 1$,
(ii) *symmetric* if $y \in p(x)$ if and only if $x \in p(y)$.

The graph of a set-valued function p will be denoted by $G(p)$, i.e., $(x, y) \in G(p)$ if and only if $y \in p(x)$.

Definition 5. A set-valued function $p\colon [0,1] \longrightarrow \mathcal{P}([0,1])$ is called *u-surjective* if for all $y \in [0,1]$ there exists an $x \in [0,1]$ such that $y \in p(x)$,

Lemma 5 *([26]). A symmetric set-valued function $p\colon [0,1] \longrightarrow \mathcal{P}([0,1])$ is u-surjective if and only if we have $p(x) \neq \emptyset$ for all $x \in [0,1]$.*

The graph of a symmetric, u-surjective, non-increasing set-valued function $p\colon [0,1] \longrightarrow \mathcal{P}([0,1])$ is a connected line (i.e., a connected set with no interior) containing points $(0,1)$ and $(1,0)$ (see [27]).

We will denote the set of all uninorms $U\colon [0,1]^2 \longrightarrow [0,1]$ such that U is continuous on $[0,1]^2 \setminus R$, where $R = G(r)$ and r is a symmetric, u-surjective, non-increasing set-valued function such that $U(x,y) = e$ implies $(x,y) \in R$, by $\mathcal{UR}$. The function r will be called the characterizing set-valued function of U.

Theorem 7 *([26]). Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm. Then $U \in \mathcal{U}$ if and only if $U \in \mathcal{UR}$ and in each point $(x,y) \in [0,1]^2$ the uninorm U is either left-continuous or right-continuous.*

Remark 3. Note that although for $U \in \mathcal{U}$ the previous theorem implies that U is continuous on $[0,1]^2 \setminus R$ for $R = G(r)$ it does not mean that all points of R are points of discontinuity of U. In fact, in [26] it was shown that for a uninorm $U \in \mathcal{U}$ either $U(x,y) = e$ implies $x = y = e$ or there exists a non-empty interval $]a,d[$ such that $U(x,y) = e$ if and only if $x, y \in]a,d[$. In the later case U is continuous in all points from $[0,1]^2 \setminus ([0,a] \cup [d,1])^2$. Moreover, for a conjunctive uninorm $U \in \mathcal{U}$ (similarly for a disjunctive uninorm $U \in \mathcal{U}$) we have either $U(x,1) = 1$ for all $x > 0$ or $U(x,1) < e$ for some $0 < x < e$. In the later case U is continuous in all points from $[0,x[\times [0,1] \cup [0,1] \times [0,x[$.

Definition 6. Let $U \in \mathcal{U}$ and let $r\colon [0,1] \longrightarrow \mathcal{P}([0,1])$ be its characterizing set-valued function. Then

(i) the set $I \subset [0,1]$ is called a maximal horizontal segment of r if $\mathrm{Card}(I) > 1$ and there exists a $y \in [0,1]$ such that $y \in p(x)$ if and only if $x \in I$,
(ii) if for $x \in [0,1]$ there is $\mathrm{Card}(r(x)) > 1$ then the set $\{x\}$ is called a maximal vertical segment of r,
(iii) the interval $[a,b]$ is called a strictly decreasing segment of r if for all $x \in]a,b[$ we have

$$\mathrm{Card}(r(x)) = 1, \mathrm{Card}(r(\max(r(x))) = 1, \tag{3}$$

(iv) the interval $[a,b]$ is called a maximal strictly decreasing segment of r if there is no interval $[c,d]$ which is a strictly decreasing segment of r such that $[a,b] \subsetneq [c,d]$.

The monotonicity of r implies that all maximal segments are intervals. Further, a subinterval of a maximal horizontal segment will be called a horizontal segment.

The symmetry of r implies that a maximal horizontal segment I can be paired with a maximal vertical segment $\{y\}$ for which we have

$$y \in r(x) \text{ for all } x \in I.$$

Then $I \times \{y\}$ as well as $\{y\} \times I$ belong to the graph of r.

Lemma 6 *([27]). Let $U \in \mathcal{U}$ and let $r\colon [0,1] \longrightarrow \mathcal{P}([0,1])$ be its characterizing set-valued function. Then all maximal segments of r are closed intervals.*

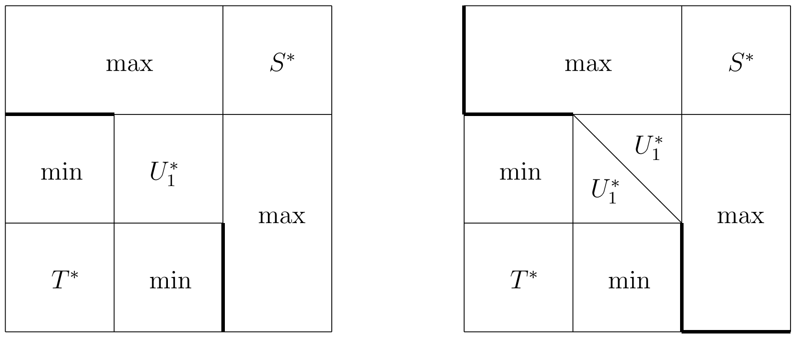

Fig. 1. The uninorm U from Example 3. Left: the bold lines denote the points of discontinuity of U. Right: the oblique and bold lines denote the characterizing set-valued function of U.

Due to the symmetry of r the previous result implies that for every $x \in [0,1]$ the set $r(x)$ is a closed interval and therefore $\min(r(x))$ and $\max(r(x))$ always exist. Let us now recall an example from [26].

Further we will denote by S_r the set of end points of all maximal segments of r and by $\bar{S_r}$ its closure. Note that there is a countable number of maximal horizontal and strictly decreasing segments and due to the symmetry of r there is also a countable number of maximal vertical segments. Therefore $\bar{S_r}$ is countable. Then we have the following result.

Proposition 9 *([27]). Let $U \in \mathcal{U}$, let $r\colon [0,1] \longrightarrow \mathcal{P}([0,1])$ be its characterizing set-valued function and assume $x \in [0,1]$. Then either $x \in \bar{S_r}$ or x is an interior point of exactly one maximal segment of r.*

Example 3. Assume a representable uninorm $U_1\colon [0,1]^2 \longrightarrow [0,1]$ and a continuous t-norm $T\colon [0,1]^2 \longrightarrow [0,1]$ and a continuous t-conorm $S\colon [0,1]^2 \longrightarrow [0,1]$. Let U_1^* be a linear transformation of U_1 to $\left[\frac{1}{3}, \frac{2}{3}\right]^2$, T^* a linear transformation of $T|_{[0,1[^2}$ to $\left[0, \frac{1}{3}\right[^2$ and S^* a linear transformation of $S|_{]0,1]^2}$ to $\left]\frac{2}{3}, 1\right]^2$. Then the ordinal sum of $G_{a_1} = \left(\left[0, \frac{1}{3}\right[, T^*\right)$, $G_{a_2} = \left(\left[\frac{1}{3}, \frac{2}{3}\right], U_1^*\right)$ and $G_{a_3} = \left(\left]\frac{2}{3}, 1\right], S^*\right)$ with the order $a_3 \prec a_1 \prec a_2$ is a uninorm $U \in \mathcal{U}$. For simplicity we will assume that $\frac{1}{2}$ is the neutral element of U_1 and that $U_1(x, 1-x) = \frac{1}{2}$ for all $x \in]0,1[$. On Fig. 1 we can see the characterizing set-valued function r of U as well as its set of discontinuity points.

Remark 4. From Theorems 2, 3, 4 and 5 we see that if $U \in \mathcal{U}$ is such that both T_U and S_U are Archimedean then either its characterizing set-valued function is strictly decreasing on $[0,1]$ – in which case it is a representable uninorm, or the interval $[0,e]$ ($[e,1]$) is a horizontal segment, i.e., there is $y \in [0,1]$ such that $r(x) = \{y\}$ for all $x \in]0,e[$ ($x \in]e,1[$). For the value y we then have $y \in \{1,e\}$ ($y \in \{0,e\}$).

Now we recall the result which will be important for the division of $U \in \mathcal{U}$ into respective semigroups.

Proposition 10 *([27]). Let $U \in \mathcal{U}$ and let r be its characterizing set-valued function. Then end points of all types of maximal segments of r are idempotent points.*

Further we will recall a relation between nilpotent components of U and maximal horizontal segments of the characterizing set-valued function.

Lemma 7 *([27]). Let $U \in \mathcal{U}$ and let r be its characterizing set-valued function. Then if $a, b \in [0,1]$, $a < b$, are idempotent elements of U such that there is no idempotent in $]a,b[$ and there exists $x \in]a,b[$ such that $U(x,x) = a$ ($U(x,x) = b$) then r on $[a,b]$ corresponds to a horizontal segment.*

In the following two lemmas we recall how does a maximal strictly decreasing segment of the characterizing set-valued function relate the components of $[0,e]$ to the components of $[e,1]$.

Lemma 8 *([27]). Let $U \in \mathcal{U}$ and let r be its characterizing set-valued function.*

(i) If $a, b \in [0,1]$, $a < b \le e$, are idempotent elements of U such that there is no idempotent in $]a,b[$ and r on $[a,b]$ corresponds to a strictly decreasing segment then $d = \min(r(a))$ and $c = \max(r(b))$ are idempotent elements of U such that there is no idempotent in $]c,d[$ and r on $[c,d]$ corresponds to a strictly decreasing segment. Further, $a = \max(r(\min(r(a))))$ and $b = \min(r(\max(r(b))))$.
(ii) If $c, d \in [0,1]$, $e \le c < d$, are idempotent elements of U such that there is no idempotent in $]c,d[$ and r on $[c,d]$ corresponds to a strictly decreasing segment then $b = \min(r(c))$ and $a = \max(r(d))$ are idempotent elements of U such that there is no idempotent in $]a,b[$ and r on $[a,b]$ corresponds to a strictly decreasing segment. Further, $c = \max(r(\min(r(c))))$ and $d = \min(r(\max(r(d))))$.

Lemma 9 *([27]). Let $U \in \mathcal{U}$ and let r be its characterizing set-valued function.*

(i) If $a, b \in [0, 1]$, $a < b \leq e$, are such that $U(x, x) = x$ for all $x \in [a, b]$ and r on $[a, b]$ corresponds to a strictly decreasing segment then for $d = \min(r(a))$ and $c = \max(r(b))$ we have $U(y, y) = y$ for all $y \in [c, d]$ and r on $[c, d]$ corresponds to a strictly decreasing segment. Further, $a = \max(r(\min(r(a))))$ and $b = \min(r(\max(r(b))))$.

(ii) If $c, d \in [0, 1]$, $e \leq c < d$, are such that $U(y, y) = y$ for all $y \in [c, d]$ and r on $[c, d]$ corresponds to a strictly decreasing segment then for $b = \min(r(c))$ and $a = \max(r(d))$ we have $U(x, x) = x$ for all $x \in [a, b]$ and r on $[a, b]$ corresponds to a strictly decreasing segment. Further, $c = \max(r(\min(r(c))))$ and $d = \min(r(\max(r(d))))$.

We conclude this section with the following useful result.

Lemma 10. *Let $U \in \mathcal{U}$.*

(i) If $U(x, y) = x$ for some $x \in]0, e[$, $y \in]e, 1[$ then $U(x_1, y_1) = x_1$ for all $x_1 \in [0, x]$, $y_1 \in [e, y]$.
(ii) If $U(x, y) = y$ for some $x \in]0, e[$, $y \in]e, 1[$ then $U(x_2, y_2) = y_2$ for all $x_2 \in [x, e]$, $y_2 \in [y, 1]$.

Proof. We will show only the first part as the second is analogous. If $U(x, y) = x$ for some $x \in]0, e[$, $y \in]e, 1[$ then since $U(x, e) = x$ the monotonicity of U implies $U(x, y_1) = x$ for all $y_1 \in [e, y]$. Since $U \in \mathcal{U}$ for each $x_1 \in [0, x]$ there exists a $q \in [0, e]$ such that $U(q, x) = x_1$. Then we have

$$U(x_1, y_1) = U(U(q, x), y_1) = U(q, U(x, y_1)) = U(q, x) = x_1. \qquad \square$$

Now we are ready to show the main result.

5. Decomposition of uninorm with continuous underlying functions via ordinal sum

In this section we will successively show that each uninorm $U \in \mathcal{U}$ can be decomposed into an ordinal sum, with a countable number of summands, of the nine types of semigroups, eight from Definition 3 plus semigroups defined on singletons.

In the following definition we will define a partition of $[0, 1]$ related to the given $U \in \mathcal{U}$.

Definition 7. Let $U \in \mathcal{U}$ and let r be its characterizing set-valued function. We will proceed in three steps.

Step 1: Definition of Archimedean segments.

Since U is continuous on the diagonal $u: [0, 1] \longrightarrow [0, 1]$ given by $u(x) = U(x, x)$, then the set of all idempotent points of U is a closed set – we will denote it by I_U. Then $[0, e] \setminus I_U = \bigcup_{k \in K}]a_k, b_k[$, where $(]a_k, b_k[)_{k \in K}$ is a system of a countable number of open and disjoint subintervals of $[0, e]$ for some index set K (see [13]). Similarly, $[e, 1] \setminus I_U = \bigcup_{l \in L}]c_l, d_l[$, where $(]c_l, d_l[)_{l \in L}$ is a system of a countable number of open and disjoint subintervals of $[e, 1]$ for some index set L. Note that we will select L in such a way that $K \cap L = \emptyset$.

Now we denote $K_1 = \{k \in K \mid a_k = U(x, x) \text{ for some } x \in [0, 1], x \neq a_k\}$ and $K_2 = \{k \in K \mid r \text{ on } [a_k, b_k] \text{ corresponds to a strictly decreasing segment }\}$, Lemma 7 implies that $K_1 \cap K_2 = \emptyset$. Let $K_3 = K \setminus (K_1 \cup K_2)$. Further $L_1 = \{l \in L \mid d_k = U(x, x) \text{ for some } x \in [0, 1], x \neq d_k\}$ and $L_2 = \{l \in L \mid r \text{ on } [c_l, d_l] \text{ corresponds to a strictly decreasing segment }\}$ and $L_3 = L \setminus (L_1 \cup L_2)$. Similarly as above, Lemma 7 implies that $L_1 \cap L_2 = \emptyset$.

Due to Lemma 8 each $k \in K_2$ can be paired with some $l \in L_2$ and vice-versa. Therefore instead of $l \in L_2$ we will assume the corresponding $k \in K_2$.

Step 2: Definition of idempotent segments.

Denote

$$X = \{x \in [0, 1] \mid x \text{ is an end point of a maximal segment of } r\}$$

and

$$B = \bigcup_{k \in K}]a_k, b_k[\cup \bigcup_{k \in K_1} \{a_k\} \cup \bigcup_{k \in K_2} \{U(b_k, c_k)\}.$$

Let

$$[0, e] \setminus (B \cup X) = \bigcup_{m \in M} Y_m,$$

where the sets Y_m are components of $[0, e] \setminus (B \cup X)$ with respect to connectedness. Note that we can select such an M that K, L, M are mutually disjoint. We denote

$$A^* = \{\sup Y_m, \inf Y_m \mid m \in M\} \setminus (B \cup \{e\})$$

and define $Z_m = Y_m \setminus A^*$ for all $m \in M$, i.e., $Z_m =]a_m, b_m[$ for some $a_m, b_m \in [0, e]$ for all $m \in M$. Denote $M_* = \{m \in M \mid Z_m \neq \emptyset\}$,

$$M_1 = \{m \in M_* \mid r \text{ on }]a_m, b_m[\text{ corresponds to a horizontal segment}\},$$

$M_2 = \{m \in M_* \mid r$ on $]a_m, b_m[$ corresponds to a strictly decreasing segment$\}$. Then M_* is countable. Further denote

$$A = [0, e[\setminus (B \cup \bigcup_{m \in M_*} Z_m).$$

Then A is countable since between any two points from A there is a rational number from $[0, e]$. Therefore there exists an isomorphism i between A and some countable index set M_3 which again can be selected to be mutually disjoint with all previous index sets. We set $Z_m = \{i(m)\}$ for all $m \in M_3$, i.e., $Z_m = \{b_m\}$ for some $b_m \in [0, e]$ for all $m \in M_3$.

Similarly, let

$$C = \bigcup_{l \in L}]c_l, d_l[\cup \bigcup_{l \in L_1} \{d_k\} \cup \bigcup_{k \in K_2} \{U(b_k, c_k)\}$$

and let

$$[e, 1] \setminus (C \cup X) = \bigcup_{o \in O} Y_o,$$

where the sets Y_o are components of $[e, 1] \setminus (C \cup X)$ with respect to connectedness. Note that we can select such an O that O, K, L, M, M_3 are mutually disjoint. We denote

$$D^* = \{\sup Y_o, \inf Y_o \mid o \in O\} \setminus (C \cup \{e\}).$$

We define $Z_o = Y_o \setminus D$ for all $o \in O$, i.e., $Z_o =]c_o, d_o[$ for some $c_o, d_o \in [e, 1]$ for all $o \in O$. Denote $O_* = \{o \in O \mid Z_o \neq \emptyset\}$,

$$O_1 = \{o \in O_* \mid r \text{ on }]c_o, d_o[\text{ corresponds to a horizontal segment}\},$$

$O_2 = \{o \in O_* \mid r$ on $]c_o, d_o[$ corresponds to a strictly decreasing segment$\}$. Then O_* is countable. Further, denote

$$D =]e, 1] \setminus (C \cup \bigcup_{o \in O_*} Z_o).$$

Then D is countable and there exists an isomorphism j between D and some countable index set O_3 which again can be selected to be mutually disjoint with all previous index sets. We set $Z_o = \{j(o)\}$ for all $o \in O_3$, i.e., $Z_o = \{c_o\}$ for some $c_o \in [e, 1]$ for all $o \in O_3$.

Due to Lemma 9 each $m \in M_2$ can be paired with some $o \in O_2$ and vice-versa. Therefore instead of $o \in O_2$ we will assume the corresponding $m \in M_2$.

If $U(x, y) = e$ for some $x \neq e$ then the point e is already covered in our partition, however, if $U(x, y) = e$ implies $x = y = e$ then we should add a separate set $\{e\}$.

Step 3: Summarization.

Thus we have a partition of $[0, e]$ into sets: $[a_k, b_k[$ for $k \in K_1$, $]a_k, b_k[\cup (\{U(b_k, c_k)\} \cap [0, e])$ for $k \in K_2$, $]a_k, b_k[$ for $k \in K_3$, $]a_m, b_m[$ for $m \in M_1 \cup M_2$, $\{a_m\}$ for $m \in M_3$ and eventually $\{e\}$.

Similarly, we have a partition of $[e, 1]$ into sets: $]c_l, d_l]$ for $l \in L_1$, $]c_k, d_k[\cup (\{U(b_k, c_k)\} \cap [e, 1])$ for $k \in K_2$, $]c_l, d_l[$ for $l \in L_3$, $]c_o, d_o[$ for $o \in O_1 \cup M_2$, $\{d_o\}$ for $o \in O_3$ and eventually $\{e\}$.

For simplicity we denote $P^* = K_1 \cup K_2 \cup K_3 \cup M_1 \cup M_2 \cup M_3 \cup L_1 \cup L_3 \cup O_1 \cup O_3$ and for $p \in P$ we will denote corresponding sets described above by X_p. If $U(x, y) = e$ implies $x = y = e$ then we additionally assume an index $p^* \notin P^*$ and the set $X_{p^*} = \{e\}$. Then $P = P^* \cup \{p^*\}$. In the other case we put $P = P^*$. Note that $X_{p_1} \cap X_{p_2} = \emptyset$ for $p_1, p_2 \in P$, $p_1 \neq p_2$.

Lemma 11. *Let $U \in \mathcal{U}$ and let r be its characterizing set-valued function. Assume the partition from Definition 7. Let $q\colon [0, 1] \longrightarrow [0, 1]$ be a function given by*

$$q(x) = \begin{cases} 1 & \text{if } x \in [0, e[\text{ and } u_x \text{ is continuous,} \\ y & \text{if } u_x \text{ is non-continuous in } y, \\ e & \text{if } x = e, \\ 0 & \text{if } x \in]e, 1] \text{ and } u_x \text{ is continuous.} \end{cases}$$

We will define two functions $L, H\colon P \longrightarrow [0,1]$ by

$$L(p)=\begin{cases} \frac{a_p+b_p}{2} & \text{if } p\in K_1,\\ \frac{a_p+b_p}{2} & \text{if } p\in K_2,\\ \frac{a_p+b_p}{2} & \text{if } p\in K_3,\\ \frac{a_p+b_p}{2} & \text{if } p\in M_1,\\ \frac{a_p+b_p}{2} & \text{if } p\in M_2,\\ b_p & \text{if } p\in M_3,\\ q(\frac{c_p+d_p}{2}) & \text{if } p\in L_1,\\ q(\frac{c_p+d_p}{2}) & \text{if } p\in L_3,\\ q(\frac{c_p+d_p}{2}) & \text{if } p\in O_1,\\ q(c_p) & \text{if } p\in O_3,\\ e & \text{else,}\end{cases} \qquad H(p)=\begin{cases} q(\frac{a_p+b_p}{2}) & \text{if } p\in K_1,\\ q(\frac{a_p+b_p}{2}) & \text{if } p\in K_2,\\ q(\frac{a_p+b_p}{2}) & \text{if } p\in K_3,\\ q(\frac{a_p+b_p}{2}) & \text{if } p\in M_1,\\ q(\frac{a_p+b_p}{2}) & \text{if } p\in M_2,\\ q(b_p) & \text{if } p\in M_3,\\ \frac{c_p+d_p}{2} & \text{if } p\in L_1,\\ \frac{c_p+d_p}{2} & \text{if } p\in L_3,\\ \frac{c_p+d_p}{2} & \text{if } p\in O_1,\\ c_p & \text{if } p\in O_3,\\ e & \text{else.}\end{cases}$$

Then there are no such $p_1, p_2 \in P$ that either $L(p_1) < L(p_2)$ and $H(p_1) < H(p_2)$, or $L(p_1) > L(p_2)$ and $H(p_1) > H(p_2)$. Further, if $L(p_1) = L(p_2)$ and $H(p_1) = H(p_2)$ for some $p_1 \neq p_2$ then $p_1, p_2 \in M_3 \cup O_3$.

Proof. The monotonicity of the characterizing set-valued function r ensures that $L(p_1) < L(p_2)$ implies $H(p_1) \geq H(p_2)$, $H(p_1) < H(p_2)$ implies $L(p_1) \geq L(p_2)$, $H(p_1) > H(p_2)$ implies $L(p_1) \leq L(p_2)$ and $L(p_1) > L(p_2)$ implies $H(p_1) \leq H(p_2)$ for all $p_1, p_2 \in P$. Further, if $L(p_1) = L(p_2)$ and $H(p_1) = H(p_2)$ for some $p_1 \neq p_2$ then both $L(p_1)$ and $H(p_1)$ are idempotent points and thus either $p_1 \in M_3$ and $p_2 \in O_3$, or $p_1 \in O_3$ and $p_2 \in M_3$. □

Lemma 12. *Let $U \in \mathcal{U}$ and assume functions L, H from Lemma 11. We will define a relation $\preceq$ on the set P as follows: for any $p_1, p_2 \in P$ there is $p_1 \preceq p_2$ if one of the following is fulfilled:*

(i) $p_1 = p_2$
(ii) $L(p_1) \leq L(p_2)$ and $H(p_1) > H(p_2)$,
(iii) $L(p_1) < L(p_2)$ and $H(p_1) \geq H(p_2)$,
(iv) $L(p_1) = L(p_2)$, $H(p_1) = H(p_2)$, $p_1 \neq p_2$ and $U(b, c) = b$, where $b \in X_{p_1}$ and $c \in X_{p_2}$.

Then $\preceq$ is a linear order on P.

Proof. To show that $\preceq$ is an order on P we have to show that it is reflexive, anti-symmetric and transitive. Reflexivity is evident. Assume that $p_1 \preceq p_2$ and $p_2 \preceq p_1$. If $p_1 \neq p_2$ we get $L(p_1) = L(p_2)$, $H(p_1) = H(p_2)$ and thus by (iv) we have $b = U(b, c) = c$ what is a contradiction. Therefore $p_1 = p_2$.

For transitivity assume that $p_1 \preceq p_2$ and $p_2 \preceq p_3$. If $p_1 = p_2$ or $p_2 = p_3$ the transitivity is clear. Further if $L(p_1) < L(p_2)$ or $L(p_2) < L(p_3)$, or $H(p_1) > H(p_2)$, or $H(p_2) > H(p_3)$ the transitivity is also easily shown. Suppose that $L(p_1) = L(p_2) = L(p_3)$, $H(p_1) = H(p_2) = H(p_3)$ and p_1, p_2 and p_3 are mutually different. Then $p_1, p_2, p_3 \in M_3 \cup O_3$. Let $a \in X_{p_1}$, $b \in X_{p_2}$ and $c \in X_{p_3}$. Then $U(a, b) = a$ and $U(b, c) = b$. The associativity of U then gives

$$U(a, c) = U(a, U(b, c)) = U(a, b) = a.$$

Thus $p_1 \preceq p_3$ and the relation $\preceq$ is transitive. Finally we have to show that $\preceq$ is linear, i.e., that for all $p_1, p_2 \in P$ we have either $p_1 \preceq p_2$ or $p_2 \preceq p_1$. Assume any $p_1, p_2 \in P$. If $p_1 = p_2$ then we have both $p_1 \preceq p_2$ and $p_2 \preceq p_1$. Let now $p_1 \neq p_2$. If $L(p_1) = L(p_2)$ and $H(p_1) = H(p_2)$ then $U(b, c) \in \{b, c\}$ for $b \in X_{p_1}$ and $c \in X_{p_2}$ and $p_1 \preceq p_2$ if $U(b, c) = b$ and $p_2 \preceq p_1$ if $U(b, c) = c$. In the other case we have one of the following four inequalities:

(i) $L(p_1) < L(p_2)$,
(ii) $L(p_1) > L(p_2)$,
(iii) $H(p_1) < H(p_2)$,
(iv) $H(p_1) > H(p_2)$.

In the first case Lemma 11 implies $H(p_1) \geq H(p_2)$ and thus $p_1 \preceq p_2$. Similarly, in the fourth case we get $p_1 \preceq p_2$ and in the second and the third case we get $p_2 \preceq p_1$. Thus $\preceq$ is a linear order. □

Remark 5. Note that the order $\preceq$ from previous lemma is compatible with the standard order $\leq$ on $[0, e]$ and reversed to the standard order $\leq$ on $[e, 1]$.

Before we will show the main result we introduce two useful results.

Lemma 13. *Let $U \in \mathcal{U}$ and let $q\colon [0, 1] \longrightarrow [0, 1]$, $L, H\colon P \longrightarrow [0, 1]$ be the functions from Lemma 11. Then for any $p_1, p_2 \in P$ we have:*

(i) if $H(p_1) > H(p_2)$ then for all $x \in X_{p_1} \cap [0, e]$, $y \in X_{p_2} \cap [e, 1]$ there is $q(x) > y$.
(ii) if $L(p_1) < L(p_2)$ then for all $x \in X_{p_1} \cap [0, e]$, $y \in X_{p_2} \cap [e, 1]$ there is $x < q(y)$.

Proof. We will show only the first part as the second is analogous. If $p_1 \in K_1 \cup K_3 \cup M_1 \cup M_3$ or $x = e$ then $H(p_1) = q(x)$ and $q(x)$ is an idempotent point of U. On the other hand, $H(p_2), y \in X_{p_2} \cap [e, 1]$. Thus $H(p_1) > H(p_2)$ implies $q(x) > y$. If $p_1 \in K_2 \cup M_2$ then $H(p_1), q(x) \in X_{p_1} \cap [e, 1]$ and since $H(p_2), y \in X_{p_2} \cap [e, 1]$ the inequality $H(p_1) > H(p_2)$ implies $q(x) > y$. $\square$

Lemma 14. *Let $U \in \mathcal{U}$ and assume the partition from Definition 7. Then for all $p \in P$*

(i) $x \in X_p \cap [0, e]$ and $y \in [e, 1] \setminus X_p$ implies $U(x, y) \in \{x, y\}$.
(ii) $x \in X_p \cap [e, 1]$ and $y \in [0, e] \setminus X_p$ implies $U(x, y) \in \{x, y\}$.

Proof. We will show only the first part as the second is analogous. If $x = e$ then the claim is trivial. Otherwise $p \in K_1 \cup K_2 \cup K_3 \cup M_1 \cup M_2 \cup M_3$. If $p \in M_1 \cup M_2 \cup M_3$ then x is an idempotent point and the result is trivial. Assume $p \in K_2$. Then $X_p = \left]a_p, b_p\right[\cup \{U(b_p, c_p)\} \cup \left]c_p, d_p\right[$ and for $x \in \left]a_p, b_p\right[$ we have $e > U(x, c_p) \in \{x, c_p\}$, i.e., $U(x, c_p) = x$. Moreover, $e < U(x, d_p) \in \{x, d_p\}$, i.e., $U(x, d_p) = d_p$. Then Lemma 10 implies $U(x, y) \in \{x, y\}$ for all $y \in [e, 1] \setminus X_p$. If $x \in \{U(b_p, c_p)\} \cap [0, e]$, i.e., $x = b_p = U(b_p, c_p)$ then x is an idempotent element and the claim is trivial.

Finally assume $p \in K_1 \cup K_3$. Then $X_p = \left]a_p, b_p\right[$ or $X_p = \left[a_p, b_p\right[$ and either u_z is continuous for all $z \in \left[0, b_p\right[$ or there exists an idempotent point $y \in [e, 1]$ such that u_z is non-continuous in y for all $z \in \left]a_p, b_p\right[$. In the first case we take $y = 1$.

Let $y_1 = \sup\{z \in [e, y[\mid U(z, z) = z\}$ and let $y_2 = \inf\{z \in]y, 1] \mid U(z, z) = z\}$. Then Lemma 10 implies that $U(x, z) \in \{x, z\}$ for all $z \in [e, y_1] \cup [y_2, 1]$. If $y_1 = y = y_2$ the proof is finished. Suppose the opposite.

First we will show that $\left[a_p, b_p\right[\cup [e, 1]$ is closed under U. From Theorem 6 we know that $\left[a_p, b_p\right] \cup [e, 1]$ is closed under U. It is evident that if $U(x, z) = b_p$ for some $x, z \in \left[a_p, b_p\right[\cup [e, 1]$ then we can select $x < b_p$ and $z > e$. Then $z < y$. However, then also $U(x_2, z) = b_p$ for all $x_2 \in \left]x, b_p\right[$. Since T_U is continuous there exists a $q \in \left]a_p, b_p\right[$ such that $U(x_2, q) = x$. Then

$$q = U(q, b_p) = U(x_2, q, z) = U(x, z) = b_p$$

what is a contradiction.

If $y_1 < y$ then $\left[a_p, b_p\right[\cup [y_1, y]$ is closed under U and $U(x, y_1) = x$ for all $x \in \left[a_p, b_p\right[\cup [y_1, y]$. Thus U restricted to $(\left[a_p, b_p\right[\cup [y_1, y])^2$ is isomorphic to a uninorm with continuous Archimedean underlying operations and thus $U(x, z) \in \{x, z\}$ for all $x \in X_p$, $z \in [y_1, y]$.

Now assume $y < y_2$. Here we will proceed as above. We first show that $\left[a_p, b_p\right] \cup]y, y_2]$ is closed under U and then since $U(b_p, z) = z$ for all $z \in \left[a_p, b_p\right] \cup]y, y_2]$ by isomorphism with a uninorm with continuous Archimedean underlying operations we can show that $U(x, z) \in \{x, z\}$ for all $x \in X_p$, $z \in [y, y_2]$. $\square$

Proposition 11. *Let $U \in \mathcal{U}$ and let r be its characterizing set-valued function. Assume the partition from Definition 7. Then U is an ordinal sum of the following semigroups:*

(i) a t-nilpotent semigroup on $[a_k, b_k[$ for all $k \in K_1$,
(ii) a representable semigroup on $]a_k, b_k[\cup \{U(b_k, c_k)\} \cup]c_k, d_k[$ for all $k \in K_2$,
(iii) a t-strict semigroup on $]a_k, b_k[$ for all $k \in K_3$,
(iv) an s-nilpotent semigroup on $]c_l, d_l]$ for all $l \in L_1$,
(v) an s-strict semigroup on $]c_l, d_l[$ for all $l \in L_3$,
(vi) a t-internal semigroup on $]a_m, b_m[$ for all $m \in M_1$,
(vii) a d-internal semigroup on $]a_m, b_m[\cup]c_m, d_m[$ for all $m \in M_2$,
(viii) a semigroup defined on $\{a_m\}$ for all $m \in M_3$,
(ix) an s-internal semigroup on $]c_o, d_o[$ for all $o \in O_1$,
(x) a semigroup defined on $\{d_o\}$ for all $o \in O_3$,
(xi) eventually a semigroup defined on $\{e\}$.

Proof. It is evident the U restricted to

(i) $([a_k, b_k[)^2$ for all $k \in K_1$ is a t-nilpotent semigroup,
(ii) $(]a_k, b_k[)^2$ for all $k \in K_3$ is a t-strict semigroup,
(iii) $(]c_l, d_l])^2$ for all $l \in L_1$ is an s-nilpotent semigroup,
(iv) $(]c_l, d_l[)^2$ for all $l \in L_3$ is an s-strict semigroup,
(v) $(]a_m, b_m[)^2$ for all $m \in M_1$ is a t-internal semigroup,
(vi) $(\{b_m\})^2$ for all $m \in M_3$ is a singleton semigroup,
(vii) $(]c_o, d_o[)^2$ for all $o \in O_1$ is an s-internal semigroup,
(viii) $(\{c_o\})^2$ for all $o \in O_3$ is a singleton semigroup,
(ix) $(\{e\})^2$ is a singleton semigroup.

Further we will focus on K_2 and M_2. For any $k \in K_2$ Lemmas 2, 3 and 8 imply that U restricted to $([a_k, b_k[\cup \{U(b_k, c_k)\} \cup]c_k, d_k])^2$ is isomorphic to a representable uninorm. Thus U restricted to $(]a_k, b_k[\cup \{U(b_k, c_k)\} \cup]c_k, d_k[)^2$ is a representable semigroup.

Further, for any $m \in M_2$ Lemmas 2 and 9 imply that U restricted to $([a_m, b_m[\cup \{e\} \cup]c_m, d_m])^2$ is isomorphic to a d-internal uninorm, i.e., U restricted to $(]a_m, b_m[\cup]c_m, d_m[)^2$ is a d-internal semigroup.

Now assume the order from Lemma 12. To conclude the proof we should show that if $p_1, p_2 \in P$, $p_1 \neq p_2$, with $p_1 \preceq p_2$ then for any $x \in X_{p_1}$ and any $y \in X_{p_2}$ we have $U(x, y) = x$. If both $x, y \in [0, e]$ then $x < y$ since $L(p_1) \leq L(p_2)$ and since there is an idempotent point in $[x, y]$ then continuity of T_U implies $U(x, y) = \min(x, y) = x$. Similarly, if $x, y \in [e, 1]$ then $U(x, y) = x$.

Now suppose that $x \in [0, e]$ and $y \in [e, 1]$ (the case when $x \in [e, 1]$ and $y \in [0, e]$ can be shown analogously). If $L(p_1) = L(p_2)$ and $H(p_1) = H(p_2)$ then $U(x, y) = x$.

If $H(p_1) > H(p_2)$ then Lemma 13 implies $q(x) > y$ and Lemma 14 implies $U(x, y) \in \{x, y\}$. Thus $U(x, y) = x$.

Finally suppose $L(p_1) < L(p_2)$. Then Lemma 13 implies $x < q(y)$ and Lemma 14 implies $U(x, y) \in \{x, y\}$. Thus $U(x, y) = x$. □

Remark 6. Due to the ordinal sum structure and monotonicity of U it is easy to see that restriction of U

(i) to $[a_k, b_k]$ for all $k \in K_1$ is isomorphic to a nilpotent t-norm,
(ii) to $[a_k, b_k[\cup \{U(b_k, c_k)\} \cup]c_k, d_k]$ for all $k \in K_2$ is isomorphic to a representable uninorm,
(iii) to $[a_k, b_k]$ for all $k \in K_3$ is isomorphic to a strict t-norm,
(iv) to $[c_l, d_l]$ for all $l \in L_1$ is isomorphic to a nilpotent t-conorm,
(v) to $[c_l, d_l]$ for all $l \in L_3$ is isomorphic to a strict t-conorm,
(vi) to $[a_m, b_m]$ for all $m \in M_1$ is isomorphic to the minimum t-norm,
(vii) to $[a_m, b_m[\cup \{U(b_m, c_m)\} \cup]c_m, d_m]$ for all $m \in M_2$, is isomorphic to a d-internal uninorm.
(viii) to $[c_o, d_o]$ for all $o \in O_1$ is isomorphic to the maximum t-conorm.

Despite this fact, U need not to be an ordinal sum of representable uninorms (including continuous Archimedean t-norms) and internal uninorms (including the min and the max). Indeed, it can happen that the end point of the set where the respective uninorm is transformed is 'separated' from the remainder of the semigroup support and there exists a $p_3 \in P$ such that $p_1 < p_3 < p_2$ (or $p_2 < p_3 < p_1$), where p_1 corresponds to the singleton semigroup of the end point and p_2 to the remaining semigroup. Due to the given order this is possible only if $p_3 \notin K_2 \cup M_2$. Sketch of this situation can be seen on Fig. 2.

To conclude our characterization we have to show also an opposite result.

Proposition 12. *Assume $e \in]0, 1[$ and let $K_1, K_2, K_3, L_1, L_3, M_1, M_2, M_3, O_1, O_3$ be mutually disjoint countable index sets and let $P^* = K_1 \cup K_2 \cup K_3 \cup L_1 \cup L_3 \cup M_1 \cup M_2 \cup M_3 \cup O_1 \cup O_3$. Further assume*

(i) a t-nilpotent semigroup on $[a_k, b_k[$ for all $k \in K_1$,
(ii) a representable semigroup on $]a_k, b_k[\cup \{U(b_k, c_k)\} \cup]c_k, d_k[$ for all $k \in K_2$,
(iii) a t-strict semigroup on $]a_k, b_k[$ for all $k \in K_3$,
(iv) an s-nilpotent semigroup on $]c_l, d_l]$ for all $l \in L_1$,
(v) an s-strict semigroup on $]c_l, d_l[$ for all $l \in L_3$,
(vi) a t-internal semigroup on $]a_m, b_m[$ for all $m \in M_1$,
(vii) a d-internal semigroup on $]a_m, b_m[\cup]c_m, d_m[$ for all $m \in M_2$,
(viii) a semigroup defined on $\{a_m\}$ for all $m \in M_3$,
(ix) an s-internal semigroup on $]c_o, d_o[$ for all $o \in O_1$,
(x) a semigroup defined on $\{d_o\}$ for all $o \in O_3$,

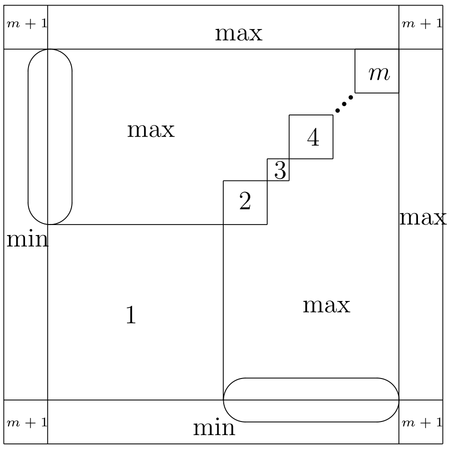

Fig. 2. Sketch of a uninorm $U \in \mathcal{U}$ which is an ordinal sum with $m+1$ non-singleton summands, $P = \{1, \dots, m, m+1\} \cup P_s$, where P_s are indices related to singleton semigroups. Here $1, m+1 \in K_2 \cup M_2$ and $2, \dots, m \notin K_2 \cup M_2$. The rounded area (the line in the center) designates the place where U can differ from an ordinal sum of representable and internal uninorms.

and if $e \neq U(b_k, c_k)$ for all $k \in K_2$ also a semigroup defined on $\{e\}$, such that $[0, e]$ is partitioned into sets: $[a_k, b_k[$ for $k \in K_1$, $]a_k, b_k[\cup (\{U(b_k, c_k)\} \cap [0, e])$ for $k \in K_2$, $]a_k, b_k[$ for $k \in K_3$, $]a_m, b_m[$ for $m \in M_1 \cup M_2$, $\{a_m\}$ for $m \in M_3$ and eventually $\{e\}$, and $[e, 1]$ is partitioned into sets: $]c_l, d_l]$ for $l \in L_1$, $]c_k, d_k[\cup (\{U(b_k, c_k)\} \cap [e, 1])$ for $k \in K_2$, $]c_l, d_l[$ for $l \in L_3$, $]c_o, d_o[$ for $o \in O_1 \cup M_2$, $\{d_o\}$ for $o \in O_3$ and eventually $\{e\}$.

If there is a separate semigroup on $\{e\}$ we will take $p^* \notin P^*$ and set $P = P^* \cup \{p^*\}$. Otherwise $P = P^*$.

Let $L, H \colon P \longrightarrow [0, 1]$ be two functions partially given by

$$L(p) = \begin{cases} \frac{a_p+b_p}{2} & \text{if } p \in K_1, \\ \frac{a_p+b_p}{2} & \text{if } p \in K_2, \\ \frac{a_p+b_p}{2} & \text{if } p \in K_3, \\ \frac{a_p+b_p}{2} & \text{if } p \in M_1, \\ \frac{a_p+b_p}{2} & \text{if } p \in M_2, \\ b_p & \text{if } p \in M_3, \\ e & \text{if } p = p^*, \end{cases} \qquad H(p) = \begin{cases} \frac{c_p+d_p}{2} & \text{if } p \in K_2, \\ \frac{c_p+d_p}{2} & \text{if } p \in M_2, \\ \frac{c_p+d_p}{2} & \text{if } p \in L_1, \\ \frac{c_p+d_p}{2} & \text{if } p \in L_3, \\ \frac{c_p+d_p}{2} & \text{if } p \in O_1, \\ c_p & \text{if } p \in O_3, \\ e & \text{if } p = p^* \end{cases}$$

such that for all $p_1, p_2 \in P$ there is neither $L(p_1) < L(p_2)$ and $H(p_1) < H(p_2)$, nor $L(p_1) > L(p_2)$ and $H(p_1) > H(p_2)$. Assume a linear order $\preceq$ on P such that if $p_1 \preceq p_2$ one of the following is satisfied:

(i) $p_1 = p_2$
(ii) $L(p_1) \leq L(p_2)$ *and* $H(p_1) > H(p_2)$,
(iii) $L(p_1) < L(p_2)$ *and* $H(p_1) \geq H(p_2)$,
(iv) $L(p_1) = L(p_2)$ *and* $H(p_1) = H(p_2)$.

Then the ordinal sum of the above described semigroups with the order $\preceq$ on the set P is a uninorm $U \in \mathcal{U}$.

Proof. Let $([0, 1], U)$ be an ordinal sum of the above described semigroups. First observe that U restricted to $[0, e]^2$ is isomorphic to an ordinal sum of continuous t-norms and U restricted to $[e, 1]^2$ is isomorphic to an ordinal sum of continuous t-conorms.

Since each of the respective semigroups is commutative also U is commutative. The associativity follows from Theorem 1. Further, if $P \neq P^*$ Then $p^* = \max(P)$ and thus $U(e, x) = x$ for all $x \in [0, 1]$. If $P = P^*$ then there is a $p \in K_2$ such that $b_p = c_p = e$, i.e., $X_p = \left]a_p, d_p\right[$. Then e is the neutral point of the semigroup corresponding to p and $p = \max(P)$. Thus $U(e, x) = x$ for all $x \in [0, 1]$.

Now we will focus on monotonicity. Since U is monotone on $[0, e]^2 \cup [e, 1]^2$ the monotonicity has to be shown on $[0, e] \times [e, 1]$ and on $[e, 1] \times [0, e]$. We will focus on $[0, e] \times [e, 1]$ as the other part is analogous. Due to commutativity of U it is enough to show that for all $x \in [0, e]$ and all $y_1, y_2 \in [e, 1]$, $y_1 < y_2$ we have $U(x, y_1) \leq U(x, y_2)$. Let $x \in X_p$, $y_1 \in X_{p_1}$ and $y_2 \in X_{p_2}$. If $p = p_1 = p_2$ then monotonicity follows from monotonicity of the semigroup corresponding to p.

Let $p = p_1 \neq p_2$. Since $y_1 < y_2$ we have $p_2 \prec p_1 = p$, i.e., $U(x, y_2) = y_2 > y_1 \geq U(x, y_1)$. If $p = p_2 \neq p_1$ then $y_1 < y_2$ implies $p = p_2 \prec p_1$, i.e., $U(x, y_2) \geq x = U(x, y_1)$. Suppose $p \neq p_1 = p_2$ then either $U(x, y_1) = x = U(x, y_2)$ or $U(x, y_2) =$

$y_2 > y_1 = U(x, y_1)$. Finally suppose that p, p_1 and p_2 are mutually different. Then $y_1 < y_2$ implies $p_2 \prec p_1$ and we have the following possibilities:

(i) $p \prec p_2 \prec p_1$,
(ii) $p_2 \prec p \prec p_1$,
(iii) $p_2 \prec p_1 \prec p$.

In the first case we get $U(x, y_2) = x = U(x, y_1)$. In the second case we get $U(x, y_2) = y_2 > x = U(x, y_1)$. Finally, in the third case we get $U(x, y_2) = y_2 > y_1 = U(x, y_1)$. Thus in all possible cases we get $U(x, y_1) \leq U(x, y_2)$ and the monotonicity holds. $\square$

6. Conclusions

Each continuous t-norm (t-conorm) is equal to an ordinal sum of continuous Archimedean t-norms (t-conorms). In this paper we have extended this characterization onto uninorms with continuous underlying t-norm and t-conorm. Using the characterizing set-valued function we have shown that such a uninorm can be decomposed into an ordinal sum of a countable number of semigroups related to representable uninorms, continuous Archimedean t-norms, continuous Archimedean t-conorms and internal uninorms (including the min and the max). However, we have shown that not every uninorm with continuous underlying t-norm and t-conorm can be decomposed into an ordinal sum of representable uninorms(including continuous Archimedean t-norms and t-conorms) and internal uninorms (including the min and the max). This result together with the properties of the characterizing set-valued function offer a complete characterization of uninorms from $\mathcal{U}$, i.e., of uninorms with continuous underlying t-norm and t-conorm. The applications of these results are expected in all domains where uninorms are used.

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Characterization of Uninorms With Continuous Underlying T-norm and T-conorm by Their Set of Discontinuity Points

Andrea Mesiarová-Zemánková

***Abstract*—Uninorms with continuous underlying t-norm and t-conorm are discussed and properties of the set of discontinuity points of such a uninorm are shown. This set is proved to be a subset of the graph of a special symmetric, u-surjective, nonincreasing set-valued function, which gives us a necessary condition for a uninorm to have continuous underlying functions. A sufficient condition for a uninorm to have continuous underlying operations is also given. Several examples are included.**

***Index Terms*—Continuous t-conorm, continuous t-norm, ordinal sum, set-valued function, uninorm.**

I. Introduction

The (left-continuous) t-norms and their dual t-conorms have an indispensable role in many domains [9], [31], [32]. Generalizations of t-norms and t-conorms that can model bipolar behavior are uninorms (see [7], [23], [33]). The class of uninorms is widely used both in theory [18], [29] and in applications [13], [34]. The complete characterization of uninorms with continuous underlying t-norm and t-conorm has been in the center of the interest for a long time, however, only partial results were achieved (see [4]–[6], [8], [10], [15]–[17], [19]–[21], [28], [30]).

In [24], we have introduced ordinal sum of uninorms and in [25] we have characterized uninorms that are ordinal sums of representable uninorms. We would like to characterize all uninorms with continuous underlying functions and obtain a similar representation as in the case of t-norms and t-conorms. In this paper, we will show that underlying operations of a uninorm U are continuous if and only if U is continuous on $[0,1]^2 \setminus G(r)$, where $G(r)$ is the graph of a special symmetric, u-surjective, nonincreasing set-valued function, and U is in each point $(x,y) \in [0,1]^2$ either left-continuous or right-continuous (or both, in which case it is continuous). We will then continue and in [26] and [27] we will show that each uninorm with continuous underlying t-norm and t-conorm can be decomposed into an ordinal sum of semigroups related to representable uninorms, continuous Archimedean t-norms and t-conorms, internal uninorms, and singleton semigroups.

In Section II, we will recall all necessary basic notions and results. We will characterize uninorms with continuous underlying functions via the properties of their set of discontinuity points (see Section III). We give our conclusions in Section IV.

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The author is with the Mathematical Institute of Slovak Academy of Sciences, Bratislava 814 73, Slovakia (e-mail: zemankova@mat.savba.sk).

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II. Basic Notions and Results

Let us now recall all necessary basic notions.

A triangular norm is a function $T : [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, nondecreasing in both variables and 1 is its neutral element. Note that in this paper we stick to the definition from [11], where a nondecreasing function means an increasing function that need not to be strictly increasing. Due to the associativity, n-ary form of any t-norm is uniquely given and thus it can be extended to an aggregation function working on $\bigcup_{n\in\mathbb{N}} [0,1]^n$. Dual functions to t-norms are t-conorms. A triangular conorm is a function $S : [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, nondecreasing in both variables and 0 is its neutral element. The duality between t-norms and t-conorms is expressed by the fact that from any t-norm T we can obtain its dual t-conorm S by the equation $S(x,y) = 1 - T(1-x, 1-y)$ and vice-versa.

Proposition 1 ([11]): Let $t : [0,1] \longrightarrow [0,\infty]$ ($s : [0,1] \longrightarrow [0,\infty]$) be a continuous strictly decreasing (increasing) function, such that $t(1) = 0$ ($s(0) = 0$). Then, the operation $T : [0,1]^2 \longrightarrow [0,1]$ ($S : [0,1]^2 \longrightarrow [0,1]$) given by

$$T(x,y) = t^{-1}(\min(t(0), t(x) + t(y)))$$

$$S(x,y) = s^{-1}(\min(s(1), s(x) + s(y)))$$

is a continuous t-norm (t-conorm). The function t (s) is called an *additive generator* of T (S).

An additive generator of an Archimedean continuous t-norm T (t-conorm S) is uniquely determined up to a positive multiplicative constant. Each continuous t-norm (t-conorm) is equal to an ordinal sum of continuous Archimedean t-norms (t-conorms). Note that a continuous t-norm (t-conorm) is Archimedean if and only if it has only trivial idempotent points 0 and 1. A continuous Archimedean t-norm T (t-conorm S) is either strict, that is, strictly increasing on $]0,1]^2$ (on $[0,1[^2$), or nilpotent, that is, there exists $(x,y) \in]0,1[^2$, such that $T(x,y) = 0$ ($S(x,y) = 1$). Moreover, each continuous Archimedean t-norm (t-conorm) has a continuous additive generator. More details on t-norms and t-conorms can be found in [1] and [11].

A uninorm (introduced in [33]) is a function $U : [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, nondecreasing in both variables, and has a neutral element $e \in]0,1[$ (see also [7]). If we take a uninorm in a broader sense, that is, if for a neutral element we have $e \in [0,1]$, then the class of uninorms covers also the class of t-norms and the class of t-conorms. In order to stress that we assume a uninorm with $e \in]0,1[$, we will call such a uninorm *proper*. For each uninorm, the value $U(1,0) \in \{0,1\}$ is the annihilator of U. A uninorm is called *conjunctive* (*disjunctive*) if $U(1,0) = 0$ ($U(1,0) = 1$). Due to the associativity, we can uniquely define n-ary form of any uninorm for any $n \in \mathbb{N}$ and therefore in some proofs we will use ternary form instead of binary, where suitable.

For each uninorm U with the neutral element $e \in [0,1]$, the restriction of U to $[0,e]^2$ is a t-norm on $[0,e]^2$, that is, a linear transformation of some t-norm T_U on $[0,1]^2$ and the restriction of U to $[e,1]^2$ is a t-conorm on $[e,1]^2$, that is, a linear transformation of some t-conorm S_U on $[0,1]^2$. Moreover, $\min(x,y) \le U(x,y) \le \max(x,y)$ for all $(x,y) \in [0,e] \times [e,1] \cup [e,1] \times [0,e]$. We will denote the set of all uninorms U, such that T_U and S_U are continuous by $\mathcal{U}$.

From any pair of a t-norm and a t-conorm, we can construct the minimal and the maximal uninorm with the given underlying functions.

Proposition 2 ([14]): Let $T : [0,1]^2 \longrightarrow [0,1]$ be a t-norm and $S : [0,1]^2 \longrightarrow [0,1]$ a t-conorm and assume $e \in]0,1[$. Then, the two functions $U_{\min}, U_{\max} : [0,1]^2 \longrightarrow [0,1]$ given by

$$U_{\min}(x,y) = \begin{cases} e \cdot T\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if (x,y)} \in [0,e]^2 \\ e + (1-e) \cdot S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if(x,y)} \in [e,1]^2 \\ \min(x,y) & \text{otherwise} \end{cases}$$

and

$$U_{\max}(x,y) = \begin{cases} e \cdot T\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if}(x,y) \in [0,e]^2 \\ e + (1-e) \cdot S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if}(x,y) \in [e,1]^2 \\ \max(x,y) & \text{otherwise} \end{cases}$$

are uninorms. We will denote the set of all uninorms of the first type by $\mathcal{U}_{\min}$ and of the second type by $\mathcal{U}_{\max}$.

Similarly as in the case of t-norms and t-conorms, we can construct uninorms using additive generators (see [7]).

Proposition 3 ([7]): Let $f : [0,1] \longrightarrow [-\infty,\infty]$, $f(0) = -\infty$, $f(1) = \infty$ be a continuous strictly increasing function. Then, a function $U : [0,1]^2 \longrightarrow [0,1]$ given by

$$U(x,y) = f^{-1}(f(x) + f(y))$$

where $f^{-1} : [-\infty,\infty] \longrightarrow [0,1]$ is an inverse function to f, with the convention $\infty + (-\infty) = \infty$ (or $\infty + (-\infty) = -\infty$), is a uninorm, which will be called a representable uninorm.

Since f is continuous and $f(0) = -\infty$, $f(1) = \infty$, there exists an $e \in]0,1[$ such that $f(e) = 0$. The point e is then the neutral element of the uninorm U. Note that if we relax the strict monotonicity of the additive generator, then the neutral element will be lost and by relaxing the condition $f(0) = -\infty$, $f(1) = \infty$ the associativity will be lost (if $f(0) < 0$ and $f(1) > 0$). In [29] (see also [23]) we can find the following result.

Proposition 4 ([29]): Let $U : [0,1]^2 \longrightarrow [0,1]$ be a uninorm continuous everywhere on the unit square except of the two points $(0,1)$ and $(1,0)$. Then, U is representable.

For our examples we will use the following ordinal sum construction introduced by Clifford.

Theorem 1 ([3]): Let $A \neq \emptyset$ be a totally ordered set and $(G_\alpha)_{\alpha \in A}$ with $G_\alpha = (X_\alpha, *_\alpha)$ be a family of semigroups. Assume that for all $\alpha, \beta \in A$ with $\alpha < \beta$ the sets X_α and X_β are either disjoint or that $X_\alpha \cap X_\beta = \{x_{\alpha,\beta}\}$, where $x_{\alpha,\beta}$ is both the neutral element of G_α and the annihilator of G_β and where for each $\gamma \in A$ with $\alpha < \gamma < \beta$ we have $X_\gamma = \{x_{\alpha,\beta}\}$. Put $X = \bigcup_{\alpha \in A} X_\alpha$ and define the binary operation $*$ on X by

$$x * y = \begin{cases} x *_\alpha y & \text{if } (x,y) \in X_\alpha \times X_\alpha, \\ x & \text{if } (x,y) \in X_\alpha \times X_\beta \text{ and } \alpha < \beta, \\ y & \text{if } (x,y) \in X_\alpha \times X_\beta \text{ and } \alpha > \beta. \end{cases}$$

Then, $G = (X,*)$ is a semigroup. The semigroup G is commutative if and only if for each $\alpha \in A$ the semigroup G_α is commutative.

Therefore, in our examples, the commutativity and the associativity of the corresponding ordinal sum uninorm will follow from Theorem 1. Monotonicity and the neutral element can be then easily checked by the reader.

Furthermore, we will use the following transformation. For any $0 \le a < b \le c < d \le 1$, $v \in [b,c]$, and a uninorm U with the neutral element $e \in]0,1[$ let $f : [0,1] \longrightarrow [a,b[\cup\{v\}\cup]c,d]$ be given by

$$f(x) = \begin{cases} (b-a) \cdot \frac{x}{e} + a & \text{if } x \in [0,e[, \\ v & \text{if } x = e, \\ d - \frac{(1-x)(d-c)}{(1-e)} & \text{otherwise.} \end{cases} \tag{1}$$

Then, f is linear on $[0,e[$ and on $]e,1]$ and thus it is a piecewise linear isomorphism of $[0,1]$ to $([a,b] [\cup\{v\}\cup] [c,d])$ and a function $U_v^{a,b,c,d} : ([a,b[\cup\{v\}\cup]c,d])^2 \longrightarrow ([a,b[\cup\{v\}\cup]c,d])$ given by

$$U_v^{a,b,c,d}(x,y) = f(U(f^{-1}(x), f^{-1}(y))) \tag{2}$$

is an operation on $([a,b[\cup\{v\}\cup]c,d])^2$ which is commutative, associative, nondecreasing in both variables (with respect to the standard order) and v is its neutral element.

Example 1: Assume $U_1 \in \mathcal{U}_{\min}$ and $U_2 \in \mathcal{U}_{\max}$ with respective neutral elements e_1, e_2. Then, U_1 is an ordinal sum of semigroups $G_\alpha = ([0,e[, T^*_{U_1})$ and $G_\beta = ([e,1], S^*_{U_1})$ with $\alpha < \beta$, where $T^*_{U_1} = U_1|_{[0,e_1]^2}$ and $S^*_{U_1} = U_1|_{[e_1,1]^2}$. Similarly, U_2 is an ordinal sum of semigroups $G_\alpha = ([0,e], T^*_{U_2})$ and $G_\beta = (]e,1], S^*_{U_2})$ with $\alpha > \beta$. If all underlying operations are continuous, then the set of discontinuity points of U_1 is equal to the set $S_1 = \{e\}\times]e,1]\cup]e,1] \times \{e\}$ and the set of discontinu-

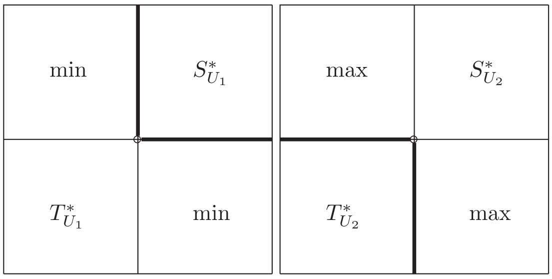

Fig. 1. Uninorm U_1 (left) and the uninorm U_2 (right) from Example 1. The bold lines denote the points of discontinuity of U_1 and U_2.

ity points of U_2 is equal to the set $S_2 = \{e\} \times [0, e[\cup[0, e[\times\{e\}$. Both uninorms can be seen in Fig. 1 .

More detailed discussion on the ordinal sum construction for uninorms can be found in [26].

III. Characterization of Uninorms $U \in \mathcal{U}$ by Means of Special Set-Valued Functions

In this section, we will show that for a uninorm U we have $U \in \mathcal{U}$ if and only if U is continuous on $[0,1]^2 \setminus G(r)$, where $G(r)$ is the graph of a special symmetric, u-surjective, nonincreasing set-valued function r and U is in each point $(x, y) \in [0,1]^2$ either left-continuous, or right-continuous (or both, in which case it is continuous). In the first part, we will focus on the necessity part, that is, we will show that each uninorm $U \in \mathcal{U}$ is continuous on $[0,1]^2 \setminus G(r)$, where $G(r)$ is the graph of some symmetric, u-surjective, nonincreasing set-valued function r (see Theorem 2). We will also show (see Theorem 3) that $U \in \mathcal{U}$ implies that U is in each point $(x, y) \in [0,1]^2$ either left-continuous, or right-continuous (or both, in which case it is continuous).

A. Necessity Part

The following lemmas and propositions are necessary for the proofs of Theorems 2 and 3.

Lemma 1 ([25]): Each uninorm $U : [0,1]^2 \longrightarrow [0,1]$, $U \in \mathcal{U}$, is continuous in (e, e).

Next, we show that for $x, y \in [0,1]$ we have $U(x, y) = \min(x, y)$ or $U(x, y) = \max(x, y)$ if x is an idempotent element of U.

Lemma 2: Let $U : [0,1]^2 \longrightarrow [0,1]$ be a uninorm and let $U \in \mathcal{U}$. If $a \in [0,1]$ is an idempotent point of U then U is internal on $\{a\} \times [0,1]$, that is, $U(a, x) \in \{x, a\}$ for all $x \in [0,1]$.

Proof: If $a = e$ the result is obvious. Suppose $a < e$ (the case when $a > e$ is analogous). Since T_U is continuous, we have $U(a, x) = \min(a, x)$ if $x \in [0, e]$. Suppose that there exists $y \in [e, 1]$, such that $U(a, y) = c \in]a, y[$. Then, $U(a, c) = U(a, a, y) = U(a, y) = c$ and if $c \le e$ then $c = U(a, c) \le a$ which is a contradiction. Thus, $y > c > e$. Then, since S_U is continuous, there exists a y_1 such that $U(c, y_1) = y$. Then, however,

$$U(a, y) = U(a, c, y_1) = U(c, y_1) = y$$

which is again a contradiction. Thus, U is internal on $\{a\} \times [0, 1]$. ■

For a given uninorm $U : [0,1]^2 \longrightarrow [0,1]$ and each $x \in [0,1]$, we define a function $u_x : [0,1] \longrightarrow [0,1]$ by $u_x(z) = U(x, z)$ for $z \in [0,1]$.

Lemma 3: Let $U : [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{U}$, and assume $x \in [0,1]$. The function u_x is continuous if and only if one of the following conditions:

1) $u_x(1) < e$;
2) $u_x(0) > e$;
3) $e \in \mathrm{Ran}(u_x)$

is satisfied.

Proof: If $e \in \mathrm{Ran}(u_x)$, then there exists a $y \in [0,1]$, such that $U(x, y) = e$. Since U is monotone continuity of u_x is equivalent with the equality $\mathrm{Ran}(u_x) = [a, b]$ for some $a = U(0, x)$ and $b = U(1, x)$. Assume $c \in [0,1]$. Then, $U(x, y, c) = c$ and for $z = U(y, c)$ we have $u_x(z) = c$, that is, $\mathrm{Ran}(u_x) = [0,1]$. If $u_x(1) = v < e$ (the case when $u_x(0) > e$ can be shown similarly), then due to the monotonicity the continuity of u_x is equivalent with the equality $\mathrm{Ran}(u_x) = [0, v]$. Assume $w \in [0, v]$. Since T_U is continuous there exists a $q \in [0, e]$ such that $U(v, q) = w$, that is, $U(x, 1, q) = w$ and then $u_x(U(1, q)) = w$. Therefore, $\mathrm{Ran}(u_x) = [0, v]$.

Vice versa, if u_x is continuous and $u_x(0) \le e \le u_x(1)$ then evidently $e \in \mathrm{Ran}(u_x)$. ■

Example 2: For a representable uninorm U, the function u_x is continuous for all $x \in]0,1[$. Moreover, if U is conjunctive (disjunctive), then u_0 (u_1) is continuous and u_1 (u_0) is noncontinuous in 0 (1). For a uninorm $U \in \mathcal{U}_{\max}$ ($U \in \mathcal{U}_{\min}$) u_x is continuous for all $x \in [e, 1]$ ($x \in [0, e]$) and u_x is noncontinuous in e for all $x \in [0, e[$ ($x \in]e, 1]$).

Now we recall a result [12, Proposition 1] which shows a connection between continuity on cuts and joint continuity of a monotone function.

Proposition 5: Let $f(x, y)$ be a real-valued function defined on an open set G in the plane. Suppose that $f(x, y)$ is continuous in x and y separately and is monotone in x for each y. Then, $f(x, y)$ is (jointly) continuous on the set G.

Proposition 6: Let $U : [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{U}$. Then, for each $x \in [0,1]$, there is at most one point of discontinuity of u_x. Furthermore, if u_x is noncontinuous in $y \in [0,1]$, then $U(x, z) < e$ for all $z < y$ and $U(x, z) > e$ for all $z > y$.

Proof: If u_x is noncontinuous, then Lemma 3 implies $e \notin \mathrm{Ran}(u_x)$, $u_x(0) < e$ and $u_x(1) > e$. We will denote

$$f = \sup\{U(x, y) \mid y \in [0,1], U(x, y) \le e\}$$

and

$$g = \inf\{U(x, y) \mid y \in [0,1], U(x, y) \ge e\}.$$

Note that the inequality $u_x(0) < e$ ($u_x(1) > e$) implies that f is the supremum (g is the infimum) of a nonempty set. Fix arbitrary $f_1 < f$. Then, there exist an $s > 0$ and y_f, such that $f_1 \le f - s \le U(x, y_f) \le f < e$ because f is the supremum. Since $U(U(x, y_f), 0) = 0$, $U(U(x, y_f), e) = U(x, y_f)$ and T_U is continuous, there exists an f_3 such that $U(U(x, y_f), f_3) = f_1$. Therefore, $U(x, U(y_f, f_3)) = f_1$ and $f_1 \in \mathrm{Ran}(u_x)$. Similarly, for each $g_1 > g$, there is $g_1 \in \mathrm{Ran}(u_x)$. Therefore, $[0,1] \setminus \mathrm{Ran}(u_x)$ is a connected set. Since u_x is monotone it has only one point of discontinuity. Also, if u_x is noncontinuous in $y \in [0,1]$, then $U(x,z) < e$ for all $z < y$ and $U(x,z) > e$ for all $z > y$. ∎

The following result shows that if $U(a,b) = e$, then U is continuous in the point (a,b). First, however, we introduce three useful lemmas.

Lemma 4: Let $U : [0,1]^2 \longrightarrow [0,1]$ be a uninorm with the neutral element $e \in [0,1]$. Then, if $U(a,b) = e$, for some $a, b \in [0,1]$, there is either $a = b = e$, or a and b are not idempotent elements of U.

Proof: If a is an idempotent point (similarly for b), then

$$e = U(a,b) = U(a, U(a,b)) = U(a,e) = a$$

and

$$e = U(a,b) = U(e,b) = b$$

i.e., $a = b = e$. ∎

Lemma 5: Let $U : [0,1]^2 \longrightarrow [0,1]$ be a uninorm with the neutral element $e \in [0,1]$. Then, if $U(a,b) = e$, for some $a, b \in [0,1]$, there is either $a = b = e$, or $a < e, b > e$, or $a > e, b < e$.

Proof: If $a = e$, then evidently also $b = e$. If $a < e$, then $b \neq e$ and we have

$$e = U(a,b) \leq U(e,b) = b$$

i.e., $e < b$. Finally, if $a > e$, then $b \neq e$ and we have

$$e = U(a,b) \geq U(e,b) = b$$

i.e., $e > b$. ∎

Lemma 6: Let $U : [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{U}$, with the neutral element $e \in [0,1]$. If $u_x(y) < e$ ($u_x(y) > e$) for some $x, y \in [0,1]$, then u_x is continuous on $[0,y]$ ($[y,1]$).

Proof: We will show only that $u_x(y) < e$ implies the continuity of u_x on $[0,y]$. The continuity of u_x on $[y,1]$ following from $u_x(y) > e$ can be shown analogously. If $u_x(y) < e$ the continuity of T_U ensures (similarly as in Proposition 6) that the range of u_x on $[0,y]$ is a connected set. Since u_x is monotone, this means that u_x is continuous on $[0,y]$. ∎

Proposition 7: Let $U : [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{U}$. If $U(a,b) = e$ for some $a, b \in [0,1]$, $a < e$, then U is continuous on $[0,1]^2 \setminus ([0,a] \cup [b,1])^2$. ∎

Proof: Since $a < e$ Lemma 4 implies that a and b are not idempotent elements of U and Lemma 5 implies that $b > e$. From Lemma 3 we know that u_a and u_b are continuous functions. Next, we will show that for all $f \in]a,b[$ there exists a $v^f \in [0,1]$, such that $U(f, v^f) = e$. Assume $f \in]a,e]$ (for $f \in [e,b[$ the proof is analogous). Since T_U is continuous and $U(a,f) \leq a$, $U(f,e) = f$ there exists an $a^f \in [0,e]$ such that $U(f, a^f) = a$. Then

$$e = U(a,b) = U(f, a^f, b)$$

and if $v^f = U(a^f, b)$, then $U(f, v^f) = e$. Summarizing, we get that for all $x \in [a,b]$ the function u_x is continuous. Now, since a and b are not idempotents, we have $U(a,a) = p < a$, $U(b,b) = q > b$ and $U(a,a,b,b) = e$. Therefore, also all u_x for $x \in [p,q]$ are continuous and thus U is continuous separately in x and in y on $[p,q]^2$. Moreover, since $U(p - \varepsilon, q) < e$ for all $\varepsilon > 0$ with $\varepsilon \leq p$, Lemma 6 implies that U is continuous separately in x and in y also on $[0,p] \times [p,q]$ (and by symmetry also on $[p,q] \times [0,p]$). Furthermore, $U(p, q + \varepsilon) > e$ for all $\varepsilon > 0$ with $\varepsilon \leq 1 - q$ and Lemma 6 implies that U is continuous separately in x and in y on $[p,q] \times [q,1]$ (and by symmetry also on $[q,1] \times [p,q]$). Summarizing, U is continuous separately in x and in y on $[p,q] \times [0,1] \cup [0,1] \times [p,q]$ and then Proposition 5 implies the result.

Proposition 8: Let $U : [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{U}$. Then, there exist idempotent points $a, d \in [0,1]$, $a \leq e \leq d$, such that for an $x \in [0,1]$ there exists a $y \in [0,1]$ with $U(x,y) = e$ if and only if $x \in]a,d[\cup \{e\}$. Moreover, U is continuous on $]a,d[\times [0,1] \cup [0,1] \times]a,d[$.

Proof: For $U \in \mathcal{U}$ there either $U(x,y) = e$ implies $x = y = e$, or there exists an $x \neq e$ such that $U(x,y) = e$ for some $y \in [0,1]$. We will focus on the second case. Then, Lemma 5 implies that either $x < e$, $y > e$, or $x > e$, $y < e$. We will suppose that $x < e$ and $y > e$ (as the other case is analogous). Lemma 4 shows that the points x and y are not idempotents. Since both underlying functions are continuous, the points $a, d \in [0,1]$, given by

$$a = \lim_{n \longrightarrow \infty} U(\underbrace{x, \ldots, x}_{n\text{-times}})$$

and

$$d = \lim_{n \longrightarrow \infty} U(\underbrace{y, \ldots, y}_{n\text{-times}})$$

are idempotent. Furthermore, the monotonicity of U implies $U(\underbrace{x, \ldots, x}_{n\text{-times}}) < e < U(\underbrace{y, \ldots, y}_{n\text{-times}})$ for all $n \in \mathbb{N}$ and therefore we have $a < e < d$.

First we will show that if $b \in]a,d[\cup \{e\}$, then there exists a $c \in [0,1]$ such that $U(b,c) = e$. Since $b \in]a,d[\cup \{e\}$ there exists an $n \in \mathbb{N}$ such that

$$b \in \left[U(\underbrace{x, \ldots, x}_{n\text{-times}}), U(\underbrace{y, \ldots, y}_{n\text{-times}}) \right].$$

Furthermore, associativity implies

$$U(\underbrace{x, \ldots, x}_{n\text{-times}}, \underbrace{y, \ldots, y}_{n\text{-times}}) = e$$

for all $n \in \mathbb{N}$ and similarly as in the proof of Proposition 7 we can show that for all

$$z \in \left[U(\underbrace{x, \ldots, x}_{n\text{-times}}), U(\underbrace{y, \ldots, y}_{n\text{-times}}) \right]$$

there exists a $q \in [0,1]$ such that $U(z,q) = e$. Thus, there exists a $c \in [0,1]$, such that $U(b,c) = e$.

Now we will show that if for an $b \in [0,1]$ there exists a $c \in [0,1]$ such that $U(b,c) = e$, then $b \in]a,d[\cup \{e\}$. Suppose that $b < a$ (the case when $b > d$ is analogous). Then, similarly as in the proof of Proposition 7 we can show that there exists a $z \in [0,1]$ such that $U(a,z) = e$, which is not possible due to Lemma 4. Therefore, $b \in]a,d[\cup \{e\}$.

Proposition 7 implies that U is continuous on $]x,y[\times [0,1] \cup [0,1] \times]x,y[$ for any $x, y \in]a,d[$ and thus, taking the union, we

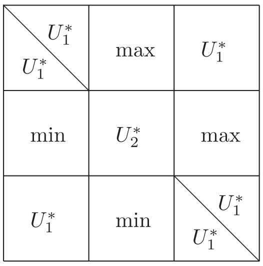

Fig. 2. Uninorm U from Example 3. The oblique lines denote the points of discontinuity of U.

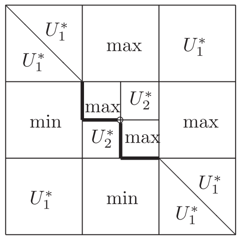

Fig. 3. Uninorm U from Example 4. The oblique and bold lines denote the points of discontinuity of U.

see that U is continuous on $]a,d[\times[0,1]\cup[0,1]\times]a,d[$. In order to include also the case when $U(x,y)=e$ implies $x=y=e$, we can generally say that for an $x\in[0,1]$ there exists a $y\in[0,1]$ such that $U(x,y)=e$ if and only if $x\in]a,d[\cup\{e\}$. Note that in the case when $U(x,y)=e$ implies $x=y=e$, we take $a=e=d$. ■

Example 3: Assume two representable uninorms $U_1,U_2:[0,1]^2\longrightarrow[0,1]$ with respective neutral elements e_1, e_2. Let U_1^* be a transformation of U_1 to $([0,\frac{1}{3}[\cup\{v\}\cup]\frac{2}{3},1])^2$ given by (2), where $v=\frac{1}{3}$ $(v=\frac{2}{3})$ if U_2 is conjunctive (disjunctive), and let U_2^* be a linear transformation of U_2 to $\left[\frac{1}{3},\frac{2}{3}\right]^2$. Then, the ordinal sum of semigroups $G_\alpha=([0,\frac{1}{3}[\cup\{v\}\cup]\frac{2}{3},1],U_1^*)$,$G_\beta=(\left[\frac{1}{3},\frac{2}{3}\right],U_2^*)$, with $\alpha<\beta$, is a semigroup $([0,1],U)$, where U is a uninorm with the neutral element $e=\frac{e_2+1}{3}$. We can find the structure of U on Fig. 2 . All points of discontinuity of U except $(0,1),(0,1)$ correspond to the transformation of the points $(x,y)\in[0,1]^2$ such that $U_1(x,y)=e_1$. For simplicity, we will assume that $U_1(x,1-x)=e_1=\frac{1}{2}$ for all $x\in[0,1]$. Moreover, for every $a\in]\frac{1}{3},\frac{2}{3}[$ there exists a $b\in]\frac{1}{3},\frac{2}{3}[$ such that $U(a,b)=e$. The previous result then implies that U is continuous in every point from $]\frac{1}{3},\frac{2}{3}[\times[0,1]$ and from $[0,1]\times]\frac{1}{3},\frac{2}{3}[$.

In the following results we will continue to investigate properties of the function u_x.

Proposition 9: Let $U:[0,1]^2\longrightarrow[0,1]$ be a uninorm, $U\in\mathcal{U}$. Then, for all $x\in[0,1]$ the function u_x is either left-continuous or right-continuous (or both, in which case it is continuous).

Proof: Assume $x\in[0,1]$. From Proposition 6 we know that u_x is noncontinuous in at most one point, and thus we will suppose that u_x is noncontinuous in the point $p\in[0,1]$. Furthermore, from the proof of the same proposition, we know that $[0,1]\setminus\mathrm{Ran}(u_x)$ is a connected set, that is, an interval I, and $u_x(p)$ is an end point of the interval I. Then, it is evident that if $u_x(p)=\inf I$ then u_x is left-continuous and $u_x(p)<e$, and if $u_x(p)=\sup I$ then u_x is right-continuous and $u_x(p)>e$. ■

Remark 1: From the proof of Proposition 9 we see that if $u_x(p)<e$ for some $p\in[0,1]$, then u_x is left-continuous on $[0,p]$ and if $u_x(p)>e$ then u_x is right-continuous on $[p,1]$.

Next we will show that the points of discontinuity of u_x are nonincreasing with respect to $x\in[0,1]$.

Proposition 10: Let $U:[0,1]^2\longrightarrow[0,1]$ be a uninorm, $U\in\mathcal{U}$. Suppose that for $x,x_1\in[0,1]$, $x_1<x$, the functions u_x and u_{x_1} are noncontinuous in points y and y_1, respectively. Then $y_1\geq y$.

Proof: From the proof of Proposition 6 we see that if u_x is noncontinuous in y, then $U(x,z)<e$ for all $z<y$ and $U(x,z)>e$ for all $z>y$. The monotonicity implies $U(x_1,z)\leq U(x,z)<e$ for $z<y$. However, if u_{x_1} is noncontinuous in y_1, then by Proposition 6 there is $U(x_1,z)>e$ for all $z>y_1$ and therefore $y_1\geq y$. ■

Corollary 1: Let $U:[0,1]^2\longrightarrow[0,1]$ be a uninorm, $U\in\mathcal{U}$. If u_{x_1} is noncontinuous in y and u_{x_2} is noncontinuous in y for some $x_1<x_2$, then u_x is noncontinuous in y for all $x\in[x_1,x_2]$.

Proof: Assume $x\in[x_1,x_2]$. Since u_{x_1} is noncontinuous in y, we have $U(x_1,z)>e$ for all $z>y$ and the monotonicity gives $U(x,z)>e$ for all $z>y$. Since u_{x_2} is noncontinuous in y, we have $U(x_2,z)<e$ for all $z<y$ and the monotonicity gives $U(x,z)<e$ for all $z<y$. Thus, u_x is either noncontinuous in y or $U(x,y)=e$. Assume that $U(x,y)=e$. If $x=y=e$, then $x_1<e<x_2$ and we get

$$e<U(x_1,x_2)<e$$

which is a contradiction. Therefore, by Lemma 4, the points x and y are not idempotent elements of U and $x\neq e$, $y\neq e$.

Suppose that $y>e$. Then, $U(x,x,y,y)=e$ with $U(x_1,y,y)>e$ implies $U(x,x)<x_1<x$ and by Proposition 7 U is continuous on $[0,1]^2\setminus([0,U(x,x)]\cup[U(y,y),1])^2$. Then, however, u_{x_1} is continuous, which is a contradiction.

In the case when $y<e$ then $U(x,x,y,y)=e$ with $U(x_2,y,y)<e$ implies $x<x_2<U(x,x)$ and using Proposition 7 again we obtain that u_{x_2} is continuous, which is a contradiction. Therefore, in both cases $U(x,y)\neq e$ and thus u_x is noncontinuous in y. ■

Example 4: Assume a representable uninorm $U_1:[0,1]^2\longrightarrow[0,1]$ with the neutral element e_1 and a uninorm $U_2\in\mathcal{U}_{\max}$ with the neutral element $e_2=\frac{1}{2}$. Let U_1^* be a transformation of U_1 to $([0,\frac{1}{3}[\cup\left[\frac{2}{3},1\right])^2$ given by (2), and let U_2^* be a linear transformation of U_2 to $\left[\frac{1}{3},\frac{2}{3}\right]^2$. Then, the ordinal sum of semigroups $G_\alpha=([0,\frac{1}{3}[\cup\left[\frac{2}{3},1\right],U_1^*)$,$G_\beta=(\left[\frac{1}{3},\frac{2}{3}\right],U_2^*)$, with $\alpha<\beta$, is a semigroup $([0,1],U)$, where U is a uninorm. We can find the structure of U in Fig. 3 . Here, $u_{\frac{1}{2}}$ is continuous and u_0 (u_1) is continuous if U_1 is conjunctive (disjunctive). In all other cases, u_x is noncontinuous. Furthermore, $u_{\frac{1}{3}}$ is noncontinuous in $e=\frac{1}{2}$ and $u_{\frac{2}{3}}$ is noncontinuous in $\frac{1}{3}$.

Now we will show how can be a point of discontinuity of a uninorm U related to the noncontinuity of corresponding functions u_x.

Proposition 11: Let $U : [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{U}$. Then, U is noncontinuous in $(x_0, y_0) \in [0,1]^2$, $(x_0, y_0) \neq (e, e)$, if and only if one of the following is satisfied.

1) u_{x_0} is noncontinuous in y_0.
2) u_{y_0} is non-continuous in x_0.
3) There exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, such that u_z is noncontinuous in x_0 and u_v is noncontinuous in y_0 either for all $z \in]y_0, y_0 + \varepsilon_1]$, $v \in]x_0, x_0 + \varepsilon_2]$, or for all $z \in [y_0 - \varepsilon_1, y_0[$, $v \in [x_0 - \varepsilon_2, x_0[$.

Proof: Suppose that U is noncontinuous in $(x_0, y_0) \in [0,1]^2$. Then, due to Proposition 8 and Lemma 1 there is $U(x_0, y_0) \neq e$. Since T_U and S_U are continuous, we have $(x_0, y_0) \in [0, e] \times [e, 1] \cup [e, 1] \times [0, e]$. We will assume $(x_0, y_0) \in [0, e] \times [e, 1]$ (the other case is analogous). From Proposition 5 it follows that if U is noncontinuous in $(x_0, y_0) \in [0,1]^2$, then for all $\delta_1 > 0$ and all $\delta_2 > 0$ there exist $x \in]x_0 - \delta_1, x_0 + \delta_1[$ and $y \in]y_0 - \delta_2, y_0 + \delta_2[$ such that either u_x is noncontinuous in y or u_y is noncontinuous in x. Thus, U on $[x_0 - \delta_1, x_0 + \delta_1] \times [y_0 - \delta_2, y_0 + \delta_2]$ attain values smaller than e and bigger than e as well. Let W be a subset of $[0,1]^2$, such that $(x, y) \in W$ if $U(x_1, y_1) < e$ for all $x_1 < x, y_1 < y$ and $U(x_2, y_2) > e$ for all $x_2 > x, y_2 > y$. Then, the set $[x_0 - \delta_1, x_0 + \delta_1] \times [y_0 - \delta_2, y_0 + \delta_2] \cap W$ is nonempty for all $\delta_1 > 0$ and all $\delta_2 > 0$. Thus, $(x_0, y_0) \in W$.

If u_{x_0} is continuous in y_0, then there exists an $\varepsilon_1 > 0$, such that either $u_{x_0}(z) < e$ for all $z \in [y_0 - \varepsilon_1, y_0 + \varepsilon_1]$ or $u_{x_0}(z) > e$ for all $z \in [y_0 - \varepsilon_1, y_0 + \varepsilon_1]$. Similarly, if u_{y_0} is continuous in x_0, then there exists an $\varepsilon_2 > 0$, such that either $u_{y_0}(v) < e$ for all $v \in [x_0 - \varepsilon_2, x_0 + \varepsilon_2]$ or $u_{y_0}(v) > e$ for all $v \in [x_0 - \varepsilon_2, x_0 + \varepsilon_2]$. Since we cannot have both $U(x_0, y_0) < e$ and $U(x_0, y_0) > e$ we have either $u_{y_0}(v) < e$ and $u_{x_0}(z) < e$ for all $z \in [y_0 - \varepsilon_1, y_0 + \varepsilon_1]$ and all $v \in [x_0 - \varepsilon_2, x_0 + \varepsilon_2]$, or $u_{y_0}(v) > e$ and $u_{x_0}(z) > e$ for all $z \in [y_0 - \varepsilon_1, y_0 + \varepsilon_1]$ and all $v \in [x_0 - \varepsilon_2, x_0 + \varepsilon_2]$. As these two cases are analogous, we will assume $u_{y_0}(v) < e$ and $u_{x_0}(z) < e$ for all $z \in [y_0 - \varepsilon_1, y_0 + \varepsilon_1]$ and all $v \in [x_0 - \varepsilon_2, x_0 + \varepsilon_2]$. Then, $U(x_0, y) < e$ for $y \in [y_0 - \varepsilon_1, y_0 + \varepsilon_1]$ and $U(x, y_0) < e$ for $x \in [x_0 - \varepsilon_2, x_0 + \varepsilon_2]$. However, since $(x_0, y_0) \in W$, $U(f, g) > e$ for all $f > x_0, g > y_0$. Thus, u_z is noncontinuous in x_0 and u_v is noncontinuous in y_0 for all $z \in]y_0, y_0 + \varepsilon_1]$, $v \in]x_0, x_0 + \varepsilon_2]$.

Vice versa, if u_{x_0} is noncontinuous in y_0, or if u_{y_0} is noncontinuous in x_0, then evidently U is noncontinuous in (x_0, y_0). Suppose that there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, such that u_z is noncontinuous in x_0 and u_v is noncontinuous in y_0 either for all $z \in]y_0, y_0 + \varepsilon_1]$, $v \in]x_0, x_0 + \varepsilon_2]$, or for all $z \in [y_0 - \varepsilon_1, y_0[$, $v \in [x_0 - \varepsilon_2, x_0[$. Then, $(x_0, y_0) \in W$ and either $U(x_0, y_0) = e$, or U is noncontinuous in (x_0, y_0). However, if $U(x_0, y_0) = e$ then since $(x_0, y_0) \neq (e, e)$ Lemma 4 implies that x_0 and y_0 are not idempotents and Proposition 7 implies that U is continuous on $[0,1]^2 \setminus ([0, U(x_0, x_0)[\cup]U(y_0, y_0), 1])^2$ if $x_0 < e < y_0$ and on $[0,1]^2 \setminus ([0, U(y_0, y_0)[\cup]U(x_0, x_0), 1])^2$ if $x_0 > e > y_0$. In both cases, we obtain a contradiction with the noncontinuity of u_z and u_v. Therefore, $U(x_0, y_0) \neq e$ and thus U is noncontinuous in (x_0, y_0). ■

Example 5: Assume two t-norms $T_1, T_2 : [0,1]^2 \longrightarrow [0,1]$, such that T_2 has no zero divisors, and a t-conorm $S : [0,1]^2 \longrightarrow [0,1]$. Let T_1^* (T_2^*) be a linear transformation of T_1 (T_2) to $\left[0, \frac{1}{3}\right]^2$ ($\left[\frac{1}{3}, \frac{2}{3}\right]^2$), and let S_2^* be a linear transformation of S_2 to $\left[\frac{2}{3}, 1\right]^2$. Then, the ordinal sum of semigroups $G_\alpha = ([0, \frac{1}{3}[, T_1^*)$, $G_\beta = (\left[\frac{1}{3}, \frac{2}{3}\right], T_2^*)$, $G_\gamma = (\left[\frac{2}{3}, 1\right], S_2^*)$, with $\alpha < \gamma < \beta$, is a semigroup $([0,1], U)$, where U is a uninorm (see Fig. 4). If we define an operation $V : [0,1]^2 \longrightarrow [0,1]$ by

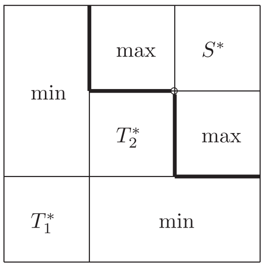

Fig. 4. Uninorm U from Example 5. The bold lines denote the points of discontinuity of U.

$$V = \begin{cases} \min(x, y) & \text{if } x = \frac{1}{3}, y \in \left[\frac{2}{3}, 1\right], \\ \min(x, y) & \text{if } y = \frac{1}{3}, x \in \left[\frac{2}{3}, 1\right], \\ U(x, y) & \text{otherwise} \end{cases}$$

then V is also a uninorm. Here, V is noncontinuous in the point $(\frac{1}{3}, \frac{2}{3})$, however, both $v_{\frac{1}{3}}$ and $v_{\frac{2}{3}}$ are continuous. Note that $([0,1], V)$ is an ordinal sum of semigroups G_α, G_γ and $G_{\beta^*} = (]\frac{1}{3}, \frac{2}{3}], T_2^*)$, $G_\delta = (\{\frac{1}{3}\}, T_2^*)$, where $\alpha < \delta < \gamma < \beta^*$.

The following two results show that the set of discontinuity points of a uninorm $U \in \mathcal{U}$ from the set $[0, e] \times [e, 1]$ ($[e, 1] \times [0, e]$) is connected.

Proposition 12: Let $U : [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{U}$. Let u_{x_1} be noncontinuous in y_1 and u_{x_2} be noncontinuous in y_2 for $x_1 < x_2 \leq e$ ($e \leq x_1 < x_2$). Then, for all $y \in [y_2, y_1]$ either there exists $x^* \in [x_1, x_2]$ such that u_{x^*} is noncontinuous in y or there is an interval $[c, d]$, where $y \in [c, d] \subset [0,1]$, and $p \in [x_1, x_2]$, such that u_z is noncontinuous in p for all $z \in [c, d]$.

Proof: If u_{x_1} is noncontinuous in y_1 and u_{x_2} is noncontinuous in y_2 for $x_1 < x_2 \leq e$ (the case when $e \leq x_1 < x_2$ is analogous), then $U(x_2, z) < e$ for all $z < y_2$ and $U(x_1, z) > e$ for all $z > y_1$ and the monotonicity implies that for all $x \in [x_1, x_2]$ the function u_x is noncontinuous in some point $z \in [y_2, y_1]$. Note that $e \notin \mathrm{Ran}(u_x)$ since otherwise by Proposition 7 either u_{x_1} or u_{x_2} would be continuous. Assume the function $g : [x_1, x_2] \longrightarrow [y_2, y_1]$ which assigns to $v \in [x_1, x_2]$ a point $w \in [y_2, y_1]$ such that u_v is noncontinuous in w. Then, by Proposition 10, the function g is nonincreasing. If $q \in [y_2, y_1] \setminus \mathrm{Ran}(g)$, then by the monotonicity there exists a $p \in [x_1, x_2]$, such that $g(d) > q$ if $d < p$ and $g(d) < q$ if $d > p$. Furthermore, since g is monotone, there exists an interval $[c, d]$, such that $q \in [c, d] \subset [y_2, y_1] \setminus \mathrm{Ran}(g)$. Then, for $z \in [c, d]$, we have $U(z, v) < e$ for all $v < p$ and $U(z, v) > e$ for all $v > p$ thus u_z has a point of discontinuity in p. ■

Lemma 7: Let $U : [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{U}$. Let u_x be noncontinuous in y_1 and u_{y_2} be noncontinuous in x for some $y_1 \neq y_2$. Then, for all $y \in [y_1, y_2]$ ($y \in [y_2, y_1]$), the function u_y is noncontinuous in x.

Proof: We will assume $y_1 < y_2$ (the case when $y_1 > y_2$ is analogous). Then, $U(x,y) > e$ for all $y > y_1$ and $U(z,y) \le U(z,y_2) < e$ for all $z < x, y \le y_2$. Then, since $U(x,y) \ne e$, the function u_y is noncontinuous in x. ■

In the following result, we show that the set of discontinuity points of a uninorm $U \in \mathcal{U}$ has a nonempty intersection with the border of the unit square.

Lemma 8: Let $U : [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{U}$. Assume $x < e$ ($x > e$) such that u_x is continuous on $[0,1]$ and let u_y be noncontinuous in x. Then, for all $q \in [y,1]$ ($q \in [0,y]$), the function u_q is noncontinuous in x.

Proof: We will assume $x < e$ (the case for $x > e$ is analogous). If $U(x,z) = e$ for some $z \in [0,1]$, then by Lemma 4 the points x, z are not idempotents and Proposition 7 implies that U is continuous on $[0,1]^2 \setminus ([0,a[\cup]b,1])^2$ for some $a < x$ and $b > z$. Therefore, for all $y \in [0,1]$, the function u_y is continuous in x, which is impossible. Since $x < e$, by Lemma 3, we have $u_x(1) < e$, that is, $U(x,z) < e$ for all $z \in [0,1]$. If u_y is noncontinuous in x, then $U(p,y) > e$ for all $p > x$ and $U(p,y) < e$ for all $p < x$. Assume any $q \in [y,1]$. Then, $U(p,q) \le U(x,q) < e$ if $p < x$ and $U(p,q) \ge U(p,y) > e$ if $p > x$, that is, u_q is noncontinuous in x. ■

Lemma 9: Let $U : [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{U}$ and let U be noncontinuous in $(x_0,y_0) \in [0,1]^2$. Then

1) for all $\varepsilon > 0$ such that $(x_0-\varepsilon, y_0-\varepsilon) \in [0,1]^2$ there is $U(x_0-\varepsilon, y_0-\varepsilon) < e$;
2) for all $\varepsilon > 0$ such that $(x_0+\varepsilon, y_0+\varepsilon) \in [0,1]^2$ there is $U(x_0+\varepsilon, y_0+\varepsilon) > e$.

Proof: The result easily follows from the monotonicity of U, Proposition 11, and the fact that if u_x is noncontinuous in y, then $u_x(y-\varepsilon) < e$ and $u_x(y+\varepsilon) > e$ for all $\varepsilon > 0$, such that $y-\varepsilon \in [0,1]$ ($y+\varepsilon \in [0,1]$). ■

Next we define a set-valued function.

Definition 1: A mapping $p : [0,1] \longrightarrow \mathcal{P}([0,1])$ is called a set-valued function on $[0,1]$ if to every $x \in [0,1]$ it assigns a subset of $[0,1]$, that is, $p(x) \subseteq [0,1]$. Assuming the standard order on $[0,1]$, a set-valued function p is called

1) *non-increasing* if for all $x_1, x_2 \in [0,1]$, $x_1 < x_2$, we have $y_1 \ge y_2$ for all $y_1 \in p(x_1)$ and all $y_2 \in p(x_2)$ and thus the cardinality $\mathrm{Card}(p(x_1) \cap p(x_2)) \le 1$,
2) *symmetric* if $y \in p(x)$ if and only if $x \in p(y)$.

The graph of a set-valued function p will be denoted by $G(p)$, that is, $(x,y) \in G(p)$ if and only if $y \in p(x)$.

Definition 2: A set-valued function $p : [0,1] \longrightarrow \mathcal{P}([0,1])$ is called *u-surjective* if for all $y \in [0,1]$ there exists an $x \in [0,1]$ such that $y \in p(x)$.

The following is evident.

Lemma 10: A symmetric set-valued function $p : [0,1] \longrightarrow \mathcal{P}([0,1])$ is u-surjective if and only if we have $p(x) \ne \emptyset$ for all $x \in [0,1]$.

The graph of a symmetric, u-surjective, nonincreasing set-valued function $p : [0,1] \longrightarrow \mathcal{P}([0,1])$ is a connected line (i.e., a connected set with no interior) containing points $(0,1)$ and $(1,0)$. Indeed, the monotonicity of such a set-valued function ensures that the graph of p has no interior. Furthermore, since p is u-surjective, monotone, and symmetric the graph of p contains points $(0,1)$ and $(1,0)$. If $G(p)$ is not a connected set, then either $p(x)$ is not a connected set for some $x \in [0,1]$, which, however, due to the monotonicity implies that p is not u-surjective, or due to the monotonicity there exists an $x \in [0,1]$ such that either

$$\inf\left(\bigcup_{q<x} p(q)\right) > \sup(p(x))$$

or

$$\sup\left(\bigcup_{q>x} p(q)\right) < \inf(p(x))$$

which, however, due to the symmetry implies that p is not u-surjective.

The previous results can be summarized in the following theorem. First, however, we introduce important remark and lemma.

Remark 2: For any uninorm $U : [0,1]^2 \longrightarrow [0,1]$, $U \in \mathcal{U}$ denote $A = \inf\{x \mid U(x,0) > 0\}$, $B = \sup\{x \mid U(x,1) < 1\}$ and let $a, d \in [0,1]$ be such that $U(x,y) = e$ for some $y \in [0,1]$ if and only if $x \in]a,d[\cup\{e\}$ (see Proposition 8). If U is conjunctive, i.e., $U(0,1) = 0$, then A is the infimum of an empty set on $[0,1]$, i.e., $A = 1$. If U is disjunctive, i.e., $U(0,1) = 1$, then B is the supremum of an empty set on $[0,1]$, i.e., $B = 0$. Therefore, we have either $A = 1, B \ne 0$, or $A \ne 1, B = 0$, or $A = 1, B = 0$. Furthermore, $U(x,0) \le e$ for some $x \in [0,1]$ implies

$$0 = U(e,0) \ge U(x,0,0) = U(x,0)$$

and thus for all $x \in [0,1]$ either $U(x,0) = 0$ or $U(x,0) > e$ (see also [22]). Therefore, U is noncontinuous in $(0,A)$ if $A \ne 1$. Similarly, $U(x,1) \ge e$ for some $x \in [0,1]$ implies

$$1 = U(e,1) \le U(x,1,1) = U(x,1)$$

and thus for all $x \in [0,1]$ either $U(x,1) = 1$ or $U(x,1) < e$ (see also [22]). Therefore U is noncontinuous in $(B,1)$ if $B \ne 0$. Finally, if $A = 1, B = 0$, then $U(x,0) = 0$ for all $x < 1$ and $U(x,1) = 1$ for all $x > 0$ and therefore U is noncontinuous in $(0,1)$. Due to Proposition 8 either $a = d = e$, or U is continuous on $]a,d[\times[0,1] \cup [0,1]\times]a,d[$ and therefore we have $0 \le B \le a \le e \le d \le A \le 1$.

Lemma 11: Let $U : [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{U}$. If U is noncontinuous in $(x_0,y_0) \in [0,1]^2$, then $(x_0,y_0) \in [B,a] \times [d,A] \cup [d,A] \times [B,a]$, where a, d are defined in Proposition 8 and A, B are defined in Remark 2.

Proof: Let U be noncontinuous in $(x_0,y_0) \in [0,1]^2$. Since $U \in \mathcal{U}$ we see that $(x_0,y_0) \notin [0,e]^2 \cup [e,1]^2$. Due to Proposition 8 $(x_0,y_0) \notin]a,d[\times[0,1] \cup [0,1]\times]a,d[$. Furthermore, due to Lemma 3, we see that u_x is continuous for all $x \in [0,1]$, such that either $x > A$ or $x < B$. Thus, Proposition 5 and Lemma 6 imply $(x_0,y_0) \notin]A,1] \times [0,1] \cup [0,1]\times]A,1]$ and $(x_0,y_0) \notin [0,B[\times[0,1] \cup [0,1] \times [0,B[$. Summarizing, $(x_0,y_0) \in [B,a] \times [d,A] \cup [d,A] \times [B,a]$. ■

Theorem 2: Let $U : [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{U}$. Then, there exists a symmetric, u-surjective, nonincreasing set-valued function r on $[0,1]$ such that U is continuous on $[0,1]^2 \setminus G(r)$ and $U(x,y) = e$ implies $(x,y) \in G(r)$ for all $(x,y) \in$

$[0,1]^2$. Note that U need not to be noncontinuous in all points from $G(r)$.

Proof: We will define the set $R^* = \{(x,y) \in [0,1]^2 \mid U$ is noncontinuous in $(x,y)\}$. Then, due to the commutativity of U the set R^* is symmetric, that is, $(x,y) \in R^*$ if and only if $(y,x) \in R^*$. If we define a set-valued function $r : [0,1] \longrightarrow \mathcal{P}([0,1])$ by

$$r(x) = \begin{cases} \{1\} & \text{if } x \in]0,B[, \\ \{0\} & \text{if } x \in]A,1[, \\ [0,B] & \text{if } x = 1, \\ [A,1] & \text{if } x = 0, \\ \{y \mid U(x,y) = e\} & \text{if } x \in]a,d[\cup\{e\}, \\ \{y \mid (x,y) \in R^*\} & \text{otherwise} \end{cases} \tag{3}$$

then r is a symmetric set-valued function (see Remark 2). Evidently, $U(x,y) = e$ implies $(x,y) \in G(r)$ for all $(x,y) \in [0,1]^2$.

Next we will show that r is u-surjective. Assume a $y \in [0,1]$. If u_y is noncontinuous in point $x \in [0,1]$, then due to the symmetry of r, we have $(x,y),(y,x) \in G(r)$, that is, $y \in r(x)$. Now suppose that u_y is continuous. Then, due to Lemma 3 and Proposition 7, there is $y \in [0,B[\cup]a,d[\cup\{e\}\cup]A,1]$. If $y \in]a,d[\cup\{e\}$, then Proposition 8 ensure the existence of an $x \in]a,d[\cup\{e\}$, such that $U(x,y) = e$ and then $y \in r(x)$. Suppose now that $y \in]A,1]$ (the case when $y \in [0,B[$ is analogous). Then, $A < 1$ and $B = 0$ and $y \in r(0)$. Summarizing all cases we see that r is u-surjective.

It is evident that if U is noncontinuous in (x_0,y_0), then $x_0 \in r(y_0)$.

Now we only have to show that r is nonincreasing. Note that $(e,e) \in G(r)$ and thus it is easy to see that if r is nonincreasing on $[0,e]$ by symmetry it is nonincreasing on the whole $[0,1]$.

1) First we will show that r is nonincreasing on $[0,B]$. If $B = 0$ the result is trivial. If $B > 0$, then due to Remark 2 there is $A = 1$ and $r(x) = \{1\}$ for all $x \in [0,B[$. Since $y \leq 1$ for all $y \in r(B)$ we see that r is nonincreasing on $[0,B]$.
2) Now we will show that r is nonincreasing on $[a,e]$. Due to Proposition 8, there is $\mathrm{Card}(r(x)) = 1$ for all $x \in]a,e[$ and since U is continuous on $]a,d[\times[0,1] \cup [0,1]\times]a,d[$ there is $\min(r(a)) = d$. The monotonicity of U implies monotonicity of r on $]a,d[$ and the fact that $y \leq d$ for all $y \in r(x)$ for all $x \in]a,d[$. Therefore, r is nonincreasing on $[a,e]$.
3) Finally, we will show that r is nonincreasing on $[B,a]$. From the definition of r, we see that if $x_1,x_2 \in [B,a]$, $x_1 < x_2$ and $y_1 \in r(x_1)$, $y_2 \in r(x_2)$, then $(x_1,y_1),(x_2,y_2) \in R^*$. Suppose that $y_1 < y_2$. Then, for any $\delta \in]0, \frac{\min(x_2-x_1,y_2-y_1)}{2}[$ Lemma 9 and the monotonicity of U give

$$e < U(x_1+\delta, y_1+\delta) \leq U(x_2-\delta, y_2-\delta) < e$$

which is a contradiction. Therefore, if $x_1 < x_2$, then $y_1 \geq y_2$ which means that r is nonincreasing on $[B,a]$.

If we summarize results for all partial intervals we obtain the main result, that is, that r is nonincreasing. ∎

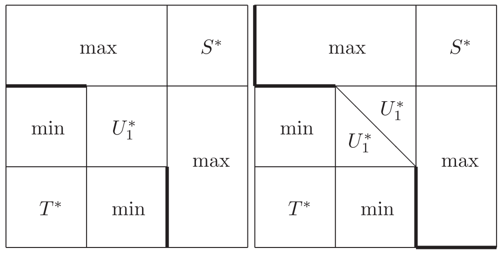

Fig. 5. Uninorm U from Example 6. Left: the bold lines denote the points of discontinuity of U. Right: the oblique and bold lines denote the characterizing set-valued function of U.

Remark 3: U need not to be noncontinuous in all points of $G(r)$. From the previous proof we see that U is continuous in all points from $\{x\} \times [0,1]$ for all $x \in [0,B[\cup]a,d[\cup\{e\}\cup]A,1]$. The symmetric nonincreasing set-valued function from the previous theorem will be called the *characterizing* set-valued function of a uninorm U for $U \in \mathcal{U}$.

Example 6: Assume a representable uninorm $U_1 : [0,1]^2 \longrightarrow [0,1]$ and a continuous t-norm $T : [0,1]^2 \longrightarrow [0,1]$ and a continuous t-conorm $S : [0,1]^2 \longrightarrow [0,1]$. For simplicity, we will assume that $\frac{1}{2}$ is the neutral element of U_1 and that $U_1(x,1-x) = \frac{1}{2}$ for all $x \in [0,1]$. Let U_1^* be a linear transformation of U_1 to $\left[\frac{1}{3},\frac{2}{3}\right]^2$, let T^* be a linear transformation of T to $\left[0,\frac{1}{3}\right]^2$ and let S^* be a linear transformation of S to $\left[\frac{2}{3},1\right]^2$. Then, the ordinal sum of semigroups $G_\alpha = (\left[0,\frac{1}{3}\right], T^*)$, $G_\beta = (\left[\frac{1}{3},\frac{2}{3}\right], U_1^*)$, $G_\gamma = (\left[\frac{2}{3},1\right], S^*)$, with $\gamma < \alpha < \beta$, is a semigroup $([0,1],U)$, where U is a uninorm, $U \in \mathcal{U}$. In Fig. 5 , we can see the characterizing set-valued function r of U as well as its set of discontinuity points.

Remark 4: It is easy to see that for $U \in \mathcal{U}$ its characterizing set-valued function r divides the uninorm U into two parts: U on points below the characterizing set-valued function attains values smaller than e, and U on points above the characterizing set-valued function attains values bigger than e.

Proposition 13: Let $U : [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{U}$. Then, in each point $(x_0,y_0) \in [0,1]^2$ the uninorm U is either left-continuous or right-continuous (or continuous).

Proof: From Proposition 9, we know that for all $x \in [0,1]$ the function u_x is either left-continuous or right-continuous. If (x_0,y_0) is the point of continuity of U the claim is trivial. Suppose that (x_0,y_0) is not the point of continuity of U. If $U(x_0,y_0) = e$, then by Proposition 7 the uninorm U is continuous in (x_0,y_0), which is impossible, and thus either $U(x_0,y_0) < e$ or $U(x_0,y_0) > e$. If $U(x_0,y_0) < e$, then for all $x \leq x_0$, $y \leq y_0$ also $U(x,y) < e$ and thus u_x is left-continuous in y and u_y is left-continuous in x (see Remark 1). Now for any $\varepsilon > 0$ there exists $\delta_1 > 0$, such that $|U(x_0-\delta_1,y_0) - U(x_0,y_0)| < \frac{\varepsilon}{2}$. Since also $u_{x_0-\delta_1}$ is left-continuous in y_0 there exists $\delta_2 > 0$, such that $|U(x_0-\delta_1,y_0-\delta_2) - U(x_0-\delta_1,y_0)| < \frac{\varepsilon}{2}$. The monotonicity of U then implies that

$$0 \leq U(x_0,y_0) - U(x_0-\delta_1,y_0-\delta_2) = U(x_0,y_0) -$$

$$U(x_0-\delta_1,y_0) + U(x_0-\delta_1,y_0) - U(x_0-\delta_1,y_0-\delta_2) < \varepsilon.$$

Taking $\delta = \min(\delta_1,\delta_2)$, by the monotonicity of U we have shown that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

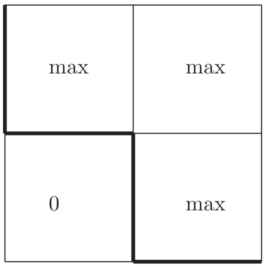

Fig. 6. Uninorm U from Example 7. The bold lines denote the characterizing set-valued function r of U.

if $x \in [x_0 - \delta, x_0]$ and $y \in [y_0 - \delta, y_0]$ we have $|U(x,y) - U(x_0, y_0)| < \varepsilon$, that is, that U is left-continuous in (x_0, y_0). Similarly, if $U(x_0, y_0) > e$, then U is right-continuous in (x_0, y_0). ∎

The previous proposition and the construction of the characterizing set-valued function r of a uninorm U implies the following.

Corollary 2: Let $U : [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{U}$. Then, there exists a symmetric, u-surjective, nonincreasing set-valued function r on $[0,1]$ such that U is continuous on $[0,1]^2 \setminus G(r)$ and if $U(x,y) = e$ then $(x,y) \in G(r)$. Moreover, in each point $(x,y) \in [0,1]^2$ the uninorm U is either left-continuous or right-continuous.

B. Sufficiency Part

In this part, we will show that if for a uninorm U there exists a symmetric, u-surjective, nonincreasing set-valued function r on $[0,1]$, such that U is continuous on $[0,1]^2 \setminus G(r)$, and $U(x,y) = e$ implies $(x,y) \in G(r)$, then $U \in \mathcal{U}$ if and only if in each point $(x,y) \in [0,1]^2$ the uninorm U is either left-continuous or right-continuous (or continuous).

We will denote the set of all uninorms $U : [0,1]^2 \longrightarrow [0,1]$ such that U is continuous on $[0,1]^2 \setminus G(r)$ and r is a symmetric, u-surjective, nonincreasing set-valued function, such that $U(x,y) = e$ implies $(x,y) \in G(r)$, by $\mathcal{UR}$. First, we will show that there exists a uninorm $U \in \mathcal{UR}$ such that $U \notin \mathcal{U}$.

Example 7: Let $U : [0,1]^2 \longrightarrow [0,1]$ be given by

$$U(x,y) = \begin{cases} 0 & \text{if } \max(x,y) < e, \\ x & \text{if } y = e, \\ y & \text{if } x = e, \\ \max(x,y) & \text{otherwise.} \end{cases}$$

Then, Proposition 2 implies that $U \in \mathcal{U}_{\max}$ is a uninorm, where the underlying t-norm is the drastic product and the underlying t-conorm is the maximum. This uninorm is noncontinuous in points from $\{e\} \times [0,e] \cup [0,e] \times \{e\}$. Thus, the corresponding set-valued function is given by (see Fig. 6)

$$r(x) = \begin{cases} [e,1] & \text{if } x = 0, \\ e & \text{if } x \in]0,e[, \\ [0,e] & \text{if } x = e, \\ 0 & \text{otherwise.} \end{cases}$$

Since $U(x,y) = e$ implies $x = y = e$, we see that U is continuous on $[0,1]^2 \setminus G(r)$ and r is a symmetric, u-surjective, nonincreasing set-valued function, such that $U(x,y) = e$ implies $(x,y) \in G(r)$. However, the drastic product t-norm is not continuous and thus $U \notin \mathcal{U}$.

Assume $U \in \mathcal{UR}$. Then, for the corresponding characterizing set-valued function r, we have $(e,e) \in G(r)$. Denote

$$D = \{e\} \times [0,1] \cup [0,1] \times \{e\}.$$

We have two possibilities: either $G(r) \cap D = \{(e,e)\}$, or $\mathrm{Card}(G(r) \cap D) > 1$. First, we will assume the case when $G(r) \cap D = \{(e,e)\}$. Then, T_U (S_U) is continuous in all points from $[0,e]^2$ ($[e,1]^2$) except possibly the point (e,e) and we have the following result.

Lemma 12: Let $T : [0,1]^2 \longrightarrow [0,1]$ be a t-norm which is continuous on $[0,1]^2 \setminus \{(1,1)\}$. Then, T is continuous on $[0,1]^2$.

Proof: Assume that T is not continuous in the point $(1,1)$. Then, there exist two sequences $\{a_n\}_{n\in\mathbb{N}}$, $a_n \in [0,1]$ and $\{b_n\}_{n\in\mathbb{N}}$, $b_n \in [0,1]$, such that $\lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty} b_n = 1$ and $\lim_{n\longrightarrow\infty} T(a_n, b_n) < 1$. Since $T(a_n, b_n) \geq T(\min(a_n, b_n), \min(a_n, b_n))$, we see that there exists a sequence $\{c_n\}_{n\in\mathbb{N}}$, $c_n \in [0,1]$, $\lim_{n\longrightarrow\infty} c_n = 1$, such that $\lim_{n\longrightarrow\infty} T(c_n, c_n) = 1 - \delta < 1$, for some $\delta > 0$. Since T is a t-norm, we have $T(1 - \frac{\delta}{2}, 1) = 1 - \frac{\delta}{2}$ and necessarily $T(1 - \frac{\delta}{2}, 1 - \frac{\delta}{2}) \leq 1 - \delta$. Since T is continuous on $[0,1]^2 \setminus \{(1,1)\}$, there exists an $\varepsilon > 0$ such that $T(1 - \frac{\delta}{2}, 1 - \varepsilon) = 1 - \frac{2\delta}{3}$ and the monotonicity of T implies $\varepsilon < \frac{\delta}{2}$. Thus,

$$1 - \frac{2\delta}{3} = T\left(1 - \frac{\delta}{2}, 1 - \varepsilon\right) \leq T(1 - \varepsilon, 1 - \varepsilon) \leq 1 - \delta$$

which is a contradiction. ∎

By duality between t-norms and t-conorms we get the following.

Lemma 13: Let $S : [0,1]^2 \longrightarrow [0,1]$ be a t-conorm which is continuous on $[0,1]^2 \setminus \{(0,0)\}$. Then, S is continuous on $[0,1]^2$.

From the two previous results, we see that if $U \in \mathcal{UR}$ and $G(r) \cap D = \{(e,e)\}$, then $U \in \mathcal{U}$.

Furthermore, we will suppose that $\mathrm{Card}(G(r) \cap D) > 1$. Then, we obtain the following result.

Lemma 14: Let $U : [0,1]^2 \longrightarrow [0,1]$ be a uninorm, $U \in \mathcal{UR}$, $U \notin \mathcal{U}$. Then there exists a point $(x,y) \in [0,1]^2$ such that U is neither left-continuous nor right-continuous in (x,y).

Proof: Since $U \notin \mathcal{U}$ Lemmas 12 and 13 imply that $\mathrm{Card}(G(r) \cap D) > 1$. Then there exists an $x_1 \in [0,1]$, $x_1 \neq e$, such that $(x_1, e) \in G(r)$. We will suppose that $x_1 < e$ (the case when $x_1 > e$ is analogous). Let

$$x_0 = \inf\{x \in [0,e] \mid (x,e) \in G(r)\}.$$

Then, the monotonicity of r implies that S_U is continuous and $]x_0, e] \times \{e\} \subset G(r)$. Moreover, $U(x,y) = e$ implies $x = y = e$ for all $x, y \in [0,1]$. Since U is continuous on $]x_0, e] \times]e,1] \cup]e,1] \times]x_0, e]$ we see that $U(x,y) > e$ for all $x \in]x_0, e]$, $y \in]e,1]$. On the other hand, the neutral element e and the monotonicity of U implies $U(x,y) \in [x,y]$ for all $x \in]x_0, e]$, $y \in]e,1]$. Thus, for all $x \in]x_0, e[$, we have $\lim_{s\longrightarrow e^+} U(x,s) = e$. Therefore, on $]x_0, e[$ the uninorm U is not right-continuous. Since $U \notin \mathcal{U}$ and T_U is continuous on $[0,1[^2$ we see that U is not left-continuous in some point (x,e) for $x \in [x_0, e]$. Now similarly as in Lemma 12 we can show that

U is not left-continuous in some point (x, e) for $x \in [x_0, e[$. Finally, the neutral element and the monotonicity of U imply that U is not left-continuous in some point (x, e) for $x \in]x_0, e[$. Summarizing, there exists a point $(x, y) \in [0, 1]^2$, such that U is neither left-continuous nor right-continuous in (x, y). ■

All previous results can be compiled into the following theorem.

Theorem 3: Let $U : [0, 1]^2 \longrightarrow [0, 1]$ be a uninorm, $U \in \mathcal{UR}$. Then, $U \in \mathcal{U}$ if and only if in each point $(x, y) \in [0, 1]^2$ the uninorm U is either left-continuous or right-continuous (or continuous).

Corollary 3: Let $U : [0, 1]^2 \longrightarrow [0, 1]$ be a uninorm. Then, $U \in \mathcal{U}$ if and only if $U \in \mathcal{UR}$ and in each point $(x, y) \in [0, 1]^2$ the uninorm U is either left-continuous or right-continuous (or continuous).

IV. Conclusion

We have shown that a uninorm with continuous underlying t-norm and t-conorm is continuous on $[0, 1]^2 \setminus G(r)$, where $G(r)$ is the graph of some symmetric, u-surjective, nonincreasing set-valued function. On the other hand, we have shown also a sufficient condition for a uninorm to have continuous underlying operations. In the follow-up papers [26], [27], we will employ these results, and using the characterizing set-valued function of a uninorm, we will show that each uninorm with continuous underlying t-norm and t-conorm can be decomposed into an ordinal sum of semigroups related to representable uninorms, continuous Archimedean t-norms, continuous Archimedean t-conorms, internal uninorms and singleton semigroups. Thus, these three papers together offer a complete characterization of uninorms from $\mathcal{U}$, that is, of uninorms with continuous underlying t-norm and t-conorm. The applications of these results are expected in all domains where uninorms are used.

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Andrea Mesiarová-Zemánková received the Master's degree in mathematics from the Faculty of Mathematics, Physics and Informatics, Comenius University, Bratislava, Slovakia, in 2002, and the Ph.D. degree in applied mathematics from the Mathematical Institute, Slovak Academy of Sciences, Bratislava, Slovakia, in 2005.

Since 2005, she has been a Researcher at the Mathematical Institute, Slovak Academy of Sciences. In years 2007–2009 and 2010–2012, she worked in the Department of Computer Science, Trinity College, Dublin, Ireland, as a Research Assistant. Her current research interests include aggregation theory, nonadditive measures and integrals, triangular norms and fuzzy systems.

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Characterizing set-valued functions of uninorms with continuous underlying t-norm and t-conorm

Andrea Mesiarová-Zemánková

Mathematical Institute, Slovak Academy of Sciences Bratislava, Slovakia

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Abstract

The uninorms with continuous underlying functions were characterized by their set of discontinuity points in the previous work of the author, using the characterizing set-valued function. In this paper properties of this characterizing set-valued function are studied. It is shown that the type of the monotonicity of such a set-valued function is always changed in idempotent points of the corresponding uninorm. Several additional properties of the characterizing set-valued function of a uninorm with continuous underlying functions are shown and an example is included. Results shown in this paper are used for a complete characterization of uninorms with continuous underlying functions via the ordinal sum construction, which is shown in the consecutive paper.

Keywords: Uninorm; Continuous t-norm; Continuous t-conorm; Set-valued function

1. Introduction

The uninorm operators generalize both the t-norm and the t-conorm operators and they can be used to model bipolar behaviour (see [1,6,9,17,25]). The uninorms with continuous underlying t-norm and t-conorm were in the centre of the interest for a long time (see [3–5,7,8,11–16,22,24]). In [18] we have introduced ordinal sum of uninorms and in [19] we have characterized uninorms that are ordinal sums of representable uninorms. Further, in [20] we have characterized uninorms with continuous underlying t-norm and t-conorm as those whose set of discontinuity points is a subset of the graph of a special symmetric, u-surjective, non-increasing set-valued function. In this paper we will investigate properties of the characterizing set-valued function of a uninorm with continuous underlying functions and we will show that the graph of this function can be divided into maximal horizontal, vertical and strictly decreasing segments, and special accumulation points, and that border points of all such maximal segments are idempotent elements of the corresponding uninorm. We will also show how does these maximal segments indicate the decomposition of the given uninorm into components. These results will be further used in [21] where we will show that each uninorm with continuous underlying t-norm and t-conorm can be decomposed into an ordinal sum

E-mail address: zemankova@mat.savba.sk.

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of semigroups related to representable uninorms, continuous Archimedean t-norms and t-conorms, internal uninorms and singleton semigroups.

We will start with several basic notions and then we will recall some useful results from [20]. The main result will be discussed in the next section. The last section then contains our conclusions.

A triangular norm is a function $T\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and 1 is its neutral element. Due to the associativity, n-ary form of any t-norm is uniquely given and thus it can be extended to an aggregation function working on $\bigcup_{n\in\mathbb{N}}[0,1]^n$. Dual functions to t-norms are t-conorms. A triangular conorm is a function $S\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and 0 is its neutral element. The duality between t-norms and t-conorms is expressed by the fact that from any t-norm T we can obtain its dual t-conorm S by the equation

$$S(x,y) = 1 - T(1-x, 1-y)$$

and vice-versa.

A uninorm (introduced in [25]) is a function $U\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and have a neutral element $e \in [0,1]$ (see also [6]). Therefore the class of uninorms covers also the class of t-norms and the class of t-conorms. If for a uninorm U there is $e \in \,]0,1[$ we will call such a uninorm *proper*. For each uninorm the value $U(1,0) \in \{0,1\}$ is the annihilator of U. A uninorm is called *conjunctive* if $U(1,0) = 0$ and *disjunctive* if $U(1,0) = 1$. Due to the associativity we can uniquely define n-ary form of any uninorm for any $n \in \mathbb{N}$ and therefore in some proofs we will use ternary form instead of binary, where suitable.

For each uninorm U with the neutral element $e \in \,]0,1[$, the restriction of U to $[0,e]^2$ is isomorphic to a t-norm, and the restriction of U to $[e,1]^2$ is isomorphic to a t-conorm.

Definition 1. Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm with the neutral element $e \in [0,1]$. If $e > 0$ then the operation $T_U\colon [0,1]^2 \longrightarrow [0,1]$ given by

$$T_U(x,y) = \frac{U(e\cdot x, e\cdot y)}{e}$$

for all $(x,y) \in [0,1]^2$ is a t-norm which will be called the *underlying t-norm of U*. If $e < 1$ then the operation $S_U\colon [0,1]^2 \longrightarrow [0,1]$ given by

$$S_U(x,y) = \frac{U(e + (1-e)\cdot x, e + (1-e)\cdot y) - e}{1-e}$$

for all $(x,y) \in [0,1]^2$ is a t-conorm which will be called the *underlying t-conorm of U*.

Definition 2. We will denote the set of all uninorms U such that T_U and S_U are continuous by $\mathcal{U}$.

From any pair of a t-norm and a t-conorm we can construct the minimal and the maximal uninorm with the given underlying functions.

Proposition 3 *([10]). Let $T\colon [0,1]^2 \longrightarrow [0,1]$ be a t-norm and $S\colon [0,1]^2 \longrightarrow [0,1]$ a t-conorm and assume $e \in [0,1]$. Then the two functions $U_{\min}, U_{\max}\colon [0,1]^2 \longrightarrow [0,1]$ given by*

$$U_{\min}(x,y) = \begin{cases} e\cdot T(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)\cdot S(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ \min(x,y) & \text{otherwise} \end{cases}$$

and

$$U_{\max}(x,y) = \begin{cases} e\cdot T(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)\cdot S(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ \max(x,y) & \text{otherwise} \end{cases}$$

are uninorms. We will denote the set of all uninorms of the first type by $\mathcal{U}_{\min}$ and of the second type by $\mathcal{U}_{\max}$.

Since the structure of continuous t-norms and t-conorms is already known, in the rest of the paper we will focus on proper uninorms.

Similarly as in the case of t-norms and t-conorms we can construct uninorms using additive generators (see [6]).

Proposition 4 *([6]). Let $f\colon [0,1] \longrightarrow [-\infty,\infty]$, $f(0) = -\infty$, $f(1) = \infty$ be a continuous strictly increasing function. Then a function $U\colon [0,1]^2 \longrightarrow [0,1]$ given by*

$$U(x,y) = f^{-1}(f(x) + f(y)),$$

where $f^{-1}\colon [-\infty,\infty] \longrightarrow [0,1]$ is an inverse function to f, with the convention $\infty + (-\infty) = \infty$ (or $\infty + (-\infty) = -\infty$), is a uninorm, which will be called a representable *uninorm.*

In [23] (see also [17]) we can find the following result.

Proposition 5 *([23]). Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm continuous everywhere on the unit square except of the two points $(0,1)$ and $(1,0)$. Then U is representable.*

For our examples we will use the following ordinal sum construction introduced by Clifford.

Theorem 6 *([2]). Let $A \neq \emptyset$ be a totally ordered set and $(G_\alpha)_{\alpha\in A}$ with $G_\alpha = (X_\alpha, *_\alpha)$ be a family of semigroups. Assume that for all $\alpha, \beta \in A$ with $\alpha < \beta$ the sets X_α and X_β are either disjoint or that $X_\alpha \cap X_\beta = \{x_{\alpha,\beta}\}$, where $x_{\alpha,\beta}$ is both the neutral element of G_α and the annihilator of G_β and where for each $\gamma \in A$ with $\alpha < \gamma < \beta$ we have $X_\gamma = \{x_{\alpha,\beta}\}$. Put $X = \bigcup_{\alpha\in A} X_\alpha$ and define the binary operation $*$ on X by*

$$x * y = \begin{cases} x *_\alpha y & \text{if } (x,y) \in X_\alpha \times X_\alpha, \\ x & \text{if } (x,y) \in X_\alpha \times X_\beta \text{ and } \alpha < \beta, \\ y & \text{if } (x,y) \in X_\alpha \times X_\beta \text{ and } \alpha > \beta. \end{cases}$$

Then $G = (X, *)$ is a semigroup. The semigroup G is commutative if and only if for each $\alpha \in A$ the semigroup G_α is commutative.

Therefore in our examples the commutativity and the associativity of the corresponding ordinal sum uninorm will follow from Theorem 6. The monotonicity and the existence of the neutral element can be easily checked by the reader.

Further we will use the following transformation. For any $0 \le a < b \le c < d \le 1$, $v \in [b,c]$, and a uninorm U with the neutral element $e \in \,]0,1[$ let $f\colon [0,1] \longrightarrow [a,b[\, \cup \{v\} \cup \,]c,d]$ be given by

$$f(x) = \begin{cases} (b-a)\cdot \frac{x}{e} + a & \text{if } x \in [0,e[, \\ v & \text{if } x = e, \\ d - \frac{(1-x)(d-c)}{(1-e)} & \text{otherwise.} \end{cases} \tag{1}$$

Then f is an isomorphism of $[0,1]$ to $([a,b[\, \cup \{v\} \cup \,]c,d])$ and a function $U_v^{a,b,c,d}\colon ([a,b[\, \cup \{v\} \cup \,]c,d])^2 \longrightarrow ([a,b[\, \cup \{v\} \cup \,]c,d])$ given by

$$U_v^{a,b,c,d}(x,y) = f(U(f^{-1}(x), f^{-1}(y))) \tag{2}$$

is an operation on $([a,b[\, \cup \{v\} \cup \,]c,d])^2$ which is commutative, associative, non-decreasing in both variables (with respect to the standard order) and v is its neutral element. Indeed, since f is increasing then the non-decreasingness of U implies the non-decreasingness of $U_v^{a,b,c,d}$. Further, the commutativity and the existence of the neutral element is preserved by isomorphism. Finally,

$$U_v^{a,b,c,d}(x, U_v^{a,b,c,d}(y,z)) = f(U(f^{-1}(x), U(f^{-1}(y), f^{-1}(z)))) =$$
$$f(U(f^{-1}(x), f^{-1}(y)), f^{-1}(z)) = U_v^{a,b,c,d}(U_v^{a,b,c,d}(x,y), z)$$

for all $x, y, z \in [0,1]$ which gives us the associativity of $U_v^{a,b,c,d}$.

Next we will recall the definition of a set-valued function from [20]. Note that $\mathcal{P}(X)$ in the following definition denotes the power set of X.

Definition 7. A mapping $p\colon [0,1] \longrightarrow \mathcal{P}([0,1])$ is called a *set-valued function on* $[0,1]$. Assuming the standard order on $[0,1]$, a set-valued function p is called

(i) *non-increasing* if for all $x_1, x_2 \in [0,1]$, $x_1 < x_2$, we have $y_1 \geq y_2$ for all $y_1 \in p(x_1)$ and all $y_2 \in p(x_2)$ and thus $p(x_1)$ and $p(x_2)$ intersect in, at most, a single point,
(ii) *symmetric* if for $x, y \in [0,1]$ there is $y \in p(x)$ if and only if $x \in p(y)$.

The graph of a set-valued function p will be denoted by $G(p)$, i.e., for $x, y \in [0,1]$ there is $(x,y) \in G(p)$ if and only if $y \in p(x)$.

Definition 8 *([20])*. A set-valued function $p\colon [0,1] \longrightarrow \mathcal{P}([0,1])$ is called *u-surjective* if for all $y \in [0,1]$ there exists an $x \in [0,1]$ such that $y \in p(x)$.

Lemma 9 *([20]). A symmetric set-valued function $p\colon [0,1] \longrightarrow \mathcal{P}([0,1])$ is u-surjective if and only if we have $p(x) \neq \emptyset$ for all $x \in [0,1]$.*

The graph of a symmetric, u-surjective, non-increasing set-valued function $p\colon [0,1] \longrightarrow \mathcal{P}([0,1])$ is a connected bounded curve (i.e., a connected bounded set with no interior) containing points $(0,1)$ and $(1,0)$ (see [20]).

Remark 10. For any uninorm $U \in \mathcal{U}$ the following results were shown in [20]:

(i) There exist idempotent points $a, d \in [0,1]$, $a \leq e \leq d$, such that if $U(x,y) = e$ for some $x, y \in [0,1]$ then $x, y \in]a,d[\cup \{e\}$. Here either $U(x,y) = e$ implies $x = y = e$, in which case $a = d = e$, or otherwise U can attain the value e only on the set $]a,d[^2$. Further, for all $x \in]a,d[\cup \{e\}$ there exists a $y \in]a,d[\cup \{e\}$ such that $U(x,y) = e$. Note that if $a < e$ then U is on $[a,d]^2$ isomorphic to a representable uninorm.
(ii) If we denote $A = \inf\{x \mid U(x,0) > 0\}$, $B = \sup\{x \mid U(x,1) < 1\}$ then A and B are idempotent elements of U and either $A = 1, B \neq 0$, or $A \neq 1, B = 0$, or $A = 1, B = 0$. In the first case U is non-continuous in $(B,1)$, in the second case U is non-continuous in $(0,A)$, and in the third case U is non-continuous in $(1,0)$. We also have

$$0 \leq B \leq a \leq e \leq d \leq A \leq 1.$$

Theorem 11 *([20]). Assume a uninorm $U \in \mathcal{U}$. Then there exists a symmetric, u-surjective, non-increasing set-valued function r on $[0,1]$ such that U is continuous on $[0,1]^2 \setminus G(r)$ and $U(x,y) = e$ implies $(x,y) \in G(r)$ for all $(x,y) \in [0,1]^2$. Note that U need not to be non-continuous in all points from $G(r)$.*

The set-valued function from Theorem 11 will be called the *characterizing* set-valued function of a uninorm U for $U \in \mathcal{U}$. This function is given by (see Remark 10)

$$r(x) = \begin{cases} \{1\} & \text{if } x \in]0,B[, \\ \{0\} & \text{if } x \in]A,1[, \\ [0,B] & \text{if } x = 1, \\ [A,1] & \text{if } x = 0, \\ \{y \mid U(x,y) = e\} & \text{if } x \in]a,d[\cup \{e\}, \\ \{y \mid (x,y) \in R^*\} & \text{otherwise,} \end{cases} \tag{3}$$

where $R^* = \{(x,y) \in [0,1]^2 \mid U \text{ is non-continuous in } (x,y)\}$.

Definition 12. For a uninorm $U\colon [0,1]^2 \longrightarrow [0,1]$ and each $x \in [0,1]$ we will define a function $u_x\colon [0,1] \longrightarrow [0,1]$ by $u_x(z) = U(x,z)$ for $z \in [0,1]$.

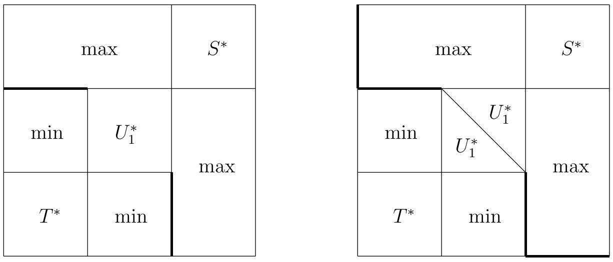

Fig. 1. The uninorm U from Example 15. Left: the bold lines denote the points of discontinuity of U. Right: the oblique and bold lines denote the characterizing set-valued function of U.

Remark 13. Let r be the characterizing set-valued function of a uninorm $U \in \mathcal{U}$. Then

(i) U is not non-continuous in all points of $G(r)$. In fact, U is continuous in all points from $\{x\} \times [0,1]$ for all $x \in [0, B[\cup]a, d[\cup \{e\} \cup]A, 1]$. This means that u_x is continuous for all $x \in [0, B[\cup]a, d[\cup \{e\} \cup]A, 1]$. On the other hand, u_x is non-continuous for all $x \in [B, a] \cup [d, A] \setminus \{e\}$,

(ii) the characterizing set-valued function r divides the uninorm U into two parts: U on points below the characterizing set-valued function attains values smaller than e, and U on points above the characterizing set-valued function attains values bigger than e.

The main result of [20] is the following theorem.

Theorem 14 *([20]). Let $U : [0,1]^2 \longrightarrow [0,1]$ be a uninorm. Then $U \in \mathcal{U}$ if and only if there exists a symmetric, u-surjective, non-increasing set-valued function r on $[0,1]$ such that U is continuous on $[0,1]^2 \setminus G(r)$, and in each point $(x, y) \in [0,1]^2$ the uninorm U is either left-continuous or right-continuous, or continuous.*

Example 15 *([20]).* Assume a representable uninorm $U_1 : [0,1]^2 \longrightarrow [0,1]$ and a continuous t-norm $T : [0,1]^2 \longrightarrow [0,1]$ and a continuous t-conorm $S : [0,1]^2 \longrightarrow [0,1]$. For simplicity we will assume that $\frac{1}{2}$ is the neutral element of U_1 and that $U_1(x, 1-x) = \frac{1}{2}$ for all $x \in]0,1[$. Let U_1^* be a linear transformation of U_1 to $\left[\frac{1}{3}, \frac{2}{3}\right]^2$, let T^* be a linear transformation of T to $\left[0, \frac{1}{3}\right]^2$ and let S^* be a linear transformation of S to $\left[\frac{2}{3}, 1\right]^2$. Then the ordinal sum of semigroups $G_\alpha = \left(\left[0, \frac{1}{3}\right], T^*\right)$, $G_\beta = \left(\left[\frac{1}{3}, \frac{2}{3}\right], U_1^*\right)$, $G_\gamma = \left(\left[\frac{2}{3}, 1\right], S^*\right)$, with $\gamma < \alpha < \beta$, is a semigroup $([0,1], U)$, where U is a uninorm, $U \in \mathcal{U}$. On Fig. 1 we can see the characterizing set-valued function r of U as well as its set of discontinuity points.

In this paper we will continue to investigate properties of the characterizing set-valued function of a uninorm $U \in \mathcal{U}$. However, before we show that main result we will recall some useful results from [20].

Definition 16. A uninorm $U : [0,1]^2 \longrightarrow [0,1]$ is called internal if $U(x, y) \in \{x, y\}$ for all $(x, y) \in [0,1]^2$.

Lemma 17 *([20]). Let $U \in \mathcal{U}$. If $x_1 \in [0,1]$ is an idempotent point of U then U is internal on $\{x_1\} \times [0,1]$, i.e., $U(x_1, x) \in \{x, x_1\}$ for all $x \in [0,1]$.*

Proposition 18 *([20]). Let $U \in \mathcal{U}$. Then for each $x \in [0,1]$ there is at most one point of discontinuity of u_x. Further, if u_x is non-continuous in $y \in [0,1]$ then $U(x, z) < e$ for all $z < y$ and $U(x, z) > e$ for all $z > y$.*

2. Main result

In this section we will show several useful results which will be used in [21]. The main aim is to study the structure of the characterizing set-valued function and to show that its graph can be divided into maximal horizontal, vertical and strictly decreasing segments (and special accumulation points), and that border points of all such maximal segments of the characterizing set-valued function r of a uninorm $U \in \mathcal{U}$ are idempotent. Indeed, it is easy to observe that the graph of the characterizing set-valued function is a connected bounded curve that contains horizontal line segments, vertical line segments, and segments on which it coincides with the graph of some strictly decreasing function. For the decomposition into the ordinal sum summands (which will be done in [21]) we are interested in maximal such segments and their projection to the first coordinate. For simplicity, in the definition we will instead of *projection of a maximal segment to the first coordinate* write just *maximal segment*.

Before we introduce the following definition let us note that for an $x \in [0,1]$ with $y \in r(x)$, and the cardinality $\mathrm{Card}(r(x))$ the following are equivalent:

(i) $r(x)$ is a singleton,
(ii) $\mathrm{Card}(r(x)) = 1$,
(iii) $r(x) = \{y\}$,
(iv) $\max(r(x)) = y$,
(v) $\min(r(x)) = y$.

Definition 19. Let $U \in \mathcal{U}$ and let $r\colon [0,1] \longrightarrow \mathcal{P}([0,1])$ be its characterizing set-valued function. Then

(i) the set $I \subset [0,1]$ is called a maximal horizontal segment of r if $\mathrm{Card}(I) > 1$, i.e., if I contains at least two different points from $[0,1]$, and there exists a $y \in [0,1]$ such that $y \in p(x)$ if and only if $x \in I$,
(ii) if for $x \in [0,1]$ there is $\mathrm{Card}(r(x)) > 1$ then the set $\{x\}$ is called a maximal vertical segment of r,
(iii) the interval $[a,b]$ is called a strictly decreasing segment of r if for all $x \in]a,b[$ we have

$$\mathrm{Card}(r(x)) = 1, \mathrm{Card}(r(\max(r(x))) = 1, \tag{4}$$

i.e., if $r(x)$ is a singleton and if $r(x) = \{y\}$ for some $y \in [0,1]$ then $r(y)$ is a singleton as well,
(iv) the interval $[a,b]$ is called a maximal strictly decreasing segment of r if there is no interval $[c,d]$ which is a strictly decreasing segment of r such that $[a,b] \subsetneq [c,d]$.

The non-increasingness of r implies that all maximal segments are intervals. Further, a subinterval of a maximal horizontal segment will be called a horizontal segment.

The symmetry of r implies that a maximal horizontal segment I can be paired with a maximal vertical segment $\{y\}$ for which we have

$$y \in r(x) \text{ for all } x \in I.$$

Then $I \times \{y\}$ as well as $\{y\} \times I$ belong to the graph of r.

Remark 20.
(i) If $I \subset [0,1]$ is a maximal horizontal segment of r then

$$\bigcup_{x \in I} (x, y),$$

for the corresponding $y \in [0,1]$ from the previous definition, is a horizontal line segment.
(ii) If $\{x\}$ for some $x \in [0,1]$ is a maximal vertical segment of r then

$$\bigcup_{y \in r(x)} (x, y)$$

is a vertical line segment.
(iii) If $[a,b]$ is a maximal strictly decreasing segment of r for some $a, b \in [0,1]$, $a < b$, then

$$(a, \min(r(a))) \cup (b, \max(r(b))) \cup \bigcup_{x \in]a,b[} (x, \max(r(x)))$$

coincides with the graph of some strictly decreasing function on $[a,b]$.

Now we will show that all maximal segments of the characterizing set-valued function of $U \in \mathcal{U}$ are closed intervals.

Lemma 21. *Let $U \in \mathcal{U}$ and let $r\colon [0,1] \longrightarrow \mathcal{P}([0,1])$ be its characterizing set-valued function. Then all maximal segments of r are closed intervals.*

Proof. Any maximal strictly decreasing segment of r is a closed interval by the definition. Similarly, any maximal vertical segment is a trivial closed interval. Further we will show that any maximal horizontal segment is a closed interval. Let $X_1 \in [0,1]$ be the left border point of some maximal horizontal segment I such that $y \in p(x)$ if and only if $x \in I$ (for the right border point the proof is analogous). Then by Proposition 18 for each $\varepsilon > 0$ (small enough) we have $U(X_1+\varepsilon, y+\varepsilon) > e$ and $U(X_1, y-\varepsilon) \le U(X_1+\varepsilon, y-\varepsilon) < e$. Thus either $U(X_1, y) = e$, or U is non-continuous in (X_1, y). In both cases we get $y \in r(X_1)$, i.e., $X_1 \in I$. □

Due to the symmetry of r the previous result implies that for every $x \in [0,1]$ the set $r(x)$ is a closed interval and therefore $\min(r(x))$ and $\max(r(x))$ always exist.

Definition 22. We will denote by S_r the set of border points of all maximal segments of r and by $\bar{S}_r$ its closure.

Note that there is a countable number of maximal horizontal and strictly decreasing segments and due to the symmetry of r there is also a countable number of maximal vertical segments. Therefore $\bar{S}_r$ is countable. Then we have the following result.

Proposition 23. *Let $U \in \mathcal{U}$ and let $r\colon [0,1] \longrightarrow \mathcal{P}([0,1])$ be its characterizing set-valued function and assume $x \in [0,1]$. Then either $x \in \bar{S}_r$ or x is an interior point of exactly one maximal segment of r.*

Proof. If x is an interior point of some maximal (horizontal or strictly decreasing) segment of r then the non-increasingness of r implies that x doesn't belong to any other maximal segment. Further, if x is a border point of some maximal segment then evidently $x \in S_r$. Suppose that x doesn't belong to any maximal segment. Then

$$\mathrm{Card}(r(x)) = 1 \text{ and } \mathrm{Card}(r(\max(r(x))) = 1.$$

Since x is not an interior point of a strictly decreasing segment there are two possibilities:

Case 1. For all $\varepsilon > 0$ (small enough) there exists a $q \in [x-\varepsilon, x+\varepsilon]$ such that $\mathrm{Card}(r(q)) > 1$. In this case x is an accumulation point of the set of border points corresponding to maximal vertical segments and therefore $x \in \bar{S}_r$.

Case 2. For all $\varepsilon > 0$ (small enough) there exists a $q \in [x-\varepsilon, x+\varepsilon]$ such that q belongs to a horizontal segment. Since x does not belong to a horizontal segment this means that in every neighbourhood of x there is a border point of a maximal horizontal segment and therefore $x \in \bar{S}_r$. □

Now we introduce two useful lemmas which will be used in the succeeding results.

Lemma 24. *Let $U \in \mathcal{U}$. If x_1 is an idempotent element of U then u_{x_1} is either continuous or it is non-continuous in point y_1, such that y_1 is an idempotent element of U.*

Proof. If x_1 is an idempotent element and u_{x_1} is non-continuous in y_1 then Proposition 18 implies that $U(x_1, y) < e$ for $y < y_1$ and $U(x_1, y) > e$ for $y > y_1$. If $x_1 = e$ then u_{x_1} is evidently continuous, thus we will assume $x_1 < e$ (the case for $x_1 > e$ is analogous). Then since $U \in \mathcal{U}$ we have $y_1 \ge e$. If y_1 is not an idempotent element then there exists a $y_2 \in [0,1]$ with $e < y_2 < y_1$ such that $U(y_2, y_2) > y_1$ and since by Lemma 17 we know that U is internal on $\{x_1\} \times [0,1]$ we have

$$e > x_1 = U(x_1, y_2) = U(x_1, y_2, y_2) = U(y_2, y_2) > y_1 \ge e$$

which is a contradiction. □

Lemma 25. *Let $U \in \mathcal{U}$ and assume an $x \in [0,1]$. If u_x is non-continuous in some $y \in [0,1]$ then for all $n \in \mathbb{N}$ the function $u_{x_U^{(n)}}$ is non-continuous in $y_U^{(n)}$, where*

$$x_U^{(n)} = U(\underbrace{x, \dots, x}_{n\text{-times}}).$$

Proof. If x is an idempotent point then the result follows from Lemma 24. Assume that x is not idempotent and that $x < e$ (the case when $x > e$ can be shown analogously). Then $U(x,z) < e$ for $z < y$ implies $U(x_U^{(n)}, z_U^{(n)}) < e$ and the continuity of S_U ensures that for all $q < y_U^{(n)}$ there exists such a $z < y$ that $z_U^{(n)} = q$. Therefore $U(x_U^{(n)}, q) < e$ for all $q < y_U^{(n)}$. Similarly we can show that $U(x_U^{(n)}, q) > e$ for all $q > y_U^{(n)}$. Thus either $U(x_U^{(n)}, y_U^{(n)}) = e$, or $u_{x_U^{(n)}}$ is non-continuous in $y_U^{(n)}$. However, since $U(x,y) \neq e$ the first case is not possible and therefore $u_{x_U^{(n)}}$ is non-continuous in $y_U^{(n)}$. □

Proposition 26. *Let $U \in \mathcal{U}$. If for some $x_1, x_2 \in [0,1]$, $x_1 < x_2$, both functions u_{x_1} and u_{x_2} are non-continuous in $y \in [0,1]$ then y is an idempotent element of U.*

Proof. Assume that y is not an idempotent element. Since $U \in \mathcal{U}$ we have either $x_1 < x_2 \leq e$ or $e \leq x_1 < x_2$. We will suppose $x_1 < x_2 \leq e$ as the other case is analogous. Then also for all $f \in [x_1, x_2]$ the function u_f is non-continuous in y and thus if there is an idempotent in $[x_1, x_2]$ Lemma 24 implies that y is an idempotent element. Assume the opposite and let $X_1, X_2 \in [0, e]$ be two idempotent elements such that $x_1, x_2 \in \,]X_1, X_2[$ and there is no idempotent element in $]X_1, X_2[$. Then since the points a, d, A, B from Remark 10 are idempotent we have $]X_1, X_2[\cap]a, d[= \emptyset$, $]X_1, X_2[\cap]0, B[= \emptyset$ and $]X_1, X_2[\cap]A, 1[= \emptyset$. Therefore Remark 13 implies that u_x is non-continuous for all $x \in \,]X_1, X_2[$. Since T_U is continuous it is possible to select a $w \in \,]X_1, X_2[$ such that $w_U^{(m)}, w_U^{(m+1)} \in [x_1, x_2]$ for some $m \in \mathbb{N}$. If u_w is non-continuous in r then by Lemma 25 we have $y = r_U^{(m)} = r_U^{(m+1)}$ which is possible only if y is an idempotent point. □

Proposition 27. *Let $U \in \mathcal{U}$ and let r be its characterizing set-valued function. Then border points of all types of maximal segments of r are idempotent points.*

Proof. Since r is symmetric Proposition 26 implies the result for all vertical segments. Further, let the interval $[X_1, X_2]$ correspond to some maximal horizontal segment of r, i.e., for some $y \in [0,1]$ we have $r(x) = \{y\}$ for all $x \in \,]X_1, X_2[$. Also $y \notin r(x)$ for all $x \in [0,1] \setminus [X_1, X_2]$. We will suppose $X_1 < X_2 \leq e$ (the case when $e \leq X_1 < X_2$ is analogous). From Proposition 26 it follows that y is an idempotent element of U. Suppose first that X_1 is not an idempotent element. Since T_U is continuous there exists $\varepsilon > 0$ (small enough) such that $X_1 + \varepsilon \in \,]X_1, X_2[$, $U(X_1 + \varepsilon, X_1 + \varepsilon) < X_1$. Then since $u_{X_1+\varepsilon}$ is non-continuous in y Lemma 25 implies that $u_{U(X_1+\varepsilon, X_1+\varepsilon)}$ is non-continuous in $U(y,y) = y$, i.e., $y \in r(U(X_1 + \varepsilon, X_1 + \varepsilon))$ which is a contradiction. Thus X_1 is an idempotent element.

Now suppose that X_2 is not idempotent. Here we have two possibilities. Either $U(p,p) = y$ for some $p \in [0,1]$, $p \neq y$, or $U(p,p) = y$ implies $p = y$ for all $p \in [0,1]$. First suppose the second case. Since T_U is continuous there exists $\varepsilon > 0$ (small enough) such that $U(X_2 + \varepsilon, X_2 + \varepsilon) \in \,]X_1, X_2[$. Similarly as before we can show that u_x is non-continuous for all $x \in \,]X_1, X_3[$, where X_3 is the smallest idempotent element bigger than X_2. Then since $X_2 + \varepsilon$ does not belong to the given horizontal segment the non-increasingness of r implies that $u_{X_2+\varepsilon}$ is non-continuous in some $p \in [0,1]$ with $p < y$. However, the function $u_{U(X_2+\varepsilon, X_2+\varepsilon)}$ is non-continuous in y and Lemma 25 then implies $U(p,p) = y$ which is a contradiction.

Finally suppose that $U(p,p) = y$ for some $p \in [0,1]$, $p < y$. Since S_U is continuous we can assume that p is the smallest point with this property. Then $U(p,z) < y$ for all $z < p$. Since y is an idempotent element for $x \in \,]X_1, X_2[$ there is either $U(x,p,p) = x$ or $U(x,p,p) = y$. Since $U(x,p) \in [x,p]$ and $U(p,p) > p$ the monotonicity of U ensures in the first case that $U(x,p) = x$. In the second case, if $U(x,p) < p$ then $U(x,p,p) < y$ and therefore $U(x,p) = p$. Further, $p < y$ and therefore we get $U(x,p) < e$, i.e., $U(x,p) = x$, for all $x \in \,]X_1, X_2[$. Let us assume an $\varepsilon > 0$ such that $U(X_2 + \varepsilon, X_2 + \varepsilon) \in \,]X_1, X_2[$. Since $X_2 + \varepsilon$ does not belong to the given horizontal segment the non-increasingness of r ensures that $U(X_2+\varepsilon, y) > e$. On the other hand we have

$$U(U(X_2+\varepsilon, X_2+\varepsilon), y) = U(X_2+\varepsilon, X_2+\varepsilon)$$

and as y is idempotent we get $U(X_2+\varepsilon, y) = X_2+\varepsilon < e$ which is a contradiction. Therefore X_2 is an idempotent element of U. We have shown that border points of all horizontal segments are idempotents.

Now assume that X_1 is a border point of a maximal strictly decreasing segment. Then either X_1 is a border point of a maximal vertical or a maximal horizontal segment as well, in which case it is an idempotent point, or it is an accumulation point of a set of border points of maximal vertical or maximal horizontal segments. Since $U \in \mathcal{U}$ the set of idempotent points of U is closed and thus also in the second case X_1 is an idempotent point of U. Summarizing, border points of all maximal segments are idempotents. □

From the previous proof it follows that all points from the set $\bar{S}_r$ are idempotent elements of U.

Further we will show the relation between nilpotent components of U and horizontal segments of the characterizing set-valued function.

Lemma 28. *Let $U \in \mathcal{U}$ and let r be its characterizing set-valued function. If $X_1, X_2 \in [0, 1]$, $X_1 < X_2$, are idempotent elements of U such that there is no idempotent in $]X_1, X_2[$ and there exists $x \in]X_1, X_2[$ such that $U(x, x) = X_1$ or $U(x, x) = X_2$ then r on $[X_1, X_2]$ corresponds to a horizontal segment.*

Proof. Let $U(x, x) = X_1$ (the case for $U(x, x) = X_2$ is analogous). Then $X_1 < X_2 \le e$ and we can assume that x is the biggest point with this property, i.e., $U(x, z) > X_1$ for all $z > x$. Since X_1 is idempotent e does not belong to the range of u_{X_1}. If u_{X_1} is continuous then $U(X_1, 1) < e$, i.e., $U(X_1, 1) = X_1$ due to Lemma 17. Then $U(x, x, 1) = X_1$ and thus $U(x, 1) \le x$. Using Lemma 17 again we get $U(x, 1) = x$ and therefore due to Proposition 27 the set-valued function r corresponds to a horizontal segment on $[0, X_2]$.

Now suppose that u_{X_1} is non-continuous in y. Then $U(X_1, y) \in \{X_1, y\}$ and we will first suppose that $U(X_1, y) = X_1$. We have $U(x, x, y) = X_1$ and therefore $U(x, y) \le x$, i.e., $U(x, y) = x$. Further, $U(X_1, z) = z$ for all $z > y$ and therefore $U(x, x, z) = z$ which implies $U(x, z) \ge z$. Since $U(x, z) \in [x, z]$ we get $U(x, z) = z$. Thus u_x is non-continuous in y and Proposition 27 implies the result.

Finally suppose that $U(X_1, y) = y$. Then $e < y$ and due to the monotonicity also $U(x, y) = y$. Further, $U(X_1, z) = X_1$ for all $e \le z < y$, i.e., $U(x, x, z) = X_1$ which implies $U(x, z) \le x$. Since $U(x, z) \in [x, z]$ we get $U(x, z) = x$. Thus again u_x is non-continuous in y and Proposition 27 implies the result. □

In the following two lemmas we show how does a strictly decreasing segment of the characterizing set-valued function relate the components of $[0, e]$ to the components of $[e, 1]$.

Lemma 29. *Let $U \in \mathcal{U}$. If $X_1, X_2 \in [0, 1]$ are idempotent elements of U such that there is no idempotent in $]X_1, X_2[$ and r on $[X_1, X_2]$ corresponds to a strictly decreasing segment then $Y_2 = \min(r(X_1))$ and $Y_1 = \max(r(X_2))$ are idempotent elements of U such that there is no idempotent in $]Y_1, Y_2[$ and r on $[Y_1, Y_2]$ corresponds to a strictly decreasing segment. Further, $X_1 = \max(r(\min(r(X_1))))$ and $X_2 = \min(r(\max(r(X_2))))$.*

Proof. There is either $r(X_1) = \{Y_2\}$ or $\mathrm{Card}(r(X_1)) > 1$. Similarly, either $r(X_2) = \{Y_1\}$ or $\mathrm{Card}(r(X_2)) > 1$. If $\mathrm{Card}(r(X_1)) > 1$ then r on $r(X_1)$ corresponds to a horizontal segment, which has a left border point Y_2 and therefore Proposition 27 implies that Y_2 is an idempotent element. If $\mathrm{Card}(r(X_1)) = 1$ then either $Y_2 = 1$ or Y_2 is an idempotent element of U due to Lemma 24. Similarly we can show that Y_1 is an idempotent element.

The commutativity of U implies that r on $[Y_1, Y_2]$ corresponds to a strictly decreasing segment. If there is an idempotent element x in $]Y_1, Y_2[$ then u_x is non-continuous in y for some $y \in]X_1, X_2[$. Lemma 24 implies that y is an idempotent element which is a contradiction.

Finally, since $Y_2 = \min(r(X_1))$ and $Y_1 = \max(r(X_2))$ then (X_1, Y_2) and (X_2, Y_1) belong to the graph of the characterizing set-valued function r. Then also (Y_2, X_1) and (Y_1, X_2) belong to the graph of the characterizing set-valued function r and since r is non-increasing we have $X_1 = \max(r(\min(r(X_1))))$ and $X_2 = \min(r(\max(r(X_2))))$. □

Lemma 30. *Let $U \in \mathcal{U}$. If $X_1, X_2 \in [0,1]$ are such that $U(x,x) = x$ for all $x \in [X_1, X_2]$ and r on $[X_1, X_2]$ corresponds to a strictly decreasing segment then for $Y_2 = \min(r(X_1))$ and $Y_1 = \max(r(X_2))$ we have $U(y,y) = y$ for all $y \in [Y_1, Y_2]$ and r on $[Y_1, Y_2]$ corresponds to a strictly decreasing segment. Further, $X_1 = \max(r(\min(r(X_1))))$ and $X_2 = \min(r(\max(r(X_2))))$.*

Proof. Similarly as in the previous lemma Y_1 and Y_2 are idempotents. Further, since for each $y \in]Y_1, Y_2[$ there exists an $x \in]X_1, X_2[$ such that u_x is non-continuous in y Lemma 24 implies that y is an idempotent element. The rest follows from the commutativity of U. The symmetry and the non-increasingness of r further imply $X_1 = \max(r(\min(r(X_1))))$ and $X_2 = \min(r(\max(r(X_2))))$. □

3. Conclusions

We have investigated properties of the characterizing set-valued function of a uninorm with continuous underlying functions. We have shown that the graph of this function can be divided into maximal horizontal, vertical and strictly decreasing segments, and special accumulation points. Further, we have shown that border points of all such maximal segments are idempotent elements of the corresponding uninorm. We have also shown how does horizontal and strictly decreasing segments indicate the decomposition of the given uninorm into components. These results will be further used in [21] where we will show that each uninorm with continuous underlying t-norm and t-conorm can be decomposed into an ordinal sum of semigroups related to representable uninorms, continuous Archimedean t-norms and t-conorms, internal uninorms and singleton semigroups.

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Uninorms continuous on $[0, e[^2 \cup]e, 1]^2$

Andrea Mesiarová-Zemánková

Mathematical Institute,Slovak Academy of Sciences, Bratislava, Slovakia

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ABSTRACT

Uninorms continuous on $[0, e[^2 \cup]e, 1]^2$ are discussed. Archimedean uninorms continuous and cancellative on $]0, e[^2 \cup]e, 1[^2$ are characterized and related non-Archimedean uninorms are also discussed. The problem when is the border-continuous projection of a t-norm associative, i.e., a t-subnorm is also answered.

1. Introduction

The (left-continuous) t-norms and their dual t-conorms have an indispensable role in many domains [8,28,29]. Each continuous t-norm (t-conorm) can be expressed as an ordinal sum of continuous Archimedean t-norms (t-conorms), while each Archimedean t-norm (t-conorm) is generated by an additive generator (see [2,11]). Generalizations of t-norms and t-conorms that can model bipolar behaviour are uninorms (see [6,21,30]). The class of uninorms is widely used both in theory [18,27] and in applications [26,31]. The complete characterization of uninorms with continuous underlying t-norm and t-conorm has been an open problem for a long time. We plan to solve this problem in our future work in [23,24] (see also [5,15]). In this paper we would like to focus on uninorms continuous on $[0, e[^2 \cup]e, 1]^2$.

In Section 2 we will first recall all important notions and results and we will discuss general uninorms continuous on $[0, e[^2 \cup]e, 1]^2$ in Section 3 . In Section 4 we will recall several known results and show some new results on continuous cancellative t-subnorms. We will show the characterization of Archimedean uninorms continuous and cancellative on $]0, e[^2 \cup]e, 1[^2$ (Section 5) and discuss related non-Archimedean uninorms. We give our conclusions in Section 6.

2. Basic notions and results

Let us now recall all necessary basic notions.

A triangular norm is a binary function $T : [0, 1]^2 \longrightarrow [0, 1]$ which is commutative, associative, non-decreasing in both variables and 1 is its neutral element. Due to the associativity, n-ary form of any t-norm is uniquely given and thus it can be extended to an aggregation function working on $\bigcup_{n \in \mathbb{N}} [0, 1]^n$. Dual functions to t-norms are t-conorms. A triangular conorm is a binary function $C : [0, 1]^2 \longrightarrow [0, 1]$ which is commutative, associative, non-decreasing in both variables and 0 is its neutral element. The duality between t-norms and t-conorms is expressed by the fact that from any t-norm T we can

E-mail address: zemankova@mat.savba.sk

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obtain its dual t-conorm C by the equation

$$C(x, y) = 1 - T(1 - x, 1 - y)$$

and vice-versa.

Proposition 1 ([11]). *Let $t : [0, 1] \longrightarrow [0, \infty]$ ($c : [0, 1] \longrightarrow [0, \infty]$) be a continuous strictly decreasing (increasing) function such that $t(1) = 0$ ($c(0) = 0$). Then the binary operation $T : [0, 1]^2 \longrightarrow [0, 1]$ ($C : [0, 1]^2 \longrightarrow [0, 1]$) given by*

$$T(x, y) = t^{-1}(\min(t(0), t(x) + t(y)))$$

$$C(x, y) = c^{-1}(\min(c(1), c(x) + c(y)))$$

is a continuous t-norm (t-conorm). The function t (c) is called an additive generator *of T (C).*

Proposition 2 ([11]). *Let K be a finite or countably infinite index set and let $(]a_k, b_k[)_{k \in K}$ ($(]c_k, d_k[)_{k \in K}$) be a disjoint system of open subintervals of $[0, 1]$. Let $(T_k)_{k \in K}$ ($(C_k)_{k \in K}$) be a system of t-norms (t-conorms). Then the ordinal sum $T = (\langle a_k, b_k, T_k\rangle \mid k \in K)$ ($C = (\langle c_k, d_k, C_k\rangle \mid k \in K)$) given by*

$$T(x, y) = \begin{cases} a_k + (b_k - a_k)T_k(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}) & \text{if } (x, y) \in [a_k, b_k[^2, \\ \min(x, y) & \text{else} \end{cases}$$

and

$$C(x, y) = \begin{cases} c_k + (d_k - c_k)C_k(\frac{x-c_k}{d_k-c_k}, \frac{y-c_k}{d_k-c_k}) & \text{if } (x, y) \in]c_k, d_k]^2, \\ \max(x, y) & \text{else} \end{cases}$$

is a t-norm (t-conorm). The t-norm T (t-conorm C) is continuous if and only if all summands T_k (C_k) for $k \in K$ are continuous.

An additive generator of a continuous t-norm T (t-conorm C) is uniquely determined up to a positive multiplicative constant. Each continuous t-norm (t-conorm) is equal to an ordinal sum of continuous Archimedean t-norms (t-conorms). Note that a continuous t-norm (t-conorm) is Archimedean if and only if it has only trivial idempotent points 0 and 1. A continuous Archimedean t-norm T (t-conorm C) is either strict, i.e., strictly increasing on $]0, 1]^2$ (on $[0, 1[^2$), or nilpotent, i.e., there exists $(x, y) \in]0, 1[^2$ such that $T(x, y) = 0$ ($C(x, y) = 1$). Moreover, each continuous Archimedean t-norm (t-conorm) has a continuous additive generator. More details on t-norms and t-conorms can be found in [2,11].

A uninorm (introduced in [30]) is a binary function $U : [0, 1]^2 \longrightarrow [0, 1]$ which is commutative, associative, non-decreasing in both variables and has the neutral element $e \in]0, 1[$ (see also [6]). If we take a uninorm in a broader sense, i.e., if for a neutral element we have $e \in [0, 1]$, then the class of uninorms covers also the class of t-norms and the class of t-conorms. In order to stress that we assume a uninorm with $e \in]0, 1[$ we will call such a uninorm *proper*. For each uninorm the value $U(1, 0) \in \{0, 1\}$ is the annihilator of U. A uninorm is called *conjunctive* (*disjunctive*) if $U(1, 0) = 0$ ($U(1, 0) = 1$). Due to the associativity we can uniquely define n-ary form of a uninorm for any $n \in \mathbb{N}$ and therefore in some proofs we will use ternary form instead of binary, where it is suitable.

For each uninorm U with the neutral element $e \in [0, 1]$, the restriction of U to $[0, e]^2$ is a t-norm on $[0, e]^2$, i.e., a linear transformation of some t-norm T_U on $[0, 1]^2$ and the restriction of U to $[e, 1]^2$ is a t-conorm on $[e, 1]^2$, i.e., a linear transformation of some t-conorm C_U on $[0, 1]^2$. Moreover, $\min(x, y) \leq U(x, y) \leq \max(x, y)$ for all $(x, y) \in [0, e] \times [e, 1] \cup [e, 1] \times [0, e]$.

From any pair of a t-norm and a t-conorm we can construct the minimal and the maximal uninorm with the given underlying functions.

Proposition 3 ([16]). *Let $T : [0, 1]^2 \longrightarrow [0, 1]$ be a t-norm and $C : [0, 1]^2 \longrightarrow [0, 1]$ a t-conorm and assume $e \in [0, 1]$. Then the two functions $U_{\min}, U_{\max} : [0, 1]^2 \longrightarrow [0, 1]$ given by*

$$U_{\min}(x, y) = \begin{cases} e \cdot T(\frac{x}{e}, \frac{y}{e}) & \text{if } (x, y) \in [0, e]^2, \\ e + (1 - e) \cdot C(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x, y) \in [e, 1]^2, \\ \min(x, y) & \text{otherwise} \end{cases}$$

and

$$U_{\max}(x, y) = \begin{cases} e \cdot T(\frac{x}{e}, \frac{y}{e}) & \text{if } (x, y) \in [0, e]^2, \\ e + (1 - e) \cdot C(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x, y) \in [e, 1]^2, \\ \max(x, y) & \text{otherwise} \end{cases}$$

are uninorms. We will denote the set of all uninorms of the first type by $\mathcal{U}_{\min}$ and of the second type by $\mathcal{U}_{\max}$.

Proposition 4 ([6]). *Let $f : [0, 1] \longrightarrow [-\infty, \infty]$, $f(0) = -\infty$, $f(1) = \infty$ be a continuous strictly increasing function. Then a binary function $U : [0, 1]^2 \longrightarrow [0, 1]$ given by*

$$U(x, y) = f^{-1}(f(x) + f(y)),$$

where $f^{-1}: [-\infty, \infty] \longrightarrow [0, 1]$ is an inverse function to f, is a uninorm, which will be called a representable *uninorm. This uninorm is conjunctive if we take the convention $\infty + (-\infty) = -\infty$ and it is disjunctive if we take the convention $\infty + (-\infty) = \infty$.*

Note that if U is a representable uninorm with the neutral element e then for every $x \in]0, 1[$ there exists a $y \in]0, 1[$ such that $U(x, y) = e$.

3. Uninorms continuous on $[0, e[^2 \cup]e, 1]^2$

Let $U : [0, 1]^2 \longrightarrow [0, 1]$ be a uninorm continuous on $[0, e[^2 \cup]e, 1]^2$. First we will focus on T_U (similar observations can be obtained for C_U by duality) and we will now show that T_U can be obtained by lifting of a continuous t-subnorm (see [10]) to a t-norm (see [9]). Let us recall that a binary operation $M : [0, 1]^2 \longrightarrow [0, 1]$ is a t-subnorm if it is commutative, associative, non-decreasing in both variables and $M(x, y) \le \min(x, y)$ for all $(x, y) \in [0, 1]^2$. Evidently, each t-norm is also a t-subnorm. In order to stress that a t-subnorm is not a t-norm we will call such a t-subnorm *proper*. A dual operation to a t-subnorm is t-superconorm. A binary operation $R : [0, 1]^2 \longrightarrow [0, 1]$ is a t-superconorm if it is commutative, associative, non-decreasing in both variables and $R(x, y) \ge \max(x, y)$ for all $(x, y) \in [0, 1]^2$. A t-subnorm $M : [0, 1]^2 \longrightarrow [0, 1]$ (t-superconorm $R : [0, 1]^2 \longrightarrow [0, 1]$) is cancellative if $M(x, y) = M(x, z)$ ($R(x, y) = R(x, z)$) implies $y = z$ for all $x > 0$ ($x < 1$), $x, y, z \in [0, 1]$. Further, a continuous t-subnorm M is proper and Archimedean if and only if 0 is the unique idempotent element of M. More details on t-subnorms can be found in [19,20].

From each t-subnorm $M : [0, 1]^2 \longrightarrow [0, 1]$ we can define a function $T : [0, 1]^2 \longrightarrow [0, 1]$ by

$$T(x, y) = \begin{cases} M(x, y) & \text{if } (x, y) \in [0, 1[^2, \\ \min(x, y) & \text{otherwise} \end{cases}$$

for all $(x, y) \in [0, 1]^2$. Then the function T is a t-norm. This process is called lifting of a t-subnorm to a t-norm. Then $T = M$ if and only if M is a t-norm.

Vice versa, in [9] a border-continuous projection $M_T : [0, 1]^2 \longrightarrow [0, 1]$ of a t-norm was defined by

$$M_T(x, y) = \begin{cases} T(x, y) & \text{if } (x, y) \in [0, 1[^2, \\ \lim\limits_{\substack{u \longrightarrow x^- \\ v \longrightarrow y^-}} T(u, v) & \text{if } \max(x, y) = 1. \end{cases}$$

The idea of this border-continuous projection was to obtain a reverse process to the lifting of a t-subnorm to a t-norm. However, such a border-continuous projection need not to be monotone.

Example 1. Let $T : [0, 1]^2 \longrightarrow [0, 1]$ be given by

$$T(x, y) = \begin{cases} \min(x, y) & \text{if } \max(x, y) = 1, \\ \max(0, \frac{2}{3}(x + y) - \frac{5}{6}) & \text{if } (x, y) \in [\frac{1}{2}, 1]^2, \max(x, y) < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Evidently, T is commutative, non-decreasing in both variables and 1 is its neutral element. Further, for any $x, y, z \in [0, 1]$, if $\max(x, y, z) = 1$ then since 1 is the neutral element of T we easily get

$$T(x, T(y, z)) = T(T(x, y), z).$$

If $\max(x, y, z) < 1$ then $T(y, z) < \frac{1}{2}$, $T(x, y) < \frac{1}{2}$ and

$$T(x, T(y, z)) = T(T(x, y), z) = 0.$$

Thus T is associative and therefore it is a t-norm.

For M_T given above we then obtain $M_T(\frac{1}{2}, 1) = 0$ and $M_T(\frac{1}{2}, \frac{7}{8}) = \frac{1}{12}$, i.e., M_T is not non-decreasing in both variables.

Therefore the proper definition of a border-continuous projection is

$$M_T(x, y) = \begin{cases} T(x, y) & \text{if } (x, y) \in [0, 1[^2, \\ \lim\limits_{u \longrightarrow 1^-} T(u, y) & \text{if } x = 1, y < 1, \\ \lim\limits_{u \longrightarrow 1^-} T(x, u) & \text{if } x < 1, y = 1, \\ \lim\limits_{\substack{u \longrightarrow 1^- \\ v \longrightarrow 1^-}} T(u, v) & \text{if } x = y = 1. \end{cases}$$

For a t-conorm C the border-continuous projection of C is given by

$$M_C(x,y) = \begin{cases} C(x,y) & \text{if } (x,y) \in]0,1]^2, \\ \lim\limits_{u \longrightarrow 0^+} C(u,y) & \text{if } x=0, y>0, \\ \lim\limits_{u \longrightarrow 0^+} C(x,u) & \text{if } x>0, y=0, \\ \lim\limits_{\substack{u \longrightarrow 0^+ \\ v \longrightarrow 0^+}} T(u,v) & \text{if } x=y=0. \end{cases}$$

In this case the border-continuous projection of a t-norm is commutative, bounded by minimum and non-decreasing in both variables. Note that a border-continuous projection is not border-continuous in the sense that each point from the border of the unit square is a point of continuity of M_T (as it is defined in [11]), but in the sense that $\lim\limits_{u \longrightarrow 1^-} M_T(x,u) = M_T(x,1)$, while the function $M_T(\,\cdot, 1)$ can be non-continuous.

Example 2. If we take the t-norm T from Example 1 then

$$M_T(x,y) = \begin{cases} \max(0, \frac{2}{3}(x+y) - \frac{5}{6}) & \text{if } (x,y) \in [\frac{1}{2}, 1]^2, \\ 0 & \text{otherwise} \end{cases}$$

and $M_T(M_T(1,1),x) = \frac{2 \cdot x}{3} - \frac{1}{2}$ for all $x \geq \frac{3}{4}$, however, $M_T(1, M_T(1,x)) = 0$ for all $x < 1$. Therefore M_T is not associative, i.e., not a t-subnorm.

The previous example shows that there is a need to characterize all t-norms such that their border-continuous projection is a t-subnorm.

It is evident that if T is border-continuous then $M_T = T$.

Lemma 1. *For a t-norm $T : [0,1]^2 \longrightarrow [0,1]$ and $x, y \in [0, 1[$ there is*

$$M_T(1, M_T(x,y)) = M_T(M_T(1,x), y)$$

if and only if there is: either

$$T(u_0, x) = \lim_{u \longrightarrow 1^-} T(u,x)$$

for some $u_0 \in [0, 1[$, or

$$T(a,y) = \lim_{v \longrightarrow a^-} T(v,y)$$

where $a = \lim\limits_{u \longrightarrow 1^-} T(u,x)$.

Proof. Assume any $x, y \in [0, 1[$ and denote $a = \lim\limits_{u \longrightarrow 1^-} T(u,x) = M_T(1,x)$. Then

$$M_T(1, M_T(x,y)) = \lim_{u \longrightarrow 1^-} T(u, T(x,y)) = \lim_{u \longrightarrow 1^-} T(T(u,x),y).$$

The monotonicity of T implies

$$\lim_{u \longrightarrow 1^-} T(T(u,x),y) \leq T(\lim_{u \longrightarrow 1^-} T(u,x), y) = M_T(M_T(1,x),y).$$

Case 1: Suppose that $T(u_0, x) = a$ for some $u_0 \in [0, 1[$. Then

$$\lim_{u \longrightarrow 1^-} T(T(u,x),y) \geq T(T(u_0,x),y) = T(a,y) = T(\lim_{u \longrightarrow 1^-} T(u,x), y),$$

i.e., the two opposite inequalities give us $M_T(1, M_T(x,y)) = M_T(M_T(1,x),y)$.

Case 2: Suppose that $T(u, x) < a$ for all $u \in [0, 1[$. Then due to the monotonicity of T there is $T(a,y) = \lim\limits_{v \longrightarrow a^-} T(v,y)$ if and only if

$$T(\lim_{u \longrightarrow 1^-} T(u,x), y) = T(a,y) = \lim_{v \longrightarrow a^-} T(v,y) = \lim_{u \longrightarrow 1^-} T(T(u,x),y),$$

which means that $M_T(1, M_T(x,y)) = M_T(M_T(1,x),y)$. □

Proposition 5. *For a t-norm $T : [0,1]^2 \longrightarrow [0,1]$ its border-continuous projection $M_T : [0,1]^2 \longrightarrow [0,1]$ is a t-subnorm if and only if the following two conditions are satisfied:*

(i) *for all $x, y \in [0, 1[$ either $T(u_0, x) = \lim\limits_{u \longrightarrow 1^-} T(u,x)$ for some $u_0 \in [0, 1[$, or $T(a,y) = \lim\limits_{v \longrightarrow a^-} T(v,y)$, where $a = \lim\limits_{u \longrightarrow 1^-} T(u,x)$,*

(ii) *either $\lim\limits_{u \longrightarrow 1^-} T(u,u) = 1$, or $T(u_0, v_0) = \lim\limits_{u \longrightarrow 1^-} T(u,u)$ for some $u_0, v_0 \in [0, 1[$, or for all $x \in [0, 1[$ there is $T(b,x) = \lim\limits_{v \longrightarrow b^-} T(v,x)$, where $b = \lim\limits_{u \longrightarrow 1^-} T(u,u)$.*

Proof. The commutativity, monotonicity and boundedness by the minimum of M_T are obvious. The associativity of M_T for $x, y, z \in [0, 1]$ with $\max(x, y, z) < 1$ follows from the associativity of T. Further, from Lemma 1 it follows that for all $x, y \in [0, 1[$ there is $M_T(1, M_T(x, y)) = M_T(M_T(1, x), y)$ and thus also $M_T(1, M_T(x, y)) = M_T(M_T(1, y), x)$ if and only if the condition (i) is satisfied. If $x = y = z = 1$ the associativity is clear.

Finally, assume $x \in [0, 1[$, $y = z = 1$ and denote

$$b = \lim_{u \longrightarrow 1^-} T(u, u) = M_T(1, 1) = \lim_{\substack{u \longrightarrow 1^- \\ v \longrightarrow 1^-}} T(u, v).$$

We have to show that $M_T(1, M_T(1, x)) = M_T(M_T(1, 1), x)$. Here

$$M_T(1, M_T(1, x)) = \lim_{u \longrightarrow 1^-} T(u, \lim_{v \longrightarrow 1^-} T(v, x))$$

and

$$M_T(M_T(1, 1), x) = T(\lim_{\substack{u \longrightarrow 1^- \\ v \longrightarrow 1^-}} T(u, v), x).$$

Due to Lemma 1 we have

$$T(u, \lim_{v \longrightarrow 1^-} T(v, x)) = \lim_{v \longrightarrow 1^-} T(T(u, v), x)$$

for all $u \in [0, 1[$ and thus

$$\lim_{u \longrightarrow 1^-} T(u, \lim_{v \longrightarrow 1^-} T(v, x)) = \lim_{\substack{u \longrightarrow 1^- \\ v \longrightarrow 1^-}} T(T(u, v), x). \tag{1}$$

Case 1: Suppose that $M_T(1, 1) < 1$ and $T(u_0, v_0) = \lim_{\substack{u \longrightarrow 1^- \\ v \longrightarrow 1^-}} T(u, v) = M_T(1, 1)$ for some $u_0, v_0 \in [0, 1[$. Then

$$\lim_{\substack{u \longrightarrow 1^- \\ v \longrightarrow 1^-}} T(T(u, v), x) = M_T(M_T(1, 1), x).$$

Case 2: Suppose that $M_T(1, 1) < 1$ and $T(u, v) < M_T(1, 1)$ for all $u, v \in [0, 1[$. Then

$$\lim_{\substack{u \longrightarrow 1^- \\ v \longrightarrow 1^-}} T(T(u, v), x) = T(\lim_{\substack{u \longrightarrow 1^- \\ v \longrightarrow 1^-}} T(u, v), x)$$

if and only if $T(b, x) = \lim_{w \longrightarrow b^-} T(w, x)$.

Case 3: Suppose that $M_T(1, 1) = 1 = \lim_{u \longrightarrow 1^-} T(u, u)$. Then

$$M_T(M_T(1, 1), x) = M_T(1, x)$$

and we have to show that

$$\lim_{u \longrightarrow 1^-} T(u, \lim_{v \longrightarrow 1^-} T(v, x)) = M_T(1, x) = \lim_{u \longrightarrow 1^-} T(u, x).$$

It is evident that $M_T(1, M_T(1, x)) \leq M_T(1, x)$. Thus if $M_T(1, M_T(1, x)) \neq M_T(M_T(1, 1), x)$ then (1) implies

$$\lim_{\substack{u \longrightarrow 1^- \\ v \longrightarrow 1^-}} T(T(u, v), x) = \lim_{u \longrightarrow 1^-} T(u, \lim_{v \longrightarrow 1^-} T(v, x)) < \lim_{u \longrightarrow 1^-} T(u, x) = c.$$

In such a case there exists an $\varepsilon > 0$ such that for all $u, v \in [0, 1[$ we have $T(T(u, v), x) < c - \varepsilon$. On the other hand, there exists a $u_0 \in [0, 1[$ such that $T(u_0, x) \geq c - \frac{\varepsilon}{2}$. Since $\lim_{u \longrightarrow 1^-} T(u, u) = 1$ there exists a $u_1 \in [0, 1[$ such that $T(u_1, u_1) > u_0$ which implies

$$c - \varepsilon > T(T(u_1, u_1), x) \geq T(u_0, x) \geq c - \frac{\varepsilon}{2}$$

what is a contradiction. Thus $M_T(1, M_T(1, x)) = M_T(1, x)$. □

As an easy corollary of the previous result we get the following.

Corollary 1. *Let $T : [0, 1]^2 \longrightarrow [0, 1]$ be a t-norm left-continuous on $[0, 1[^2$. Then its border-continuous projection $M_T : [0, 1]^2 \longrightarrow [0, 1]$ is a t-subnorm.*

Since we focus on uninorms which are continuous on $[0, e[^2 \cup]e, 1]^2$ then M_{T_U} is always a t-subnorm and similar result can be obtained also for C_U.

Further we recall a result from [20] (see also [10]).

Theorem 1. *A mapping $M : [0,1]^2 \longrightarrow [0,1]$ is a continuous proper t-subnorm if and only if it is an ordinal sum of continuous Archimedean t-norms and a continuous Archimedean proper t-subnorm, $M = (\langle a_k, b_k, M_k\rangle \mid k \in K)$, where $(]a_k, b_k[)_{k\in K}$ is a disjoint system of open subintervals of $[0, 1]$ with $b_{k_0} = 1$ for some $k_0 \in K$, and M_{k_0} is a continuous Archimedean proper t-subnorm and M_k is a continuous Archimedean t-norm for all $k \neq k_0$, i.e.,*

$$M(x,y) = \begin{cases} a_k + (b_k - a_k)M_k(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}) & \text{if } (x,y) \in]a_k, b_k]^2, \\ \min(x,y) & \text{else.} \end{cases}$$

Remark 1. If U is continuous on $[0, e[^2 \cup]e, 1]^2$ then M_{T_U} is an ordinal sum of continuous Archimedean t-norms and the last continuous Archimedean t-subnorm (which can be also a t-norm if T_U is a continuous t-norm) and M_{C_U} is an ordinal sum of the first continuous Archimedean t-superconorm (which can be also a t-conorm if C_U is a continuous t-conorm) and continuous Archimedean t-conorms. Thus there exist a $b^* \in [0, e]$ and a $c^* \in [e, 1]$ such that U is Archimedean on $[b^*, e]^2$ and on $[c, b^*]^2$ (note that on singletons a uninorm is trivially Archimedean). Here, if T_U (C_U) is not continuous then $b^* < e$ ($c^* > e$) Further, M_{T_U} on $[0, b^*]^2$ is a continuous t-norm (on $[0, b^*]^2$) and M_{C_U} on $[c^*, 1]^2$ is a continuous t-conorm (on $[c^*, 1]^2$).

As the first step we will focus on Archimedean uninorms, i.e., uninorms with Archimedean underlying functions, which are continuous on $[0, e[^2 \cup]e, 1]^2$. The set of continuous Archimedean t-subnorms can be divided into three parts: continuous cancellative t-subnorms, continuous nilpotent t-subnorms, continuous t-subnorms with no nilpotent element which are not cancellative. Here continuous nilpotent t-subnorms are such that posses a nilpotent element $x \in]0, 1]$ with $M(x, x) = 0$. Although the structure of uninorms such that M_{T_U} (M_{C_U}) is not cancellative is quite complicated, in the case that M_{T_U} (M_{C_U}) is a continuous cancellative t-subnorm (t-superconorm) several results similar as in the case of uninorms with strict underlying functions can be shown. First we recall the result for uninorms with strict underlying functions from [7], which was later corrected in [17].

Theorem 2. *Let $U : [0,1]^2 \longrightarrow [0,1]$ be a uninorm with the neutral element $e \in]0, 1[$ such that both T_U and C_U are strict then one of the following seven statements holds:*

(i) $U \in \mathcal{U}_{\min}$,

(ii)

$$U(x,y) = \begin{cases} e \cdot T_U(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e) \cdot C_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ 1 & \text{if } x = 1 \text{ or } y = 1, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

(iii)

$$U(x,y) = \begin{cases} e \cdot T_U(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e) \cdot C_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ 1 & \text{if } x = 1, y > 0 \text{ or } y = 1, x > 0, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

(iv) $U \in \mathcal{U}_{\max}$,

(v)

$$U(x,y) = \begin{cases} e \cdot T_U(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e) \cdot C_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ 0 & \text{if } x = 0 \text{ or } y = 0, \\ \max(x,y) & \text{otherwise,} \end{cases}$$

(vi)

$$U(x,y) = \begin{cases} e \cdot T_U(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e) \cdot C_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ 0 & \text{if } x = 0, y < 1 \text{ or } y = 0, x < 1, \\ \max(x,y) & \text{otherwise,} \end{cases}$$

(vii) *U is representable.*

In Section 5 we will show that a similar representation holds also for Archimedean uninorms continuous and cancellative on $]0, e[^2 \cup]e, 1[^2$.

Let us now recall the fundamental result of Clifford [3].

Theorem 3. *Let $A \neq \emptyset$ be a totally ordered set and $(G_\alpha)_{\alpha \in A}$ with $G_\alpha = (X_\alpha, *_\alpha)$ be a family of semigroups. Assume that for all $\alpha, \beta \in A$ with $\alpha < \beta$ the sets X_α and X_β are either disjoint or that $X_\alpha \cap X_\beta = \{x_{\alpha,\beta}\}$, where $x_{\alpha,\beta}$ is both the neutral element of G_α and the annihilator of G_β and where for each $\gamma \in A$ with $\alpha < \gamma < \beta$ we have $X_\gamma = \{x_{\alpha,\beta}\}$. Put $X = \bigcup_{\alpha \in A} X_\alpha$ and define the binary operation $*$ on X by*

$$x * y = \begin{cases} x *_\alpha y & \text{if } (x, y) \in X_\alpha \times X_\alpha, \\ x & \text{if } (x, y) \in X_\alpha \times X_\beta \text{ and } \alpha < \beta, \\ y & \text{if } (x, y) \in X_\alpha \times X_\beta \text{ and } \alpha > \beta. \end{cases}$$

Then $G = (X, *)$ is a semigroup. The semigroup G is commutative if and only if for each $\alpha \in A$ the semigroup G_α is commutative.

Note that ordinal sum constructions of t-norms, t-conorms and uninorms are all based on this result (see [12,13,22]).

Example 3. In the case when for a uninorm U operations T_U and C_U are strict we have two possibilities: either $U(x, y) = e$ implies $x = y = e$ for all $(x, y) \in [0, 1]^2$, or there exist $x, y \in [0, 1]$, $x < e < y$ such that $U(x, y) = e$. From the previous result we see that then U is representable. In such a case for all $x \in]0, 1[$ there exists a $y \in]0, 1[$ such that $U(x, y) = e$. If we assume a division into sets on which U is closed then the finest possible partition that we can get is to divide $[0, 1]$ into $\{0\}$, $]0, 1[$, and $\{1\}$. Thus we have three semigroups $G_{a_1} = (\{0\}, U)$, $G_{a_2} = (]0, 1[, U)$, and $G_{a_3} = (\{1\}, U)$. Let $\preceq$ be an order on the set $A = \{a_1, a_2, a_3\}$. Then the monotonicity implies $a_1 \prec a_2$ and $a_3 \prec a_2$. Thus there are two possible orders on the set A: either $a_1 \prec a_3 \prec a_2$, which corresponds to a conjunctive representable uninorm, or $a_3 \prec a_1 \prec a_2$, which corresponds to a disjunctive representable uninorm. Thus in both cases, i.e., whether U is conjunctive or disjunctive, it is easy to see that U is equal to an ordinal sum of G_{a_1}, G_{a_2} and G_{a_3}.

Assume now that $U(x, y) = e$ implies $x = y = e$ for all $(x, y) \in [0, 1]^2$. Similarly as above we can show that then the finest partition which we can make is to divide $[0, 1]$ into $\{0\}$, $]0, e[$, $\{e\}$, $]e, 1[$ and $\{1\}$. Thus we have five semigroups $G_{a_1} = (\{0\}, U)$, $G_{a_2} = (]0, e[, U)$, $G_{a_3} = (\{e\}, U)$, $G_{a_4} = (]e, 1[, U)$ and $G_{a_5} = (\{1\}, U)$. Let $\preceq$ be an order on the set $A = \{a_1, a_2, a_3, a_4, a_5\}$. Since e is the neutral element we have $a_i \prec a_3$ for $i = 1, 2, 4, 5$. Further the monotonicity implies $a_1 \prec a_2$ and $a_5 \prec a_4$. Then we have the following six possible orders on the set A:

(i) $a_1 \prec a_2 \prec a_5 \prec a_4 \prec a_3$,
(ii) $a_1 \prec a_5 \prec a_2 \prec a_4 \prec a_3$,
(iii) $a_1 \prec a_5 \prec a_4 \prec a_2 \prec a_3$,
(iv) $a_5 \prec a_1 \prec a_2 \prec a_4 \prec a_3$,
(v) $a_5 \prec a_1 \prec a_4 \prec a_2 \prec a_3$,
(vi) $a_5 \prec a_4 \prec a_1 \prec a_2 \prec a_3$.

Again it is easy to see that an ordinal sum of G_{a_1}, G_{a_2}, G_{a_3}, G_{a_4} and G_{a_5} with the first order corresponds to the form (i) from Theorem 2, the second to the form (iii), the third to the form (v), the fourth to the form (ii), the fifth to the form (vi) and the last to the form (iv).

Theorem 3 and Example 3 show that all uninorms with strict underlying functions can be expressed as an ordinal sum of semigroups related to trivial uninorms (t-norms and t-conorms), representable uninorms and singletons. The same construction can be used also for uninorms continuous on $[0, e[^2 \cup]e, 1]^2$.

4. Continuous cancellative t-subnorms

In this section we will recall several known results and show some new results on continuous cancellative t-subnorms that we will use in the next section.

From [20] we know that a continuous t-subnorm M is proper if and only if $M(1, 1) < 1$. Moreover, a continuous t-subnorm M is proper and Archimedean if and only if $M(1, x) < x$ for all $x > 0$.

The following result [19, Theorem 27] (see also [25]) is based on results of Aczél [1].

Theorem 4. *Let $S : [0, 1]^2 \longrightarrow [0, 1]$ be a continuous, Archimedean, proper t-subnorm which is cancellative on $]0, 1]^2$. Then S has a continuous additive generator.*

The proof of this result is based on the definition of powers $x_S^{(m)} = \underbrace{S(x, S(x, \ldots))}_{m\text{-times}}$, $x_S^{(\frac{1}{n})} = y$ if and only if $y_S^{(n)} = x$, $x_S^{(\frac{m}{n})} = z$ if and only if $z_S^{(n)} = x_S^{(m)}$, for all $n, m \in \mathbb{N}$. Since $S(1, 1) < 1$ we will start from $c = 1$ and then due to continuity of S and its cancellativity $c_S^{(\frac{m}{n})}$ is well defined in the case that $\frac{m}{n} \geq 1$. Thus we can define a strictly decreasing function $s^* : [1, \infty[\cap \mathbb{Q} \longrightarrow [0, 1]$, where $s^*(\frac{m}{n}) = c_S^{(\frac{m}{n})}$. This function can be uniquely extended to a continuous, strictly decreasing function $s_* : [1, \infty] \longrightarrow [0, 1]$. The inverse function s of s_* will be then the continuous, strictly decreasing additive generator of S.

Note that a continuous, cancellative t-subnorm is Archimedean. Indeed, in the opposite case the continuity implies existence of a non-trivial idempotent point which, however, violates the cancellativity.

Now we will show that each proper, continuous, cancellative t-subnorm is isomorphic to the t-subnorm $S_P(x, y) = \frac{x \cdot y}{2}$.

Proposition 6. *Let $S : [0, 1]^2 \longrightarrow [0, 1]$ be a proper, continuous, cancellative t-subnorm. Then there exists an increasing isomorphism $\varphi : [0, 1] \longrightarrow [0, 1]$ such that $S(x, y) = \varphi^{-1}(\frac{\varphi(x) \cdot \varphi(y)}{2})$ for all $(x, y) \in [0, 1]^2$.*

Proof. Due to Theorem (4) S has an additive generator s^* which is unique up to a positive multiplicative constant. Then s^* is continuous and strictly decreasing with $s^*(0) = \infty$, $s^*(1) > 0$. Moreover, the t-subnorm S_P has an additive generator $s(x) = -\ln(\frac{x}{2})$. Let m be an additive generator of S such that $m(1) = s(1)$, and let s^{-1} be an inverse function to s. Then the function $\varphi : [0, 1] \longrightarrow [0, 1]$ given by $\varphi(x) = s^{-1}(m(x))$ is continuous, since both s, m are continuous, increasing, since both s, m are strictly decreasing and $\varphi(0) = 0$, $\varphi(1) = 1$. Thus φ is an increasing isomorphism on [0, 1]. Further,

$$\varphi^{-1}(\frac{\varphi(x) \cdot \varphi(y)}{2}) = \varphi^{-1}(s^{-1}(s(\varphi(x)) + s(\varphi(y)))) = m^{-1}(m(x) + m(y)) = S(x, y).$$

□

By duality between t-subnorms and t-superconorms we can show the following result.

Proposition 7. *Let $R : [0, 1]^2 \longrightarrow [0, 1]$ be a proper, continuous, cancellative t-superconorm. Then there exists an increasing isomorphism $\varphi : [0, 1] \longrightarrow [0, 1]$ such that*

$$R(x, y) = \varphi^{-1}\left(\frac{1 + \varphi(x) + \varphi(y) - \varphi(x) \cdot \varphi(y)}{2}\right)$$

for all $(x, y) \in [0, 1]^2$.

Note that the t-subnorm S_P is dual to the t-superconorm R_P given by $R_P = \frac{1+x+y-x \cdot y}{2}$. It is also easy to see the following corollary.

Corollary 2.

(i) *Let T be a t-norm such that M_T is a proper, continuous, cancellative t-subnorm. Then T is isomorphic to T^P, where $T^P : [0, 1]^2 \longrightarrow [0, 1]$ is a t-norm given for all $(x, y) \in [0, 1]^2$ by*

$$T^P(x, y) = \begin{cases} \frac{x \cdot y}{2} & \text{if } \max(x, y) < 1, \\ \min(x, y) & \text{else.} \end{cases}$$

(ii) *Let C be a t-conorm such that M_C is a proper, continuous, cancellative t-superconorm. Then C is isomorphic to C^P, where $C^P : [0, 1]^2 \longrightarrow [0, 1]$ is a t-conorm given for all $(x, y) \in [0, 1]^2$ by*

$$C^P(x, y) = \begin{cases} \frac{1+x+y-x \cdot y}{2} & \text{if } \min(x, y) > 0, \\ \max(x, y) & \text{else.} \end{cases}$$

Based on these results we can show the following.

Proposition 8. *Let $U : [0, 1]^2 \longrightarrow [0, 1]$ be a uninorm such that M_{T_U} is a proper, continuous, cancellative t-subnorm and M_{C_U} is a proper, continuous, cancellative t-superconorm. Then there exists an increasing isomorphism $\varphi : [0, 1] \longrightarrow [0, 1]$ such that $U(x, y) = \varphi^{-1}(UP(\varphi(x), \varphi(y)))$ for all $(x, y) \in [0, 1]^2$, where UP is a uninorm such that $M_{T_{UP}} = \frac{x \cdot y}{2}$ and $M_{C_{UP}} = \frac{1+x+y-x \cdot y}{2}$.*

Proof. Due to Corollary 2 we know that there exist two increasing isomorphisms ψ, ϕ on [0, 1] such that $T_U(x, y) = \psi^{-1}(T^P(\psi(x), \psi(y)))$, $C_U(x, y) = \phi^{-1}(C^P(\phi(x), \phi(y)))$. For any $e_1 \in]0, 1[$ and the neutral element $e \in]0, 1[$ of U we define an increasing isomorphism $\varphi : [0, 1] \longrightarrow [0, 1]$ by

$$\varphi(x) = \begin{cases} e \cdot \psi(\frac{x}{e_1}) & \text{if } x \leq e_1, \\ e + (1 - e) \cdot \phi(\frac{x - e_1}{1 - e_1}) & \text{otherwise.} \end{cases}$$

Then $\varphi(0) = 0$, $\varphi(1) = 1$ and $\varphi(e_1) = e$. Since isomorphic transformation of a uninorm is again a uninorm then $UP(x, y) = \varphi(U(\varphi^{-1}(x), \varphi^{-1}(y)))$ is a uninorm. Moreover, it is easy to see that $M_{T_{UP}} = \frac{x \cdot y}{2}$ and $M_{C_{UP}} = \frac{1+x+y-x \cdot y}{2}$. □

Similarly we can show the following results.

Proposition 9. *Let $U : [0, 1]^2 \longrightarrow [0, 1]$ be a uninorm such that M_{T_U} is a proper, continuous, cancellative t-subnorm and M_{C_U} is a continuous, cancellative t-conorm. Then there exists an increasing isomorphism $\varphi : [0, 1] \longrightarrow [0, 1]$ such that $U(x, y) = \varphi^{-1}(UPT(\varphi(x), \varphi(y)))$ for all $(x, y) \in [0, 1]^2$, where UPT is a uninorm such that $M_{T_{UPT}} = \frac{x \cdot y}{2}$ and $M_{C_{UPT}} = x + y - x \cdot y$.*

Proposition 10. *Let $U : [0, 1]^2 \longrightarrow [0, 1]$ be a uninorm such that M_{T_U} is a continuous, cancellative t-norm and M_{C_U} is a proper, continuous, cancellative t-superconorm. Then there exists an increasing isomorphism $\varphi : [0, 1] \longrightarrow [0, 1]$ such that $U(x, y) = \varphi^{-1}(UPC(\varphi(x), \varphi(y)))$ for all $(x, y) \in [0, 1]^2$, where UPC is a uninorm such that $M_{T_{UPC}} = x \cdot y$ and $M_{C_{UPC}} = \frac{1+x+y-x \cdot y}{2}$.*

5. Uninorms continuous and cancellative on $]0, e[^2 \cup]e, 1[^2$

First let us observe that due to the boundedness by the minimum (maximum) from above (from below) of a t-subnorm (t-superconorm) a uninorm which is continuous on $]0, e[^2 \cup]e, 1[^2$ is continuous on $[0, e[^2 \cup]e, 1]^2$. If a uninorm is cancellative and continuous on $]0, e[^2 \cup]e, 1[^2$ then it is Archimedean. First we will focus on these uninorms and then we will discuss related non-Archimedean (i.e., non-cancellative) uninorms.

5.1. Archimedean uninorms

We will now proceed similarly as in [7], although in our case there are few differences that should be modified.

Lemma 2. *Let $U : [0,1]^2 \longrightarrow [0,1]$ be a uninorm with the neutral element $e \in]0, 1[$, such that M_{T_U} and M_{C_U} are continuous and cancellative. Then*

(i) *if $U(x_0, y_0) = x_0$ for some $x_0 \in]0, e[$, $y_0 \in]e, 1[$ then $U(x, y_0) = x$ for all $x \in [0, e[$.*
(ii) *if $U(x_0, y_0) = y_0$ for some $x_0 \in]0, e[$, $y_0 \in]e, 1[$ then $U(x_0, y) = y$ for all $y \in]e, 1]$.*

Proof. We will show only the first part as the second part is analogous. Let $U(x_0, y_0) = x_0$. Since M_{T_U} is cancellative we have $U(x_0, u) > 0$ for all $u > 0$. Then since M_{T_U} is continuous for all $u \in [0, U(e/2, x_0)]$ there exists a $t \in [0, e]$ such that $U(t, x_0) = u$. Then

$$U(u, y_0) = U(t, U(x_0, y_0)) = U(t, x_0) = u.$$

Now assume any $x \in [0, e[$: then there exists a $q \in [0, e]$ such that $U(x, q) \in [0, U(e/2, x_0)]$. Then $U(x, y_0) \geq x$ and if $U(x, y_0) > x$ we have

$$U(q, x) = U(U(q, x), y_0) = U(q, U(x, y_0)) > U(q, x)$$

what is a contradiction. The last inequality follows from the cancellativity of M_{T_U} and from $U(q, U(x, y_0)) \geq q > U(q, x)$ in the case that $U(x, y_0) \geq e$. Summarizing, $U(x, y_0) = x$ for all $x \in [0, e[$. □

Lemma 3. *Let $U : [0,1]^2 \longrightarrow [0,1]$ be a uninorm with the neutral element $e \in]0, 1[$, such that M_{T_U} and M_{C_U} are continuous and cancellative. Then*

(i) *if $U(x_0, y_0) = x_0$ for some $x_0 \in]0, e[$, $y_0 \in]e, 1[$ then $U(x, y) = x$ for all $x \in [0, e[$, $y \in [e, 1[$.*
(ii) *if $U(x_0, y_0) = y_0$ for some $x_0 \in]0, e[$, $y_0 \in]e, 1[$ then $U(x, y) = y$ for all $x \in]0, e]$, $y \in]e, 1]$.*

Proof. We will again show only the first part as the second part is analogous. From the previous lemma we know that $U(x, y_0) = x$ for all $x \in [0, e[$. Let $U(x, z) > x$ for some $x \in]0, e[$, $z \in]e, 1[$. Then $U(x, y_0) = x$ and $U(x, e) = x$, i.e., $z > y_0$. Further,

$$U(x, z) = U(U(x, y_0), z) = U(U(x, z), y_0) = \cdots = U(U(x, z), U(\underbrace{y_0, \ldots, y_0}_{n\text{-times}})).$$

Since M_{C_U} is Archimedean we have $\lim_{n \longrightarrow \infty} U(\underbrace{y_0, \ldots, y_0}_{n\text{-times}}) = 1$ and thus $U(U(x, z), t) = U(x, z)$ for all $t \in [e, 1[$. Particularly, $U(U(x, z), z) = U(x, z)$ and thus if $U(x, z) < e$ the previous lemma implies $U(x, z) = x$. In the case when $U(x, z) \geq e$ then $U(U(x, z), t) = U(U(x, z), e)$ for all $t \in [e, 1[$ violates the cancellativity of M_{C_U}. Therefore $U(x, y) = x$ for all $x \in]0, e[$, $y \in]e, 1[$. The monotonicity and the neutral element of U then imply the result. □

We see that if for a single point (x, y) from $]0, e[\times]e, 1[$ we have $U(x, y) = \min(x, y)$ ($U(x, y) = \max(x, y)$) then $U = \min$ ($U = \max$) on the whole set $]0, e[\times]e, 1[$. We will now check the values $U(0, x)$ and $U(1, x)$ for $x \in [0, 1]$.

Lemma 4. *Let $U : [0,1]^2 \longrightarrow [0,1]$ be a uninorm with the neutral element $e \in]0, 1[$, such that M_{T_U} and M_{C_U} are continuous and cancellative. Then $U(0, x) \in \{0, x\}$ and $U(1, x) \in \{1, x\}$ for all $x \in [0, 1]$.*

Proof. We will show only $U(0, x) \in \{0, x\}$ for all $x \in [0, 1]$ as the proof of the second part is analogous. It is evident that $U(0, x) = 0$ for all $x \in [0, e]$. Now suppose that $U(0, x) = c \in]0, x[$ for some $x \in]e, 1]$. Then the associativity implies $U(0, c) = c$ and if $c \leq e$ we get $c = 0$ what is a contradiction. Thus $c > e$. Then due to the continuity of M_{C_U} there exist a, $b \in]e, 1]$ such that $U(x, a) = U(c, b)$, where $a \neq b$ due to the cancellativity of M_{C_U}. Then

$$U(c, b) = U(0, U(c, b)) = U(0, U(x, a)) = U(c, a)$$

what is a contradiction with the cancellativity of M_{C_U}. □

Lemma 5. *Let $U : [0,1]^2 \longrightarrow [0,1]$ be a uninorm with the neutral element $e \in]0, 1[$, such that M_{T_U} and M_{C_U} are continuous and cancellative. Then if there exists a $p \in [0, 1]$ such that $U(0, x) = 0$ for all $x < p$ and $U(0, x) = x$ for all $x > p$ then p is an idempotent point of U. Similarly, if there exists a $q \in [0, 1]$ such that $U(1, x) = x$ for all $x < q$ and $U(1, x) = 1$ for all $x > q$ then q is an idempotent point of U.*

Proof. We will again show only the first part of the claim as the second part is analogous. Let $U(0, x) = 0$ for all $x < p$ and $U(0, x) = x$ for all $x > p$. Then $p \geq e$. If $p = e$ we are done, and therefore we will suppose that $p > e$. If p is not an idempotent element then there exists $p_1 \in]e, 1[$ such that $p_1 < p$ and $U(p_1, p_1) > p$. Then we have $0 = U(0, p_1) = U(U(0, p_1), p_1)$, however, $U(0, U(p_1, p_1)) = U(p_1, p_1) > p$, what is a contradiction. □

Remark 2. The previous results show that if for a uninorm U, such that M_{T_U} and M_{C_U} are continuous and cancellative, there is $U(x, y) = x$ for some $(x, y) \in]0, e[\times]e, 1[$ then U has one of the forms (i), (ii), or (iii) from Theorem 2. If $U(x, y) = y$ for some $(x, y) \in]0, e[\times]e, 1[$ then U has one of the forms (iv), (v), or (vi) from Theorem 2.

We will denote by $\mathcal{U}_{nr}$ all uninorms of the form (i), (ii), (iii), (iv), (v), or (vi) from Theorem 2. In the following we will investigate uninorms, such that M_{T_U} (M_{C_U}) is a continuous cancellative t-subnorm (t-superconorm), which are not from the set $\mathcal{U}_{nr}$.

Lemma 6. *Let $U : [0, 1]^2 \longrightarrow [0, 1]$ be a uninorm with the neutral element $e \in]0, 1[$, such that M_{T_U} and M_{C_U} are continuous and cancellative, however, at least one of the underlying functions of U is non-continuous. Then*

(i) *there does not exist $x_0 \in]0, e[$ such that $x_0 < U(x_0, y) < e$ for all $y \in]e, 1[$,*
(ii) *there does not exist $y_0 \in]e, 1[$ such that $e < U(x, y_0) < y_0$ for all $x \in]0, e[$.*

Proof. We will again show only the first part as the second part is analogous. Suppose the contrary, i.e., that there exists an $x_0 \in]0, e[$ such that $x_0 < U(x_0, y) < e$ for all $y \in]e, 1[$. Let

$$x_1 = \sup\{x \in]0, e[\mid U(x, y) < e \text{ for all } y \in]e, 1[\}.$$

Then $e \geq x_1 > 0$ and since M_{T_U} is cancellative either $U(x_1, x_1) < x_1$, or $x_1 = e$. If $U(x_2, y_2) = x_2$ for some $x_2 \in]0, e[$, $y_2 \in]e, 1[$, then by Lemma 3 also $U(x_0, y_2) = x_0$, what is a contradiction. Thus

$$U(x, y) > x \text{ for all } x \in]0, e[, y \in]e, 1[. \tag{2}$$

First we will show that $\mathbf{x_1 = e}$. Suppose that $x_1 < e$. Then for all $\varepsilon > 0$ (small enough) there exists a $y_\varepsilon \in]e, 1[$ such that $U(x_1 + \varepsilon, y_\varepsilon) \geq e$. Let $y_1 \in]e, 1[$. Then (2) implies $U(x_1, y_1) > x_1$ and for $\varepsilon_y = U(x_1, y_1) - x_1$ we get

$$U(x_1, U(y_1, y_{\varepsilon_y})) = U(U(x_1, y_1), y_{\varepsilon_y}) = U(x_1 + \varepsilon_y, y_{\varepsilon_y}) \geq e.$$

Suppose that $U(x_3, y_3) \geq x_1$ for some $x_3 < x_1$, $y_3 \in]e, 1[$. Then for $z = U(y_1, y_{\varepsilon_y})$ we get $U(x_3, U(y_3, z)) = U(U(x_3, y_3), z) \geq e$. Since M_{C_U} is cancellative and $y_1, y_{\varepsilon_y}, y_3 \in]e, 1[$ also $U(y_3, z) \in]e, 1[$ which is a contradiction since from the definition of x_1 it follows that $U(x_3, q) < e$ for all $q \in]e, 1[$, i.e.,

$$U(x_3, U(y_3, z)) < e.$$

Thus $U(x, y) < x_1$ for all $x < x_1$, $y \in]e, 1[$. Further, $U(x_1, z) \geq e$ implies $U(U(x_1, x_1), z) \geq x_1$, however, since $U(x_1, x_1) < x_1$ we have also $U(U(x_1, x_1), z) < x_1$ what is a contradiction. **Therefore $\mathbf{x_1 = e}$.**

This means that we have $U(x, y) < e$ for all $x \in]0, e[$, $y \in]e, 1[$. Together with (2) we get

$$x < U(x, y) < e \text{ for all } x \in]0, e[, y \in]e, 1[. \tag{3}$$

Since M_{T_U} (M_{C_U}) is a continuous cancellative t-subnorm (t-superconorm), but the underlying functions of U are not continuous, then U is isomorphic to one of the uninorms from Propositions 8, 9 and 10. We will focus only on the first two cases as the remaining case is analogous. This means that there exists an isomorphism $\varphi : [0, 1] \longrightarrow [0, 1]$ such that $U^\varphi : [0, 1]^2 \longrightarrow [0, 1]$ given by $U^\varphi(x, y) = \varphi^{-1}(U(\varphi(x), \varphi(y)))$ is a uninorm with the neutral element $e = \frac{1}{2}$ such that $U^\varphi(x, y) = x \cdot y$ for all $x, y \in [0, \frac{1}{2}[$. Then (2) implies for all $x \in]0, \frac{1}{2}[$, $y \in]\frac{1}{2}, 1[$

$$x < U^\varphi(x, y) < \frac{1}{2}.$$

We will show that such a uninorm does not exist. Assume any $x_4 \in]0, \frac{1}{2}[$, $y_4 \in]\frac{1}{2}, 1[$. Then $U^\varphi(x_4, y_4) = a$ for some $a \in]x_4, \frac{1}{2}[$. The associativity gives

$$U^\varphi(x_4 \cdot u, y_4) = U^\varphi(U^\varphi(x_4, u), y_4) = u \cdot a$$

for all $u \in]0, \frac{1}{2}[$. For all $x \in]0, \frac{1}{2}[$ there exist $x_5, x_6 \in]0, \frac{1}{2}[$ such that $x = \frac{x_4 \cdot x_6}{x_5}$. We have

$$x_5 \cdot U^\varphi(x, y_4) = U^\varphi(x \cdot x_5, y_4) = U^\varphi(U^\varphi(x_4, x_6), y_4) = x_6 \cdot a = \frac{x_5 \cdot x \cdot a}{x_4}.$$

Thus $U^\varphi(x, y_4) = \frac{x \cdot a}{x_4}$ for all $x \in]0, \frac{1}{2}[$. Since $a \in]x_4, \frac{1}{2}[$ we have $a - x_4 = q > 0$ and since $U^\varphi(x, y_4) < \frac{1}{2}$ for all $x \in]0, \frac{1}{2}[$ we get $\frac{x \cdot (x_4 + q)}{x_4} < \frac{1}{2}$, what is a contradiction since $\lim_{x \longrightarrow \frac{1}{2}^-} \frac{x \cdot (x_4 + q)}{x_4} > \frac{1}{2}$.

□

Now we recall [14, Proposition 1].

Proposition 11. *Let f(x, y) be a real valued function defined on an open set G in the plane. Suppose that f(x, y) is continuous in x and y separately and is monotone in x for each y. Then f(x, y) is (jointly) continuous on the set G.*

Lemma 7. *Let* $U : [0,1]^2 \longrightarrow [0,1]$ *be a uninorm with neutral element* $e \in]0, 1[$, *such that* M_{T_U} *and* M_{C_U} *are continuous and cancellative. If* $U \notin \mathcal{U}_{nr}$ *then U is a representable uninorm, i.e., a uninorm with continuous underlying functions.*

Proof. Since $U \notin \mathcal{U}_{nr}$ Remark 2 implies that $y > U(x, y) > x$ for all $x \in]0, e[$, $y \in]e, 1[$.

First assume that $\mathbf{U(x_0, y_0) = e}$ **for some** $\mathbf{x_0 \in]0, e[, y_0 \in]e, 1[}$**.** Then also

$$U(U(x_0, x_0), U(y_0, y_0)) = e$$

and since U is Archimedean if we repeat this several times we see that we can find an arbitrarily small $\varepsilon > 0$ and $x_\varepsilon \in]0, \varepsilon[$ ($y_\varepsilon \in]1-\varepsilon, 1[$) such that $U(x_\varepsilon, y_\varepsilon) = e$ for some $y_\varepsilon \in]e, 1[$ ($x_\varepsilon \in]0, e[$). Then since M_{T_U} (M_{C_U}) is continuous for all $x \in]0, e[$ ($y \in]e, 1[$) there exist an $\varepsilon > 0$ such that $U(x, z) = x_\varepsilon$ for some $z \in]0, e[$ ($U(y, z) = y_\varepsilon$ for some $z \in]e, 1[$). Then $U(U(x, z), y_\varepsilon) = e$ ($U(U(y, z), x_\varepsilon) = e$) and thus for all $x \in]0, e[$ and all $y \in]e, 1[$ there exists $u, v \in [0, 1]$ such that $U(x, u) = e = U(y, v)$. Since the uninorm U is monotone for every $x \in [0, 1]$ the function $u_x : [0, 1] \longrightarrow [0, 1]$ given by $u_x(z) = U(x, z)$ for $z \in [0, 1]$, is continuous if and only if its range is a connected set. However, if $U(x, u) = e$ for some $u \in [0, 1]$ then

$$U(x, U(u, a)) = U(U(x, u), a) = a$$

for all $a \in [0, 1]$ and thus $\mathrm{Ran}(u_x) = [0, 1]$ for all $x \in]0, 1[$. Due to Proposition 11 U is continuous on $]0, 1[^2$. However, then T_U and C_U are continuous, i.e., strict and since $U \notin \mathcal{U}_{nr}$ according to Theorem 2 U is representable.

Now assume that $\mathbf{U(x, y) \neq e}$ **for all** $\mathbf{x \in]0, e[, y \in]e, 1[}$**.** Then U is not representable and since $U \notin \mathcal{U}_{nr}$ we know that at least one of the underlying functions of U is not continuous. Then Lemma 6 and Remark 2 imply that there exist $x_1 \in]0, e[$, $y_1 \in]e, 1[$ such that

$$x_1 < U(x_1, y_1) < e.$$

If $U(U(x_1, y_1), z) \geq x_1$ for some $z \in]0, e[$ then since M_{T_U} is continuous there exists a $z_1 \in]0, e[$ such that

$$U(x_1, U(y_1, z_1)) = U(U(x_1, y_1), z_1) = x_1.$$

Then the cancellativity of M_{T_U} and the fact that $y > U(x, y) > x$ for all $x \in]0, e[$, $y \in]e, 1[$, implies $U(y_1, z_1) = e$ what is a contradiction. Thus $U(U(x_1, y_1), z) < x_1$ for all $z \in]0, e[$. Then by associativity $U(x_1, U(y_1, z)) < x_1$ which implies by monotonicity $U(y_1, z) < e$ for all $z \in]0, e[$.

Now let

$$a = sup\{y \in]e, 1[\mid U(y, z) < e \text{ for all } z \in]0, e[\}.$$

Then $a > e$ and we will show that a is an idempotent point: otherwise there exists $a_1 \in]e, 1[$ such that $a_1 < a < U(a_1, a_1)$. Then there exists $z_2 \in]0, e[$ such that $U(U(a_1, a_1), z_2) \geq e$. However, $U(a_1, z_2) < e$ and thus $U(a_1, U(a_1, z_2)) < e$ what is a contradiction. Thus $a \in]e, 1]$ is an idempotent element and since M_{C_U} is cancellative we get $a = 1$. Therefore $z < U(y, z) < e$ for all $z \in]0, e[$, $y \in]e, 1[$, which is a contradiction with Lemma 6. □

We have now characterized all uninorms continuous and cancellative on $]0, e[^2 \cup]e, 1[^2$. For such a uninorm we have either $U \in \mathcal{U}_{nr}$ or U is representable.

5.2. Related non-Archimedean uninorms

Further we will focus on related non-Archimedean uninorms continuous on $[0, e[^2 \cup]e, 1]^2$. We plan to solve the case when U has continuous underlying functions in [23,24]. Here we will suppose that at least one of the underlying functions is non-continuous.

Definition 1. Let $\mathcal{U}_{lcc}$ be the set of all uninorms continuous on $[0, e[^2 \cup]e, 1]^2$ such that there exists an idempotent point $b_0 \in [0, 1]$, $b_0 < e$, such that U is continuous and cancellative on $]b_0, e[^2$, and let $\mathcal{U}_{rcc}$ be the set of all uninorms continuous on $[0, e[^2 \cup]e, 1]^2$ such that there exists an idempotent point $c_0 \in [0, 1]$, $c_0 > e$, such that U is continuous and cancellative on $]e, c_0[^2$.

Observe that if $U \in \mathcal{U}_{lcc}$ then for the corresponding b_0 we have $b_0 = \sup\{b \in [0, e[\mid b \text{ is an idempotent point}\}$ and if $U \in \mathcal{U}_{rcc}$ then for the corresponding c_0 we have $c_0 = \inf\{c \in]e, 1] \mid c \text{ is an idempotent point}\}$.

We will discuss three classes:

Class I: uninorms U from $\mathcal{U}_{lcc} \cap \mathcal{U}_{rcc}$, such that T_U and C_U are non-continuous.

Class II: uninorms U from $\mathcal{U}_{lcc}$ such that T_U is non-continuous and C_U is a continuous t-conorm.

Class III: uninorms U from $\mathcal{U}_{rcc}$ such that C_U is non-continuous and T_U is a continuous t-norm.

We will denote the union of all three classes by $\mathcal{C}$.

First we will show the structure of a uninorm from $\mathcal{C}$ on idempotents.

Proposition 12. *Let* $U \in \mathcal{C}$. *Then for all idempotent elements* $a \in [0, 1]$ *of U we have* $U(a, x) \in \{a, x\}$ *for all* $x \in [0, 1]$.

Proof. If $a = e$ the result is trivial. Suppose that $a < e$ (the case when $a > e$ is analogous). If $x \in [0, e]$ then the claim follows from the ordinal sum structure of M_{T_U}. Suppose that $x > e$ and $U(a, x) = z \in]a, x[$. If $z \leq e$ then

$$a = U(a, z) = U(a, U(a, x)) = z$$

what is a contradiction. Thus $z > e$.

If there exists an idempotent point $v \in [z, x]$ of U then

$$z = U(a, x) = U(a, U(v, x)) = U(z, v) = v,$$

i.e., $z = v$ which implies

$$z = U(z, z) = U(U(a, x), z) = U(x, z) = x,$$

what is a contradiction. Therefore there is no idempotent point of U in $[z, x]$. Since c_0 is an idempotent point we have $x \neq c_0$. Now there are two cases.

Case 1: If $x \in]e, c_0[$ then

$$U(z, z) = U(U(a, x), U(a, x)) = U(a, U(x, x)) = U(x, z)$$

what is a contradiction with the cancellativity of M_{C_U} on $]e, c_0[^2$.

Case 2: If $x \in]c_0, 1]$ then since M_{C_U} is a continuous t-conorm on $[c_0, 1]^2$ there exists a z_1 such that $U(z, z_1) = x$. Then

$$z = U(a, x) = U(a, U(z, z_1)) = U(z, z_1) = x,$$

what is a contradiction. Summarizing, $U(a, x) \in \{a, x\}$ for all $x \in [0, 1]$. □

Further we will show the structure of a uninorm from $\mathcal{C}$ on some special subregions of $[0, 1]^2$. For this we first recall [4, Theorem 5.1]. Here $\mathcal{U}(e) = \{U : [0, 1]^2 \longrightarrow [0, 1] \mid U$ is associative, non-decreasing, with the neutral element $e \in [0, 1]\}$. Thus $U \in \mathcal{U}(e)$ is a uninorm if it is commutative.

Theorem 5. *Let $U \in \mathcal{U}(e)$ and $a, b, c, d \in [0, 1]$, $a \leq b \leq e \leq c \leq d$, be such that $U|_{[a,b]^2}$ is associative, non-decreasing, with the neutral element b and $U|_{[c,d]^2}$ is associative, non-decreasing, with the neutral element c. Then the set $([a, b] \cup [c, d])^2$ is closed under U.*

Proposition 13. *Let $U : [0, 1]^2 \longrightarrow [0, 1]$ be a uninorm, $U \in \mathcal{C}$ and let $a, b \in [0, e]$ and $c, d \in [e, 1]$ be idempotent elements of U, $a \leq b$ and $c \leq d$. Then $([a, b[\cup \{U(b, c)\} \cup]c, d])^2$ is closed under U.*

Proof. From Theorem 5 we know that the set $([a, b] \cup [c, d])^2$ is closed under U. Further, Proposition 12 implies that $U(b, c) \in \{b, c\}$. If $b = c$ we are finished and therefore we will suppose that $b < c$. Assume that $U(b, c) = b$ (the case when $U(b, c) = c$ is analogous). Then if there exist $x, y \in [a, b[\cup \{U(b, c)\} \cup]c, d]$ such that $U(x, y) = c$ the monotonicity implies that x and y can be selected in such a way that $x \in [a, b]$ and $y \in]c, d]$. Then

$$c = U(x, y) = U(U(x, b), y) = U(c, b) = b,$$

what is a contradiction. Thus the set $([a, b[\cup \{U(b, c)\} \cup]c, d])^2$ is closed under U. □

Proposition 14. *Let $U : [0, 1]^2 \longrightarrow [0, 1]$ be a uninorm, $U \in \mathcal{C}$ and let $a, b \in [0, e]$ and $c, d \in [e, 1]$ be idempotent elements of U, $a \leq b$ and $c \leq d$, such that there is no idempotent element of U in $]a, b[\cup]c, d[$. If $([a, b[\cup]c, d])^2$ is not closed under U then $U(b, c)$ is the neutral element of U on $([a, b[\cup \{U(b, c)\} \cup]c, d])^2$.*

Proof. If $([a, b[\cup]c, d])^2$ is not closed under U then due to the monotonicity there exist a $x_1 \in]a, b[$ and a $y_1 \in]c, d[$ such that $U(x_1, y_1) = U(b, c)$. Assume that $U(b, c) = b$ (the case when $U(b, c) = c$ is analogous). Then $U(x, b) = x$ for all $x \in [a, b]$. Further, $U(y, b) \in \{y, b\}$ for all $y \in]c, d]$. If $U(y_2, b) = b$ for some $y_2 \in]c, d]$ then also $b = U(b, U(\underbrace{y_2, \dots, y_2}_{n\text{-times}}))$ and together with monotonicity we get $U(y, b) = b$ for all $y \in]c, d[$. Then also $U(b, y_1) = b$ and we get

$$b = U(b, U(b, b)) = U(U(x_1, b), U(y_1, b)) = U(x_1, b) = x_1$$

what is a contradiction. Thus $U(b, y) = y$ for all $y \in]c, d[$. From monotonicity then also $U(b, d) = d$. Thus $U(b, c)$ is the neutral element of U on $([a, b[\cup \{U(b, c)\} \cup]c, d])^2$. □

Remark 3. If for $U \in \mathcal{C}$ the set $([a, b[\cup]c, d])^2$ is closed under U, where $a, b \in [0, e]$ and $c, d \in [e, 1]$ are idempotent elements of U, $a \leq b$ and $c \leq d$, such that there is no idempotent element of U in $]a, b[\cup]c, d[$ then U on $([a, b[\cup \{e\} \cup]c, d])^2$ is isomorphic to a uninorm with Archimedean underlying functions. These underlying functions can be either strict, or nilpotent, or non-continuous such that their border-continuous projection is continuous and cancellative. If the set $([a, b[\cup]c, d])^2$ is not closed under U and $a, b \in [0, e]$ and $c, d \in [e, 1]$ are idempotent elements of U, $a \leq b$ and $c \leq d$, such that there is no idempotent element of U in $]a, b[\cup]c, d[$ then U on $([a, b[\cup \{U(b, c)\} \cup]c, d])^2$ is isomorphic to a representable uninorm.

To conclude our investigation we should characterize all uninorms with Archimedean underlying functions, which are either strict, or nilpotent, or non-continuous such that their border-continuous projection is continuous and cancellative.

The case when both underlying functions are cancellative was covered in the previous section. If both underlying functions are nilpotent then we have the following result.

Theorem 6 ([17]). *Let $U : [0,1] \longrightarrow [0,1]^2$ be a uninorm with the neutral element $e \in\,]0, 1[$ such that both T_U and C_U are nilpotent. Then either one of the following two statements holds:*

(i) $U \in \mathcal{U}_{\min}$,
(ii) $U \in \mathcal{U}_{\max}$.

Finally, we add the two remaining combinations.

Proposition 15. *Let $U : [0,1] \longrightarrow [0,1]^2$ be a uninorm with the neutral element $e \in\,]0, 1[$ such that M_{T_U} is continuous and cancellative and C_U is a nilpotent t-conorm. Then either one of the following two statements holds:*

(i) $U \in \mathcal{U}_{\min}$,
(ii) $U \in \mathcal{U}_{\max}$,
(iii)

$$U(x,y) = \begin{cases} e \cdot T_U(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e) \cdot C_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ 0 & \text{if } x = 0, \\ \max(x,y) & \text{otherwise.} \end{cases}$$

Proof. If C_U is a nilpotent t-conorm then for each $y \in\,]e, 1[$ there exists exactly one $y^U \in\,]e, 1[$ such that $U(y, y^U) = 1$ and $U(q, y^U) < 1$ for all $q < y$. Assume a $x_1 \in [0, e[$. Then Proposition 12 implies $U(x_1, 1) \in \{1, x_1\}$. If $U(x_1, 1) = 1$ then for all $y \in\,]e, 1[$ we have $U(x_1, y) \in [x_1, y]$ and

$$1 = U(U(y, y^U), x_1) = U(U(y, x_1), y^U)$$

which means that $U(y, x_1) = y$. If $U(1, x_1) = x_1$ then the monotonicity implies $U(y, x_1) = x_1$ for all $y \in\,]e, 1[$.

Further, if $U(1, x) = 1$ for some $x \in\,]0, e[$ then $1 = U(1, U(\underbrace{x, \ldots, x}_{n\text{-times}}))$ for all $n \in \mathbb{N}$. Since M_{T_U} is Archimedean for each $q \in\,]0, e[$ there exists an $N \in \mathbb{N}$ such that $U(\underbrace{x, \ldots, x}_{N\text{-times}}) < q$ and thus the monotonicity of U implies $U(1, q) = 1$ for all $q \in\,]0, e[$. Therefore either $U(x, y) = \max(x, y)$ for all $x \in\,]0, e[$, $y \in\,]e, 1]$, or $U(x, y) = \min(x, y)$ for all $x \in\,]0, e[$, $y \in\,]e, 1]$. Moreover, either $U(0, y) = 0$ for all $y \in [e, 1]$, or $U(0, y) = y$ for all $y \in [e, 1]$. Summarising we get the result. □

Similarly we can show the following result.

Proposition 16. *Let $U : [0,1] \longrightarrow [0,1]^2$ be a uninorm with the neutral element $e \in\,]0, 1[$ such that M_{C_U} is continuous and cancellative and T_U is a nilpotent t-norm. Then either one of the following two statements holds:*

(i) $U \in \mathcal{U}_{\min}$,
(ii) $U \in \mathcal{U}_{\max}$,
(iii)

$$U(x,y) = \begin{cases} e \cdot T_U(\frac{x}{e}, \frac{y}{e}) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e) \cdot C_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } (x,y) \in [e,1]^2, \\ 1 & \text{if } x = 1, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

Remark 4. Summarising all previous results: for every $U \in \mathcal{C}$ we know its structure on idempotents and on every closed subset. Therefore we know all possible structures of U on the whole $[0, 1]^2$. Moreover, it is possible to show that the structure of a uninorm from $\mathcal{C}$ is similar to that of a uninorm with continuous underlying functions, i.e., that it is an ordinal sum of continuous t-norms and t-conorms, representable uninorms and cancellative continuous t-subnorms and t-superconorms (possibly without border points). However, the construction of the order in the ordinal sum is quite lengthy and therefore we will not go into details and we recommend interested readers to our future work in [23,24].

6. Conclusions

We have investigated uninorms continuous on $[0, e[^2 \cup]e, 1]^2$ and we have shown that these are related to continuous proper t-subnorms (t-superconorms). We have characterised uninorms continuous on $[0, e[^2 \cup]e, 1]^2$ such that M_{T_U} (M_{C_U}) is a continuous cancellative t-subnorm (t-superconorm). We have also discussed related non-Archimedean uninorms continuous on $[0, e[^2 \cup]e, 1]^2$. Our results indicate that in the context of this paper cancellativity of the underlying functions of a uninorm can substitute continuity. Beside our main aim we have also shown when a border-continuous projection of a t-norm is a t-subnorm.

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Chapter 5

n-uninorms with continuous underlying functions

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Characterization of idempotent n-uninorms

Andrea Mesiarová-Zemánková

Mathematical Institute, Slovak Academy of Sciences, Bratislava, Slovakia

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Abstract

The structure of idempotent n-uninorms is studied. We show that each idempotent 2-uninorm can be expressed as an ordinal sum of an idempotent uninorm (possibly also of a countable number of idempotent semigroups with operations min and max) and a 2-uninorm from Class 1 (possibly restricted to open or half-open unit square). Similar results are shown also for idempotent n-uninorms. Further, it is shown that idempotent n-uninorms are in one-to-one correspondence with special lower semi-lattices defined on the unit interval. The z-ordinal sum construction for partially ordered semigroups is also defined.

Keywords: n-Uninorm; Uninorm; Ordinal sum; t-Norm; t-Conorm; Nullnorm

1. Introduction

The aggregation operators on the unit interval have applications in many domains such as data fusion, control systems, multi-criteria decision making, image processing, expert systems and many others. The associativity of a binary function brings an advantage of an easy extension to any finite number $n > 2$ of inputs, particularly to an aggregation operator. Moreover, additional input is easily combined with the previous output by an associative function. This is the reason why associative functions, especially associative aggregation operators were widely studied in the past decades. At the beginning the biggest attention was given to t-norms and t-conorms [3,9], which were later generalized into uninorms capable of representing bipolar aggregation (see [7,11,22]). These functions represent the class of associative aggregation operators that are commutative and posses a neutral element. Another generalization of t-norms and t-conorms yields nullnorms (also called t-operators) [5,10]. These functions represent the class of associative aggregation operators that are commutative and posses an annihilator.

The class of continuous t-norms was easily characterized, showing that each continuous t-norm is an ordinal sum of continuous Archimedean t-norms, which can be further divided into strict and nilpotent, while each pair of strict (nilpotent) t-norms is isomorphic. Since t-conorms are dual operations to t-norms the characterization of continuous t-conorms is straightforward. In the case of uninorms, several characterizations of uninorms were obtained under partial

E-mail address: zemankova@mat.savba.sk.

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continuity conditions (such as continuity on the open unit square [7,8,16], or continuity of the underlying functions [4,13,15]).

The above generalizations bring together t-norms and t-conorms. In the second step a notion that brings together uninorms and nullnorms was introduced by Akella [1]. These special aggregation operators are called n-uninorms and each n-uninorm possesses n local neutral elements. The basic structure of n-uninorms was described by Akella in [1]. Further, 5 possible classes, defined by values of $U^2(0,1)$, $U^2(0,z_1)$ and $U^2(z_1,1)$ were characterized in [23]. If for a 2-uninorm there is $e_2 = 1$ we obtain a uni-nullnorm and if $e_1 = 0$ we obtain a null-uninorm [18]. The migrativity and the distributivity of uni-nullnorms were studied in [20,21].

In this paper we focus on idempotent n-uninorms. Since idempotent uninorms were fully characterized in [12,17], we would like to extend these results also for n-uninorms. This paper is the first work towards the characterization of n-uninorms with continuous underlying functions. In the next papers we would like to characterize the basic structure of n-uninorms with continuous underlying functions, study characterizing functions of such n-uninorms and afterwards describe their decomposition into irreducible semigroups. Note that the first step towards this end was done by Sun, Wang and Qu in [19], where uni-nullnorms with continuous Archimedean underlying functions were characterized.

In the following section we recall all necessary basic notions and results. In Section 3 we will study the basic structure of 2-uninorms and n-uninorms. In Section 4 we define the z-ordinal sum construction for partially ordered semigroups and we show that idempotent n-uninorms are in one-to-one correspondence with special partial orders on $[0,1]$. Finally, we give our conclusions in Section 5.

2. Basic notions

A triangular norm ([9]) is a binary function $T\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and 1 is its neutral element. Due to the associativity, n-ary form of any t-norm is uniquely given and thus it can be extended to an aggregation function working on $\bigcup_{n\in\mathbb{N}}[0,1]^n$. Dual functions to t-norms are t-conorms. A triangular conorm ([9]) is a binary function $S\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and 0 is its neutral element. The duality between t-norms and t-conorms is expressed by the fact that from any t-norm T we can obtain its dual t-conorm S by the equation

$$S(x,y) = 1 - T(1-x, 1-y)$$

and vice-versa.

Since we will work in this paper with ordinal sums of semigroups we recall the fundamental result of Clifford [6].

Theorem 2.1. *Let $A \neq \emptyset$ be a totally ordered set and $(G_\alpha)_{\alpha\in A}$ with $G_\alpha = (X_\alpha, *_\alpha)$ be a family of semigroups. Assume that for all $\alpha, \beta \in A$ with $\alpha < \beta$ the sets X_α and X_β are either disjoint or that $X_\alpha \cap X_\beta = \{x_{\alpha,\beta}\}$, where $x_{\alpha,\beta}$ is both the neutral element of G_α and the annihilator of G_β and where for each $\gamma \in A$ with $\alpha < \gamma < \beta$ we have $X_\gamma = \{x_{\alpha,\beta}\}$. Put $X = \bigcup\limits_{\alpha\in A} X_\alpha$ and define the binary operation $*$ on X by*

$$x * y = \begin{cases} x *_\alpha y & \text{if } (x,y) \in X_\alpha \times X_\alpha, \\ x & \text{if } (x,y) \in X_\alpha \times X_\beta \text{ and } \alpha < \beta, \\ y & \text{if } (x,y) \in X_\alpha \times X_\beta \text{ and } \alpha > \beta. \end{cases}$$

Then $G = (X, *)$ is a semigroup. The semigroup G is commutative if and only if for each $\alpha \in A$ the semigroup G_α is commutative.

The ordinal sum construction for t-norms and t-conorms reduces to the following proposition [9].

Proposition 2.2. *Let K be a finite or countably infinite index set and let $(]a_k, b_k[)_{k\in K}$ ($(]c_k, d_k[)_{k\in K}$) be a system of open, disjoint subintervals of $[0,1]$. Let $(T_k)_{k\in K}$ ($(S_k)_{k\in K}$) be a system of t-norms (t-conorms). Then the ordinal sum $T = (\langle a_k, b_k, T_k\rangle \mid k \in K)$ ($S = (\langle a_k, b_k, S_k\rangle \mid k \in K)$) given by*

$$T(x,y) = \begin{cases} a_k + (b_k - a_k) T_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right) & \text{if } (x,y) \in [a_k, b_k[^2, \\ \min(x,y) & \text{else} \end{cases}$$

and

$$S(x,y)=\begin{cases} c_k+(d_k-c_k)S_k(\frac{x-c_k}{d_k-c_k},\frac{y-c_k}{d_k-c_k}) & \text{if } (x,y)\in\,]c_k,d_k]^2,\\ \max(x,y) & \text{else}\end{cases}$$

is a t-norm (t-conorm). The t-norm T (t-conorm S) is continuous if and only if all summands T_k (S_k) for $k\in K$ are continuous.

Each continuous t-norm (t-conorm) is equal to an ordinal sum of continuous Archimedean t-norms (t-conorms). Note that a continuous t-norm (t-conorm) is Archimedean if and only if it has only trivial idempotent points 0 and 1. A continuous Archimedean t-norm T (t-conorm S) is either strict, i.e., strictly increasing on $]0,1]^2$ (on $[0,1[^2$), or nilpotent, i.e., there exists $(x,y)\in\,]0,1[^2$ such that $T(x,y)=0$ ($S(x,y)=1$). Moreover, each continuous Archimedean t-norm (t-conorm) has a continuous additive generator, which is uniquely determined up to a positive multiplicative constant. More details on t-norms and t-conorms can be found in [3,9].

A uninorm (introduced in [22]) is a binary function $U\colon [0,1]^2\longrightarrow[0,1]$ which is commutative, associative, non-decreasing in both variables and have a neutral element $e\in[0,1]$ (see also [7]). Evidently, if $e=1$ ($e=0$) then we retrieve a t-norm (t-conorm).

For each uninorm the value $U(1,0)\in\{0,1\}$ is the annihilator of U. A uninorm is called conjunctive (disjunctive) if $U(1,0)=0$ ($U(1,0)=1$). For each uninorm U with the neutral element $e\in\,]0,1[$, the restriction of U to $[0,e]^2$ is a t-norm on $[0,e]^2$, i.e., a linear transformation of some t-norm T_U on $[0,1]^2$ and the restriction of U to $[e,1]^2$ is a t-conorm on $[e,1]^2$, i.e., a linear transformation of some t-conorm S_U. Moreover, $\min(x,y)\le U(x,y)\le\max(x,y)$ for all $(x,y)\in[0,e]\times[e,1]\cup[e,1]\times[0,e]$.

Similarly as in the case of t-norms and t-conorms we can construct uninorms using additive generators (see [7]). A uninorm which possesses a continuous additive generator is called representable. Note that in [16] (see also [11]) it was shown that a uninorm is representable if and only if it is continuous on $[0,1]^2\setminus\{(0,1),(1,0)\}$.

Definition 2.3. A uninorm $U\colon[0,1]^2\longrightarrow[0,1]$ is called internal if $U(x,y)\in\{x,y\}$ for all $(x,y)\in[0,1]^2$; and it is called idempotent if $U(x,x)=x$ for all $x\in[0,1]$.

Observe that if a uninorm U is internal then it is also idempotent and vice-versa.

Let us recall the basic result from [17] that characterizes idempotent uninorms.

Theorem 2.4. *Let $U\colon[0,1]^2\longrightarrow[0,1]$ be a binary function. Then U is an idempotent uninorm with the neutral element $e\in\,]0,1[$ if and only if there exists a non-increasing function $g\colon[0,1]\longrightarrow[0,1]$, symmetric with respect to the main diagonal, with $g(e)=e$, such that*

$$U(x,y)=\begin{cases}\min(x,y) & \text{if } y<g(x) \text{ or } (y=g(x) \text{ and } x<g(g(x))),\\ \max(x,y) & \text{if } y>g(x) \text{ or } (y=g(x) \text{ and } x>g(g(x))),\\ x \text{ or } y & \text{if } y=g(x) \text{ and } x=g(g(x)),\end{cases}$$

being commutative in the points (x,y) such that $y=g(x)$ with $x=g(g(x))$. This class of uninorms is denoted by $\mathcal{U}_{\mathrm{ide}}$.

Note that the graph of the function g from Theorem 2.4 is a subset of the graph of the characterizing set-valued function of an idempotent uninorm (for more details see [14,15]). Therefore the completed graph of the function g divides the idempotent uninorm U into two parts: below the completed graph of g we have $U(x,y)=\min(x,y)$, i.e., $U(x,y)<e$, and above the completed graph of g there is $U(x,y)=\max(x,y)$, i.e., $U(x,y)>e$.

Uninorms with continuous underlying functions were completely characterized in [13,15]. In [13] it was shown that each uninorm with continuous underlying functions can be decomposed into an ordinal sum of a countable number of semigroups related to representable uninorms, continuous Archimedean t-norms, continuous Archimedean t-conorms and internal uninorms (including the min and the max operator). In [15] it was shown that the set of all points of discontinuity of a uninorm with continuous underlying functions is a subset of the graph of the characterizing set-valued function of such a uninorm.

Now let us recall the definition of an n-uninorm (see [1]).

Definition 2.5. Assume an $n \in \mathbb{N} \setminus \{1\}$. Let $V\colon [0,1]^2 \longrightarrow [0,1]$ be a commutative binary function. Then $\{e_1, \ldots, e_n\}_{z_1,\ldots,z_{n-1}}$ is called an n-neutral element of V if for $0 = z_0 < z_1 < \cdots < z_n = 1$ and $e_i \in [z_{i-1}, z_i]$, $i = 1, \ldots, n$ we have $V(e_i, x) = x$ for all $x \in [z_{i-1}, z_i]$.

Definition 2.6. A binary function $U^n\colon [0,1]^2 \longrightarrow [0,1]$ is an n-uninorm if it is associative, non-decreasing in each variable, commutative and has an n-neutral element $\{e_1, \ldots, e_n\}_{z_1,\ldots,z_{n-1}}$.

The basic structure of n-uninorms was described by Akella in [1] and the characterizations of the main five classes of 2-uninorms was given in [23]. Now we will recall these five exhaustive and mutually exclusive classes:

- Class 1: 2-uninorms with $U^2(0,1) = z_1$.
- Class 2a: 2-uninorms with $U^2(0,1) = 0$, $U^2(1, z_1) = z_1$.
- Class 2b: 2-uninorms with $U^2(0,1) = 1$, $U^2(0, z_1) = z_1$.
- Class 3a: 2-uninorms with $U^2(0,1) = 0$, $U^2(1, z_1) = 1$.
- Class 3b: 2-uninorms with $U^2(0,1) = 1$, $U^2(0, z_1) = 0$.

An idempotent 2-uninorm U^2 from Class 1 has a very simple structure: on $[0, z_1]^2$ ($[z_1, 1]^2$) it is isomorphic to an idempotent uninorm and $U^2(x, y) = z_1$ on $[0, z_1] \times [z_1, 1]$ and $[z_1, 1] \times [0, z_1]$.

Each n-uninorm has the following building blocks around the main diagonal.

Proposition 2.7. *Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be an n-uninorm with the n-neutral element $\{e_1, \ldots, e_n\}_{z_1,\ldots,z_{n-1}}$. Then*

(i) U^n restricted to $[z_{i-1}, e_i]^2$, for $i = 1, \ldots, n$, is isomorphic to a t-norm. We will denote this t-norm by T_i.
(ii) U^n restricted to $[e_i, z_i]^2$ for $i = 1, \ldots, n$, is isomorphic to a t-conorm. We will denote this t-conorm by S_i.
(iii) U^n restricted to $[z_{i-1}, z_i]^2$ for $i = 1, \ldots, n$, is isomorphic to a uninorm. We will denote this uninorm by U_i.
(iv) U^n restricted to $[z_i, z_j]^2$ for $i, j \in \{0, 1, \ldots, n\}$, $i < j$, is isomorphic to a $(j - i)$-uninorm.

Before we proceed with the main results of the paper we will recall several notions that we will use.

Since we will use ordinal sums of trivial semigroups, let us recall that there exists only one operation on a trivial semigroup, namely the function $\mathrm{Id}\colon \{x\}^2 \longrightarrow \{x\}$, which is simply defined by $\mathrm{Id}(x, x) = x$.

If we will talk about linear transformation from interval $[a, b]$ to interval $[c, d]$ we mean a linear function $\varphi\colon [a, b] \longrightarrow [c, d]$ given by

$$\varphi(x) = \frac{(x - a) \cdot (d - c)}{b - a} + c,$$

which transforms a unary function $f\colon [a, b] \longrightarrow [a, b]$ to a function $g\colon [c, d] \longrightarrow [c, d]$ given by $g(x) = \varphi(f(\varphi^{-1}(x)))$, and transforms a binary function $V\colon [a, b]^2 \longrightarrow [a, b]$ to a function $U\colon [c, d]^2 \longrightarrow [c, d]$ given by $U(x, y) = \varphi(V(\varphi^{-1}(x), \varphi^{-1}(y)))$. Further, for any $0 \le a < b \le c < d \le 1$, $v \in [b, c]$ and a uninorm $U\colon [0,1]^2 \longrightarrow [0,1]$ with the neutral element $e \in]0,1[$ we will use the transformation $f\colon [0,1] \longrightarrow [a, b[\cup \{v\} \cup]c, d]$, given by

$$f(x) = \begin{cases} (b - a) \cdot \frac{x}{e} + a & \text{if } x \in [0, e[, \\ v & \text{if } x = e, \\ d - \frac{(1-x)(d-c)}{(1-e)} & \text{otherwise.} \end{cases} \tag{1}$$

Then f is linear on $[0, e[$ and on $]e, 1]$ and thus it is an increasing, piece-wise linear bijection from $[0,1]$ to $([a, b[\cup \{v\} \cup]c, d])$ which preserves the commutativity, the associativity, the monotonicity, the idempotency and the neutral element; and the binary function $U_v^{a,b,c,d}\colon ([a, b[\cup \{v\} \cup]c, d])^2 \longrightarrow ([a, b[\cup \{v\} \cup]c, d])$ given by

$$U_v^{a,b,c,d}(x, y) = f(U(f^{-1}(x), f^{-1}(y))) \tag{2}$$

is a uninorm on $([a, b[\cup \{v\} \cup]c, d])^2$. The backward transformation f^{-1} then transforms a uninorm defined on $([a, b[\cup \{v\} \cup]c, d])^2$ to a uninorm defined on $[0,1]^2$.

For the rest of the paper if we say that two semigroups (X_1, F_1) and (X_2, F_2) are isomorphic we assume that there exists an increasing bijection $\varphi\colon X_1 \longrightarrow X_2$ such that $F_1(x, y) = \varphi^{-1}(F_2(\varphi(x), \varphi(y)))$ for all $x, y \in X_1$. Note that such a bijection preserves the commutativity, the associativity, the monotonicity, the idempotency, the (local) neutral element and the annihilator, as well.

3. Idempotent n-uninorms

First let us settle for this paper that if we say that a function is an n-uninorm we will suppose that it possesses the n-neutral element $\{e_1, \dots, e_n\}_{z_1,\dots,z_{n-1}}$.

For idempotent n-uninorms it holds $U^n(x, x) = x$ for all $x \in [0, 1]$. Thus we get $T_i = \min$ and $S_i = \max$ for all $i = 1, \dots, n$.

We will divide this section into results on 2-uninorms and then similar results will be presented for n-uninorms for $n \in \mathbb{N}, n > 2$.

3.1. Basic characterization of 2-uninorms

For every idempotent uninorm U there is $U(x, y) \in \{x, y\}$. From [2, Theorem 2] we get a similar result also for 2-uninorms.

Lemma 3.1. *Let $U^2\colon [0, 1]^2 \longrightarrow [0, 1]$ be an idempotent 2-uninorm. Then $U^2(x, y) \in \{x, y, z_1\}$.*

Remark 3.2. Since $U^2(x, z_1) \in \{x, z_1\}$ for all $x \in [0, 1]$ the monotonicity of U^2 implies that there exists an $x_0 \in [0, e_1]$ and a $y_0 \in [e_2, 1]$ such that $U^2(x, z_1) = x$ for all $x < x_0$ and $U^2(x, z_1) = z_1$ for all $x_0 < x \le z_1$, $U^2(y, z_1) = y$ for all $y > y_0$ and $U^2(y, z_1) = z_1$ for all $z_1 \le y < y_0$. Then for all $x_0 < x \le z_1$ and $z_1 \le y < y_0$ there is $U^2(x, y) = z_1$. The structure of a 2-uninorm U^2 on $]x_0, y_0[^2$ is given as follows:

(i) U^2 restricted to $]x_0, z_1]^2$ is isomorphic to an idempotent uninorm restricted to $]0, 1]^2$.
(ii) U^2 restricted to $[z_1, y_0[^2$ is isomorphic to an idempotent uninorm restricted to $[0, 1[^2$.
(iii) U^2 restricted to $]x_0, z_1] \times [z_1, y_0[$ or to $[z_1, y_0[\times]x_0, z_1]$ is equal to z_1.

Summarizing, U^2 restricted to $]x_0, y_0[^2$ is isomorphic to an idempotent 2-uninorm from Class 1, restricted to $]0, 1[^2$. Observe that a 2-uninorm U^2 restricted to $[x_0, y_0]^2$ is isomorphic to a 2-uninorm, however, depending on the values of U^2 on $\{x_0, y_0\} \times [x_0, y_0]$, this 2-uninorm can belong to any of the 5 classes. Note that if $x_0 = 0$, $y_0 = 1$ and $U^2(x_0, y_0) = z_1$ then U^2 is an idempotent 2-uninorm from Class 1.

Example 3.3. Let $U_1, U_2\colon [0, 1]^2 \longrightarrow [0, 1]$ be a disjunctive and a conjunctive idempotent uninorm, respectively, with the neutral element $e = \frac{1}{2}$ and let $U^{[a,b]}$ denote the linear transformation of the uninorm U to the interval $[a, b]^2$. Then we can assume the following functions.

(i) The function $U_1^2\colon [0, 1]^2 \longrightarrow [0, 1]$ given by $U_1^2(x, y) = U_1^{\left[0,\frac{1}{2}\right]}(x, y)$ if $x, y \in \left[0, \frac{1}{2}\right]$, $U_1^2(x, y) = U_2^{\left[\frac{1}{2},1\right]}(x, y)$ if $x, y \in \left[\frac{1}{2}, 1\right]$ and $U_1^2(x, y) = \frac{1}{2}$ otherwise, is a 2-uninorm from Class 1 such that $e_1 = \frac{1}{4}$, $e_2 = \frac{3}{4}$ and $z_1 = \frac{1}{2}$. Here $x_0 = 0$ and $y_0 = 1$.
(ii) Assume the function $U_2^2\colon [0, 1]^2 \longrightarrow [0, 1]$, which is on $\left[\frac{1}{4}, \frac{3}{4}\right]^2$ a linear transformation of the 2-uninorm U_1^2 and $U_2^2(x, y) = \min(x, y)$ if $\min(x, y) < \frac{1}{4}$, $U_2^2(x, y) = \max(x, y)$ if $\min(x, y) \ge \frac{1}{4}$ and $\max(x, y) > \frac{3}{4}$. Then U_2^2 is a 2-uninorm from Class 3a and $e_1 = \frac{3}{8}$, $e_2 = \frac{5}{8}$ and $z_1 = \frac{1}{2}$. Here $x_0 = \frac{1}{4}$ and $y_0 = \frac{3}{4}$.
(iii) Similarly we can assume the function $U_3^2\colon [0, 1]^2 \longrightarrow [0, 1]$, which coincides with U_2^2 on $\left[\frac{1}{4}, \frac{3}{4}\right]^2$ and $U_3^2(x, y) = \max(x, y)$ if $\max(x, y) > \frac{3}{4}$, $U_3^2(x, y) = \min(x, y)$ if $\max(x, y) \le \frac{3}{4}$ and $\min(x, y) < \frac{1}{4}$. Then U_3^2 is a 2-uninorm from Class 3b and $e_1 = \frac{3}{8}$, $e_2 = \frac{5}{8}$ and $z_1 = \frac{1}{2}$. Here $x_0 = \frac{1}{4}$ and $y_0 = \frac{3}{4}$.

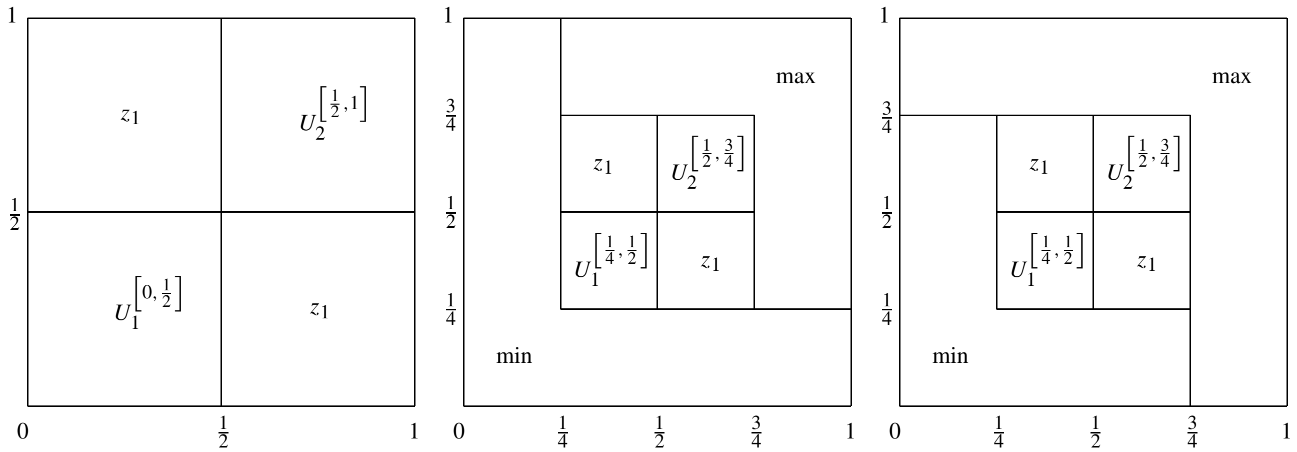


Fig. 1. The 2-uninorms U_1^2 (left), U_2^2 (center) and U_3^2 (right) from Example 3.3.

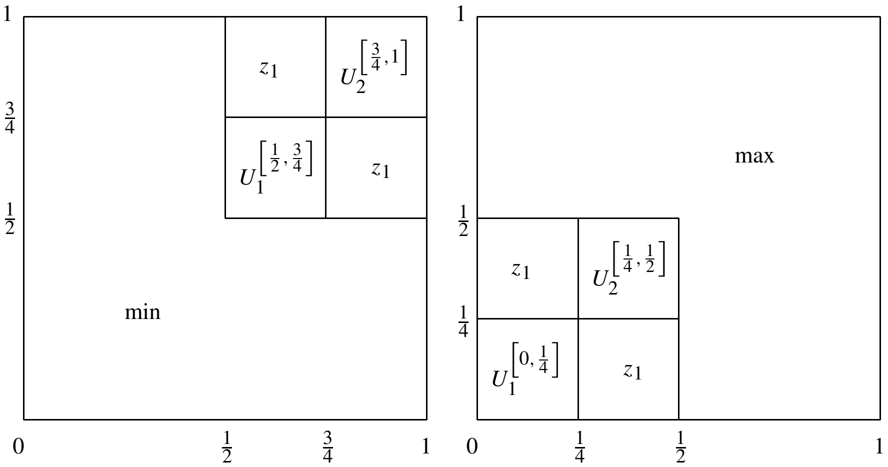


Fig. 2. The 2-uninorms U_4^2 (left) and U_5^2 (right) from Example 3.3.

(iv) Assume the function $U_4^2\colon [0,1]^2 \longrightarrow [0,1]$, which is on $\left[\frac{1}{2},1\right]^2$ a linear transformation of the 2-uninorm U_1^2 and $U_4^2(x,y) = \min(x,y)$ if $\min(x,y) < \frac{1}{2}$. Then U_4^2 is a 2-uninorm from Class 2a and $e_1 = \frac{5}{8}$, $e_2 = \frac{7}{8}$ and $z_1 = \frac{3}{4}$. Here $x_0 = \frac{1}{2}$ and $y_0 = 1$.

(v) Assume the function $U_5^2\colon [0,1]^2 \longrightarrow [0,1]$, which is on $\left[0,\frac{1}{2}\right]^2$ a linear transformation of the 2-uninorm U_1^2 and $U_5^2(x,y) = \max(x,y)$ if $\max(x,y) > \frac{1}{2}$. Then U_5^2 is a 2-uninorm from Class 2b and $e_1 = \frac{1}{8}$, $e_2 = \frac{3}{8}$ and $z_1 = \frac{1}{4}$. Here $x_0 = 0$ and $y_0 = \frac{1}{2}$.

The 2-uninorms from this example are depicted on Figs. 1 and 2.

From now on we will distinguish five different cases

- if $U^2(x_0, y_0) = z_1$,
- if $U^2(x_0, y_0) = x_0$, $U^2(x_0, y) = y$ for all $y > y_0$,
- if $U^2(x_0, y_0) = x_0$, $U^2(x_0, y) \neq y$ for some $y > y_0$,
- if $U^2(x_0, y_0) = y_0$, $U^2(y_0, x) = x$ for all $x < x_0$,
- if $U^2(x_0, y_0) = y_0$, $U^2(y_0, x) \neq x$ for some $x < x_0$.

Since the second and the fourth (the third and the fifth) cases are analogous we will focus just on the first three cases. Observe that due to the monotonicity the value $U^2(x_0, y_0)$ is the annihilator of U^2 on $[x_0, y_0]$. At first we will suppose that $U^2(x_0, y_0) = z_1$.

Lemma 3.4. *Let $U^2: [0, 1] \longrightarrow [0, 1]$ be an idempotent 2-uninorm and let $U^2(x_0, y_0) = z_1$. If $U^2(x, y) = z_1$ for some $x, y \in [0, 1]$ then $x, y \in [x_0, y_0]$.*

Proof. If $U^2(x, y) = z_1$ then $U^2(x, z_1) = U^2(x, U^2(x, y)) = U^2(U^2(x, x), y) = U^2(x, y) = z_1$ and similarly $U^2(y, z_1) = z_1$. Thus $x, y \in [x_0, y_0]$. □

Proposition 3.5. *Let $U^2: [0, 1] \longrightarrow [0, 1]$ be an idempotent 2-uninorm and let $U^2(x_0, y_0) = z_1$. Then U^2 is an ordinal sum of semigroups $G_1 = ([0, x_0[\cup \{z_1\} \cup]y_0, 1], U^2)$ and $G_2 = ([x_0, y_0], U^2)$, where the order in the ordinal sum construction is $1 < 2$, and G_1 is isomorphic to $([0, 1], U)$, where U is an idempotent uninorm and G_2 is isomorphic to $([0, 1], V^2)$, where V^2 is an idempotent 2-uninorm from Class 1.*

Proof. Due to Lemmas 3.1 and 3.4 and the definition of x_0 and y_0 in Remark 3.2, we know that U^2 is closed on $([0, x_0[\cup \{z_1\} \cup]y_0, 1])^2$ and z_1 is the neutral element of this restriction. Therefore U^2 is on $([0, x_0[\cup \{z_1\} \cup]y_0, 1])^2$ isomorphic to an idempotent uninorm. Further, since $U^2(x_0, y_0) = z_1$ the restriction of U^2 to $[x_0, y_0]^2$ is isomorphic to an idempotent 2-uninorm from Class 1; and z_1 is the annihilator of U^2 on $[x_0, y_0]^2$. For any $x \in [0, x_0[\cup \{z_1\} \cup]y_0, 1]$ and $y \in [x_0, y_0]$ there is $U^2(x, z_1) = x$ and $U^2(y, z_1) = z_1$. Thus

$$U^2(x, y) = U^2(U^2(x, z_1), y) = U^2(x, U^2(z_1, y)) = U^2(x, z_1) = x,$$

i.e., $U^2(x, y) = x$ for all $x \in [0, x_0[\cup \{z_1\} \cup]y_0, 1]$ and $y \in [x_0, y_0]$. The commutativity of U^2 then shows that $([0, 1], U^2)$ is an ordinal sum of semigroups G_1 and G_2 with the corresponding order $1 < 2$ (see Fig. 3). □

Remark 3.6. The previous result can be also reverted. Assume $x_0, y_0, e_1, e_2, z_1 \in [0, 1]$ such that $0 \le x_0 \le e_1 \le z_1 \le e_2 \le y_0 \le 1$ and $0 < z_1 < 1$. Let $V^2: [0, 1]^2 \longrightarrow [0, 1]$ be an idempotent 2-uninorm with the 2-neutral element $\{e_1^*, e_2^*\}_{z_1^*}$ and $U: [0, 1]^2 \longrightarrow [0, 1]$ be an idempotent uninorm. Let $G_1 = ([0, x_0[\cup \{z_1\} \cup]y_0, 1], F)$ and $G_2 = ([x_0, y_0], H)$, where G_2 is isomorphic with $([0, 1], V^2)$ via an increasing isomorphism $\varphi: [0, 1] \longrightarrow [x_0, y_0]$ such that $\varphi(e_1^*) = e_1$, $\varphi(e_2^*) = e_2$, and $\varphi(z_1^*) = z_1$, and G_1 is isomorphic with $([0, 1], U)$ via (1). Then the ordinal sum of G_1 and G_2 with $1 < 2$ is an idempotent 2-uninorm denoted by U^2. The commutativity and the associativity follows from Theorem 2.1 and U^2 is evidently idempotent. The points e_1 and e_2 are local neutral elements on $[x_0, y_0]^2$ and their extension to $[0, 1]^2$ follows from the ordinal sum construction. Therefore the only non-trivial property is the monotonicity. Assume $x \in [0, 1]$. Then we have the following cases:

(i) If $x \in [0, x_0[$. Then $U^2(x, \cdot)$ is non-decreasing on $[0, x_0[$ and $U^2(x, y) \le x$ for all $y \in [0, x_0[$. For $y \in [x_0, y_0]$ we have $U^2(x, y) = x$. Finally, for $y \in]y_0, 1]$ we know that $U^2(x, y) \in [x, y]$ and $U^2(x, \cdot)$ is non-decreasing on $]y_0, 1]$. Together, for all $x \in [0, x_0[$ the cut $U^2(x, \cdot)$ is non-decreasing.

(ii) If $x \in [x_0, y_0]$. Then $U^2(x, y) = y$ for all $y \in [0, x_0[$ and $U^2(x, x_0) \ge x_0$. Further, $U^2(x, \cdot)$ is non-decreasing on $[x_0, y_0]$ and $U^2(x, y_0) \le y_0$. Finally, for $y \in]y_0, 1]$ there is $U^2(x, y) = y$. Thus the cut $U^2(x, \cdot)$ is non-decreasing for all $x \in [x_0, y_0]$.

(iii) If $x \in]y_0, 1]$ then $U^2(x, \cdot)$ is non-decreasing on $[0, x_0[$ and $U^2(x, y) \in [y, x]$ for all $y \in [0, x_0[$. Further, $U^2(x, y) = x$ for $y \in [x_0, y_0]$. Finally, $U^2(x, \cdot)$ is non-decreasing on $]y_0, 1]$ and $U^2(x, y) \ge x$ for all $y \in]y_0, 1]$. Summarizing, $U^2(x, \cdot)$ is non-decreasing for all $x \in]y_0, 1]$.

Similar observations can be done in all following results (including the results on idempotent n-uninorms).

Further we will suppose that $U^2(x_0, y_0) = x_0$. Then x_0 is the annihilator of U^2 on $[x_0, y_0]^2$. However, x_0 need not to be the neutral element on the rest of the unit interval. We have a similar result as in the previous case.

Lemma 3.7. *Let $U^2: [0, 1] \longrightarrow [0, 1]$ be an idempotent 2-uninorm and let $U^2(x_0, y_0) = x_0$. If $U^2(x, y) = z_1$ then $x, y \in]x_0, y_0]$.*

Proof. If $U^2(x,y)=z_1$ then $U^2(x,z_1)=U^2(x,U^2(x,y))=U^2(x,y)=z_1$ and similarly $U^2(y,z_1)=z_1$. Thus $x,y\in\,]x_0,y_0]$. □

First we will discuss the case when $U^2(x_0,y)=y$ for all $y\in\,]y_0,1]$.

Proposition 3.8. *Let $U^2\colon[0,1]\longrightarrow[0,1]$ be an idempotent 2-uninorm such that $U^2(x_0,y_0)=x_0$ and for all $y\in\,]y_0,1]$ there is $U^2(x_0,y)=y$. Then U^2 is an ordinal sum of semigroups $G_1=([0,x_0]\cup\,]y_0,1],U^2)$, and $G_2=(]x_0,y_0],U^2)$, where $1<2$. Further, G_1 is isomorphic to $([0,1],U)$, where U is an idempotent uninorm and G_2 is either isomorphic to $(]0,1],V^2)$, where V^2 is an idempotent 2-uninorm from Class 1 restricted to $]0,1]^2$, or G_2 is an ordinal sum of two semigroups, $(\{y_0\},\mathrm{Id})$ and $(]x_0,y_0[,U^2)$, while U^2 is on $]x_0,y_0[^2$ isomorphic to an idempotent 2-uninorm from Class 1 restricted to $]0,1[^2$.*

Proof. Due to Lemma 3.7 we know that U^2 is closed on $([0,x_0]\cup\,]y_0,1])^2$. Since $U^2(x_0,y)=y$ for all $y\in\,]y_0,1]$ and $U^2\equiv\min$ on $[0,x_0]^2$, i.e., $U^2(x,x_0)=x$ for all $x\in[0,x_0]$, we know that x_0 is the neutral element of U^2 on $[0,x_0]\cup\,]y_0,1]$. Thus U^2 is on $([0,x_0]\cup\,]y_0,1])^2$ isomorphic to an idempotent uninorm. Further we have to distinguish two cases. If $U^2(z_1,y_0)=z_1$ then U^2 is on $]x_0,y_0]^2$ isomorphic to a 2-uninorm from Class 1 restricted to $]0,1]^2$. If $U^2(z_1,y_0)=y_0$ then $U^2(y_0,x)=y_0$ for all $x\in\,]x_0,y_0[$ and due to Lemma 3.1 U^2 is closed on $]x_0,y_0[^2$. Thus U^2 on $]x_0,y_0]^2$ is an ordinal sum of a trivial semigroup and $(]x_0,y_0[,U^2)$, while U^2 restricted to $]x_0,y_0[^2$ is isomorphic to an idempotent 2-uninorm from Class 1 restricted to $]0,1[^2$.

To finish the proof we have to investigate the order of the two semigroups in the ordinal sum construction. Due to the commutativity it is enough to take an $x\in\,]x_0,y_0]$ and a $y\in[0,x_0]\cup\,]y_0,1]$. Here we have

$$U^2(x,y)=U^2(x,U^2(x_0,y))=U^2(U^2(x,x_0),y)=U^2(x_0,y)=y,$$

i.e., $1<2$. Thus U^2 is an ordinal sum of semigroups G_1 and G_2 with $1<2$. □

Finally, in this subsection we will investigate the case when $U^2(x_0,y_0)=x_0$ and there exists a $y\in\,]y_0,1]$ such that $U^2(x_0,y)\neq y$.

Lemma 3.9. *Let $U^2\colon[0,1]\longrightarrow[0,1]$ be an idempotent 2-uninorm such that $U^2(x_0,y_0)=x_0$. If there is $U^2(x_0,v)\neq v$ for some $v\in\,]y_0,1]$ then $U^2(x_0,v)=x_0$.*

Proof. Lemma 3.1 implies $U^2(x_0,v)\in\{x_0,v,z_1\}$. If $U^2(x_0,v)=z_1$ then

$$z_1=U^2(x_0,v)=U^2(x_0,U^2(v,v))=U^2(U^2(x_0,v),v)=U^2(z_1,v),$$

which is a contradiction since $U^2(y,z_1)=y$ for all $y\in\,]y_0,1]$. Since $U^2(x_0,v)\neq v$ we obtain $U^2(x_0,v)=x_0$. □

Further we will denote

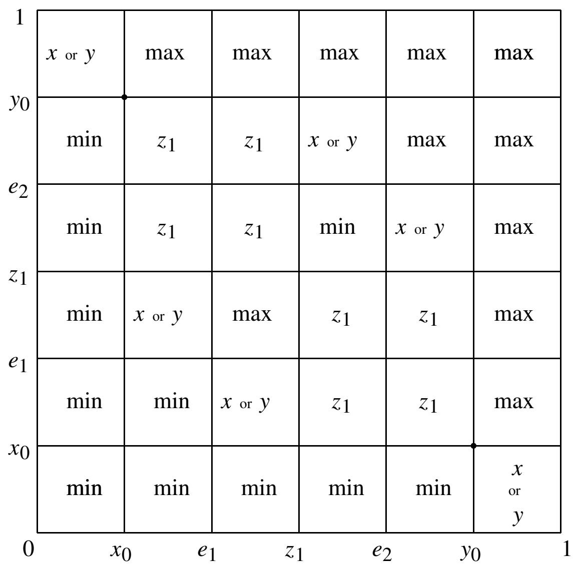
$$y_1=\sup\{y\in[y_0,1]\mid U^2(x_0,y)=x_0\}.$$

Proposition 3.10. *Let $U^2\colon[0,1]\longrightarrow[0,1]$ be an idempotent 2-uninorm such that $U^2(x_0,y_0)=x_0$. If $y_1>y_0$ and $U^2(x_0,y_1)=x_0$ then U^2 can be expressed as an ordinal sum of $G_1=([0,x_0]\cup\,]y_1,1],U^2)$, $G_2=(]x_0,y_0],U^2)$ described in Proposition 3.8 and $G_3=(]y_0,y_1],\max)$. Further, G_1 is isomorphic to $([0,1],U)$, where U is an idempotent uninorm and the order in the ordinal sum construction is $1<3<2$.*

Proof. Due to Lemma 3.7 we know that U^2 is closed on $([0,x_0]\cup\,]y_1,1])^2$. Since $U^2(x_0,x)=x$ for all $x\le x_0$ and $U^2(x_0,y)=y$ for all $y>y_1$ we see that x_0 is the neutral element of U^2 on $([0,x_0]\cup\,]y_1,1])^2$. Thus U^2 is on $([0,x_0]\cup\,]y_1,1])^2$ isomorphic to an idempotent uninorm.

Assume $x\in\,]x_0,y_0]$ and $y\in\,]y_0,y_1]$. Then due to Lemma 3.7 we have $U^2(x,y)\in\{x,y\}$ and since $U^2(x,z_1)\ge z_1$, $U^2(z_1,y)=y$, due to the monotonicity we get $U^2(x,y)=y$. Thus $3<2$. Further, since x_0 is the annihilator of U^2 on $[x_0,y_0]^2$ and the neutral element of U^2 on $([0,x_0]\cup\,]y_1,1])^2$ we get

$$U^2(x,y)=U^2(U^2(x,x_0),y)=U^2(x,U^2(x_0,y))=U^2(x,x_0)=x$$



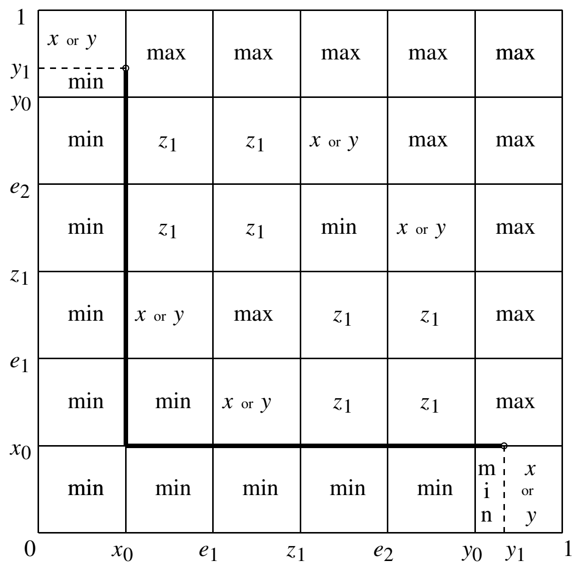


Fig. 3. A 2-uninorm with $U^2(x_0, y_0) = z_1$ (left) and with $U^2(x_0, y_0) = x_0$, $y_1 > y_0$ (right).

for all $x \in [0, x_0] \cup]y_1, 1]$ and $y \in]x_0, y_0]$, i.e., $1 < 2$. Finally, since $U^2(x_0, y) = x_0$ for all $y \in]y_0, y_1]$ we get

$$U^2(x, y) = U^2(U^2(x, x_0), y) = U^2(x, U^2(x_0, y)) = U^2(x, x_0) = x$$

for all $x \in [0, x_0] \cup]y_1, 1]$ and $y \in]y_0, y_1]$, i.e., $1 < 3$. Therefore U^2 is an ordinal sum of G_1, G_2 and G_3 with the order $1 < 3 < 2$ (see Fig. 3). □

Remark 3.11. Assume $U^2(x_0, y_0) = x_0$ and $U^2(x_0, v) \neq v$ for some $v \in]y_0, 1]$. Lemma 3.9 implies that $U^2(x_0, y) = y$ for all $y > y_1$. However, it can happen that $U^2(x_0, y_1) = y_1$. In such a case the point y_1 behaves differently than the rest of the semigroup defined on $]y_0, y_1]$ and therefore we cannot use the same construction as above.

We define $x_1 = \inf\{x \in [0, x_0] \mid U^2(y_1, x) = y_1\}$ and we can continue like this by the induction: for $n \in \mathbb{N}$ we define

$$y_n = \sup\{y \in \left[y_{n-1}, 1\right] \mid U^2(x_{n-1}, y) = x_{n-1}\}$$

and

$$x_n = \inf\{x \in \left[0, x_{n-1}\right] \mid U^2(y_n, x) = y_n\}.$$

It can happen that $y_{n_0} = y_{n_0-1}$ (and then $x_{n_0} = x_{n_0-1}$) for some $n_0 \in \mathbb{N}$, however it is also possible that $(y_i)_{i\in\mathbb{N}}$ ($(x_i)_{i\in\mathbb{N}}$) is an increasing (decreasing) sequence. Therefore we see that the structure of U^2 on $([0, x_0] \cup]y_0, 1])^2$ can be rather peculiar. That is why it needs not to be easy to express U^2 as an ordinal sum of a uninorm, a 2-uninorm from Class 1 (restricted to $]0, 1]^2$) and few other semigroups. Therefore we adopt a different approach.

Lemma 3.12. *Let $U^2 \colon [0, 1] \longrightarrow [0, 1]$ be an idempotent 2-uninorm and let $U^2(x_0, y_0) = x_0$. Then U^2 restricted to $([0, x_0[\cup \{z_1\} \cup]y_0, 1])^2$ is isomorphic to an idempotent uninorm.*

Proof. Lemma 3.7 implies that U^2 is internal on $([0, x_0[\cup]y_0, 1])^2$. Therefore U^2 is closed on $([0, x_0[\cup \{z_1\} \cup]y_0, 1])^2$. Further, the definition of x_0 and y_0 implies that z_1 is the neutral element of U^2 on $([0, x_0[\cup \{z_1\} \cup]y_0, 1])^2$. Thus U^2 restricted to $([0, x_0[\cup \{z_1\} \cup]y_0, 1])^2$ is isomorphic to an idempotent uninorm. □

Remark 3.13. Assume that $y_1 > y_0$ and $U^2(x_0, y_1) = y_1$. Due to results from [13] (see also [12]) we know that each idempotent uninorm is an ordinal sum of a countable number of semigroups with the operation min, semigroups with the operation max and semigroups corresponding to idempotent uninorms (such that the related function g from Theorem 2.4 is strictly decreasing) possibly restricted to open or half-open unit square. Thus also $([0, x_0[\cup \{z_1\} \cup$

$]y_0, 1]$, U^2) is an ordinal sum of a countable number of such semigroups. Since each idempotent uninorm is internal, also the semigroup $G = ([0, x_0[\cup]y_0, 1], U^2)$ can be expressed as an ordinal sum of a countable number of the above mentioned semigroups.

Further we know that $U^2(x_0, y) = x_0$ for all $y \in]y_0, y_1[$ and $U^2(x_0, x) = x$ for all $x \in [0, x_0[\cup[y_1, 1]$. Therefore we split the semigroup G into two parts: $G_1 = ([0, x_0[\cup[y_1, 1], U^2)$ and $G_2 = (]y_0, y_1[, \max)$ since U^2 on $(]y_0, y_1[)^2$ is given by the maximum operator. Then G_1 can be again expressed as an ordinal sum of a countable number of the above mentioned semigroups, however, y_1 need not to be the neutral element of G_1. Note that due to Lemma 3.12 we can find a similar characterization of the semigroup G_1 using a non-decreasing function g as in Theorem 2.4. Now U^2 on $([0, x_0]\cup]y_0, 1])^2$ can be expressed as an ordinal sum of semigroups G_1, G_2 and $G_3 = (\{x_0\}, \mathrm{Id})$, with order $1 < 3 < 2$. Indeed, from the previous we know that $3 < 2$ and $1 < 3$ and for an $x \in [0, x_0[\cup[y_1, 1]$ and a $y \in]y_0, y_1[$ we have

$$U^2(x, y) = U^2(U^2(x, x_0), y) = U^2(x, U^2(x_0, y)) = U^2(x, x_0) = x,$$

i.e., $1 < 2$.

Proposition 3.14. *Let $U^2\colon [0, 1] \longrightarrow [0, 1]$ be an idempotent 2-uninorm and let $U^2(x_0, y_0) = x_0$. If $y_1 > y_0$ and $U^2(x_0, y_1) = y_1$ then U^2 can be expressed as an ordinal sum of the semigroups $G_1 = ([0, x_0[\cup[y_1, 1], U_2)$, $G_2 = (]x_0, y_0], U^2)$ described in Proposition 3.8, $G_3 = (]y_0, y_1[, \max)$ and $G_4 = (\{x_0\}, \mathrm{Id})$. Further, G_1 can be expressed as an ordinal sum of a countable number of idempotent semigroups described in Remark 3.13 and $1 < 4 < 3 < 2$.*

Proof. Similarly as in Proposition 3.10 we can show that $U^2(x, y) = y$ for all $y \in]y_0, y_1[$ and $x \in]x_0, y_0]$. Therefore $3 < 2$. Further, $U^2(x_0, y) = x_0$ for all $y \in]y_0, y_1[$. Thus $4 < 3$. Finally, $U^2(x_0, x) = x$ for all $x \in [0, x_0[\cup[y_1, 1]$, i.e., $1 < 4$. Since the associativity of U^2 implies the transitivity of the relation $\le$ we can easily obtain all other comparisons. Therefore $1 < 4 < 3 < 2$. The rest follows from Remark 3.13. □

3.2. *Basic characterization of n-uninorms*

The basic structure of idempotent n-uninorms is very similar to that of 2-uninorms. Therefore we will proceed with the results analogous to those in the previous subsection. From [2, Theorem 2] we obtain the following result.

Lemma 3.15. *Let $U^n\colon [0, 1]^2 \longrightarrow [0, 1]$ be an idempotent n-uninorm. Then $U^n(x, y) \in \{x, y\} \cup \{z_i \mid z_i \in]x, y[\}$.*

Next we will investigate the value of $U^n(e_1, e_n)$.

Lemma 3.16. *Let $U^n\colon [0, 1] \longrightarrow [0, 1]$ be an idempotent n-uninorm. Then $U^n(e_1, e_n) = z_k$ for some $k \in \{1, \dots, n-1\}$ and z_k is the annihilator of U^n on $[e_1, e_n]^2$.*

Proof. The restriction of U^n to $\big[z_1, z_{n-1}\big]^2$ is an $(n-2)$-uninorm and thus $U^n(z_1, z_{n-1}) \in \{z_i \mid z_i \in \big[z_1, z_{n-1}\big]\}$. Let us denote $U^n(z_1, z_{n-1}) = z_k$. We can easily show that $U^n(z_1, z_k) = z_k = U^n(z_k, z_{n-1})$ and then the monotonicity implies that z_k is the annihilator of U^n on $\big[z_1, z_{n-1}\big]$. Since $U^n(e_1, z_1) = z_1$ and $U^n(z_{n-1}, e_n) = z_{n-1}$ we get $U^n(e_1, z_k) = U^n(e_1, U^n(z_1, z_k)) = U^n(z_1, z_k) = z_k$ and similarly $U^n(e_n, z_k) = z_k$. Thus

$$z_k = U^n(e_1, z_k) \le U^n(e_1, e_n) \le U^n(z_k, e_n) = z_k,$$

i.e., $U^n(e_1, e_n) = z_k$ and the monotonicity implies that z_k is the annihilator of U^n on $[e_1, e_n]^2$ (see Fig. 4). □

From now on we will denote the value $U^n(e_1, e_n)$ by z_k.

Lemma 3.17. *Let $U^n\colon [0, 1] \longrightarrow [0, 1]$ be an idempotent n-uninorm. Then $U^n(x, z_k) \in \{x, z_k\}$ for all $x \in [0, 1]$.*

Proof. If $x \in [e_1, e_n]$ then the claim follows from Lemma 3.16. Assume that $x < e_1$ (the case when $x > e_n$ is analogous). If $U^n(x, z_k) = z_i$ for some $z_i \in]x, z_k[$ then $k > i > 0$ and $z_i = U^n(x, z_k) = U^n(x, U^n(z_k, z_k)) = U^n(U^n(x, z_k), z_k) = U^n(z_i, z_k) = z_k$, which is a contradiction. Thus $U^n(x, z_k) \in \{x, z_k\}$ for all $x \in [0, 1]$. □

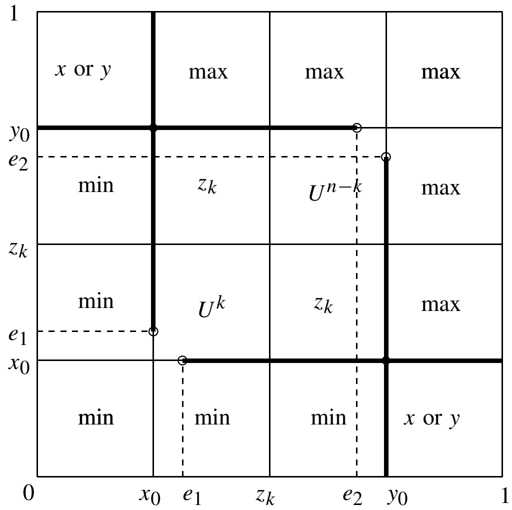


Fig. 4. A sketch of an idempotent n-uninorm, where $U^n(e_1, e_n) = z_k$. The bold lines denote the area where the values of U^n differ for distinct cases.

We say that an idempotent n-uninorm U^n belongs to the Class 1 if and only if $U^n(0, 1) = z_k$ for some $k \in \{1, \dots, n-1\}$. The structure of an idempotent n-uninorm U^n from Class 1 is again very simple. If $U^n(0, 1) = z_k$ then U^n is isomorphic to an idempotent k-uninorm on $[0, z_k]^2$, to an idempotent $(n-k)$-uninorm on $[z_k, 1]^2$, and otherwise $U^n(x, y) = z_k$.

Remark 3.18. Lemma 3.17 and the monotonicity of U^n imply that there exists an $x_0 \in [0, e_1]$ and a $y_0 \in [e_n, 1]$ such that $U^n(x, z_k) = x$ for all $x < x_0$ and $U^n(x, z_k) = z_k$ for all $x_0 < x \leq z_k$, $U^n(y, z_k) = y$ for all $y > y_0$ and $U^n(y, z_k) = z_k$ for all $z_k \leq y < y_0$. Then for all $x_0 < x \leq z_k$ and $z_k \leq y < y_0$ there is $U^n(x, y) = z_k$. The structure of an idempotent n-uninorm U^n on $]x_0, y_0[^2$ is given as follows:

(i) U^n restricted to $]x_0, z_k]^2$ is isomorphic to an idempotent k-uninorm restricted to $]0, 1]^2$.
(ii) U^n restricted to $[z_k, y_0[^2$ is isomorphic to an idempotent $(n-k)$-uninorm restricted to $[0, 1[^2$.
(iii) U^n restricted to $]x_0, z_k] \times [z_k, y_0[$ or to $[z_k, y_0[\times]x_0, z_k]$ is equal to z_k.

Summarizing, U^n restricted to $]x_0, y_0[^2$ is isomorphic to an idempotent n-uninorm from Class 1, restricted to $]0, 1[^2$.

Now all results for 2-uninorms can be analogously shown also for n-uninorms. Therefore we introduce them without proofs.

Proposition 3.19. *Let $U^n : [0, 1] \longrightarrow [0, 1]$ be an idempotent n-uninorm and let $U^n(x_0, y_0) = z_k$. Then U^n is an ordinal sum of semigroups $G_1 = ([0, x_0[\cup \{z_k\} \cup]y_0, 1], U^n)$ and $G_2 = ([x_0, y_0], U^n)$, where the order in the ordinal sum construction is $1 < 2$; and G_1 is isomorphic to $([0, 1], U)$, where U is an idempotent uninorm and G_2 is isomorphic to $([0, 1], V^n)$, where V^n is an idempotent n-uninorm from Class 1.*

Proposition 3.20. *Let $U^n : [0, 1] \longrightarrow [0, 1]$ be an idempotent n-uninorm such that $U^n(x_0, y_0) = x_0$ and for all $y \in]y_0, 1]$ there is $U^n(x_0, y) = y$. Then U^n is an ordinal sum of semigroups $G_1 = ([0, x_0] \cup]y_0, 1], U^n)$ and $G_2 = (]x_0, y_0], U^n)$, where $1 < 2$. Further, G_1 is isomorphic to $([0, 1], U)$, where U is an idempotent uninorm and G_2 is either isomorphic to $(]0, 1], V^n)$, where V^n is an idempotent n-uninorm from Class 1 restricted to $]0, 1]^2$, or G_2 is an ordinal sum of two semigroups, $(\{y_0\}, \mathrm{Id})$ and $(]x_0, y_0[, U^n)$, while U^n is on $]x_0, y_0[^2$ isomorphic to an idempotent n-uninorm from Class 1 restricted to $]0, 1[^2$.*

We will again denote $y_1 = \sup\{y \in [y_0, 1] \mid U^n(x_0, y) = x_0\}$.

Proposition 3.21. *Let $U^n : [0, 1] \longrightarrow [0, 1]$ be an idempotent n-uninorm such that $U^n(x_0, y_0) = x_0$. If there is $y_1 > y_0$ and $U^n(x_0, y_1) = x_0$ then U^n can be expressed as an ordinal sum of $G_1 = ([0, x_0] \cup]y_1, 1], U^n)$, $G_2 = (]x_0, y_0], U^n)$*

described in Proposition 3.20 and $G_3 = (]y_0, y_1], \max)$. *Further,* G_1 *is isomorphic to* $([0,1], U)$, *where* U *is an idempotent uninorm and* $1 < 3 < 2$.

Proposition 3.22. *Let* $U^n: [0,1] \longrightarrow [0,1]$ *be an idempotent* n*-uninorm and let* $U^n(x_0, y_0) = x_0$. *If* $y_1 > y_0$ *and* $U^n(x_0, y_1) = y_1$ *then* U^n *can be expressed as an ordinal sum of semigroups* $G_1 = ([0, x_0[\cup [y_1, 1], U^n)$, $G_2 = (]x_0, y_0], U^n)$ *described in Proposition 3.20,* $G_3 = (]y_0, y_1[, \max)$ *and* $G_4 = (\{x_0\}, \mathrm{Id})$. *Further,* $1 < 4 < 3 < 2$ *and* G_1 *can be expressed as an ordinal sum of a countable number of semigroups with the operation* min*, semigroups with the operation* max *and semigroups corresponding to idempotent uninorms (such that the related function g from Theorem 2.4 is strictly decreasing) possibly restricted to open or half-open unit square.*

The above results show us that n-uninorms are constructed from blocks which contain uninorms of lower orders. These blocks are glued together either by the ordinal sum construction, or by the constant value z_i for $i \in \{1, \dots, n-1\}$.

4. Idempotent n-uninorms as partially ordered ordinal sums of trivial semigroups

In this section we will use a special construction method, similar to the ordinal sum construction, which, however, covers also partially ordered semigroups, in the case that the partial order on the respective index set corresponds to a lower semi-lattice.

Definition 4.1. A meet semi-lattice (or lower semi-lattice) is a partially ordered set which has a meet (or greatest lower bound) for any non-empty finite subset.

Note that since the existence of the meet is required only for non-empty finite subsets this is equivalent to the existence of the meet between all pairs of arguments.

The following theorem describes the z-ordinal sum construction for semigroups.

Theorem 4.2. *Let* A *and* B *be two index sets such that* $A \cap B = \emptyset$ *and* $C = A \cup B \neq \emptyset$. *Let* $(G_\alpha)_{\alpha \in C}$ *with* $G_\alpha = (X_\alpha, *_\alpha)$ *be a family of semigroups and let the set* C *be partially ordered by the binary relation* $\preceq$ *such that* $(C, \preceq)$ *is a meet semi-lattice. Further suppose that each semigroup* G_α *for* $\alpha \in A$ *possesses an annihilator* z_α, *and for all* $\alpha, \beta \in C$ *such that* α *and* β *are incomparable there is* $\alpha \wedge \beta \in A$. *Assume that for all* $\alpha, \beta \in C$ *the sets* X_α *and* X_β *are either disjoint or that* $X_\alpha \cap X_\beta = \{x_{\alpha,\beta}\}$. *In the second case suppose that for all* $\gamma \in C$ *which is incomparable with* $\alpha \wedge \beta$ *there is* $\alpha \wedge \gamma = \beta \wedge \gamma$ *and for each* $\gamma \in C$ *with* $\alpha \wedge \beta \prec \gamma \prec \alpha$ *or* $\alpha \wedge \beta \prec \gamma \prec \beta$ *we have* $X_\gamma = \{x_{\alpha,\beta}\}$. *Further,*

(i) *in the case that* $\alpha \wedge \beta \in A$ *then* $x_{\alpha,\beta} = z_{\alpha \wedge \beta}$ *is the annihilator of both* G_β *and* G_α*;*
(ii) *in the case that* $\alpha \wedge \beta = \alpha \in B$ *then* $x_{\alpha,\beta}$ *is both the annihilator of* G_β *and the neutral element of* G_α.

Put $X = \bigcup_{\alpha \in C} X_\alpha$ *and define the binary operation* $*$ *on* X *by*

$$x * y = \begin{cases} x *_\alpha y & \text{if } (x, y) \in X_\alpha \times X_\alpha, \\ x & \text{if } (x, y) \in X_\alpha \times X_\beta,\ \alpha \neq \beta, \text{ and } \alpha \wedge \beta = \alpha \in B, \\ y & \text{if } (x, y) \in X_\alpha \times X_\beta,\ \alpha \neq \beta, \text{ and } \alpha \wedge \beta = \beta \in B, \\ z_\gamma & \text{if } (x, y) \in X_\alpha \times X_\beta,\ \alpha \neq \beta, \text{ and } \alpha \wedge \beta = \gamma \in A. \end{cases}$$

Then $G = (X, *)$ *is a semigroup. The semigroup* G *is commutative if and only if for each* $\alpha \in C$ *the semigroup* G_α *is commutative.*

Proof. First note that if $\alpha \wedge \beta \in B$ then α and β are comparable. Obviously, G is commutative if and only if for each $\alpha \in C$ the semigroup G_α is commutative. Further, we have to show that the operation $*$ is well-defined and associative.

If $x \in X_\alpha \cap X_\beta$ then for each $\gamma \in C$ and $y \in X_\gamma$ we can define $x * y$ as an operation on $X_\alpha \times X_\gamma$, or as an operation on $X_\beta \times X_\gamma$. We are going to show that in all cases this two definitions coincide. First assume that γ is incomparable with $\alpha \wedge \beta$. Then the assumption implies $\alpha \wedge \gamma = \beta \wedge \gamma$ and thus $\alpha \wedge \gamma = \beta \wedge \gamma = \alpha \wedge \beta \wedge \gamma$ which belongs to A since γ and $\alpha \wedge \beta$ are incomparable. Then in both cases $x * y = z_{\alpha\wedge\beta\wedge\gamma}$. Now assume that γ is comparable with $\alpha \wedge \beta$. If $\gamma \preceq \alpha \wedge \beta$ then in both cases $x * y = y$ if $\gamma \in B$ ($x * y = z_\gamma$ if $\gamma \in A$). Finally assume that $\alpha \wedge \beta \prec \gamma$. If $\alpha \wedge \beta \in A$ then $x = z_{\alpha\wedge\beta}$, $\alpha \wedge \beta \preceq \alpha \wedge \gamma \preceq \alpha$, $\alpha \wedge \beta \preceq \beta \wedge \gamma \preceq \beta$, and it is easy to check that in both cases $x * y = z_{\alpha\wedge\beta}$. If $\alpha \wedge \beta \in B$ then α and β are comparable and it is again easy to check that in both cases $x * y = x$.

Now we will show that $*$ is associative. Assume any $x, y, z \in X$. If $x, y, z \in X_\alpha$ for some $\alpha \in A$ then the associativity follows from the associativity of $*_\alpha$. Let $x, y \in X_\alpha$, $z \in X_\beta$ for some $\alpha, \beta \in C$, $\alpha \neq \beta$. First assume that $\alpha \wedge \beta \in B$, i.e., α and β are comparable. Then the associativity follows from Theorem 2.1. If $\alpha \wedge \beta = \gamma \in A$ then

$$(x * y) * z = z_\gamma = x * z_\gamma = x * (y * z),$$
$$(x * z) * y = z_\gamma * y = z_\gamma = x * z_\gamma = x * (z * y),$$
$$(z * x) * y = z_\gamma * y = z_\gamma = z * (x * y).$$

Further suppose that $x \in X_\alpha$, $y \in X_\beta$, $z \in X_\delta$. We denote $\alpha \wedge \beta = \gamma$, $\alpha \wedge \delta = \pi$ and $\beta \wedge \delta = \rho$. Then $\theta = \gamma \wedge \pi = \gamma \wedge \rho = \pi \wedge \rho$ is the greatest lower bound of α, β and γ.

Now we have 4 possible cases.

Case 1: If $\gamma, \pi, \rho \in A$. Then $\theta \in A$ and we leave as an easy exercise for the reader to check that $(a * b) * c = a * (b * c) = z_\theta$ whenever (a, b, c) is a permutation of (x, y, z).

Case 2: If $\gamma, \pi \in A$ and $\rho \in B$. Here β and δ are comparable and $\theta \in A$. If $\beta \succ \delta$ then $\theta = \pi \preceq \gamma$. If $\beta \prec \delta$ then $\theta = \gamma \preceq \pi$. We again leave the proof of this case as an easy exercise for the reader. Note that similarly as in the previous case we always obtain $(a * b) * c = a * (b * c) = z_\theta$.

Case 3: If $\gamma \in A$ and $\pi, \rho \in B$. Here both α and β are comparable with δ. If α and β are comparable then $\gamma \in A$ and $\pi, \rho \in B$ implies $\delta \prec \alpha \wedge \beta$ and similarly as above it is easy to show that $(a * b) * c = a * (b * c) = z$ whenever (a, b, c) is a permutation of (x, y, z). Assume that α and β are incomparable. Then either $\delta \succ \alpha$ and $\delta \succ \beta$, or $\delta \prec \alpha$, $\delta \prec \beta$. First suppose that $\delta \succ \alpha$ and $\delta \succ \beta$. Then also $\delta \succ \gamma$. Since $\pi, \rho \in B$ we know that $\alpha, \beta \in B$. We have

$$(x * y) * z = z_\gamma * z = z_\gamma = x * y = x * (y * z),$$
$$(y * x) * z = z_\gamma * z = z_\gamma = y * x = y * (x * z),$$
$$(x * z) * y = x * y = z_\gamma = x * y = x * (z * y),$$
$$(z * x) * y = x * y = z_\gamma = z * z_\gamma = z * (x * y),$$
$$(y * z) * x = y * x = z_\gamma = y * x = y * (z * x),$$
$$(z * y) * x = y * x = z_\gamma = z * z_\gamma = z * (y * x).$$

Now suppose that $\delta \prec \alpha$, $\delta \prec \beta$. Since $\pi, \rho \in B$ we know that $\delta \in B$. Since $\gamma \in A$ we know that $\delta \prec \gamma$. We have

$$(x * y) * z = z_\gamma * z = z = x * z = x * (y * z),$$
$$(y * x) * z = z_\gamma * z = z = y * z = y * (x * z),$$
$$(x * z) * y = z * y = z = x * z = x * (z * y),$$
$$(z * x) * y = z * y = z = z * z_\gamma = z * (x * y),$$
$$(y * z) * x = z * x = z = y * z = y * (z * x),$$
$$(z * y) * x = z * x = z = z * z_\gamma = z * (y * x).$$

Case 4: If $\gamma, \pi, \rho \in B$ then α, β, δ are all mutually comparable and the claim follows from Theorem 2.1. □

Note that if in the previous theorem $A = \emptyset$ then the set $C = B$ is linearly ordered and the z-ordinal sum reduces to the standard ordinal sum construction. Further, if each semigroup G_α for $\alpha \in C$ is trivial and $A = C$ then the z-ordinal sum of G_α is given by $x * y = x \wedge^* y$, where the order $\leq^*$ is given for $x \in X_\alpha$ and $y \in X_\beta$ by $x \leq^* y$ if $\alpha \preceq \beta$.

Remark 4.3. Assume that in the previous theorem $x \in X_\alpha \cap X_\beta$. Then the condition that $\alpha \wedge \gamma = \beta \wedge \gamma$ for all $\gamma \in C$ incomparable with $\alpha \wedge \beta$ is necessary. Indeed, if γ is incomparable with $\alpha \wedge \beta$ and for example $\gamma \prec \beta$ then $\alpha \wedge \gamma \neq \beta \wedge \gamma$ implies that α and γ are incomparable. Here on $X_\beta \times X_\gamma$ there is $x * y = y$ if $\gamma \in B$ ($x * y = z_\gamma$ if $\gamma \in A$) however, on $X_\alpha \times X_\gamma$ there is $x * y = z_{\alpha\wedge\beta\wedge\gamma}$. Note that alternatively we can require that if $x \in X_\alpha \cap X_\beta$ and for some $\gamma \in C$, which is incomparable with $\alpha \wedge \beta$, there is $\alpha \wedge \gamma \neq \beta \wedge \gamma$ then $X_\gamma = \{z_{\alpha\wedge\beta\wedge\gamma}\}$.

Example 4.4. Let us assume the only idempotent nullnorm $V \colon [0,1]^2 \longrightarrow [0,1]$ with the annihilator $z_1 \in]0,1[$, which is a special case of a 2-uninorm with local neutral elements $e_1 = 0$ and $e_2 = 1$. Then $V(x,y) = \max(x,y)$ if $x, y \in [0, z_1]$, $V(x,y) = \min(x,y)$ if $x, y \in [z_1, 1]$ and $V(x,y) = z_1$ otherwise. The semigroup $([0,1], V)$ is a z-ordinal sum of trivial semigroups $G_x = (\{x\}, \mathrm{Id})_{x\in[0,1]}$ with $A = \{z_1\}$, where $z_1 \preceq x$ for all $x \in [0,1]$ and $x_1 \preceq x_2$ if $x_1, x_2 \in [0, z_1]$ and $x_1 \geq x_2$, or $x_1, x_2 \in [z_1, 1]$ and $x_1 \leq x_2$. Further, x and y are incomparable for all $x \in [0, z_1[$ and $y \in]z_1, 1]$. On the other hand, V can be expressed also as a z-ordinal sum of 3 semigroups $G_1 = ([0, z_1[\,, \max)$, $G_2 = (]z_1, 1]\,, \min)$ and $G_3 = (\{z_1\}, \mathrm{Id})$, where $3 \prec 1$, $3 \prec 2$ and 2 is incomparable with 1.

Let us now recall two results from [12].

Proposition 4.5. *Let $U \colon [0,1]^2 \longrightarrow [0,1]$ be an idempotent uninorm. Then $([0,1], U)$ is an ordinal sum of singleton semigroups $(\{x\}, \mathrm{Id})$ for $x \in [0,1]$.*

Proposition 4.6. *Let P be an index set isomorphic with $[0,1]$ via the isomorphism i. For all $p \in P$ we put $X_p = \{x\}$ if $i(p) = x$. Let $e \in [0,1]$ and let $\preceq$ be a linear order on P. If $([0,1], U)$ is the ordinal sum of $\{(X_p, \mathrm{Id})\}_{p\in P}$ with the linear order $\preceq$ then U is an idempotent uninorm with the neutral element e if and only if the following two conditions are fulfilled:*

(i) *$p_1 \prec p_2$ for all $p_1, p_2 \in P$ such that $X_{p_1} = \{x_1\}$, $X_{p_2} = \{x_2\}$, $x_1 < x_2$ and $x_1, x_2 \in [0, e]$,*
(ii) *$p_1 \prec p_2$ for all $p_1, p_2 \in P$ such that $X_{p_1} = \{y_1\}$, $X_{p_2} = \{y_2\}$, $y_1 > y_2$ and $y_1, y_2 \in [e, 1]$.*

Example 4.7. For the construction of t-norms and t-conorms, an ordinal sum with respect to a countable index set is usually taken, which is due to the fact that the supports of respective summands need not cover the whole interval $[0,1]$ and the remaining part is covered by the minimum (maximum). However, in the case of uninorms this is no longer true and ordinal sums with uncountable index sets can be also used (see [12]). Assume the isomorphism i from the previous proposition, a family of semigroups $\{(X_p, \mathrm{Id})\}_{p\in P}$ and the relation $\preceq$ on P given for $p_1, p_2 \in P$ with $X_{p_1} = \{x\}$, $X_{p_2} = \{y\}$ by

$$p_1 \preceq p_2 \qquad \text{if} \qquad (x + y < 1 \text{ and } x \leq y) \text{ or } (x + y \geq 1 \text{ and } x \geq y).$$

Then the relation $\preceq$ is reflexive since $x = x$ for all $x \in [0,1]$, whether $2 \cdot x < 1$ or $2 \cdot x \geq 1$. Further, $\preceq$ is antisymmetric, since for $x + y < 1$ there $p_1 \preceq p_2$ and $p_2 \preceq p_1$ implies $x \leq y$ and $y \leq x$, i.e., $x = y$ and similarly for $x + y \geq 1$ there $p_1 \preceq p_2$ and $p_2 \preceq p_1$ implies $x \leq y$ and $y \leq x$, i.e., $x = y$. Finally we have to show that $\preceq$ is transitive. Assume that $p_1 \preceq p_2$ and $p_2 \preceq p_3$ for some $p_1, p_2, p_3 \in P$ with $X_{p_1} = \{x\}$, $X_{p_2} = \{y\}$ and $X_{p_3} = \{z\}$. Then we have the following possibilities:

1. If $x + y < 1$, $y + z < 1$ then $x \leq y$ and $y \leq z$, i.e., $x \leq z$ and $x + z \leq y + z < 1$ which means that $p_1 \preceq p_3$.
2. If $x + y < 1$, $y + z \geq 1$ then $x \leq y$ and $y \geq z$. Then $x \leq z$ since $x < 1 - y \leq z$ and $x + z \leq x + y < 1$, i.e., $p_1 \preceq p_3$.
3. If $x + y \geq 1$, $y + z < 1$ then $x \geq y$ and $y \leq z$. Then $x \geq z$ since $x \geq 1 - y > z$ and $1 \leq x + y \leq x + z$, i.e., $p_1 \preceq p_3$.
4. If $x + y \geq 1$, $y + z \geq 1$ then $x \geq y$ and $y \geq z$, i.e., $x \geq z$ and $x + z \geq y + z \geq 1$ which means that $p_1 \preceq p_3$.

Therefore $\preceq$ is transitive and thus it is a partial order. It is easy to see that $\preceq$ is also a linear order since for any $x, y \in [0,1]$ there is $(x + y < 1)$ or $(1 \leq x + y)$ and there is $(x \leq y)$ or $(y \leq x)$.

If $([0,1], U)$ is the ordinal sum of $\{(X_p, \mathrm{Id})\}_{p\in P}$ with the linear order $\preceq$ then U is given by $U(x, y) = \min(x, y)$ if $x + y < 1$ and $U(x, y) = \max(x, y)$ if $1 \le x + y$. It is easy to check that U is an idempotent uninorm with the neutral element $\frac{1}{2}$.

Thus uninorm U is an ordinal sum of trivial semigroups with respect to an index set which is uncountable. Note that this uninorm cannot be expressed as an ordinal sum of a countable number of non-trivial semigroups with supports that are connected sets.

We would like to show similar results for idempotent 2-uninorms and then generally for idempotent n-uninorms. We will start with 2-uninorms.

Definition 4.8. Let P be an index set isomorphic with $[0,1]$ via the isomorphism i. For all $p \in P$ we put $X_p = \{x\}$ if $i(p) = x$. Assume a commutative binary function $F\colon [0,1]^2 \longrightarrow [0,1]$. On the set P we define a relation $\preceq$ by $p_1 \preceq p_2$ if $F(x, y) = x$, where $X_{p_1} = \{x\}$ and $X_{p_2} = \{y\}$.

Lemma 4.9. *Let $U^2\colon [0,1]^2 \longrightarrow [0,1]$ be an idempotent 2-uninorm. Then $\preceq$ defined in Definition 4.8 is a partial order on the set P and the set P partially ordered by the binary relation $\preceq$ is a meet semi-lattice. Moreover, if p_1 and p_2 are incomparable for $X_{p_1} = \{x\}$ and $X_{p_2} = \{y\}$ then the meet of p_1 and p_2 is q_1, i.e., $p_1 \wedge p_2 = q_1$, where $X_{q_1} = \{z_1\}$.*

Proof. Since U^2 is idempotent $\preceq$ is reflexive. The transitivity follows from the associativity since for $X_{p_1} = \{x\}$, $X_{p_2} = \{y\}$ and $X_{p_3} = \{z\}$ there $p_1 \preceq p_2$ and $p_2 \preceq p_3$ implies $U^2(x, z) = U^2(U^2(x, y), z) = U^2(x, U^2(y, z)) = U^2(x, y) = x$, i.e., $p_1 \preceq p_3$. Finally, if $p_1 \preceq p_2$ and $p_2 \preceq p_1$ the result is implied by the commutativity. Thus $\preceq$ is a partial order.

From Proposition 4.6 we know that all semigroups corresponding to points from $[0, z_1]$ ($[z_1, 1]$) are linearly ordered by $\preceq$. Therefore the definition of $\preceq$ and Lemma 3.1 implies that p_1 and p_2 are incomparable for $X_{p_1} = \{x\}$, $X_{p_2} = \{y\}$, $x < y$, if and only if $U^2(x, y) = z_1$ and $z_1 \in \,]x, y[$.

Now we will show that each pair of our trivial semigroups have the meet. Assume the semigroups $X_{p_1} = \{x\}$ and $X_{p_2} = \{y\}$. If p_1 and p_2 are comparable, i.e., if $U^2(x, y) \in \{x, y\}$ then we are done. Assume that $U^2(x, y) = z_1$, $x \neq z_1$ and $y \neq z_1$. Then $z_1 \in \,]x, y[$ and we claim that q_1 is the meet of p_1 and p_2, where $X_{q_1} = \{z_1\}$. First we have to show that $q_1 \preceq p_1$ and $q_1 \preceq p_2$. We have

$$z_1 = U^2(x, y) = U^2(U^2(x, x), y) = U^2(x, U^2(x, y)) = U^2(x, z_1)$$

and similarly $U^2(y, z_1) = z_1$. Thus $q_1 \preceq p_1$ and $q_1 \preceq p_2$. Now we know that q_1 is the lower bound of p_1 and p_2, however, we have to show that it is the greatest lower bound. Assume that there exists a $p_j \in P$ such that $p_j \preceq p_1$, $p_j \preceq p_2$ and $q_1 \preceq p_j$ for $X_{p_j} = \{w\}$. Then we have

$$U^2(x, w) = w = U^2(y, w)$$

and $U^2(w, z_1) = z_1$. However, since $z_1 \in \,]x, y[$ the monotonicity implies $U^2(z_1, w) = w$ and thus $w = z_1$, i.e., $q_1 = p_j$. Therefore q_1 is the meet of p_1 and p_2. Thus the set P partially ordered by the binary relation $\preceq$ is a meet semi-lattice. $\square$

Remark 4.10. Assume $x_0, y_0 \in [0,1]$ from Remark 3.2. Then $q_1 \prec p_3$ and $q_1 \prec p_4$, where $X_{q_1} = \{z_1\}$, $X_{p_3} = \{x_1\}$ for $x_1 \in \,]x_0, z_1[$ and $X_{p_4} = \{y_1\}$ for $y_1 \in \,]z_1, y_0[$. Similarly, $p_5 \prec q_1$ and $p_6 \prec q_1$, where $X_{p_5} = \{x_2\}$ for $x_2 \in [0, x_0[$ and $X_{p_6} = \{y_2\}$ for $y_2 \in \,]y_0, 1]$.

To simplify the notation hereinafter we will denote $X_{q_i} = \{z_i\}$, $X_{w_i} = \{e_i\}$, $X_{a_i} = \{x_i\}$, $X_{b_i} = \{y_i\}$ for $i \in \mathbb{N}$, $X_a = \{x\}$, $X_b = \{y\}$ and $X_{p_0} = \{U^2(0, 1)\}$. Note that since $U^2(0, 1)$ is the annihilator of U^2 then p_0 is the bottom element of the meet semi-lattice $(P, \preceq)$.

Proposition 4.11. *Let $U^2\colon [0,1]^2 \longrightarrow [0,1]$ be an idempotent 2-uninorm. Then $([0,1], U^2)$ is a z-ordinal sum of singleton semigroups $(\{x\}, \mathrm{Id})$ for $x \in [0,1]$.*

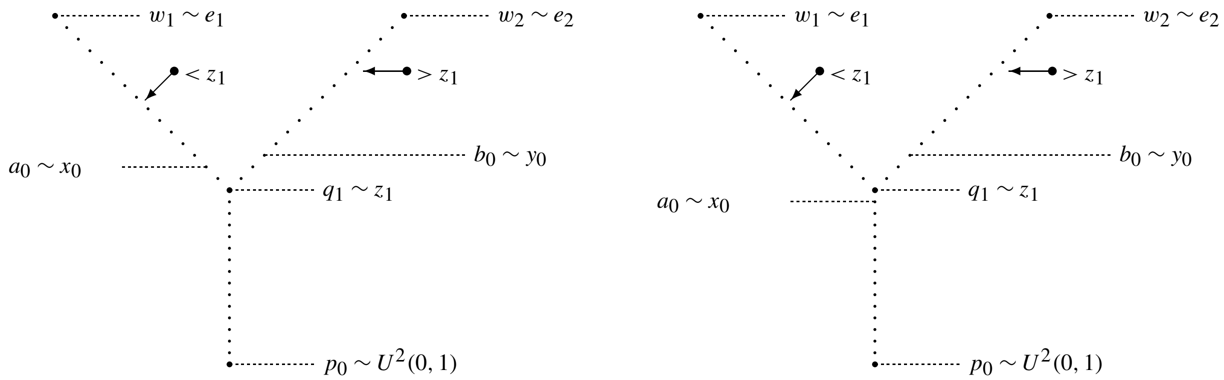


Fig. 5. A lower semi-lattice corresponding to an idempotent 2-uninorm with $U^2(x_0, y_0) = z_1$ (left) and with $U^2(x_0, y_0) = x_0$, $U^2(y_0, z_1) = z_1$ (right).

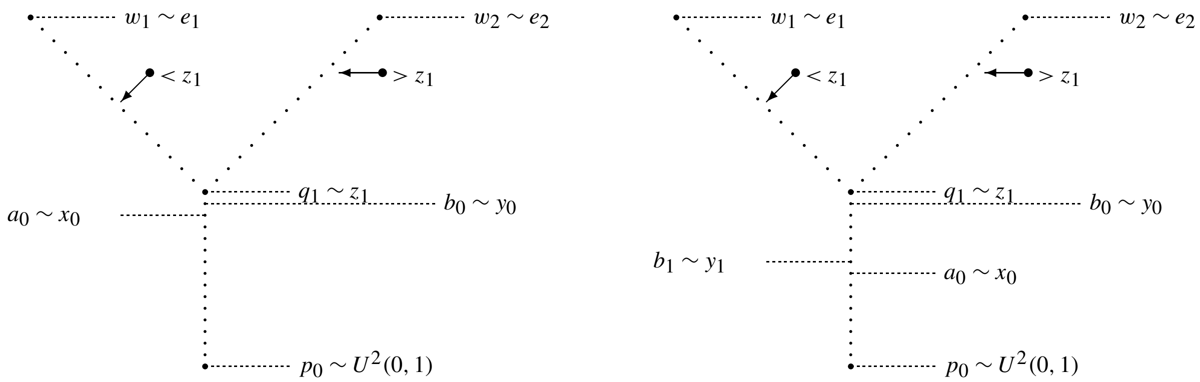


Fig. 6. A lower semi-lattice corresponding to an idempotent 2-uninorm with $U^2(x_0, y_0) = x_0$, $U^2(z_1, y_0) = y_0$, $U^2(x_0, y) = y$ for all $y > y_0$ (left) and with $U^2(x_0, y_0) = x_0$, $U^2(y_0, z_1) = y_0$, $U^2(x_0, y_1) = x_0$ (right).

Proof. Assume the partial order from Definition 4.8. Let $([0, 1], U^*)$ be the z-ordinal sum of $\{(X_p, \mathrm{Id})\}_{p\in P}$ with the partial order $\preceq$, such that $A = \{q_1\}$. Assume $x, y \in [0, 1]$. If $x = y$ evidently $U^2(x, x) = U^*(x, x) = x$. If $U^2(x, y) = x$ then $a \preceq b$. If $x \neq z_1$ then $a \in B$ and $U^*(x, y) = x$. If $x = z_1$ then $a \in A$ and $U^*(x, y) = z_1 = x$. Thus $U^2(x, y) = U^*(x, y) = x$. Similarly, if $U^2(x, y) = y$ then $U^2(x, y) = U^*(x, y) = y$.

Finally assume that $U^2(x, y) = z_1$, $z_1 \in]x, y[$. Then a and b are incomparable and therefore $U^*(x, y) = z_1$. Thus $U^2(x, y) = U^*(x, y) = z_1$. Summarizing, $U^2(x, y) = U^*(x, y)$ for all $x, y \in [0, 1]$ (see Fig. 5 and Fig. 6). □

Proposition 4.12. *Let P be an index set isomorphic with $[0, 1]$ via the isomorphism i. For all $p \in P$ we put $X_p = \{x\}$ if $i(p) = x$. Let $z_1 \in]0, 1[$, $e_1, e_2 \in [0, 1]$, $e_1 \leq z_1 \leq e_2$. Denote $A = \{q_1\}$, where $X_{q_1} = \{z_1\}$ and $B = P \setminus \{q_1\}$. Let $\preceq$ be a partial order on P such that all requirements of Theorem 4.2 are fulfilled. If $([0, 1], U^2)$ is the z-ordinal sum of $\{(X_p, \mathrm{Id})\}_{p\in P}$ with the partial order $\preceq$ then U^2 is an idempotent 2-uninorm with the 2-neutral element $\{e_1, e_2\}_{z_1}$ if and only if the following conditions are fulfilled:*

(i) $a_1 \prec a_2$ for all $a_1, a_2 \in P$ such that $X_{a_1} = \{x_1\}$, $X_{a_2} = \{x_2\}$, $x_1 < x_2$ and $x_1, x_2 \in [0, e_1] \cup [z_1, e_2]$,
(ii) $b_1 \prec b_2$ for all $b_1, b_2 \in P$ such that $X_{b_1} = \{y_1\}$, $X_{b_2} = \{y_2\}$, $y_1 > y_2$ and $y_1, y_2 \in [e_1, z_1] \cup [e_2, 1]$.
(iii) a and b are incomparable if and only if $q_1 \prec a$, $q_1 \prec b$ and $z_1 \in]x, y[$, where $X_a = \{x\}$, $X_b = \{y\}$.
(iv) a_1 and a_2 are comparable for all $a_1, a_2 \in P$ such that $X_{a_1} = \{x_1\}$, $X_{a_2} = \{x_2\}$, where $(x_1, x_2) \in [0, z_1]^2 \cup [z_1, 1]^2$.

Proof. The necessity follows from Proposition 4.6 and the monotonicity as $U^2(x, z_1) = z_1 = U^2(y, z_1)$ implies $U^2(x, y) = z_1$ if $z_1 \in]x, y[$.

Now we will show the sufficiency part. The associativity and the commutativity of U^2 follows from Theorem 4.2. Further, U^2 is evidently idempotent. From (i) it follows that if $x \le e_1$ then $U^2(x, e_1) = x$. Further, from (ii) it follows that if $x \in [e_1, z_1]$ then $U^2(x, e_1) = x$. Thus e_1 is the neutral element of U^2 on $[0, z_1]^2$. Similarly we can show that e_2 is the neutral element of U^2 on $[z_1, 1]^2$.

To finish the proof we have to show that U^2 is non-decreasing. Proposition 4.6 implies that U^2 is non-decreasing on $[0, z_1]^2$ and on $[z_1, 1]^2$.

Let $x, y_1, y_2 \in [0, 1]$. We will assume $x \in [0, z_1]$ as the case when $x \in [z_1, 1]$ is analogous. Then it is enough to check the monotonicity for $y_1, y_2 \in [z_1, 1]$, $y_1 < y_2$. If $x = z_1$ then the monotonicity follows from Proposition 4.6. Thus we will suppose that $x < z_1$. Further, if $y_2 = z_1$ then also $y_1 = z_1$ and therefore we will suppose that $y_2 > z_1$. From (ii) we know that $b_2 \preceq b_1$. Now we have the following 6 cases:

Case 1: If $a \preceq b_2 \preceq b_1$. Since $x \neq z_1$ we get

$$U^2(x, y_1) = x = U^2(x, y_2).$$

Case 2: If $b_2 \preceq a \preceq b_1$. Since $y_2 \neq z_1$ and $x \neq z_1$ we have

$$U^2(x, y_1) = x < y_2 = U^2(x, y_2).$$

Case 3: If $b_2 \preceq b_1 \preceq a$ We have $U^2(x, y_1) = y_1$ in both cases, when $y_1 = z_1$ and when $y_1 > z_1$ and thus we have

$$U^2(x, y_1) = y_1 < y_2 = U^2(x, y_2).$$

Case 4: If a is incomparable with b_1 and $b_2 \preceq a$ we get

$$U^2(x, y_1) = z_1 < y_2 = U^2(x, y_2).$$

Case 5: If a is incomparable with b_2 and $a \preceq b_1$ we get

$$U^2(x, y_1) = x < z_1 = U^2(x, y_2).$$

Case 6: If a and b_1 are incomparable and a and b_2 are incomparable we get

$$U^2(x, y_1) = z_1 = U^2(x, y_2).$$

Thus in all cases $U^2(x, y_1) \le U^2(x, y_2)$. $\square$

Example 4.13. Assume a 2-uninorm U_1^2 from Example 3.3. Then U_1^2 is a z-ordinal sum of semigroups $G_1 = ([0, \frac{1}{2}], U_1^{[0,\frac{1}{2}]})$, $G_2 = ([\frac{1}{2}, 1], U_2^{[\frac{1}{2},1]})$ and $G_3 = (\{\frac{1}{2}\}, \mathrm{Id})$, where $A = \{3\}$, 1 and 2 are incomparable and $1 \wedge 2 = 3$. Observe that $\{\frac{1}{2}\} \in X_1 \cap X_2 \cap X_3$ and $\frac{1}{2}$ is the annihilator of all three semigroups. Since $a \wedge b = 3$ for all $a, b \in \{1, 2, 3\}$, $a \neq b$ and $3 \preceq p$ for all $p \in \{1, 2, 3\}$ we see that there are no $\alpha, \beta, \gamma \in \{1, 2, 3\}$ such that γ is incomparable with $\alpha \wedge \beta$.

We can also show that U_1^2 is a z-ordinal sum of trivial semigroups. From [12] we know that each uninorm $U : [0, 1]^2 \longrightarrow [0, 1]$ corresponds to an ordinal sum of trivial semigroups, where the order in the ordinal sum construction is given by $a \le_U b$ for $X_a = \{x\}$ and $X_b = \{y\}$ if $U(x, y) = x$. Thus it is easy to show that U_1^2 is a z-ordinal sum of trivial semigroups, where $A = \{q_1\}$ for $X_{q_1} = \{\frac{1}{2}\}$ and the corresponding order is given by:

- for $x, y \in [0, \frac{1}{2}]$ the respective a, b are ordered by the corresponding order induced by U_1,
- for $x, y \in [\frac{1}{2}, 1]$ the respective a, b are ordered by the corresponding order induced by U_2
- for $x \in [0, \frac{1}{2}[$ and $y \in]\frac{1}{2}, 1]$ the respective a, b are incomparable.

Note that since U_1 is disjunctive and U_2 is conjunctive we get $q_1 \preceq p$ for all $p \in P$ and $X_{p_0} = \{\frac{1}{2}\}$, i.e., q_1 is the bottom element of the meet semi-lattice $(P, \preceq)$.

Assume a 2-uninorm U_2^2 from Example 3.3. Then U_2^2 is an ordinal sum of semigroups $G_1 = ([0, \frac{1}{4}[, \min)$, $G_2 = (]\frac{3}{4}, 1], \max)$ and $G_3 = ([\frac{1}{4}, \frac{3}{4}], U_2^2)$ with the respective order $1 < 2 < 3$. From the previous we know that G_3 can be further decomposed to the z-ordinal sum of semigroups $G_4 = ([\frac{1}{4}, \frac{1}{2}], U_1^{[\frac{1}{4},\frac{1}{2}]})$, $G_5 = ([\frac{1}{2}, \frac{3}{4}], U_2^{[\frac{1}{2},\frac{3}{4}]})$ and $G_6 =$

$(\{\frac{1}{2}\}, \mathrm{Id})$, where $A = \{6\}$. Thus U_2^2 can be also expressed as a z-ordinal sum of semigroups G_1, G_2, G_4, G_5 and G_6, where $A = \{6\}$ and the corresponding order in the z-ordinal sum construction is $1 \prec 2 \prec 6$, 4 and 5 are incomparable and $4 \wedge 5 = 6$. We can also show that U_2^2 is a z-ordinal sum of trivial semigroups. The respective order is given by:

- for $x \in \left[0, \frac{1}{4}\right[$, $y \in [0, 1]$ the respective a, b fulfill $a \preceq b$ if $\min(x, y) = x$,
- for $x \in \left]\frac{3}{4}, 1\right]$, $y \in \left[\frac{1}{4}, 1\right]$ the respective a, b fulfill $a \preceq b$ if $\max(x, y) = x$,
- for $x, y \in \left[\frac{1}{4}, \frac{1}{2}\right]$ the respective a, b are ordered by the corresponding order induced by U_1,
- for $x, y \in \left[\frac{1}{2}, \frac{3}{4}\right]$ the respective a, b are ordered by the corresponding order induced by U_2,
- for $x \in \left[\frac{1}{4}, \frac{1}{2}\right[$ and $y \in \left]\frac{1}{2}, \frac{3}{4}\right]$ the respective a and b are incomparable.

Observe that $X_{p_0} = \{0\}$.

Similarly we can use the z-ordinal sum construction to decompose 2-uninorms U_3^2, U_4^2 and U_5^2 from Example 3.3.

Now we will show the above results for idempotent n-uninorms for $n \in \mathbb{N}$ with $n > 2$.

Lemma 4.14. *Let P be an index set isomorphic with $[0, 1]$ via the isomorphism i. For all $p \in P$ we put $X_p = \{x\}$ if $i(p) = x$. Further, let U^n be an idempotent n-uninorm. On the set P we define a relation $\preceq$ by $a \preceq b$ if $U^n(x, y) = x$, where $X_a = \{x\}$ and $X_b = \{y\}$. Then $\preceq$ is a partial order on the set P and the set P partially ordered by the binary relation $\preceq$ is a meet semi-lattice. Moreover, if a and b are incomparable for $X_a = \{x\}$ and $X_b = \{y\}$ then $a \wedge b = q_i$, where $X_{q_i} = \{z_i\}$ for some $i \in \{1, \ldots, z_{n-1}\}$ and $z_i \in]\min(x, y), \max(x, y)[$.*

Proof. Similarly as in Lemma 4.9 we can show that $\preceq$ is a well-defined partial order, where its reflexivity follows from the idempotency of U^n, its transitivity from the associativity of U^n and its anti-symmetry from the commutativity of U^n. From Proposition 4.6 we know that all semigroups corresponding to points from $\left[z_i, z_{i+1}\right]$ for all $i = 0, \ldots, n-1$ are linearly ordered by $\preceq$. From the definition of $\preceq$ and Lemma 3.15 we see that a and b are incomparable for $x < y$ if and only if $U^n(x, y) = z_i$ for some $z_i \in]x, y[$. Further, for $z_k = U^n(e_1, e_n)$ we know from Lemma 3.17 that $U^n(z_k, x) \in \{x, z_k\}$ for all $x \in [0, 1]$ and thus q_k is comparable with every $p \in P$. Note that $q_k \preceq q_i$ for all $i \in \{1, \ldots, n-1\}$.

Now we will show that each pair of points from P have the meet. Assume that $X_a = \{x\}$ and $X_b = \{y\}$. If a and b are comparable, i.e., if $U^n(x, y) \in \{x, y\}$ then we are done. Assume that $U^n(x, y) = z_i$ for some $z_i \in]x, y[$. We claim that then q_i is the meet of a and b. First we have to show that $q_i \preceq a$ and $q_i \preceq b$. We have

$$z_i = U^n(x, y) = U^n(U^n(x, x), y) = U^n(x, U^n(x, y)) = U^n(x, z_i)$$

and similarly $U^n(y, z_i) = z_i$. Thus $q_i \preceq a$ and $q_i \preceq b$. Now we know that q_i is the lower bound of a and b, however, we have to show that it is the greatest lower bound. Assume that there exists a $p_j \in P$ such that $p_j \preceq a$, $p_j \preceq b$, $q_i \preceq p_j$ and $X_{p_j} = \{v\}$. Then we have

$$U^n(x, v) = v = U^n(y, v)$$

and $U^n(v, z_i) = z_i$. However, since $z_i \in]x, y[$ the monotonicity implies $U^n(z_i, v) = v$ and thus $v = z_i$, i.e., $q_i = p_j$. Therefore q_i is the meet of a and b. Thus the set P partially ordered by the binary relation $\preceq$ is a meet semi-lattice. $\square$

Remark 4.15. For an idempotent n-uninorm there is also $q_k \prec a_1$ and $q_k \prec b_1$, where $X_{a_1} = \{x_1\}$ for $x_1 \in]x_0, z_k[$ and $X_{b_1} = \{y_1\}$ for $y_1 \in]z_k, y_0[$. Similarly, $a_2 \prec q_k$ and $b_2 \prec q_k$, where $X_{a_2} = \{x_2\}$ for $x_2 \in [0, x_0[$ and $X_{b_2} = \{y_2\}$ for $y_2 \in]y_0, 1]$. Further, a and b are incomparable for $X_a = \{x\}$ and $X_b = \{y\}$ if and only if there exists a $q_i \in A$ such that $z_i \in]x, y[$ and $q_i \preceq a$, $q_i \preceq b$. The necessity of this claim follows from Lemma 4.14 and the sufficiency follows from the monotonicity as $q_i \preceq a$, $q_i \preceq b$ implies $U^n(x, z_i) = z_i = U^n(z_i, y)$ and $z_i \in]x, y[$ then implies $U^n(x, y) = z_i$.

Proposition 4.16. *Let $U^n \colon [0, 1]^2 \longrightarrow [0, 1]$ be an idempotent n-uninorm. Then $([0, 1], U^n)$ is a z-ordinal sum of singleton semigroups $(\{x\}, \mathrm{Id})$ for $x \in [0, 1]$.*

Proof. We will define a partial order as in Lemma 4.14. Denote $Z = \{z_1, \ldots, z_{n-1}\}$. Now let $([0, 1], U^*)$ be the z-ordinal sum of $\{(X_p, \mathrm{Id})\}_{p \in P}$ with the partial order $\preceq$, such that $A = \{q_1, \ldots, q_{n-1}\}$, where $X_{q_i} = \{z_i\}$ for $i =$

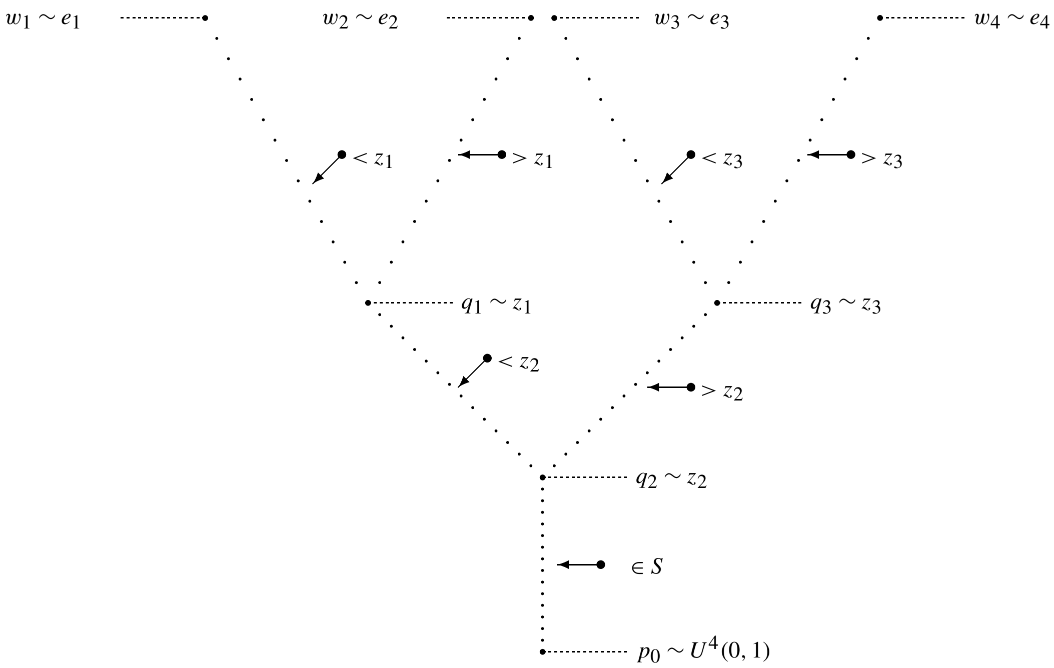


Fig. 7. A lower semi-lattice corresponding to an idempotent 4-uninorm, where $k = 2$. Here $S = [0, x_0[\cup]y_0, 1]$ if $U^4(x_0, y_0) = z_1$, $S = [0, x_0] \cup]y_0, 1]$ if $U^4(x_0, y_0) = x_0$, $U^4(z_1, y_0) = z_1$ and $S = [0, x_0] \cup [y_0, 1]$ if $U^4(x_0, y_0) = x_0$, $U^4(z_1, y_0) = y_0$.

$1, \dots, n-1$. Assume $x, y \in [0, 1]$. If $x = y$ evidently $U^n(x, x) = U^*(x, x) = x$. If $U^n(x, y) = x$ then $a \preceq b$. If $x \notin Z$ then $a \in B$ and thus $U^*(x, y) = x$. If $x \in Z$ then $a \in A$ and $x = z_i$ for some $i \in \{1, \dots, n-1\}$. Therefore $U^*(x, y) = z_i = x$ and $U^n(x, y) = U^*(x, y) = x$. Similarly, if $U^n(x, y) = y$ then $U^n(x, y) = U^*(x, y) = y$.

Finally assume that $U^2(x, y) = z_i$ for some $z_i \in]x, y[$. Then a and b are incomparable and similarly as in the proof of Lemma 4.14 we can show that q_i is the meet of a and b. Therefore $U^*(x, y) = z_i$. Thus $U^n(x, y) = U^*(x, y) = z_i$. Summarizing, $U^n(x, y) = U^*(x, y)$ for all $x, y \in [0, 1]$ (see Fig. 7 and Fig. 8). □

Proposition 4.17. *Let P be an index set isomorphic with $[0, 1]$ via the isomorphism i. For all $p \in P$ we put $X_p = \{x\}$ if $i(p) = x$. Let $e_1, \dots, e_n, z_1, \dots, z_{n-1} \in [0, 1]$, $0 = z_0 < z_1 < \dots < z_n = 1$, $e_i \in [z_{i-1}, z_i]$ for $i = 1, \dots, n$. Denote $A = \{q_1, \dots, q_{n-1}\}$, where $X_{q_i} = \{z_i\}$ for $i = 1, \dots, n-1$ and $B = P \setminus A$. Let $\preceq$ be a partial order on P such that all requirements of Theorem 4.2 are fulfilled. If $([0, 1], U^n)$ is the z-ordinal sum of $\{(X_p, \mathrm{Id})\}_{p \in P}$ with the partial order $\preceq$ then U^n is an idempotent n-uninorm with the n-neutral element $\{e_1, \dots, e_n\}_{z_1, \dots, z_{n-1}}$ if and only if the following conditions are fulfilled:*

(i) *$a_1 \prec a_2$ for all $a_1, a_2 \in P$ such that $X_{a_1} = \{x_1\}$, $X_{a_2} = \{x_2\}$, $x_1 < x_2$ and $x_1, x_2 \in [z_{i-1}, e_i]$, for $i = 1, \dots, n$.*
(ii) *$b_1 \prec b_2$ for all $b_1, b_2 \in P$ such that $X_{b_1} = \{y_1\}$, $X_{b_2} = \{y_2\}$, $y_1 > y_2$ and $y_1, y_2 \in [e_i, z_i]$ for $i = 1, \dots, n$.*
(iii) *For $a, b \in P$, $X_a = \{x\}$, $X_b = \{y\}$, are a and b incomparable if and only if there exists an $i \in \{1, \dots, n-1\}$ such that $q_i \preceq a$, $q_i \preceq b$ and $z_i \in]x, y[$, where $X_{q_i} = \{z_i\}$.*
(iv) *a_1 and a_2 are comparable for all $a_1, a_2 \in P$ such that $X_{a_1} = \{x_1\}$, $X_{a_2} = \{x_2\}$, where $(x_1, x_2) \in [z_{i-1}, z_i]^2$ for $i = 1, \dots, n$.*

Proof. The necessity follows from Proposition 4.6 and Remark 4.15.

Now we will show the sufficiency part. The associativity and the commutativity of U^n follow from Theorem 4.2. Further, U^n is evidently idempotent. From (i) it follows that if $x \in [z_{i-1}, e_i]$ for $i = 1, \dots, n$ then $U^n(x, e_i) = x$. Further, from (ii) it follows that if $x \in [e_i, z_i]$ for $i = 1, \dots, n$ then $U^n(x, e_i) = x$. Thus e_i is the neutral element of U^n on $[z_{i-1}, z_i]^2$.

To finish the proof we have to show that U^n is non-decreasing, however, first we will show two important points.

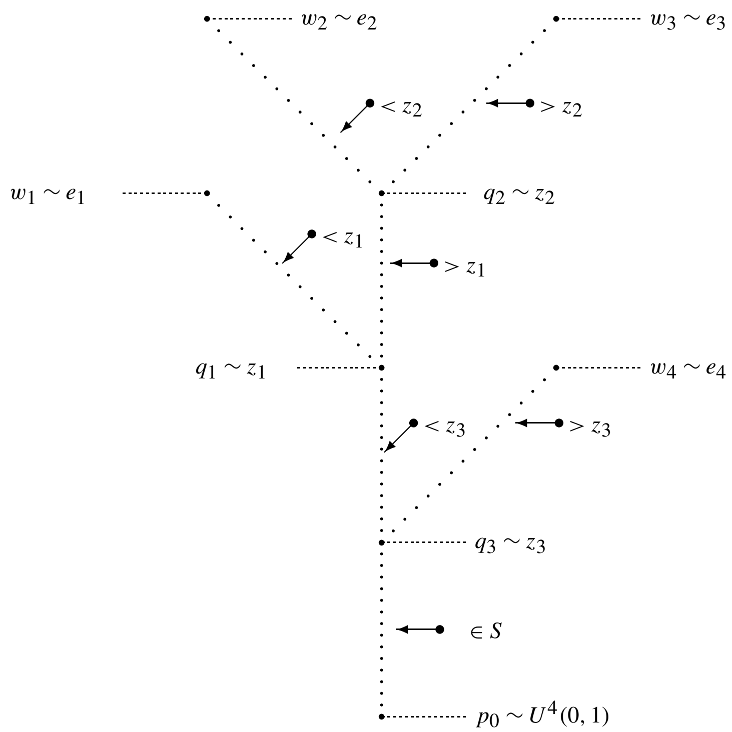


Fig. 8. A lower semi-lattice corresponding to an idempotent 4-uninorm, where $k = 3$. Here $S = [0, x_0[\cup]y_0, 1]$ if $U^4(x_0, y_0) = z_1$, $S = [0, x_0] \cup]y_0, 1]$ if $U^4(x_0, y_0) = x_0$, $U^4(z_1, y_0) = z_1$ and $S = [0, x_0] \cup [y_0, 1]$ if $U^4(x_0, y_0) = x_0$, $U^4(z_1, y_0) = y_0$.

Point 1: If $q_i \preceq a$ and $q_i \preceq b$, for some $X_a = \{x\}$, $X_b = \{y\}$, $x < y$, $i \in \{1, \dots, n-1\}$ and $z_i \in]x, y[$ then $U^n(x, y) = z_i$.
Point 2: $U^n(e_1, e_n) = z_k$ for some $k \in \{1, \dots, n-1\}$.

To show Point 1 first observe that if $q_i \preceq a$ and $q_i \preceq b$ for some $i \in \{1, \dots, n-1\}$ and $z_i \in]x, y[$ then (iii) implies that a and b are incomparable. Suppose that $a \wedge b = q_j$, where $j \neq i$, i.e., $U^n(x, y) = z_j$. Then either $z_j < z_i$, or $z_j > z_i$. We will assume that $z_j < z_i$ (the case when $z_j > z_i$ is analogous). Then $q_i \prec q_j$ and $z_i \in]z_j, y[$, i.e., by (iii) q_j and y are incomparable, which is a contradiction. Thus $U^n(x, y) = z_i$.

To show Point 2 first observe that $q_1 \wedge \cdots \wedge q_{n-1} \in A$ and thus

$$q_1 \wedge \cdots \wedge q_{n-1} = q_k$$

for some $k \in \{1, \dots, n-1\}$. Then q_k is comparable with all $p \in P$. Indeed, if q_k is incomparable with some $p \in P$ then $p \wedge q_k = q_i$ for some $i \in \{1, \dots, n-1\}$, however, since $q_k \wedge q_i = q_k$ we get $q_k \preceq p$, which is a contradiction. Thus $U^n(x, z_k) \in \{x, z_k\}$ for all $x \in [0, 1]$. Since $q_k \preceq q_1 \preceq w_1$ and $q_k \preceq q_{n-1} \preceq w_n$ we have $U^n(e_1, z_k) = z_k$, $U^n(e_n, z_k) = z_k$. If $z_k \in]e_1, e_n[$ we get $U^n(e_1, e_n) = z_k$. If $z_k = e_1$ ($z_k = e_n$) then $w_1 \preceq w_n$ ($w_n \preceq w_1$) and $U^n(e_1, e_n) = e_1 = z_k$ ($U^n(e_1, e_n) = e_n = z_k$) as in the previous case.

Now we are ready to show that U^n is non-decreasing. The proof will be done by induction. Since for 2-uninorms and uninorms (see Propositions 4.12 and 4.6) the claim holds we will suppose that the claim of this proposition holds for all $(n-1)$-uninorms. Now we will show that it holds also for all n-uninorms. Due to the induction assumption U^n is non-decreasing on $[0, z_{n-1}]^2$ as well as on $[z_1, 1]^2$. Thus we only have to check the monotonicity on $[0, z_1] \times [z_{n-1}, 1]$ and on $[z_{n-1}, 1] \times [0, z_1]$. We will take $X_a = \{x\}$, $X_{b_1} = \{y_1\}$ and $X_{b_2} = \{y_2\}$ and we will assume $x \in [0, z_1]$ (the case when $x \in [z_{n-1}, 1]$ is analogous). Then it is enough to check the monotonicity for $y_1, y_2 \in [z_{n-1}, 1]$, $y_1 < y_2$. If $x = z_1$ then the monotonicity follows from the induction assumption. Thus we will suppose that $x < z_1$. Further, if $y_2 = z_{n-1}$ then also $y_1 = z_{n-1}$ and therefore we will suppose that $y_2 > z_{n-1}$. Note that the conditions (i) and (ii) imply $a \preceq w_1$ and $b_1 \preceq w_n$, $b_2 \preceq w_n$ and thus from the monotonicity of the meet we know that $a \wedge b_1 \preceq w_1 \wedge w_n = q_k$ and similarly $a \wedge b_2 \preceq q_k$. Therefore if a and b_1 (b_2) are incomparable then their meet is equal to q_k. From (ii) we know that $b_2 \preceq b_1$. Now we have the following 6 cases:

Case 1: If $a \preceq b_2 \preceq b_1$. Since $x \neq z_1$ we get

$$U^n(x, y_1) = x = U^n(x, y_2).$$

Case 2: If $b_2 \preceq a \preceq b_1$. Since $y_2 \neq z_{n-1}$ and $x \neq z_1$ we have

$$U^n(x, y_1) = x < y_2 = U^n(x, y_2).$$

Case 3: If $b_2 \preceq b_1 \preceq a$. We have $U^n(x, y_1) = y_1$ in both cases, when $y_1 = z_{n-1}$ and when $y_1 > z_{n-1}$ and thus we have

$$U^n(x, y_1) = y_1 < y_2 = U^n(x, y_2).$$

Case 4: If a is incomparable with b_1 and $b_2 \preceq a$ we get

$$U^n(x, y_1) = z_k < y_2 = U^n(x, y_2).$$

Case 5: If a is incomparable with b_2 and $a \preceq b_1$ we get

$$U^n(x, y_1) = x < z_k = U^n(x, y_2).$$

Case 6: If a and b_1 are incomparable and a and b_2 are incomparable we get

$$U^n(x, y_1) = z_k = U^n(x, y_2).$$

Thus in all cases $U^n(x, y_1) \leq U^n(x, y_2)$. □

If we summarize the results obtained in this section we will see that the partial order on P induced by an idempotent n-uninorm resembles a tree, where $q_i \sim z_i$ for $i = 1, \ldots, n-1$ are nodes of this tree. More precisely, if p_1 and p_2 are incomparable for $p_1, p_2 \in P$ then p_1 and p_2 have no join (the least upper bound).

5. Conclusions

We have described the structure of idempotent 2-uninorms and n-uninorms showing that each idempotent 2-uninorm (n-uninorm) can be expressed as an ordinal sum of an idempotent uninorm (possibly also of a countable number of idempotent semigroups with operations min and max) and an idempotent 2-uninorm (n-uninorm) such that $U^2(0, 1) = z_1$ ($U^n(0, 1) = z_k$), possibly restricted to $]0, 1[^2$, or $[0, 1[^2$, or $]0, 1]^2$.

Further, we have shown that idempotent n-uninorms are in one-to-one correspondence with special lower semi-lattices defined on the unit interval, where these semi-lattices have a tree-like structure. We have also defined the z-ordinal sum construction for partially ordered semigroups. This construction extends the ordinal sum construction of Clifford and our aim is to show that similarly as continuous t-norms (t-conorms) can be expressed as an ordinal sum of continuous Archimedean t-norms (t-conorms) also n-uninorms with continuous underlying functions can be decomposed to Archimedean and idempotent semigroups using the z-ordinal sum construction.

As we mentioned in the introduction, in the future work we would like to introduce the notion of the characterizing function defined for uninorms with continuous underlying functions also for n-uninorms, showing that each n-uninorm with continuous underlying functions possesses n characterizing functions and that graphs of these characterizing functions cover all points of discontinuity of such an n-uninorm. Finally, we would like to show that each n-uninorm with continuous underlying functions is a z-ordinal sum of a countable number of semigroups related to representable uninorms, continuous Archimedean t-norms, continuous Archimedean t-conorms and internal uninorms (including the min and the max operator).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Andrea Mesiarová-Zemánková

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The n-uninorms with continuous underlying t-norms and t-conorms

Andrea Mesiarová-Zemánková

Mathematical Institute, Slovak Academy of Sciences, Bratislava, Slovakia

ABSTRACT

The n-uninorms with continuous underlying t-norms and t-conorms are studied. We show that each 2-uninorm with continuous underlying functions can be expressed as an ordinal sum of a uninorm with continuous underlying functions (possibly also of a countable number of semigroups related to representable uninorms, continuous Archimedean t-norms, continuous Archimedean t-conorms and internal uninorms) and a 2-uninorm with continuous underlying functions such that $U^2(0,1) = z_1$ (possibly restricted to open or half-open unit square $[0,1[^2$, $]0,1]^2$). Similar results are shown for n-uninorms with continuous underlying functions, where $n \in \mathbb{N}$, $n > 2$.

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n-uninorm; uninorm; ordinal sum; t-norm; t-conorm

1. Introduction

Triangular norms and conorms (see Klement, Mesiar, and Pap 2000; Alsina, Frank, and Schweizer 2006) represent the prominent classes of associative aggregation functions on the unit interval which are, due to their nice properties, used in many applications and many theoretical studies. The class of continuous t-norms was completely characterized using the ordinal sum construction and additive generators. Since t-conorms are dual operations to t-norms the characterization of continuous t-conorms is straightforward. When focusing on the neutral element of an aggregation function, t-norms and t-conorms can be generalized into uninorms capable of representing bipolar aggregation (see Yager and Rybalov 1996; Fodor, Yager, and Rybalov 1997; Mesiarová-Zemánková 2015). If we focus on the annihilator, t-norms and t-conorms can be generalized into nullnorms (also called t-operators) (Mas, Mayor, and Torrens 1999; Calvo, De Baets, and Fodor 2001). The above generalizations bring together t-norms and t-conorms. In the second step a notion that brings together uninorms and nullnorms was introduced by Akella (2007). These special aggregation functions are called n-uninorms and each n-uninorm possesses n local neutral elements. The basic structure of n-uninorms was described in Akella (2007). Further, 5 possible classes, defined by values of $U^2(0,1)$, $U^2(0,z_1)$ and $U^2(z_1,1)$ were characterized in Zong et al. (2018). If for a 2-uninorm there is $e_2 = 1$ we obtain a uni-nullnorm and if $e_1 = 0$ we obtain a null-uninorm (Sun, Wang, and Qu 2017). The migrativity and distributivity of uni-nullnorms were studied in Sun, Qu, and Zhu (2019); Wang, Qin, and

CONTACT Andrea Mesiarová-Zemánková zemankova@mat.savba.sk, vicandy27@yahoo.ca

Li (2019). Moreover, uni-nullnorms with continuous Archimedean underlying functions were characterized in Sun, Wang, and Qu (2018).

In this paper we focus on n-uninorms with continuous underlying functions. The idempotent n-uninorms were fully characterized in Mesiarová-Zemánková (forthcoming) and we would like to obtain similar results for all n-uninorms with continuous underlying functions. In the next papers we would like to study characterizing functions of such n-uninorms and afterwards describe their decomposition into irreducible semigroups.

The paper is structured as follows. In the following section we recall all necessary basic notions and results. Section 3 gives basic results on 2-uninorms with continuous underlying functions which will be used in the following section. In Section 4 we will study the structure of 2-uninorms and n-uninorms are examined in Section 5. We give our conclusions in Section 6.

2. Basic notions

A triangular norm is a binary function $T\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and 1 is its neutral element. Due to the associativity, n-ary form of any t-norm is uniquely given and thus it can be extended to an aggregation function working on $\bigcup_{n\in\mathbb{N}}[0,1]^n$. Dual functions to t-norms are t-conorms. A triangular conorm is a binary function $S\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and 0 is its neutral element. The duality between t-norms and t-conorms is expressed by the fact that from any t-norm T we can obtain its dual t-conorm S by the equation

$$S(x,y) = 1 - T(1-x, 1-y)$$

and vice-versa.

Each continuous t-norm (t-conorm) is equal to an ordinal sum of continuous Archimedean t-norms (t-conorms). Note that a continuous t-norm (t-conorm) is Archimedean if and only if it has only trivial idempotent points 0 and 1. A continuous Archimedean t-norm T (t-conorm S) is either strict, i.e. strictly increasing on $]0,1]^2$ (on $[0,1[^2$), or nilpotent, i.e. there exists $(x,y) \in]0,1[^2$ such that $T(x,y) = 0$ ($S(x,y) = 1$). Moreover, each continuous Archimedean t-norm (t-conorm) has a continuous additive generator, which is uniquely determined up to a positive multiplicative constant. More details on t-norms and t-conorms can be found in Klement, Mesiar, and Pap (2000); Alsina, Frank, and Schweizer (2006).

A uninorm (introduced in Yager and Rybalov 1996) is a binary function $U\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and have the neutral element $e \in [0,1]$ (see also Fodor, Yager, and Rybalov 1997). Evidently, if $e = 1$ ($e = 0$) then we retrieve a t-norm (t-conorm).

Due to the monotonicity, for each uninorm the value $U(1,0) \in \{0,1\}$ is the annihilator of U. A uninorm is called *conjunctive* (*disjunctive*) if $U(1,0) = 0$ ($U(1,0) = 1$).

For each uninorm U with the neutral element $e \in]0,1[$, the restriction of U to $[0,e]^2$ is a t-norm on $[0,e]^2$, i.e.a linear transformation of some t-norm T_U on $[0,1]^2$ and the restriction of U to $[e,1]^2$ is a t-conorm on $[e,1]^2$, i.e. a linear transformation of some t-conorm S_U. Moreover, $\min(x,y) \le U(x,y) \le \max(x,y)$ for all $(x,y) \in [0,e] \times [e,1] \cup [e,1] \times [0,e]$.

Similarly as in the case of t-norms and t-conorms we can construct uninorms using additive generators (see Fodor, Yager, and Rybalov 1997). A uninorm which possesses an additive generator is called representable. In Ruiz and Torrens 2006 (see also Mesiarová-Zemánková 2015) it was shown that U is representable if and only if it is continuous on $[0,1]^2 \setminus \{(0,1),(1,0)\}$. This result completely characterizes the set of representable uninorms.

Definition 2.1: A uninorm $U\colon [0,1]^2 \longrightarrow [0,1]$ is called *internal* if $U(x,y) \in \{x,y\}$ for all $(x,y) \in [0,1]^2$. Further, U is called locally internal on $A(e)$ if U is internal on $A(e) = [0,e] \times [e,1] \cup [e,1] \times [0,e]$.

Several results on internal and locally internal uninorms can be found in Czogala and Drewniak (1984), De Baets (1998), Martín, Mayor, and Torrens (2003), Ruiz-Aguilera et al. (2010) and Drygas, Ruiz-Aguilera, and Torrens (2016). Note that if a uninorm U is internal then it is also idempotent, i.e. $U(x,x) = x$ for all $x \in [0,1]$ and vice versa.

Ordinal sums of uninorms were defined in Mesiarová-Zemánková (2016). Uninorms with continuous underlying functions were completely characterized in Mesiarová-Zemánková (2017, 2018). In Mesiarová-Zemánková (2017) it was shown that each uninorm with continuous underlying functions can be decomposed into an ordinal sum of a countable number of semigroups related to representable uninorms, continuous Archimedean t-norms, continuous Archimedean t-conorms and internal uninorms (including the min and the max operator). In Mesiarová-Zemánková (2018) it was shown that the set of all points of discontinuity of a uninorm with continuous underlying functions is a subset of the graph of the characterizing function of such a uninorm.

The following is the definition of a nullnorm (Calvo, De Baets, and Fodor 2001). Note that t-operators were independently defined in Mas, Mayor, and Torrens (1999) and in Mas, Mayor, and Torrens (2002) it was shown that t-operators and nullnorms coincide.

Definition 2.2: A binary function $V\colon [0,1]^2 \longrightarrow [0,1]$ is called a nullnorm if it is commutative, associative, non-decreasing in each variable and has an annihilator $z \in [0,1]$ such that $V(0,x) = x$ for all $x \le z$ and $V(1,x) = x$ for all $x \ge z$.

If $z = 0$ ($z = 1$) then V is a t-norm (t-conorm). Note that for a commutative, associative and non-decreasing function $F\colon [0,1]^2 \longrightarrow [0,1]$ with $F(0,0) = 0$, $F(1,1) = 1$, the value $F(0,1)$ is always an annihilator of F. Thus for a nullnorm $z = V(0,1)$. In Calvo, De Baets, and Fodor (2001) the following result was shown.

Theorem 2.3: *Let $z \in]0,1[$. Then $V\colon [0,1]^2 \longrightarrow [0,1]$ is a nullnorm with the annihilator z if and only if there exists a t-norm T_V and a t-conorm S_V such that*

$$V(x,y) = \begin{cases} z \cdot S_V(\frac{x}{z}, \frac{y}{z}) & \text{if } x,y \in [0,z]^2\,, \\ z + (1-z) \cdot T_V\left(\dfrac{x-z}{1-z}, \dfrac{y-z}{1-z}\right) & \text{if } x,y \in [z,1]^2\,, \\ z & \text{otherwise.} \end{cases}$$

Now let us recall the definition of an n-uninorm (see Akella 2007).

Definition 2.4: Assume an $n \in \mathbb{N} \setminus \{1\}$. Let $V\colon [0,1]^2 \longrightarrow [0,1]$ be a commutative binary function. Then $\{e_1,\dots,e_n\}_{z_1,\dots,z_{n-1}}$ is called an n-neutral element of V if for $0 = z_0 < z_1 < \cdots < z_n = 1$ and $e_i \in [z_{i-1}, z_i]$, $i = 1,\dots,n$ we have $V(e_i, x) = x$ for all $x \in [z_{i-1}, z_i]$.

Definition 2.5: A binary function $U^n\colon [0,1]^2 \longrightarrow [0,1]$ is an n-uninorm if it is associative, non-decreasing in each variable, commutative and has an n-neutral element $\{e_1,\dots,e_n\}_{z_1,\dots,z_{n-1}}$.

The basic structure of n-uninorms was described in Akella (2007) and the characterization of the main five classes of 2-uninorms was given in Zong et al. (2018). Now we will recall these five exhaustive and mutually exclusive classes:

- Class 1: 2-uninorms with $U^2(0,1) = z_1$.
- Class 2a: 2-uninorms with $U^2(0,1) = 0$, $U^2(1,z_1) = z_1$.
- Class 2b: 2-uninorms with $U^2(0,1) = 1$, $U^2(0,z_1) = z_1$.
- Class 3a: 2-uninorms with $U^2(0,1) = 0$, $U^2(1,z_1) = 1$.
- Class 3b: 2-uninorms with $U^2(0,1) = 1$, $U^2(0,z_1) = 0$.

Each n-uninorm has the following building blocks around the main diagonal.

Proposition 2.6: *Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be an n-uninorm with the n-neutral element $\{e_1,\dots,e_n\}_{z_1,\dots,z_{n-1}}$. Then*

(i) *U^n restricted to $[z_{i-1}, e_i]^2$, for $i = 1,\dots,n$, is isomorphic to a t-norm. We will denote this t-norm by T_i.*
(ii) *U^n restricted to $[e_i, z_i]^2$ for $i = 1,\dots,n$, is isomorphic to a t-conorm. We will denote this t-conorm by S_i.*
(iii) *U^n restricted to $[z_{i-1}, z_i]^2$ for $i = 1,\dots,n$, is isomorphic to a uninorm. We will denote this uninorm by U_i.*
(iv) *U^n restricted to $\left[z_i, z_j\right]^2$ for $i, j \in \{0,1,\dots,n\}$, $i < j$, is isomorphic to a $(j-i)$-uninorm.*

For $n \in \mathbb{N}$ we will denote the set of all n-uninorms U^n such that $T_1,\dots,T_n$ and $S_1,\dots,S_n$ are continuous by $\mathcal{U}_n$.

Since we will work in this paper with ordinal sums of semigroups we recall a fundamental result of Clifford (1954).

Theorem 2.7: *Let $A \neq \emptyset$ be a totally ordered set and $(G_\alpha)_{\alpha\in A}$ with $G_\alpha = (X_\alpha, *_\alpha)$ be a family of semigroups. Assume that for all $\alpha, \beta \in A$ with $\alpha < \beta$ the sets X_α and X_β are either disjoint or that $X_\alpha \cap X_\beta = \{x_{\alpha,\beta}\}$, where $x_{\alpha,\beta}$ is both the neutral element of G_α and the annihilator of G_β and where for each $\gamma \in A$ with $\alpha < \gamma < \beta$ we have $X_\gamma = \{x_{\alpha,\beta}\}$. Put $X = \bigcup_{\alpha\in A} X_\alpha$ and define the binary operation $*$ on X by*

$$x * y = \begin{cases} x *_\alpha y & \text{if } (x,y) \in X_\alpha \times X_\alpha, \\ x & \text{if } (x,y) \in X_\alpha \times X_\beta \text{ and } \alpha < \beta, \\ y & \text{if } (x,y) \in X_\alpha \times X_\beta \text{ and } \alpha > \beta. \end{cases}$$

Then $G = (X, *)$ is a semigroup. The semigroup G is commutative if and only if for each $\alpha \in A$ the semigroup G_α is commutative.

Remark 2.8: As we see in the previous theorem the ordinal sum construction assumes that the index set A is totally (linearly) ordered and therefore if we say that semigroups G_α for $\alpha \in A$ are ordered we are speaking about the order defined on the index set A. Then, if for some $\alpha, \beta \in A$ we have $X_\alpha \neq X_\beta$ and $x * y = x$ for all $x \in X_\alpha$ and all $y \in X_\beta$, necessarily $\alpha < \beta$.

Vice versa, assume a commutative, associative function $F\colon [0,1]^2 \longrightarrow [0,1]$, an index set A and semigroups $(G_\alpha)_{\alpha\in A}$ with $G_\alpha = (X_\alpha, F|_{X_\alpha})$, where $F|_{X_\alpha}\colon X_\alpha^2 \longrightarrow X_\alpha$ is the restriction of F to X_α, such that $[0,1] = \bigcup_{\alpha\in A} X_\alpha$, and for $\alpha, \beta \in A$ the sets X_α and X_β are either disjoint, or $X_\alpha \cap X_\beta = \{x_{\alpha,\beta}\}$, and $X_\alpha \neq X_\beta$ whenever $\alpha \neq \beta$. We define a partial order on A by $\alpha \leq_A \beta$ for $\alpha, \beta \in A$ if either $\alpha = \beta$, or $F(x,y) = x$ for all $x \in X_\alpha$ and all $y \in X_\beta$. Then $\leq_A$ is evidently reflexive, the antisymmetry of $\leq_A$ follows from the commutativity of F and the fact that $X_\alpha \neq X_\beta$ whenever $\alpha \neq \beta$, and the transitivity of $\leq_A$ follows from the associativity of F. In the case when $\leq_A$ is a total (linear) order it is easy to check that $([0,1], F)$ is an ordinal sum of $(G_\alpha)_{\alpha\in A}$ with respect to order $\leq_A$.

Therefore, in order to show that F is an ordinal sum of semigroups $(G_\alpha)_{\alpha\in A}$ it is enough to show that these semigroups are totally ordered by the order $\leq_A$ defined above.

Note that when we speak about ordinal sums of t-norms (t-conorms) we work either with trivial semigroups, or with semigroups acting on subintervals $[a_i, b_i]$ of the unit interval, $a_i < b_i$, with operations which are equal to t-norms (t-conorms) linearly transformed to $[a_i, b_i]$. In the case of uninorms we use uninorms transformed to the set $[a_i, b_i[\cup \{v_i\} \cup]c_i, d_i]$ by a piece-wise linear, increasing isomorphism.

Therefore, if we will speak about linear transformation from interval $[a,b]$ to interval $[c,d]$ we mean a linear function $\varphi\colon [a,b] \longrightarrow [c,d]$ given by

$$\varphi(x) = \frac{(x-a)\cdot(d-c)}{b-a} + c,$$

which transforms a unary function $f\colon [a,b] \longrightarrow [a,b]$ to a function $g\colon [c,d] \longrightarrow [c,d]$ given by $g(x) = \varphi(f(\varphi^{-1}(x)))$, and transforms a binary function $V\colon [a,b]^2 \longrightarrow [a,b]$ to a function $U\colon [c,d]^2 \longrightarrow [c,d]$ given by $U(x,y) = \varphi(V(\varphi^{-1}(x), \varphi^{-1}(y)))$. Further, for any $0 \leq a < b < c < d \leq 1$, $v \in [b,c]$ and a uninorm $U\colon [0,1]^2 \longrightarrow [0,1]$ with the neutral element $e \in]0,1[$ we use the transformation $f\colon [0,1] \longrightarrow [a,b[\cup \{v\} \cup]c,d]$, given by

$$f(x) = \begin{cases} (b-a)\cdot \dfrac{x}{e} + a & \text{if } x \in [0,e[\,, \\ v & \text{if } x = e, \\ d - \dfrac{(1-x)(d-c)}{(1-e)} & \text{otherwise.} \end{cases} \tag{1}$$

Then f is linear on $[0,e[$ and on $]e,1]$ and thus it is a piece-wise linear isomorphism of $[0,1]$ to $([a,b[\cup \{v\} \cup]c,d])$ and the binary function $U_v^{a,b,c,d}\colon ([a,b[\cup \{v\} \cup]c,d])^2 \longrightarrow$

$([a,b[\cup\{v\}\cup]c,d])$ given by

$$U_v^{a,b,c,d}(x,y)=f(U(f^{-1}(x),f^{-1}(y))) \tag{2}$$

is a uninorm on $([a,b[\cup\{v\}\cup]c,d])^2$. The backward transformation f^{-1} then transforms a uninorm defined on $([a,b[\cup\{v\}\cup]c,d])^2$ to a uninorm defined on $[0,1]^2$.

For the rest of the paper if we say that two semigroups (X_1,F_1) and (X_2,F_2) are isomorphic we assume that there exists an increasing isomorphism $\varphi\colon X_1\longrightarrow X_2$ such that $F_1(x,y)=\varphi^{-1}(F_2(\varphi(x),\varphi(y)))$ for all $x,y\in X_1$. Note that such an isomorphism preserves the commutativity, the associativity, the monotonicity, the (local) neutral element and the annihilator, as well.

Since we will use ordinal sums of trivial semigroups, let us recall that there exists only one operation on a trivial semigroup, namely the function $\mathrm{Id}\colon\{x\}^2\longrightarrow\{x\}$, which is simply defined by $\mathrm{Id}(x,x)=x$.

3. Basic results on 2-uninorms with continuous underlying functions

Let us settle for this paper that if we say that a function is an n-uninorm we will suppose that it possesses the n-neutral element $\{e_1,\dots,e_n\}_{z_1,\dots,z_{n-1}}$. In this section we will focus on 2-uninorms $U^2\colon[0,1]^2\longrightarrow[0,1]$ such that T_1,T_2 and S_1,S_2 are continuous. First we introduce several useful examples of 2-uninorms related to a 2-uninorm from Class 1, i.e. such that $U^2(0,1)=z_1$. Afterwards we will describe the structure of 2-uninorms with continuous underlying functions.

Example 3.1: Uninorms with nilpotent (strict) underlying t-norm and t-conorm were described in Fodor and De Baets (2012) and Li, Liu, and Fodor (2014). In (Mesiarová-Zemánková 2017, Example 1,Example 2) it was shown that these uninorms are in fact ordinal sums of semigroups defined on $[0,e[$, $\{e\}$ and $]e,1]$ in the nilpotent case and on $\{0\}$, $]0,e[$, $\{e\}$, $]e,1[$, $\{1\}$ in the strict case. The order of these semigroups then determines the class of the corresponding uninorm. A similar construction can be used to construct special examples of 2-uninorms related to a 2-uninorm U^2 from the Class 1 which will be used later for the characterization of 2-uninorms with continuous underlying functions. Assume a 2-uninorm $U_1^2\colon[0,1]^2\longrightarrow[0,1]$, $U_1^2\in\mathcal{U}_2$, such that $U_1^2(1,0)=z_1$. Then U_1^2 is isomorphic to a uninorm with continuous underlying functions on $[0,z_1]^2$ ($[z_1,1]^2$) and $U^2(x,y)=z_1$ for all $x,y\in[0,1]$ such that $z_1\in\left[x,y\right]$.

In the case when T_1 has no zero divisors then U_1^2 is closed on $]0,1]^2$, i.e. $U_1^2(x,y)\in\,]0,1]^2$ for all $x,y\in\,]0,1]^2$, and we can define a function $U_2^2\colon[0,1]^2\longrightarrow[0,1]$ by

$$U_2^2(x,y)=\begin{cases}U_1^2(x,y) & \text{if } x,y\in\,]0,1],\\ 0 & \text{otherwise.}\end{cases}$$

Then U_2^2 is an ordinal sum of two semigroups $G_1=(]0,1],U_1^2)$ and $G_2=(0,\mathrm{Id})$, where the order of semigroups in the ordinal sum construction is $2<1$. Therefore U_2^2 is associative and commutative. The monotonicity and the 2-neutral element of U_2^2 are easily verified and thus we see that U_2^2 is a 2-uninorm.

Similarly, if S_2 has no divisors of 1 then U_1^2 is closed on $[0,1[^2$ and we can define a function $U_3^2\colon [0,1]^2 \longrightarrow [0,1]$ by

$$U_3^2(x,y) = \begin{cases} U_1^2(x,y) & \text{if } x,y \in [0,1[\,, \\ 1 & \text{otherwise.} \end{cases}$$

Then U_3^2 is an ordinal sum of two semigroups $G_1 = ([0,1[\,, U_1^2)$ and $G_2 = (1, \mathrm{Id})$, where the order of semigroups in the ordinal sum construction is $2 < 1$. Therefore U_3^2 is associative and commutative. The monotonicity and the 2-neutral element of U_3^2 are easily verified and thus we see that U_3^2 is a 2-uninorm.

Finally, if both T_1 has no zero divisors and S_2 has no divisors of 1 then we can define two 2-uninorms given as ordinal sums of $G_1 = (]0,1[\,, U_1^2)$, $G_2 = (0, \mathrm{Id})$ and $G_3 = (1, \mathrm{Id})$. To keep the monotonicity for the order of the semigroups we need to have $2 < 1$ and $3 < 1$, however, there are two possibilities: either $2 < 3$, or $3 < 2$. Then we obtain the following two functions

$$U_4^2(x,y) = \begin{cases} U_1^2(x,y) & \text{if } x,y \in]0,1[\,, \\ 0 & \text{if } \min(x,y) = 0, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$U_5^2(x,y) = \begin{cases} U_1^2(x,y) & \text{if } x,y \in]0,1[\,, \\ 1 & \text{if } \max(x,y) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly as in the previous cases we can easily check that U_4^2 and U_5^2 are 2-uninorms which possess the same 2-neutral element as U_1^2. Observe that U_2^2 coincides with U_1^2 on $]0,1]^2$, U_3^2 coincides with U_1^2 on $[0,1[^2$, and U_4^2, U_5^2 coincide with U_1^2 on $]0,1[^2$.

For any $U^2 \in \mathcal{U}_2$ we know that U^2 is isomorphic to a uninorm with continuous underlying functions on $[0,z_1]^2$ ($[z_1,1]^2$). In Mesiarová-Zemánková (2018) the following result was shown.

Lemma 3.2: *Let* $U\colon [0,1]^2 \longrightarrow [0,1]$ *be a uninorm with continuous underlying functions. If* $a \in [0,1]$ *is an idempotent point of* U *then* $U(a,x) \in \{x,a\}$ *for all* $x \in [0,1]$.

A similar result can be shown also for idempotent points of a 2-uninorm $U^2 \in \mathcal{U}_2$, however, here $U^2(a,x) \in \{x,a,z_1\}$ for all $x \in [0,1]$.

Lemma 3.3: *Let* $U^2\colon [0,1]^2 \longrightarrow [0,1]$ *be a 2-uninorm and let* $U^2 \in \mathcal{U}_2$. *If* $a \in [0,1]$ *is an idempotent point of* U^2 *then* $U^2(a,x) \in \{x,a,z_1\}$ *for all* $x \in [0,1]$.

Proof: Let $a \in [0,1]$ be an idempotent point of U^2. If $a = z_1$ then $U^2(z_1,x) \in \{z_1,x\}$ for all $x \in [0,z_1] \cup [z_1,1] = [0,1]$ follows from Lemma 3.2. Further we will assume that $a < z_1$ as the case when $a > z_1$ is analogous. Due to Lemma 3.2 there is $U^2(a,x) \in \{a,x\}$ for all $x \in [0,z_1]$. Assume any $y \in [z_1,1]$. Then $U^2(a,z_1) \in \{a,z_1\}$ and $U^2(y,z_1) \in \{y,z_1\}$. Therefore there are the following four possibilities:

(i) If $U^2(a, z_1) = z_1$ and $U^2(y, z_1) = z_1$. Then the monotonicity of U^2 gives us

$$z_1 = U^2(a, z_1) \le U^2(a, y) \le U^2(z_1, y) = z_1$$

and therefore $U^2(a, y) = z_1$.

(ii) If $U^2(a, z_1) = z_1$ and $U^2(y, z_1) = y$. Then

$$U^2(a, y) = U^2(a, U^2(z_1, y)) = U^2(U^2(a, z_1), y) = U^2(z_1, y) = y.$$

(iii) If $U^2(a, z_1) = a$ and $U^2(y, z_1) = z_1$. Then

$$U^2(a, y) = U^2(U^2(a, z_1), y) = U^2(a, U^2(z_1, y)) = U^2(a, z_1) = a.$$

(iv) If $U^2(a, z_1) = a$ and $U^2(y, z_1) = y$. First suppose that $U^2(a, y) \le e_2$. Then

$$a = U^2(a, z_1) = U^2(a, U^2(e_2, z_1)) = U^2(e_2, U^2(a, z_1)) = U^2(e_2, a)$$

and therefore

$$a = U^2(a, z_1) \le U^2(a, y) = U^2(U^2(a, a), y) = U^2(a, U^2(a, y)) \le U^2(a, e_2) = a,$$

i.e. $U^2(a, y) = a$.

Now suppose that $U^2(a, y) > e_2$. Since $U^2(a, y) \le U^2(z_1, y) = y$ there is $U^2(a, y) \le y$. Suppose that $U^2(a, y) < y$. Then since S_2 is continuous there exists a $y_1 \in [e_2, 1]$ such that $U^2(y_1, U^2(a, y)) = y$ and we get

$$\begin{aligned} U^2(a, y) &= U^2(a, U^2(U^2(a, y), y_1)) = U^2(U^2(a, a), U^2(y, y_1)) \\ &= U^2(U^2(a, y), y_1) = y. \end{aligned}$$

Summarizing the four cases, for any $y \in [z_1, 1]$ we get $U^2(a, y) \in \{a, y, z_1\}$.

Therefore $U^2(a, x) \in \{a, x, z_1\}$ for all $x \in [0, 1]$. ∎

Remark 3.4: Since for each $x \in [0, z_1]$ ($x \in [z_1, 1]$) there is $U^2(x, z_1) \in \{x, z_1\}$ the monotonicity of U^2 implies that there exists an $x_0 \in [0, e_1]$ and a $y_0 \in [e_2, 1]$ such that $U^2(x, z_1) = x$ for all $x < x_0$ and $U^2(x, z_1) = z_1$ for all $x_0 < x \le z_1$, and $U^2(y, z_1) = y$ for all $y > y_0$ and $U^2(y, z_1) = z_1$ for all $z_1 \le y < y_0$. Note that if $x_0 < e_1$ ($y_0 > e_2$) then there are possible both cases $U^2(x_0, z_1) = x_0$ as well as $U^2(x_0, z_1) = z_1$ ($U^2(y_0, z_1) = y_0$ as well as $U^2(y_0, z_1) = z_1$).

Further, it is clear that x_0 and y_0 are idempotent points. Otherwise, if x_0 is not an idempotent point of U^2 (and similarly for y_0) due to the continuity of T_1 there exists an $x_1 \in [0, e_1]$ such that $x_1 > x_0$ and $U^2(x_1, x_1) < x_0$. Then we get

$$z_1 = U^2(x_1, z_1) = U^2(x_1, U^2(x_1, z_1)) = U^2(U^2(x_1, x_1), z_1) = U^2(x_1, x_1) < x_0,$$

which is a contradiction.

Note that if for a 2-uninorm U^2 there is $x_0 = 0$, $y_0 = 1$ and $U^2(x_0, y_0) = z_1$ we obtain just the 2-uninorm U_1^2 from Example 3.1.

For the rest of this section we assume that x_0 and y_0 are defined as in Remark 3.4.

Since x_0 (y_0) is an idempotent point, U^2 restricted to $[x_0, e_1]^2$ ($[e_2, y_0]^2$) is isomorphic a t-norm T_0 (t-conorm S_0). The following result describes the case when T_0 has zero divisors (S_0 has divisors of 1).

Lemma 3.5: *Let $U^2\colon [0,1]^2 \longrightarrow [0,1]$ be a 2-uninorm and let $U^2 \in \mathcal{U}_2$. If there exist $x_1, x_2 \in \,]x_0, 1]$ such that $U^2(x_1, x_2) = x_0$ then $U^2(x_0, z_1) = z_1$. Similarly, if there exist $y_1, y_2 \in [0, y_0[$ such that $U^2(y_1, y_2) = y_0$ then $U^2(y_0, z_1) = z_1$.*

Proof: We will show only the first part, as the second is analogous. If there exist $x_1, x_2 \in \,]x_0, 1]$ such that $U^2(x_1, x_2) = x_0$ then evidently $x_1 \le z_1$ and $x_2 \le z_1$ and

$$U^2(x_0, z_1) = U^2(U^2(x_1, x_2), z_1) = U^2(x_1, U^2(x_2, z_1)) = U^2(x_1, z_1) = z_1.$$

■

Remark 3.6: Lemma 3.5 shows that if there exist $x_1, x_2 \in \,]x_0, 1]$ such that $U^2(x_1, x_2) = x_0$ and $y_1, y_2 \in [0, y_0[$ such that $U^2(y_1, y_2) = y_0$ then U^2 on $[x_0, y_0]^2$ is isomorphic to the 2-uninorm U_1^2 from Example 3.1. If there exist $x_1, x_2 \in \,]x_0, 1]$ such that $U^2(x_1, x_2) = x_0$ and U^2 is closed on $[0, y_0[^2$ then U^2 on $[x_0, y_0]^2$ is isomorphic to U_1^2 in the case when $U^2(z_1, y_0) = z_1$ and it is isomorphic to U_3^2 in the case when $U^2(z_1, y_0) = y_0$. If there exist $y_1, y_2 \in [0, y_0[$ such that $U^2(y_1, y_2) = y_0$ and U^2 is closed on $]x_0, 1]^2$ then U^2 on $[x_0, y_0]^2$ is isomorphic to U_1^2 in the case when $U^2(z_1, x_0) = z_1$ and it is isomorphic to U_2^2 in the case when $U^2(z_1, x_0) = x_0$. Finally, if U^2 is closed on $]x_0, y_0[^2$ then U^2 on $[x_0, y_0]^2$ is isomorphic to one of the five 2-uninorms from Example 3.1 depending on the values of $U^2(z_1, y_0)$, $U^2(z_1, x_0)$ and $U^2(x_0, y_0)$.

Since x_0 and y_0 are idempotent elements of U^2, the monotonicity ensures that U^2 is closed on $[x_0, y_0]^2$.

Now we are going to determine the structure of U^2 on $([0, x_0[\cup]y_0, 1])^2$. We will show that, similarly as in the case of uninorms with continuous underlying functions, U^2 is closed on $([0, x_0[\cup \{U^2(x_0, y_0)\} \cup]y_0, 1])^2$.

Lemma 3.7: *Let $U^2\colon [0,1]^2 \longrightarrow [0,1]$ be a 2-uninorm and let $U^2 \in \mathcal{U}_2$. Assume an $a \in [0, x_0]$ and a $b \in [y_0, 1]$, where a and b are idempotent points of U^2. Then U^2 is closed on $([0, a] \cup \{z_1\} \cup [b, 1])^2$.*

Proof: The monotonicity implies that U^2 is closed on $[0, a]^2$ as well as on $[b, 1]^2$. Further, for all $x \in [0, 1]$ there is $U^2(x, z_1) \in \{x, z_1\}$. Due to Lemma 3.3 we have $U^2(a, x) \in \{a, x, z_1\}$ and $U^2(b, x) \in \{b, x, z_1\}$ for all $x \in [0, a] \cup \{z_1\} \cup [b, 1]$. Thus, due to the commutativity, we have to check only the value of $U^2(x, y)$ for $x \in [0, a[$ and $y \in \,]b, 1]$. First assume that $U^2(x, y) \le e_1$. Then

$$U^2(x, y) = U^2(U^2(a, x), y) = U^2(a, U^2(x, y)) \le U^2(a, e_1) = a.$$

Similarly, if $U^2(x, y) \ge e_2$ then $U^2(x, y) \ge b$. Finally assume that $U^2(x, y) \in \,]e_1, e_2[$. Then

$$U^2(x, y) = U^2(U^2(x, z_1), y) = U^2(U^2(x, y), z_1) = z_1.$$

Summarizing, U^2 is closed on $([0, a] \cup \{z_1\} \cup [b, 1])^2$. ■

Lemma 3.8: *Let $U^2\colon [0,1]^2 \longrightarrow [0,1]$ be a 2-uninorm and let $U^2 \in \mathcal{U}_2$. Assume an $a \in [0,x_0]$ and a $b \in \left[y_0,1\right]$, where a and b are idempotent points of U^2. Then U^2 is either closed on $([0,a[\,\cup\,]b,1])^2$, or U^2 is closed on $([0,a[\,\cup\{U^2(a,b)\}\cup\,]b,1])^2$.*

Proof: If U^2 is not closed on $([0,a[\,\cup\,]b,1])^2$ then due to the monotonicity and Lemma 3.7 there exist an $x_1 \in [0,a[$ and an $x_2 \in\,]b,1]$ such that $U^2(x_1,x_2) \in \{a,b,z_1\}$. We will show that then $U^2(x_1,x_2) = U^2(a,b)$. First suppose that $U^2(x_1,x_2) = a$. Then

$$\begin{aligned} a = U^2(x_1,x_2) &= U^2(U^2(x_1,a),U^2(x_2,b)) = U^2(U^2(a,b),U^2(x_1,x_2)) \\ &= U^2(U^2(a,b),a) = U^2(U^2(a,a),b) = U^2(a,b). \end{aligned}$$

Analogously, if $U^2(x_1,x_2) = b$ then $U^2(a,b) = b$. Finally assume that $U^2(x_1,x_2) = z_1$. Then similarly as above we can show that $U^2(U^2(a,b),z_1) = z_1$. The associativity gives us $U^2(a,z_1) = z_1 = U^2(b,z_1)$ and then the monotonicity of U^2 implies $U^2(a,b) = z_1$. ■

Corollary 3.9: *Let $U^2\colon [0,1]^2 \longrightarrow [0,1]$ be a 2-uninorm and let $U^2 \in \mathcal{U}_2$. Then U^2 is either closed on $([0,x_0[\,\cup\,]y_0,1])^2$, or U^2 is closed on $([0,x_0[\,\cup\{U^2(x_0,y_0)\}\cup\,]y_0,1])^2$.*

From the previous result we see that U^2 is closed on $([0,x_0[\,\cup\{z_1\}\cup\,]y_0,1])^2$ if either $U^2(x_0,y_0) = z_1$, or U^2 is closed on $([0,x_0[\,\cup\,]y_0,1])^2$. For such a 2-uninorm we have the following result.

Proposition 3.10: *Let $U^2\colon [0,1]^2 \longrightarrow [0,1]$ be a 2-uninorm and let $U^2 \in \mathcal{U}_2$. If U^2 is closed on $([0,x_0[\,\cup\{z_1\}\cup\,]y_0,1])^2$ then U^2 restricted to $([0,x_0[\,\cup\{z_1\}\cup\,]y_0,1])^2$ is isomorphic to a uninorm with continuous underlying functions.*

Proof: It is enough to observe that if U^2 is closed on $([0,x_0[\,\cup\{z_1\}\cup\,]y_0,1])^2$ then z_1 is the neutral element of U^2 restricted to $([0,x_0[\,\cup\{z_1\}\cup\,]y_0,1])^2$. Since U^2 is continuous on $([0,x_0[)^2$ and on $(]y_0,1])^2$ we see that U^2 restricted to $([0,x_0[\,\cup\{z_1\}\cup\,]y_0,1])^2$ is isomorphic to a uninorm with continuous underlying functions. ■

4. Characterization of 2-uninorms with continuous underlying functions

In this section we will again assume that x_0 and y_0 are defined as in Remark 3.4. We will relate all 2-uninorms with continuous underlying functions to 2-uninorms from Class 1 (or their restrictions to open, or half-open unit square). Due to Lemma 3.3 there are three possibilities: $U^2(x_0,y_0) = x_0$, $U^2(x_0,y_0) = y_0$ and $U^2(x_0,y_0) = z_1$, while $U^2(x_0,y_0)$ is the annihilator of U^2 on $\left[x_0,y_0\right]^2$. Since the cases $U^2(x_0,y_0) = x_0$ and $U^2(x_0,y_0) = y_0$ are analogous we will focus just on the cases when $U^2(x_0,y_0) = z_1$ and when $U^2(x_0,y_0) = x_0$. In the case when $U^2(x_0,y_0) = z_1$ the structure of U^2 is simple, however, in the case when $U^2(x_0,y_0) = x_0$ ($U^2(x_0,y_0) = y_0$) the situation is much more complicated and we have to divide it into more cases.

- *Case 1*: $\mathbf{U^2(x_0,y_0) = z_1}$.

Theorem 4.1: *Let $U^2\colon [0,1]^2 \longrightarrow [0,1]$ be a 2-uninorm and let $U^2 \in \mathcal{U}_2$. If $U^2(x_0,y_0) = z_1$ then U^2 is an ordinal sum of two semigroups $G_1 = ([0,x_0[\,\cup\{z_1\}\cup\,]y_0,1], U^2)$ and $G_2 =$*

$([x_0, y_0], U^2)$, *where* G_2 *is isomorphic to* U_1^2 *from Example* 3.1 *and* G_1 *is isomorphic to a uninorm with continuous underlying functions and the order of semigroups in the ordinal sum construction is* $1 < 2$.

Proof: Proposition 3.10 shows that G_1 is isomorphic to a uninorm with continuous underlying functions. Since $U^2(x_0, y_0) = z_1$ it is easy to see that G_2 is isomorphic to U_1^2 from Example 3.1. Further, z_1 is the annihilator of U^2 on $[x_0, y_0]^2$ and the neutral element of U^2 on $([0, x_0[\cup \{z_1\} \cup]y_0, 1])^2$. Finally, if $x \in [0, x_0[\cup \{z_1\} \cup]y_0, 1]$ and $y \in [x_0, y_0]$ we get

$$U^2(x, y) = U^2(U^2((x, z_1), y) = U^2(x, U^2(z_1, y)) = U^2(x, z_1) = x,$$

i.e. $1 < 2$. ■

The case when U^2 is closed on $([0, x_0[\cup]y_0, 1])^2$ and $U^2(x_0, y_0) \neq z_1$ will be discussed later (see Remark 4.10).

- *Case 2*: $\mathbf{U^2(x_0, y_0) = x_0}$.

Since the cases when $U^2(x_0, y_0) = x_0$ and when $U^2(x_0, y_0) = y_0$ are analogous we will further focus just on the case when $U^2(x_0, y_0) = x_0$. In such a case the monotonicity implies $U^2(x_0, z_1) = x_0$. Therefore Lemma 3.5 implies that U^2 is closed on $]x_0, y_0]^2$. Here U^2 restricted to $]x_0, y_0]^2$ is isomorphic to a 2-uninorm U_1^2 restricted to $]0, 1]^2$.

For the rest of the section let us denote

$$y_1 = \sup\{y \in [y_0, 1] \mid U^2(x_0, y) = x_0\}.$$

Then $y_1 \geq y_0$ and similarly as for x_0 and y_0 in Remark 3.4 we can show that y_1 is an idempotent point of U^2. If $y_1 = y_0$ then $U^2(x_0, y_0) = x_0 = U^2(x_0, y_1)$ and if $y_1 > y_0$ we have $U^2(x_0, y_1) \in \{x_0, z_1, y_1\}$.

Lemma 4.2: *Let* $U^2\colon [0,1]^2 \longrightarrow [0,1]$ *be a* 2*-uninorm and let* $U^2 \in \mathcal{U}_2$. *If* $y_1 > y_0$ *then* $U^2(x_0, y_1) \in \{x_0, y_1\}$ *and* $U^2(x_0, y) = y$ *for all* $y > y_1$.

Proof: We know that $U^2(x_0, y_1) \in \{x_0, z_1, y_1\}$. Assume that $U^2(x_0, y_1) = z_1$. Then

$$z_1 = U^2(x_0, y_1) = U^2(U^2(x_0, x_0), y_1) = U^2(x_0, U^2(x_0, y_1)) = U^2(x_0, z_1) = x_0,$$

which is a contradiction. Thus $U^2(x_0, y_1) \in \{x_0, y_1\}$. Similarly we can show that $U^2(x_0, y) = y$ for all $y > y_1$. ■

Due to these facts we see that we have to distinguish three cases: when $y_1 = y_0$, when $y_1 > y_0$ and $U^2(x_0, y_1) = x_0$ and when $y_1 > y_0$ and $U^2(x_0, y_1) = y_1$.

- *Case 2a*: $\mathbf{U^2(x_0, y_0) = x_0, y_1 = y_0}$.

Proposition 4.3: *Let* $U^2\colon [0,1]^2 \longrightarrow [0,1]$ *be a* 2*-uninorm and let* $U^2 \in \mathcal{U}_2$. *If* $U^2(x_0, y_0) = x_0$ *and* $U^2(x_0, y) = y$ *for all* $y > y_0$ *then* U^2 *restricted to* $([0, x_0] \cup]y_0, 1])^2$ *is isomorphic to a uninorm with continuous underlying functions.*

Proof: Due to Corollary 3.9 U^2 is closed on $([0, x_0] \cup]y_0, 1])^2$. Further, $U^2(x, x_0) = x$ for all $x < x_0$ and $U^2(x_0, y) = y$ for all $y > y_0$ and therefore x_0 is the neutral element of U^2 on $([0, x_0] \cup]y_0, 1])^2$. Since U^2 is continuous on $[0, x_0]^2$ and on $]y_0, 1]^2$ we see that U^2 restricted to $([0, x_0] \cup]y_0, 1])^2$ is isomorphic to a uninorm with continuous underlying functions. ■

Theorem 4.4: *Let $U^2 \colon [0, 1]^2 \longrightarrow [0, 1]$ be a 2-uninorm and let $U^2 \in \mathcal{U}_2$. If $U^2(x_0, y_0) = x_0$, $U^2(z_1, y_0) = z_1$ and $U^2(x_0, y) = y$ for all $y > y_0$ then U^2 is an ordinal sum of two semigroups $G_1 = ([0, x_0] \cup]y_0, 1], U^2)$ and $G_2 = (]x_0, y_0], U^2)$, where G_2 is isomorphic to U_1^2 from Example 3.1 restricted to $]0, 1]^2$, G_1 is isomorphic to a uninorm with continuous underlying functions and the order of semigroups in the ordinal sum construction is $1 < 2$.*

Proof: Proposition 4.3 implies that G_1 is isomorphic to a uninorm with continuous underlying functions. Further, it is easy to see that G_2 is isomorphic to U_1^2 from Example 3.1 restricted to $]0, 1]^2$. Finally, if $x \in [0, x_0] \cup]y_0, 1]$ and $y \in]x_0, y_0]$ then

$$U^2(x, y) = U^2(U^2(x, z_1), y) = U^2(x, U^2(z_1, y)) = U^2(x, z_1) = x,$$

i.e. $1 < 2$. ■

Theorem 4.5: *Let $U^2 \colon [0, 1]^2 \longrightarrow [0, 1]$ be a 2-uninorm and let $U^2 \in \mathcal{U}_2$. If $U^2(x_0, y_0) = x_0$, $U^2(z_1, y_0) = y_0$ and $U^2(x_0, y) = y$ for all $y > y_0$ then U^2 is an ordinal sum of three semigroups $G_1 = ([0, x_0] \cup]y_0, 1], U^2)$, $G_2 = (]x_0, y_0[, U^2)$ and $G_3 = (\{y_0\}, \mathrm{Id})$, where G_1 is isomorphic to a uninorm with continuous underlying functions, G_2 is isomorphic to a 2-uninorm U_1^2 restricted to $]0, 1[^2$ and the order of semigroups in the ordinal sum construction is $1 < 3 < 2$.*

Proof: Proposition 4.3 implies that G_1 is isomorphic to a uninorm with continuous underlying functions. Further it is easy to see that G_2 is isomorphic to U_1^2 from Example 3.1 restricted to $]0, 1[^2$. Finally, if $x \in [0, x_0] \cup]y_0, 1]$ and $y \in]x_0, y_0[$ then

$$U^2(x, y) = U^2(U^2(x, z_1), y) = U^2(x, U^2(z_1, y)) = U^2(x, z_1) = x,$$

i.e. $1 < 2$. If $x \in [0, x_0] \cup]y_0, 1]$ then

$$U^2(x, y_0) = U^2(U^2(x, x_0), y_0) = U^2(x, U^2(x_0, y_0)) = U^2(x, x_0) = x,$$

i.e. $1 < 3$ and if $x \in]x_0, y_0[$ then

$$U^2(x, y_0) = U^2(x, U^2(z_1, y_0)) = U^2(U^2(x, z_1), y_0) = U^2(z_1, y_0) = y_0,$$

i.e. $3 < 2$. ■

- *Case 2b*: $\mathbf{U^2(x_0, y_0) = x_0, y_1 > y_0, U^2(x_0, y_1) = x_0}$.

From now on we will assume that $y_1 > y_0$. Since y_0 and y_1 are idempotent points of U^2 Proposition 2.6 and the structure of continuous t-conorms imply that U^2 restricted to

$\left]y_0, y_1\right[^2$ ($\left]y_0, y_1\right]^2$) is isomorphic to a t-conorm restricted to $]0,1[^2$ ($]0,1]^2$). Similarly, U^2 restricted to $\left[y_0, y_1\right]^2$ is isomorphic to a t-conorm.

Since $U^2(x_0, y) = y$ for all $y > y_1$ in the case that $U^2(x_0, y_1) = x_0$ we can show the following result.

Proposition 4.6: *Let $U^2\colon [0,1]^2 \longrightarrow [0,1]$ be a 2-uninorm and let $U^2 \in \mathcal{U}_2$. Assume $U^2(x_0, y_0) = x_0$, $y_1 > y_0$ and $U^2(x_0, y_1) = x_0$. Then U^2 restricted to $([0, x_0] \cup \left]y_1, 1\right])^2$ is isomorphic to a uninorm with continuous underlying functions.*

Proof: Due to Lemma 3.8 U^2 is closed on $([0, x_0] \cup \left]y_1, 1\right])^2$. Further, $U^2(x, x_0) = x$ for all $x < x_0$ and $U^2(x_0, y) = y$ for all $y > y_1$ and therefore x_0 is the neutral element of U^2 on $([0, x_0] \cup \left]y_1, 1\right])^2$. Since U^2 is continuous on $[0, x_0]^2$ and on $\left]y_1, 1\right]^2$ we see that U^2 restricted to $([0, x_0] \cup \left]y_1, 1\right])^2$ is isomorphic to a uninorm with continuous underlying functions. ■

Theorem 4.7: *Let $U^2\colon [0,1]^2 \longrightarrow [0,1]$ be a 2-uninorm and let $U^2 \in \mathcal{U}_2$. Assume $U^2(x_0, y_0) = x_0$, $y_1 > y_0$ and $U^2(x_0, y_1) = x_0$, $U^2(z_1, y_0) = z_1$. Then U^2 is an ordinal sum of three semigroups $G_1 = ([0, x_0] \cup \left]y_1, 1\right], U^2)$, $G_2 = (\left]x_0, y_0\right], U^2)$ and $G_3 = (\left]y_0, y_1\right], U^2)$, where G_1 is isomorphic to a uninorm with continuous underlying functions, G_2 is isomorphic to a 2-uninorm U_1^2 restricted to $]0,1]^2$ and G_3 is isomorphic to a t-conorm restricted to $]0,1]^2$. Moreover, the order of semigroups in the ordinal sum construction is $1 < 3 < 2$.*

Proof: Proposition 4.6 implies that G_1 is isomorphic to a uninorm with continuous underlying functions. Further it is easy to see that G_2 is isomorphic to U_1^2 from Example 3.1 restricted to $]0,1]^2$ and from the previous we know that U^2 on $\left]y_0, y_1\right]^2$ is isomorphic to a t-conorm restricted to $]0,1]^2$.

Finally, if $x \in [0, x_0] \cup \left]y_1, 1\right]$ and $y \in \left]x_0, y_0\right]$ then

$$U^2(x, y) = U^2(U^2(x, z_1), y) = U^2(x, U^2(z_1, y)) = U^2(x, z_1) = x,$$

i.e. $1 < 2$. If $x \in [0, x_0] \cup \left]y_1, 1\right]$, $y \in \left]y_0, y_1\right]$ then

$$U^2(x, y) = U^2(U^2(x, x_0), y) = U^2(x, U^2(x_0, y)) = U^2(x, x_0) = x,$$

i.e. $1 < 3$ and if $x \in \left]x_0, y_0\right]$ and $y \in \left]y_0, y_1\right]$ then

$$U^2(x, y) = U^2(x, U^2(z_1, y)) = U^2(U^2(x, z_1), y) = U^2(z_1, y) = y,$$

i.e. $3 < 2$. ■

Theorem 4.8: *Let $U^2\colon [0,1]^2 \longrightarrow [0,1]$ be a 2-uninorm and let $U^2 \in \mathcal{U}_2$. Assume $U^2(x_0, y_0) = x_0$, $y_1 > y_0$ and $U^2(x_0, y_1) = x_0$, $U^2(z_1, y_0) = y_0$. Then U^2 is an ordinal sum of three semigroups $G_1 = ([0, x_0] \cup \left]y_1, 1\right], U^2)$, $G_2 = (\left]x_0, y_0\right[, U^2)$ and $G_3 = (\left[y_0, y_1\right], U^2)$, where G_1 is isomorphic to a uninorm with continuous underlying functions, G_2 is isomorphic to a 2-uninorm U_1^2 restricted to $]0,1[^2$ and G_3 is isomorphic to a t-conorm. Moreover, the order of semigroups in the ordinal sum construction is $1 < 3 < 2$.*

Proof: Proposition 4.6 implies that G_1 is isomorphic to a uninorm with continuous underlying functions. Further it is easy to see that G_2 is isomorphic to U_1^2 from Example 3.1 restricted to $]0,1[^2$ and from the previous we know that U^2 on $\left[y_0, y_1\right]^2$ is isomorphic to a t-conorm.

Finally, similarly as in the previous theorem we can show that $1 < 3 < 2$. ■

Now we will show that if $U^2(x_0, y_0) = x_0$ and U^2 is not closed on $([0, x_0[\cup \left]y_0, 1\right])^2$ then $U^2(y_1, x_0) = x_0$.

Lemma 4.9: *Let $U^2 \colon [0,1]^2 \longrightarrow [0,1]$ be a 2-uninorm and let $U^2 \in \mathcal{U}_2$. Suppose that $U^2(x_0, y_0) = x_0$ and U^2 is not closed on $([0, x_0[\cup \left]y_0, 1\right])^2$. Then $U^2(y_1, x_0) = x_0$.*

Proof: Since U^2 is not closed on $([0, x_0[\cup \left]y_0, 1\right])^2$ and $U^2(x_0, y_0) = x_0$ there exist an $x_1 \in [0, x_0[$ and an $x_2 \in \left]y_0, 1\right]$ such that $U^2(x_1, x_2) = x_0$. Then $U^2(x_1, x_0) = x_1$. If $U^2(x_0, x_2) = x_0$ then we get

$$x_0 = U^2(x_1, x_2) = U^2(U^2(x_1, x_0), x_2) = U^2(x_1, U^2(x_0, x_2)) = U^2(x_1, x_0) = x_1,$$

which is a contradiction. Therefore $x_2 \geq y_1$. Since y_1 is an idempotent point of U^2 and U^2 on $[e_2, 1]^2$ is isomorphic to a continuous t-conorm we know that $U^2(y_1, x_2) = x_2$. Then we get

$$x_0 = U^2(x_1, x_2) = U^2(x_1, U^2(x_2, y_1)) = U^2(U^2(x_1, x_2), y_1) = U^2(x_0, y_1).$$

■

- *Case 2c*: $\mathbf{U^2(x_0, y_0) = x_0, y_1 > y_0, U^2(x_0, y_1) = y_1}$.

To conclude our investigation of the structure of 2-uninorms with continuous underlying functions we have to discuss the case is when $y_1 > y_0$ and $U^2(x_0, y_1) = y_1$.

Remark 4.10: Assume $U^2(x_0, y_0) = x_0$, $y_1 > y_0$ and $U^2(x_0, y_1) = y_1$. Due to Lemma 4.9 we know that in such a case U^2 is closed on $([0, x_0[\cup \left]y_0, 1\right])^2$. Since $U^2(x_0, y_1) = y_1$ the point y_1 behaves differently than the rest of the semigroup defined on $\left]y_0, y_1\right]$ ($\left[y_0, y_1\right]$) and therefore we cannot use the same construction as above. We define $x_1 = \inf\{x \in [0, x_0] \mid U^n(y_1, x) = y_1\}$ and we can continue like this by the induction: for $n \in \mathbb{N}$ we define

$$y_n = \sup\{y \in \left[y_{n-1}, 1\right] \mid U^n(x_{n-1}, y) = x_{n-1}\}$$

and

$$x_n = \inf\{x \in [0, x_{n-1}] \mid U^n(y_n, x) = y_n\}.$$

It can happen that $y_{n_0} = y_{n_0-1}$ (and then $x_{n_0} = x_{n_0-1}$) for some $n_0 \in \mathbb{N}$. In such a case U^2 is an ordinal sum of a semigroup isomorphic to a uninorm with continuous underlying functions, of a semigroup isomorphic to a U_1^2 from Example 3.1 (restricted to $]0,1[^2$, or to $]0,1]^2$) and a number of semigroups corresponding to continuous t-norms or continuous t-conorms (possibly restricted to open, or half-open unit square). However, it is also possible that $(y_i)_{i\in\mathbb{N}}$ ($(x_i)_{i\in\mathbb{N}}$) is an increasing (decreasing) sequence. Therefore we see that the structure of U^2 on $([0, x_0] \cup]y_0, 1])^2$ can be rather peculiar. That is why it need not to be easy to express U^2 as an ordinal sum of a uninorm, a 2-uninorm from Example 3.1 and few other semigroups. Therefore we adopt a different approach.

Proposition 3.10 implies that U^2 restricted to $([0, x_0[\cup \{z_1\} \cup]y_0, 1])^2$ is isomorphic to a uninorm with continuous underlying functions. Due to results from Mesiarová-Zemánková (2017) we know that each uninorm with continuous underlying functions is an ordinal sum of a countable number of semigroups related to representable uninorms, continuous Archimedean t-norms, continuous Archimedean t-conorms and internal uninorms (including the min and the max operator). At first we will suppose that $U^2(y_0, z_1) = z_1$. Then U^2 is closed on $]y_0, y_1[^2$. Further, Lemma 3.8 shows that U^2 is closed on $([0, x_0[\cup [y_1, 1])^2$. Therefore, since $([0, x_0[\cup \{z_1\} \cup]y_0, 1])^2 \cap ([0, x_0[\cup [y_1, 1])^2 = ([0, x_0[\cup [y_1, 1])^2$ we know that $([0, x_0[\cup [y_1, 1], U^2)$ can be expressed as an ordinal sum of a countable number of semigroups related to representable uninorms, continuous Archimedean t-norms, continuous Archimedean t-conorms and internal uninorms (including the min and the max operator).

For $x \in [0, x_0[\cup [y_1, 1]$ and $y \in]y_0, y_1[$ there is

$$U^2(x, y) = U^2(U^2(x, x_0), y) = U^2(x, U^2(x_0, y)) = U^2(x, x_0) = x.$$

Since U^2 is closed on $([0, x_0[\cup]y_0, 1])^2$ we see that $([0, x_0[\cup]y_0, 1], U^2)$ can be expressed as an ordinal sum of $G_1 = ([0, x_0[\cup [y_1, 1], U^2)$ and $G_2 = (]y_0, y_1[, U^2)$ with $1 < 2$. Then $([0, x_0] \cup]y_0, 1], U^2)$ is an ordinal sum of G_1, G_2 and $G_3 = (\{x_0\}, \mathrm{Id})$, with $1 < 3 < 2$. We get the following result.

Theorem 4.11: *Let $U^2\colon [0,1]^2 \longrightarrow [0,1]$ be a 2-uninorm and let $U^2 \in \mathcal{U}_2$. Assume $U^2(x_0, y_0) = x_0$, $y_1 > y_0$ and $U^2(x_0, y_1) = y_1$, $U^2(z_1, y_0) = z_1$. Then U^2 is an ordinal sum of four semigroups $G_1 = ([0, x_0[\cup [y_1, 1], U^2)$, $G_2 = (]y_0, y_1[, U^2)$, $G_3 = (\{x_0\}, \mathrm{Id})$ and $G_4 = (]x_0, y_0], U^2)$, where G_1 can be expressed as an ordinal sum of a countable number of semigroups related to representable uninorms, continuous Archimedean t-norms, continuous Archimedean t-conorms and internal uninorms, G_2 is isomorphic to a restriction of a continuous t-conorm to the open unit square and G_4 is isomorphic to a 2-uninorm U_1^2 restricted to $]0, 1]^2$. Moreover, the order of semigroups in the ordinal sum construction is $1 < 3 < 2 < 4$.*

Let us note that U^2 can be expressed also as an ordinal sum of three semigroups $H_1 = ([0, x_0] \cup [y_1, 1], U^2)$, $H_2 = (]y_0, y_1[, U^2)$ and $H_3 = (]x_0, y_0], U^2)$, $1 < 2 < 3$, where U^2 restricted to $([0, x_0] \cup [y_1, 1])^2$ is so-called generalized uninorm defined in Mesiarová-Zemánková (2016), with the neutral element x_0.

Similarly we can show the following result.

Theorem 4.12: *Let $U^2\colon [0,1]^2 \longrightarrow [0,1]$ be a 2-uninorm and let $U^2 \in \mathcal{U}_2$. Assume $U^2(x_0, y_0) = x_0$, $y_1 > y_0$ and $U^2(x_0, y_1) = y_1$, $U^2(z_1, y_0) = y_0$. Then U^2 is an ordinal sum of four semigroups $G_1 = ([0, x_0[\cup [y_1, 1], U^2)$, $G_2 = ([y_0, y_1[, U^2)$, $G_3 = (\{x_0\}, \mathrm{Id})$ and $G_4 = (]x_0, y_0[, U^2)$, where G_1 can be expressed as an ordinal sum of a countable number of semigroups related to representable uninorms, continuous Archimedean t-norms, continuous Archimedean t-conorms and internal uninorms, G_2 is isomorphic to a restriction of a*

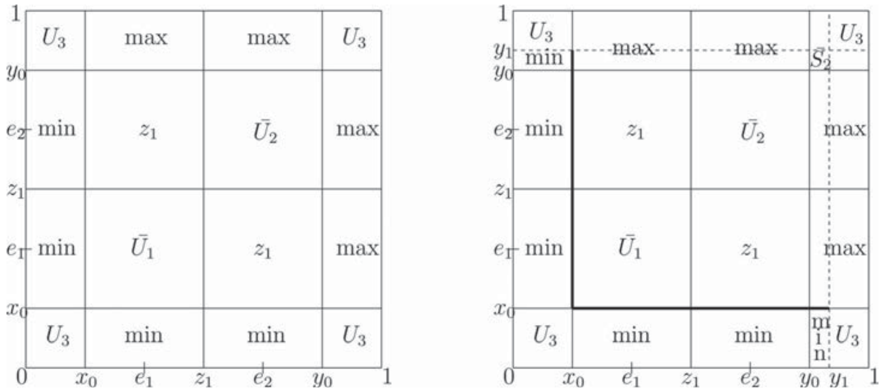

Figure 1. A 2-uninorm with $U^2(x_0, y_0) = z_1$ (left) and with $U^2(x_0, y_0) = x_0$, $y_1 > y_0$, $U^2(x_0, y_1) = x_0$ (right). $\bar{U}_1$ $(\bar{U}_2, \bar{S}_2)$ indicates that U^2 is on the given area isomorphic to a restriction of U_1 (U_2, S_2) to a subinterval of $[0, 1]$.

continuous t-conorm to $[0, 1[^2$ *and* G_4 *is isomorphic to a* 2*-uninorm* U_1^2 *restricted to* $]0, 1[^2$. *Moreover, the order of semigroups in the ordinal sum construction is* $1 < 3 < 2 < 4$.

Since uninorms with continuous underlying functions were completely characterized, the structure of any 2-uninorm with continuous underlying functions is described by one of the Theorems 4.1, 4.4, 4.5, 4.7, 4.8, 4.11, 4.12 (or an analogous result in the case when $U^2(x_0, y_0) = y_0$), see Figure 1.

Remark 4.13: Assume a 2-uninorm $U^2 \in \mathcal{U}_2$. If the underlying uninorm U_1 (U_2) of U^2 is disjunctive (conjunctive) then $x_0 = 0$ $(y_0 = 1)$. Moreover, if the underlying t-norms and t-conorm of U^2 are Archimedean then $x_0 \in \{0, e_1\}$ and $y_0 \in \{e_2, 1\}$.

Since for uni-nullnorms there is $e_2 = 1$ in this case we obtain $y_0 = 1$. Thus any uni-nullnorm $UN \in \mathcal{U}_2$ is uniquely determined on $]x_0, 1]^2$ and therefore its structure on $[0, 1]^2$ is uniquely determined by the structure of the underlying uninorm U_1 (which has continuous underlying functions). Therefore the structure of uni-nullnorms with continuous underlying functions follows from the structure of uninorms with continuous underlying functions, see for example, Li, Liu, and Fodor (2014), Li and Liu (2016) and Mesiarová-Zemánková (2017).

If we assume that underlying t-norms and t-conorm of UN are Archimedean we get $x_0 \in \{0, e_1\}$. Note that since $UN(z_1, y_0) = UN(z_1, 1) = z_1$ there is $UN(x_0, y_0) \neq y_0$.

First assume that $x_0 = e_1$. Then $UN(x_0, z_1) = UN(y_0, z_1) = z_1$ and the monotonicity gives $UN(x_0, y_0) = z_1$. Here the uninorm on $([0, x_0[\cup \{z_1\} \cup]y_0, 1])^2$ reduces to a t-norm on $([0, e_1[\cup \{z_1\})^2$. Since z_1 is the annihilator of UN on $[e_1, 1]^2$ and the neutral element of UN on $([0, e_1[\cup \{z_1\})^2$, we see that in this case UN is an ordinal sum of a uni-nullnorm from Class 1 acting on $[e_1, 1]^2$ (which is isomorphic to a nullnorm) and a (restricted) t-norm acting on the interval $[0, e_1[^2$ (compare Sun, Wang, and Qu 2018, Theorem 4.1(i)).

Now assume that $x_0 = 0$. If $UN(x_0, y_0) = z_1$ then the uninorm on $([0, x_0[\cup \{z_1\} \cup]y_0, 1])^2$ reduces to an operation on a single point $\{z_1\}$ and UN belongs to the Class 1. Here

z_1 is the annihilator of UN. This case encompasses (Sun, Wang, and Qu 2018, Theorem 4.1), cases (ii), (iii), (iv) and (v).

If $UN(x_0, y_0) = x_0$ then the uninorm on $([0, x_0] \cup]y_0, 1])^2$ reduces to an operation on a single point $\{x_0\}$ and thus UN is an ordinal sum of a (restricted) uni-nullnorm from Class 1 acting on $]0, 1]^2$ and a semigroup $(\{0\}, \mathrm{Id})$. This case encompasses (Sun, Wang, and Qu 2018, Theorem 4.1), cases (*vi*), (*vii*) and (*viii*).

5. *n*-uninorms with continuous underlying functions

In this section we will generalize the results from the previous sections for n-uninorms with continuous underlying functions, where $n \in \mathbb{N}, n > 2$.

5.1. Basic results on n-uninorms with continuous underlying functions

Recall that if $U^n\colon [0, 1]^n \longrightarrow [0, 1]$ is an n-uninorm then U^n restricted to $\left[z_i, z_j\right]^2$ for $0 \leq i < j \leq 1$, where $j–i = p$ is a p-uninorm.

Lemma 5.1: *Let $U^n\colon [0, 1]^2 \longrightarrow [0, 1]$ be an n-uninorm and let $U^n \in \mathcal{U}_n$. If $a \in [0, 1]$ is an idempotent point of U^n then $U^n(a, x) \in \{x, a\} \cup \{z_i \mid z_i \in]\min(a, x), \max(a, x)[\}$ for all $x \in [0, 1]$.*

Proof: We will do the proof by the induction. By Lemmas 3.2 and 3.3 we know that for a uninorm and a 2-uninorm the result holds. Assume $n \in \mathbb{N}, n > 2$. We will suppose that the result holds for all $(n-1)$-uninorms. Let $a \in [0, 1]$ be an idempotent point of U^n. If $a = z_{n-1}$ then by Lemma 3.2 we know that $U^n(z_{n-1}, x) = \{z_{n-1}, x\}$ for all $x \in [z_{n-1}, 1]$ and $U^n(z_{n-1}, x) \in \{x, z_{n-1}\} \cup \{z_i \mid x < z_i < z_{n-1}\}$ for $x \in [0, z_{n-1}]$ since the claim holds for all $(n-1)$-uninorms and U^n restricted to $[0, z_{n-1}]^2$ is an $(n-1)$-uninorm. Thus

$$U^n(z_{n-1}, x) \in \{x, z_{n-1}\} \cup \{z_i \mid z_i \in]\min(z_{n-1}, x), \max(z_{n-1}, x)[\}$$

for all $x \in [0, 1]$. Now we will assume that $a < z_{n-1}$ (the case when $a > z_{n-1}$ is analogous).

Then $U^n(a, x) \in \{x, a\} \cup \{z_i \mid z_i \in]\min(a, x), \max a, x[\}$ for all $x \in [0, z_{n-1}]$ since the claim holds for all $(n-1)$-uninorms. Assume an $x \in]z_{n-1}, 1]$. Further we will discuss 6 possible cases.

Case 1: When $U^n(a, z_{n-1}) = z_{n-1}$ and $U^n(x, z_{n-1}) = z_{n-1}$. Then

$$z_{n-1} = U^n(a, z_{n-1}) \leq U^n(a, x) \leq U^n(z_{n-1}, x) = z_{n-1}.$$

Thus $U^n(a, x) = z_{n-1}$.

Case 2: When $U^n(a, z_{n-1}) = z_{n-1}$ and $U^n(x, z_{n-1}) = x$. Then

$$U^n(a, x) = U^n(a, U^n(z_{n-1}, x)) = U^n(U^n(a, z_{n-1}), x) = U^n(z_{n-1}, x) = x.$$

Case 3: When $U^n(a, z_{n-1}) = z_i, a < z_i < z_{n-1}$, for some $i \in \{1, \ldots, n-2\}$ and $U^n(x, z_{n-1}) = z_{n-1}$. Then

$$U^n(z_i, z_{n-1}) = U^n(U^n(a, z_{n-1}), z_{n-1}) = U^n(a, U^n(z_{n-1}, z_{n-1})) = U^n(a, z_{n-1}) = z_i$$

and

$$U^n(x, z_i) = U^n(x, U^n(z_{n-1}, z_i)) = U^n(U^n(x, z_{n-1}), z_i) = U^n(z_{n-1}, z_i) = z_i.$$

Further, $U^n(a, z_i) = U^n(a, U^n(a, z_{n-1})) = U^n(U^n(a, a), z_{n-1}) = U^n(a, z_{n-1}) = z_i$ and we get

$$z_i = U^n(a, z_i) \leq U^n(a, x) \leq U^n(z_i, x) = z_i,$$

i.e. $U^n(a, x) = z_i$.

Case 4: When $U^n(a, z_{n-1}) = z_i$, $a < z_i < z_{n-1}$, for some $i \in \{1, \dots, n-2\}$ and $U^n(x, z_{n-1}) = x$. Then similarly as in the previous case we have $U^n(z_i, z_{n-1}) = z_i$ and thus the monotonicity implies that $U^n(z_i, z_k) = z_i$ for all $k \in \{i, \dots n-1\}$. Moreover, $U^n(a, z_i) = z_i$. Further, either $U^n(x, z_i) = x$, or $U^n(x, z_i) = z_j$ for some $j \in \{i, \dots, n-1\}$ since the claim holds for all $(n-1)$-uninorms and $i \geq 1$. If $U^n(x, z_i) = z_j$ then

$$z_j = U^n(x, z_i) = U^n(x, U^n(z_i, z_i)) = U^n(U^n(x, z_i), z_i) = U^n(z_j, z_i) = z_i$$

and similarly as before the monotonicity implies $U^n(a, x) = z_i$.

If $U^n(x, z_i) = x$ then we get

$$U^n(a, x) = U^n(a, U^n(z_i, x)) = U^n(U^n(a, z_i), x) = U^n(z_i, x) = x.$$

Case 5: When $U^n(a, z_{n-1}) = a$ and $U^n(x, z_{n-1}) = z_{n-1}$. Then

$$U^n(a, x) = U^n(U^n(a, z_{n-1}), x) = U^n(a, U^n(z_{n-1}, x)) = U^n(a, z_{n-1}) = a.$$

Case 6: When $U^n(a, z_{n-1}) = a$ and $U^n(x, z_{n-1}) = x$.

If $U^n(a, x) \leq e_n$ then $U^n(a, e_n) = U^n(U^n(a, z_{n-1}), e_n) = U^n(a, U^n(z_{n-1}, e_n)) = U^n(a, z_{n-1}) = a$ implies

$$a = U^n(a, z_{n-1}) \leq U^n(a, x) = U^n(U^n(a, a), x) = U^n(a, U^n(a, x)) \leq U^n(a, e_n) = a,$$

i.e. $U^n(a, x) = a$. Now suppose that $U^n(a, x) > e_n$. Then $U^n(a, x) \leq U^n(z_{n-1}, x) = x$. If $U^n(a, x) < x$ then since S_n is continuous there exists an $x_1 \in [e_n, 1]$ such that $U^n(U^n(a, x), x_1) = x$ and we get

$$U^n(a, x) = U^n(a, U^n(U^n(a, x), x_1)) = U^n(U^n(U^n(a, a), x), x_1) = U^n(U^n(a, x), x_1) = x.$$

Thus in both cases $U^n(a, x) \in \{a, x\}$.

If we summarize all cases we see that the claim holds for $a \leq z_{n-1}$ and for all $x \in [0, 1]$ and analogously for $a > z_{n-1}$ and for all $x \in [0, 1]$. ∎

Assume an n-uninorm $U^n \in \mathcal{U}_n$. Then Lemma 5.1 implies that for all $i, j \in \{0, 1, \dots, n\}$, $i \leq j$, there is $U^n(z_i, z_j) = z_k$, were $k \in \{i, \dots, j\}$. Then the associativity implies $U^n(z_i, z_k) = z_k$ and $U^n(z_j, z_k) = z_k$.

Lemma 5.2: *Let $U^n\colon [0, 1]^2 \longrightarrow [0, 1]$ be an n-uninorm and let $U^n \in \mathcal{U}_n$. Assume that $U^n(z_1, z_{n-1}) = z_k$ for some $k \in \{1, \dots, n-1\}$. If for an $x \in [0, z_1]$ and a $y \in [z_{n-1}, 1]$ there is $U^n(x, y) = z_m$ for some $m \in \{1, \dots, n-1\}$ then $m = k$.*

Proof: If $U^n(z_1, z_{n-1}) = z_k$ for some $k \in \{1, \dots, n-1\}$ then $U^n(z_1, z_k) = z_k = U^n(z_{n-1}, z_k)$ and due to the monotonicity of U^n we know that z_k is the annihilator of U^n on $[z_1, z_{n-1}]^2$. Moreover,

$$z_k = U^n(z_1, z_k) = U^n(U^n(e_1, z_1), z_k) = U^n(e_1, U^n(z_1, z_k)) = U^n(e_1, z_k)$$

and similarly $U^n(e_n, z_k) = z_k$. Thus the monotonicity implies $U^n(e_1, e_n) = z_k$. Assume that for an $x \in [0, z_1]$ and a $y \in [z_{n-1}, 1]$ we have $U^n(x, y) = z_m$ for some $m \in \{1, \dots, n-1\}$. Then

$$\begin{aligned} z_m &= U^n(x, y) = U^n(U^n(x, e_1), U^n(e_n, y)) \\ &= U^n(U^n(x, y), U^n(e_1, e_n)) = U^n(z_m, z_k) = z_k, \end{aligned}$$

i.e. $z_m = z_k$. ∎

From now on we will denote $z_k = U^n(z_1, z_{n-1})$, where $k \in \{1, \dots, n-1\}$.

Lemma 5.3: *Let* $U^n\colon [0,1]^2 \longrightarrow [0,1]$ *be an n-uninorm and let* $U^n \in \mathcal{U}_n$. *Then* $U^n(x, z_k) \in \{x, z_k\}$ *for all* $x \in [0, 1]$.

Proof: Since z_k is the annihilator of U^n on $[e_1, e_n]^2$ the claim holds if $x \in [e_1, e_n]$. Further we will assume that $x < e_1$ as the case when $x > e_n$ is analogous. Then $U^n(x, z_k) \in [x, z_k]$ since

$$x = U^n(x, e_1) \leq U^n(x, z_k) \leq U^n(z_k, z_k) = z_k.$$

First assume that $U^n(x, z_k) \geq e_1$. Then

$$z_1 = U^n(z_1, e_1) \leq U^n(z_1, U^n(z_k, x)) = U^n(U^n(z_1, z_k), x) = U^n(z_k, x)$$

and

$$z_k = U^n(z_k, z_1) \leq U^n(z_k, U^n(z_k, x)) = U^n(U^n(z_k, z_k), x) = U^n(z_k, x) \leq z_k,$$

i.e. $U^n(z_k, x) = z_k$. Now assume $U^n(x, z_k) < e_1$. If $U^n(x, z_k) > x$ then since T_1 is continuous there exists an $x_1 \in [0, e_1]$ such that $U^n(U^n(x, z_k), x_1) = x$. Then

$$\begin{aligned} U^n(x, z_k) &= U^n(U^n(U^n(x, z_k), x_1), z_k) = U^n(U^n(x, U^n(z_k, z_k)), x_1) \\ &= U^n(U^n(x, z_k), x_1) = x. \end{aligned}$$

Therefore $U^n(x, z_k) = x$. ∎

Remark 5.4: Since $U^n(x, z_k) \in \{x, z_k\}$ for all $x \in [0, 1]$ the monotonicity of U^n implies that there exists an $x_0 \in [0, e_1]$ and a $y_0 \in [e_n, 1]$ such that $U^n(x, z_k) = x$ for all $x < x_0$ and $U^n(x, z_k) = z_k$ for all $x_0 < x \leq z_k$, and $U^n(y, z_k) = y$ for all $y > y_0$ and $U^n(y, z_k) = z_k$ for all $z_k \leq y < y_0$. Note that if $x_0 < e_1$ ($y_0 > e_n$) then there are possible both cases $U^n(x_0, z_k) = x_0$ as well as $U^n(x_0, z_k) = z_k$ ($U^n(y_0, z_k) = y_0$ as well as $U^n(y_0, z_k) = z_k$).

Further, similarly as in the case of 2-uninorms, it is clear that x_0 and y_0 are idempotent points. If $x \in]x_0, z_k]$ and $y \in [z_k, y_0[$ then $U^n(x, z_k) = z_k$ and $U^n(y, z_k) = z_k$ and the monotonicity of U^n implies $U^n(x, y) = z_k$.

Since x_0 (y_0) is an idempotent point of U^n, then U^n restricted to $[x_0, e_1]^2$ ($[e_n, y_0]^2$) is isomorphic a t-norm T_0 (t-conorm S_0). Similarly as in Lemma 3.5 we can show the following result for the case when T_0 has zero divisors (S_0 has divisors of 1).

Lemma 5.5: *Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be an n-uninorm and let $U^n \in \mathcal{U}_n$. If there exist $x_1, x_2 \in]x_0, 1]$ such that $U^n(x_1, x_2) = x_0$ then $U^n(x_0, z_k) = z_k$. Similarly, if there exist $y_1, y_2 \in [0, y_0[$ such that $U^n(y_1, y_2) = y_0$ then $U^n(y_0, z_k) = z_k$.*

We will say that an n-uninorm belongs to Class 1 if $U^n(0, 1) = z_k$ for some $k \in \{1, \ldots, n-1\}$. Such an n-uninorm is a composition of a k-uninorm which acts on $[0, z_k]^2$, an $(n-k)$-uninorm which acts on $[z_k, 1]^2$ and $U^n(x, y) = z_k$ for all $(x, y) \in [0, z_k] \times [z_k, 1] \cup [z_k, 1] \times [0, z_k]$. In exactly the same way as in Example 3.1 we can define 5 types of n-uninorms related to an n-uninorm from Class 1 which differ only on the boundary of the unit square. Such n-uninorms will play a major role in the further investigation. Due to the lack of space we do not introduce exact definitions (as they are analogous to these in Example 3.1) and for the respective n-uninorm we will only mention that it coincides with the n-uninorm from Class 1 (possibly restricted to open or half-open unit square).

As it was in the case of 2-uninorms also in the case of n-uninorms, U^n restricted to $[x_0, y_0]^2$ is isomorphic to one of such n-uninorms.

Further we will distinguish tree cases: either $U^n(x_0, y_0) = z_k$, or $U^n(x_0, y_0) = x_0$, or $U^n(x_0, y_0) = y_0$. Since the case when $U^n(x_0, y_0) = y_0$ is analogous to the case when $U^n(x_0, y_0) = x_0$, we will discuss only the first two cases. First we show that U^n is closed on $([0, x_0[\cup \{U^n(x_0, y_0)\} \cup]y_0, 1])^2$.

Lemma 5.6: *Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be an n-uninorm and let $U^n \in \mathcal{U}_n$. Assume an $a \in [0, x_0]$ and a $b \in [y_0, 1]$, where a and b are idempotent points of U^n. Then U^n is closed on $([0, a] \cup \{z_k\} \cup [b, 1])^2$.*

Proof: The monotonicity implies that U^2 is closed on $[0, a]^2$ as well as on $[b, 1]^2$. Further, for all $x \in [0, 1]$ there is $U^2(x, z_k) \in \{x, z_k\}$. Since U^n restricted to $[0, z_1]^2$ is a uninorm with continuous underlying functions Lemma 3.2 implies $U^n(a, x) \in \{a, x\}$ for $x \in [0, a]$. Similarly $U^n(b, y) \in \{b, y\}$ for $y \in [0, b]$. Assume $y \in [0, b]$. Then

$$U^n(a, y) = U^n(U^n(a, e_1), U^n(y, e_n)) = U^n(U^n(a, y), z_k)$$

and therefore if $U^n(a, y) = z_i$ for some $i \neq k$ we get $z_i = U^n(z_i, z_k) = z_k$, which is a contradiction. Thus $U^n(a, x) \in \{a, x, z_k\}$ for all $x \in [0, a] \cup \{z_k\} \cup [b, 1]$ and similarly $U^n(b, x) \in \{b, x, z_k\}$ for all $x \in [0, a] \cup \{z_k\} \cup [b, 1]$. Due to the commutativity, what remains is to check the value of $U^n(x, y)$ for $x \in [0, a[$ and $y \in]b, 1]$.

First assume that $U^n(x, y) \leq e_1$. Then

$$U^n(x, y) = U^n(U^n(a, x), y) = U^n(a, U^n(x, y)) \leq U^n(a, e_1) = a.$$

Similarly, if $U^n(x,y) \geq e_n$ then $U^n(x,y) \geq b$. Finally assume that $U^n(x,y) \in]e_1, e_n[$. Then

$$U^n(x,y) = U^n(U^n(x,z_k),y) = U^n(U^n(x,y),z_k) = z_k.$$

Summarizing, U^n is closed on $([0,a] \cup \{z_k\} \cup [b,1])^2$. ■

Similarly as in Lemma 3.8 we can show the following result.

Lemma 5.7: *Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be an n-uninorm and let $U^n \in \mathcal{U}_n$. Assume an $a \in [0,x_0]$ and a $b \in \left[y_0,1\right]$, where a and b are idempotent points of U^n. Then U^n is either closed on $([0,a[\cup]b,1])^2$, or U^n is closed on $([0,a[\cup \{U^n(a,b)\} \cup]b,1])^2$.*

Corollary 5.8: *Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be an n-uninorm and let $U^n \in \mathcal{U}_n$. Then U^n is either closed on $([0,x_0[\cup \left]y_0,1\right])^2$, or U^n is closed on $([0,x_0[\cup \{U^n(x_0,y_0)\} \cup \left]y_0,1\right])^2$.*

Proposition 5.9: *Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be an n-uninorm and let $U^n \in \mathcal{U}_n$. If U^n is closed on $([0,x_0[\cup \{z_k\} \cup \left]y_0,1\right])^2$ then U^n restricted to $([0,x_0[\cup \{z_k\} \cup \left]y_0,1\right])^2$ is isomorphic to a uninorm with continuous underlying functions.*

The proof is analogous to the proof of Proposition 3.10.

5.2. Characterization of n-uninorms with continuous underlying functions

In this section we will suppose that x_0 and y_0 are defined as in Remark 5.4.

Due to Lemma 5.1 there are three possible cases: either $U^n(x_0,y_0) = z_k$ for some $k \in \{1,\dots,n-1\}$, or $U^n(x_0,y_0) = x_0$, or $U^n(x_0,y_0) = y_0$. Since the case when $U^n(x_0,y_0) = x_0$ and $U^n(x_0,y_0) = y_0$ are analogous we will focus only on the cases when $U^n(x_0,y_0) = z_k$ and when $U^n(x_0,y_0) = x_0$. The following results are analogous to that for 2-uninorms and therefore we introduce them without proofs.

- *Case 1:* $\mathbf{U^n(x_0,y_0) = z_k}$.

Theorem 5.10: *Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be an n-uninorm and let $U^n \in \mathcal{U}_n$. If $U^n(x_0,y_0) = z_k$ then U^n is an ordinal sum of two semigroups $G_1 = ([0,x_0[\cup \{z_k\} \cup \left]y_0,1\right], U^n)$ and $G_2 = (\left[x_0,y_0\right], U^n)$, where G_2 is isomorphic to an n-uninorm from Class 1 and G_1 is isomorphic to a uninorm with continuous underlying functions. Moreover, the order of semigroups in the ordinal sum construction is $1 < 2$.*

- *Case 2*: $\mathbf{U^n(x_0,y_0) = x_0}$.

Further we will assume that $U^n(x_0,y_0) = x_0$. Let us denote

$$y_1 = \sup\{y \in \left[y_0,1\right] \mid U^n(x_0,y) = x_0\}.$$

Then $y_1 \geq y_0$ and we can easily show that y_1 is an idempotent point of U^n. For $y > y_0$ there is $U^n(x_0,y) \in \{y, z_k, x_0\}$ and if $U^n(y,x_0) = z_k$ then

$$z_k = U^n(y,x_0) = U^n(y,U^n(x_0,x_0)) = U^n(U^n(y,x_0),x_0) = U^n(z_k,x_0) = x_0,$$

which is a contradiction. Thus $U^n(x_0,y_1) \in \{x_0,y_1\}$ and $U^n(x_0,y) = y$ for all $y > y_1$. Therefore in the following we will distinguish three cases: when $y_1 = y_0$, when $y_1 > y_0$

and $U^n(x_0, y_1) = x_0$ and when $y_1 > y_0$ and $U^n(x_0, y_1) = y_1$. We obtain the following results.

- *Case 2a*: $\mathbf{U^n(x_0, y_0) = x_0, y_1 = y_0}$.

Theorem 5.11: *Let* $U^n\colon [0,1]^2 \longrightarrow [0,1]$ *be an n-uninorm and let* $U^n \in \mathcal{U}_n$. *If* $U^n(x_0, y_0) = x_0$, $U^n(z_k, y_0) = z_k$ *and* $U^n(x_0, y) = y$ *for all* $y > y_0$ *then* U^n *is an ordinal sum of two semigroups* $G_1 = ([0, x_0] \cup \left]y_0, 1\right], U^n)$ *and* $G_2 = (\left]x_0, y_0\right], U^2)$, *where* G_2 *is isomorphic to an n-uninorm from Class* 1 *restricted to* $]0,1]^2$ *and* G_1 *is isomorphic to a uninorm with continuous underlying functions. Moreover, the order of semigroups in the ordinal sum construction is* $1 < 2$.

Theorem 5.12: *Let* $U^n\colon [0,1]^2 \longrightarrow [0,1]$ *be an n-uninorm and let* $U^n \in \mathcal{U}_n$. *If* $U^n(x_0, y_0) = x_0$, $U^n(z_k, y_0) = y_0$ *and* $U^n(x_0, y) = y$ *for all* $y > y_0$ *then* U^n *is an ordinal sum of three semigroups* $G_1 = ([0, x_0] \cup \left]y_0, 1\right], U^n)$, $G_2 = (\left]x_0, y_0\right[, U^n)$ *and* $G_3 = (\{y_0\}, \mathrm{Id})$, *where* G_1 *is isomorphic to a uninorm with continuous underlying functions and* G_2 *is isomorphic to an n-uninorm from Class* 1 *restricted to* $]0,1[^2$. *Moreover, the order of semigroups in the ordinal sum construction is* $1 < 3 < 2$.

- *Case 2b*: $\mathbf{U^n(x_0, y_0) = x_0, y_1 > y_0, U^n(x_0, y_1) = x_0}$.

Theorem 5.13: *Let* $U^n\colon [0,1]^2 \longrightarrow [0,1]$ *be an n-uninorm and let* $U^n \in \mathcal{U}_n$. *Assume* $U^n(x_0, y_0) = x_0$, $y_1 > y_0$ *and* $U^n(x_0, y_1) = x_0$, $U^n(z_k, y_0) = z_k$. *Then* U^n *is an ordinal sum of three semigroups* $G_1 = ([0, x_0] \cup \left]y_1, 1\right], U^n)$, $G_2 = (\left]x_0, y_0\right], U^n)$ *and* $G_3 = (\left]y_0, y_1\right], U^n)$, *where* G_1 *is isomorphic to a uninorm with continuous underlying functions,* G_2 *is isomorphic to an n-uninorm from Class* 1 *restricted to* $]0,1]^2$ *and* G_3 *is isomorphic to a t-conorm restricted to* $]0,1]^2$. *Moreover, the order of semigroups in the ordinal sum construction is* $1 < 3 < 2$.

Theorem 5.14: *Let* $U^n\colon [0,1]^2 \longrightarrow [0,1]$ *be an n-uninorm and let* $U^n \in \mathcal{U}_n$. *Assume* $U^n(x_0, y_0) = x_0$, $y_1 > y_0$ *and* $U^n(x_0, y_1) = x_0$, $U^n(z_k, y_0) = y_0$. *Then* U^n *is an ordinal sum of three semigroups* $G_1 = ([0, x_0] \cup \left]y_1, 1\right], U^n)$, $G_2 = (\left]x_0, y_0\right[, U^n)$ *and* $G_3 = (\left[y_0, y_1\right], U^n)$, *where* G_1 *is isomorphic to a uninorm with continuous underlying functions,* G_2 *is isomorphic to an n-uninorm from Class* 1 *restricted to* $]0,1[^2$ *and* G_3 *is isomorphic to a t-conorm. Moreover, the order of semigroups in the ordinal sum construction is* $1 < 3 < 2$.

- *Case 2c*: $\mathbf{U^n(x_0, y_0) = x_0, y_1 > y_0, U^n(x_0, y_1) = y_1}$.

Similarly as in the case of 2-uninorms if $U^n(x_0, y_0) = x_0$, $y_1 > y_0$ and $U^n(x_0, y_1) = y_1$ then the structure of U^n on $([0, x_0] \cup \left]y_0, 1\right])^2$ can be rather peculiar. However, it is possible to show that U^n on $([0, x_0] \cup \left]y_0, 1\right])^2$ can be expressed as an ordinal sum of semigroups $G_1 = ([0, x_0[\cup \left[y_1, 1\right], U^n)$, $G_2 = (\left]y_0, y_1\right[, U^n)$ and $G_3 = (\{x_0\}, \mathrm{Id})$, with $1 < 3 < 2$, where U^n on $\left]y_0, y_1\right[^2$ is isomorphic to a t-conorm restricted to $]0,1[^2$. Moreover, G_1 can be expressed as an ordinal sum of a countable number of semigroups related to representable uninorms, continuous Archimedean t-norms, continuous Archimedean t-conorms and internal uninorms (including the min and the max operator). However, y_1 need not to be the neutral element of G_1.

Theorem 5.15: *Let* $U^n\colon [0,1]^2 \longrightarrow [0,1]$ *be an n-uninorm and let* $U^n \in \mathcal{U}_n$*. Assume* $U^n(x_0, y_0) = x_0, y_1 > y_0$ *and* $U^n(x_0, y_1) = y_1$, $U^n(z_k, y_0) = z_k$*. Then* U^n *is an ordinal sum of four semigroups* $G_1 = ([0, x_0[\cup [y_1, 1], U^n)$, $G_2 = (]y_0, y_1[, U^n)$, $G_3 = (\{x_0\}, \mathrm{Id})$ *and* $G_4 = (]x_0, y_0], U^n)$*, where* G_1 *can be expressed as an ordinal sum of a countable number of semigroups related to representable uninorms, continuous Archimedean t-norms, continuous Archimedean t-conorms and internal uninorms,* G_2 *is isomorphic to a restriction of a continuous t-conorm to the open unit square and* G_4 *is isomorphic to an n-uninorm from Class* 1 *restricted to* $]0,1]^2$*. Moreover, the order of semigroups in the ordinal sum construction is* $1 < 3 < 2 < 4$.

Theorem 5.16: *Let* $U^n\colon [0,1]^2 \longrightarrow [0,1]$ *be an n-uninorm and let* $U^n \in \mathcal{U}_n$*. Assume* $U^n(x_0, y_0) = x_0, y_1 > y_0$ *and* $U^n(x_0, y_1) = y_1$, $U^n(z_k, y_0) = y_0$*. Then* U^n *is an ordinal sum of four semigroups* $G_1 = ([0, x_0[\cup [y_1, 1], U^n)$, $G_2 = ([y_0, y_1[, U^n)$, $G_3 = (\{x_0\}, \mathrm{Id})$ *and* $G_4 = (]x_0, y_0[, U^n)$*, where* G_1 *can be expressed as an ordinal sum of a countable number of semigroups related to representable uninorms, continuous Archimedean t-norms, continuous Archimedean t-conorms and internal uninorms,* G_2 *is isomorphic to a restriction of a continuous t-conorm to* $[0,1[^2$ *and* G_4 *is isomorphic to an n-uninorm from Class* 1 *restricted to* $]0,1[^2$*. Moreover, the order of semigroups in the ordinal sum construction is* $1 < 3 < 2 < 4$.

6. Conclusions

We have studied n-uninorms with continuous underlying t-norms and t-conorms and we have shown that each n-uninorm with continuous underlying functions can be expressed as an ordinal sum of a uninorm with continuous underlying functions (possibly also of a countable number of semigroups related to representable uninorms, continuous Archimedean t-norms, continuous Archimedean t-conorms and internal uninorms) and an n-uninorm such that $U^n(0,1) = z_k$ for $k \in \{1, \ldots, n-1\}$ (possibly restricted to open or half-open unit square $]0,1[^2$, $[0,1[^2$, $]0,1]^2$).

In the future work, we would like to study characterizing functions of n-uninorms with continuous underlying functions and their relation to the points of discontinuity of such an n-uninorm. We would also like to study the decomposition of n-uninorms via the z-ordinal sum construction into Archimedean and idempotent semigroups.

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Notes on contributor

Andrea Mesiarová-Zemánková received the Master's degree in mathematics from the Faculty of Mathematics, Physics and Informatics, Comenius University, Bratislava, Slovakia, in 2002, and the Ph.D. degree in applied mathematics from the Mathematical Institute of the Slovak Academy of Sciences, Bratislava, Slovakia, in 2005. Since 2005 she has been a Researcher at the Mathematical Institute, Slovak Academy of Sciences. In years 2007–2009 and 2010-2012 she worked at the Department of Computer Science, Trinity College, Dublin, Ireland as a Research Assistant. Her current research interests include the aggregation theory, associative functions on bounded lattices, multi-polar aggregation and non-additive measures and integrals.

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Characterizing functions of n-uninorms with continuous underlying functions

Andrea Mesiarová-Zemánková
Mathematical Institute, Slovak Academy of Sciences
Bratislava, SLOVAKIA
zemankova@mat.savba.sk

***Abstract*—The characterizing set-valued functions were introduced for uninorms with continuous underlying functions and these set-valued functions were useful for the complete characterization of such uninorms. In this work we study the characterizing functions of n-uninorms with continuous underlying t-norms and t-conorms. We will show that an n-uninorm with continuous underlying functions possesses n characterizing set-valued functions, where the graphs of these characterizing set-valued functions cover the set of all points of discontinuity of the respective n-uninorm. Moreover, for $i = 1, \ldots, n$, the i-th characterizing set-valued function divides the unit square into the two sets, where below the graph of the i-th characterizing set-valued function the n-uninorm attains values smaller than the local neutral element e_i and above the graph of the i-th characterizing set-valued function the n-uninorm attains values greater than the local neutral element e_i.**

***Index Terms*—n-uninorm, uninorm, representable uninorm, t-norm, t-conorm, nullnorm**

I. Introduction

The uninorm operators generalize both the t-norm and the t-conorm operators and they can be used to model bipolar behaviour (see [1], [2], [3], [4], [5]). The uninorms with continuous underlying t-norm and t-conorm were in the center of the interest for a long time and their complete characterization was given in [6], [7]. In [6] it was shown that each uninorm with continuous underlying functions can be decomposed into an ordinal sum of a countable number of semigroups related to representable uninorms, continuous Archimedean t-norms, continuous Archimedean t-conorms and internal uninorms (including the $\min$ and the $\max$ operator). In [7] it was shown that the set of all points of discontinuity of a uninorm with continuous underlying functions is a subset of the graph of the characterizing set-valued function of such a uninorm. Note that the notion of the characterizing set-valued function is closely connected with the non-increasing function from [8] which characterizes an idempotent uninorm.

Another possible generalization of t-norms and t-conorms are nullnorms (also called t-operators) [9], [11], [12], which consist of a t-conorm and a t-norm which are glued together by an annihilator $a \in \,]0, 1[$.

The above generalizations bring together t-norms and t-conorms. In the second step a notion that brings together uninorms and nullnorms was introduced by Akella [13]. These special aggregation functions are called n-uninorms and each n-uninorm possesses n local neutral elements. Note that each n-uninorm has a block structure, where the blocks around the main diagonal consist of t-norms, t-conorms, uninorms and nullnorms. Therefore, similarly as in the case of the ordinal sum construction, in applications we can assume different uninorms on distinct areas separated by division points $z_1, \ldots, z_{n-1}$. The class of n-uninorms is interesting for theoretical studies, as well as for applications. Recall for example the study of the distributivity [27], modularity conditions [28], ordering [22] and applications in neural networks [10]. The description of the basic structure of n-uninorms can be found in [13], [14].

The idempotent n-uninorms were discussed in [15] where it was shown that each idempotent n-uninorm can be expressed as an ordinal sum of an idempotent uninorm (possibly also of a countable number of idempotent semigroups with operations $\min$ and $\max$) and an idempotent n-uninorm such that $U^n(0,1) = z_k$ (possibly restricted to $]0,1[^2$, or $[0,1[^2$, or $]0,1]^2$) for some $k \in \{1, \ldots, n-1\}$. Similar results were shown in [16] for n-uninorms with continuous underlying functions.

Since the structure of uninorms with continuous underlying functions was described using the notion of the characterizing set-valued function, we would like to do the same also for the set of n-uninorms. In this paper we want to show that each n-uninorm possesses n characterizing set-valued functions which have similar properties as the characterizing set-valued function of a uninorm with continuous underlying functions, and the i-th characterizing set-valued function is connected to the i-th local neutral element. Similarly as in the case of uninorms with continuous underlying functions, in the follow-up paper we would like to describe the decomposition of n-uninorms with continuous underlying functions into irreducible semigroups using the ordinal sum and the z-ordinal sum construction (see [15]).

The paper is organized as follows. In Section II we will recall all necessary basic notions and results. We introduce the characterizing (set-valued) functions for n-uninorms and show some basic results in Section III. In Sections IV we show that graphs of these characterizing set-valued functions cover the set of all points of discontinuity of the corresponding n-uninorm and that an n-uninorm which is continuous in all points that are not covered by the graphs of the characterizing set-valued functions have continuous underlying functions whenever U^n is in each point $(x_0, y_0) \in [0,1]^2$, which is covered by exactly one characterizing set-valued function, either right-continuous, or left-continuous (or continuous). We give our conclusions in Section V.

II. Basic notions and results

Let us now recall all necessary basic notions.

A triangular norm is a function $T\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and 1 is its neutral element. Dual functions to t-norms are t-conorms. A triangular conorm is a function $S\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and 0 is its neutral element.

Each continuous t-norm (t-conorm) is equal to an ordinal sum (see [17]) of continuous Archimedean t-norms (t-conorms). Note that a continuous t-norm (t-conorm) is Archimedean if and only if it has only trivial idempotent points 0 and 1. A continuous Archimedean t-norm T (t-conorm S) is either strict, i.e., strictly increasing on $]0,1]^2$ (on $[0,1[^2$), or nilpotent, i.e., there exists $(x,y) \in]0,1[^2$ such that $T(x,y) = 0$ ($S(x,y) = 1$). Moreover, each continuous Archimedean t-norm (t-conorm) has a continuous additive generator, and an additive generator of an Archimedean continuous t-norm T (t-conorm S) is uniquely determined up to a positive multiplicative constant. More details on t-norms and t-conorms can be found in [1], [2].

A uninorm (introduced in [3]) is a function $U\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and have a neutral element $e \in [0,1]$ (see also [4]). For each uninorm the value $U(1,0) \in \{0,1\}$ is the annihilator of U. A uninorm is called conjunctive (disjunctive) if $U(1,0) = 0$ ($U(1,0) = 1$).

For each uninorm U with the neutral element $e \in [0,1]$, the restriction of U to $[0,e]^2$ is a t-norm on $[0,e]^2$, i.e., a linear transformation of some t-norm T_U on $[0,1]^2$ and the restriction of U to $[e,1]^2$ is a t-conorm on $[e,1]^2$, i.e., a linear transformation of some t-conorm S_U on $[0,1]^2$. Moreover, $\min(x,y) \le U(x,y) \le \max(x,y)$ for all $(x,y) \in [0,e] \times [e,1] \cup [e,1] \times [0,e]$. We will denote the set of all uninorms U such that T_U and S_U are continuous by $\mathcal{U}$.

Similarly as in the case of t-norms and t-conorms we can construct uninorms using additive generators (see [4]). Uninorms that are generated by a continuous, strictly increasing additive generator are called representable.

Proposition II.1 ([18])
Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a uninorm continuous everywhere on the unit square except of the two points $(0,1)$ and $(1,0)$. Then U is representable.

A uninorm $U\colon [0,1]^2 \longrightarrow [0,1]$ is called internal if $U(x,y) \in \{x,y\}$ for all $(x,y) \in [0,1]^2$. Further, U is called locally internal on $A(e)$ if U is internal on $A(e) = [0,e] \times [e,1] \cup [e,1] \times [0,e]$. More results on internal and locally internal uninorms can be found in [8], [19], [20], [21], [24].

Note that if a uninorm U is internal then it is also idempotent, i.e., $U(x,x) = x$ for all $x \in [0,1]$, and vice-versa. Let us recall the basic result from [8] that characterizes idempotent uninorms.

Theorem II.2
Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a binary function. Then U is an idempotent uninorm with the neutral element $e \in]0,1[$ if and only if there exists a non-increasing function $g\colon [0,1] \longrightarrow [0,1]$, symmetric with respect to the main diagonal, with $g(e) = e$, such that

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } y < g(x) \text{ or } (y = g(x) \\ & \quad \text{and } x < g(g(x))), \\ \max(x,y) & \text{if } y > g(x) \text{ or } (y = g(x) \\ & \quad \text{and } x > g(g(x))), \\ x \text{ or } y & \text{if } y = g(x) \text{ and } x = g(g(x)), \end{cases}$$

being commutative in the points (x,y) such that $y = g(x)$ with $x = g(g(x))$.

Note that the graph of the function g from Theorem II.2 is a subset of the graph of the characterizing set-valued function of an idempotent uninorm (for more details see [7], [25]). Therefore the completed graph of the function g divides the idempotent uninorm U into two parts: below the completed graph of g we have $U(x,y) = \min(x,y)$, i.e., $U(x,y) < e$, and above the completed graph of g there is $U(x,y) = \max(x,y)$, i.e., $U(x,y) > e$.

Next we will recall the definition of a set-valued function from [7]. Note that $\mathcal{P}(X)$ in the following definition denotes the power set of X.

Definition II.3
A mapping $p\colon [0,1] \longrightarrow \mathcal{P}([0,1])$ is called a set-valued function on $[0,1]$. Assuming the standard order on $[0,1]$, a set-valued function p is called

(i) non-increasing if for all $x_1, x_2 \in [0,1]$, $x_1 < x_2$, we have $y_1 \ge y_2$ for all $y_1 \in p(x_1)$ and all $y_2 \in p(x_2)$, i.e., $\sup p(x_2) \le \inf p(x_1)$ and thus $p(x_1)$ and $p(x_2)$ intersect in, at most, a single point,

(ii) symmetric if for all $x, y \in [0,1]$ it holds $y \in p(x)$ if and only if $x \in p(y)$,

(iii) u-surjective if for all $y \in [0,1]$ there exists an $x \in [0,1]$ such that $y \in p(x)$.

The graph of a set-valued function p will be denoted by $G(p)$, i.e., for $x, y \in [0,1]$ there is $(x,y) \in G(p)$ if and only if $y \in p(x)$.

Lemma II.4 ([7])
A symmetric set-valued function $p\colon [0,1] \longrightarrow \mathcal{P}([0,1])$ is u-surjective if and only if we have $p(x) \ne \emptyset$ for all $x \in [0,1]$.

The graph of a symmetric, u-surjective, non-increasing set-valued function $p\colon [0,1] \longrightarrow \mathcal{P}([0,1])$ is a connected bounded curve (i.e., a connected bounded set with no interior) containing points $(0,1)$ and $(1,0)$ (see [7]).

The following is the definition of a nullnorm [9]. Note that t-operators were independently defined in [12] and in [11] it was shown that t-operators and nullnorms coincide.

Definition II.5
A binary function $V\colon [0,1]^2 \longrightarrow [0,1]$ is called a nullnorm if it is commutative, associative, non-decreasing in each variable

and has an annihilator $z \in [0,1]$ such that $V(0,x) = x$ for all $x \le z$ and $V(1,x) = x$ for all $x \ge z$.

If $z = 0$ ($z = 1$) then V is a t-norm (t-conorm). Note that for a commutative, associative and non-decreasing function $F\colon [0,1]^2 \longrightarrow [0,1]$ the value $F(0,1)$ is always an annihilator of F. Thus for a nullnorm $z = V(0,1)$.

Now let us recall the definition of an n-uninorm (see [13]).

Definition II.6
Assume an $n \in \mathbb{N} \setminus \{1\}$. Let $V\colon [0,1]^2 \longrightarrow [0,1]$ be a commutative binary function. Then $\{e_1, \dots, e_n\}_{z_1,\dots,z_{n-1}}$ is called an n-neutral element of V if for $0 = z_0 < z_1 < \dots < z_n = 1$ and $e_i \in [z_{i-1}, z_i]$, $i = 1, \dots, n$, we have $V(e_i, x) = x$ for all $x \in [z_{i-1}, z_i]$.

Definition II.7
A binary function $U^n\colon [0,1]^2 \longrightarrow [0,1]$ is an n-uninorm if it is associative, non-decreasing in each variable, commutative and has an n-neutral element $\{e_1, \dots, e_n\}_{z_1,\dots,z_{n-1}}$.

The basic structure of n-uninorms was described by Akella in [13] and the characterization of the main five classes of 2-uninorms was given in [14].

Each n-uninorm has the following building blocks around the main diagonal.

Proposition II.8
Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be an n-uninorm with the n-neutral element $\{e_1, \dots, e_n\}_{z_1,\dots,z_{n-1}}$. Then
(i) U^n restricted to $[z_{i-1}, e_i]^2$, for $i = 1, \dots, n$, is isomorphic to a t-norm. We will denote this t-norm by T_i.
(ii) U^n restricted to $[e_i, z_i]^2$ for $i = 1, \dots, n$, is isomorphic to a t-conorm. We will denote this t-conorm by S_i.
(iii) U^n restricted to $[z_{i-1}, z_i]^2$ for $i = 1, \dots, n$, is isomorphic to a uninorm. We will denote this uninorm by U_i.
(iv) U^n restricted to $[e_i, e_{i+1}]^2$ for $i = 1, \dots, n-1$, is isomorphic to a nullnorm. We will denote this nullnorm by V_i.
(v) U^n restricted to $[z_i, z_j]^2$ for $i, j \in \{0, 1, \dots, n\}$, $i < j$, is isomorphic to a $(j-i)$-uninorm.

For $n \in \mathbb{N}$ we will denote the set of all n-uninorms U^n such that $T_1, \dots, T_n$ and $S_1, \dots, S_n$ are continuous by $\mathcal{U}_n$.

III. Characterizing functions for n-uninorms with continuous underlying functions

From now on, if we say that a function is an n-uninorm we will suppose that it possesses the n-neutral element $\{e_1, \dots, e_n\}_{z_1,\dots,z_{n-1}}$. Observe that it can happen that $e_i = e_{i+1}$ for some $i \in \{1, \dots, n-1\}$. However, in such a case $e_i = z_i = e_{i+1}$ and thus e_i is the neutral element of U^n on $[z_{i-1}, z_{i+1}]^2$ and the nullnorm around point z_i is degenerate to just one point. Here U^n is in fact an $(n-1)$-uninorm and we can drop e_i, z_i from the n-neutral element. Therefore in this paper we will suppose $e_i \neq e_{i+1}$ for all $i \in \{1, \dots, n-1\}$.

The notion of the characterizing set-valued function can be approached from the two sides. On one hand, the notion of the characterizing set-valued function was constructed to cover the set of all points of discontinuity of a uninorm. On the other hand, the non-increasing function g from Theorem II.2 for idempotent uninorms was constructed to separate the area where the uninorm is equal to the minimum operator and where it is equal to the maximum operator. It is perhaps not a surprise that these two functions coincide in that sense that the graph of the non-increasing function g is a subset of the graph of the characterizing set-valued function of an idempotent uninorm. However, the non-increasing function g contains more information since for points $x \in [0,1]$, for which the functional value of the characterizing set-valued function is a set which contains more than one element, one cannot see where exactly the cut $U(x,\cdot)$ is discontinuous, i.e., where the change from the minimum to the maximum operator occurs. However, the non-increasing function g is defined merely for idempotent uninorms. To keep as much information as possible we can easily define g for all uninorms with continuous underlying functions. Here instead of change from the minimum to the maximum operator we will look for a change from values smaller than the neutral element e to the values greater than the neutral element e. A similar approach can be adopted also for n-uninorms from $\mathcal{U}_n$. In order to keep as much information as possible we will use both notions, the characterizing set-valued function and the non-increasing function g which will be simply called the characterizing function.

However, there is a small difference. While in the case of uninorms for each $x \in [0,1]$ there exists at most one $y \in [0,1]$ such that $U(x,y) = e$, in the case of n-uninorms this need not be valid in the case when $e_i = z_{i-1}$, or $e_i = z_i$ for some $i \in \{1, \dots, n\}$. Assume for example that $e_i = z_i$ for some $i \in \{0, \dots, n\}$. Then $U^n(z_i, z_i) = e_i = U^n(z_i, e_{i+1})$.

Example III.1
Assume a 2-uninorm $U_1^2\colon [0,1]^2 \longrightarrow [0,1]$ with the 2-neutral element $\{\frac{1}{2}, \frac{3}{4}\}_{\frac{1}{2}}$ given by

$$U_1^2(x,y) = \begin{cases} \min(x,y) & \text{if } \min(x,y) < \frac{1}{2}, \\ \max(\frac{1}{2}, x+y-\frac{3}{4}) & \text{if } x,y \in \left[\frac{1}{2}, \frac{3}{4}\right], \\ \max(x,y) & \text{otherwise.} \end{cases}$$

Then $e_1 = z_1 = \frac{1}{2}$ and for all $(x,y) \in \left[\frac{1}{2}, \frac{3}{4}\right]^2$ with $x + y \le \frac{5}{4}$ we have $U_1^2(x,y) = e_1$. Note that U_1^2 (see Figure 1) is both the 2-uninorm and a uninorm since the local neutral element $\frac{3}{4}$ is also the global neutral element. A dual case is a 2-uninorm $U_2^2\colon [0,1]^2 \longrightarrow [0,1]$ with the 2-neutral element $\{\frac{1}{4}, \frac{1}{2}\}_{\frac{1}{2}}$ given by

$$U_2^2(x,y) = \begin{cases} \max(x,y) & \text{if } \max(x,y) > \frac{1}{2}, \\ \min(\frac{1}{2}, x+y-\frac{1}{4}) & \text{if } x,y \in \left[\frac{1}{4}, \frac{1}{2}\right], \\ \min(x,y) & \text{otherwise.} \end{cases}$$

Here $e_2 = z_1 = \frac{1}{2}$ and for all $(x,y) \in \left[\frac{1}{4}, \frac{1}{2}\right]^2$ with $x + y \ge \frac{3}{4}$ we have $U_1^2(x,y) = e_2$. Note that U_2^2 (see Figure 1) is both the 2-uninorm and a uninorm since the local neutral element $\frac{1}{4}$ is also the global neutral element.

The previous example shows that in the case of n-uninorms where $e_i = z_j$ for some $i, j \in \{0, \dots, n\}$ it is not possible

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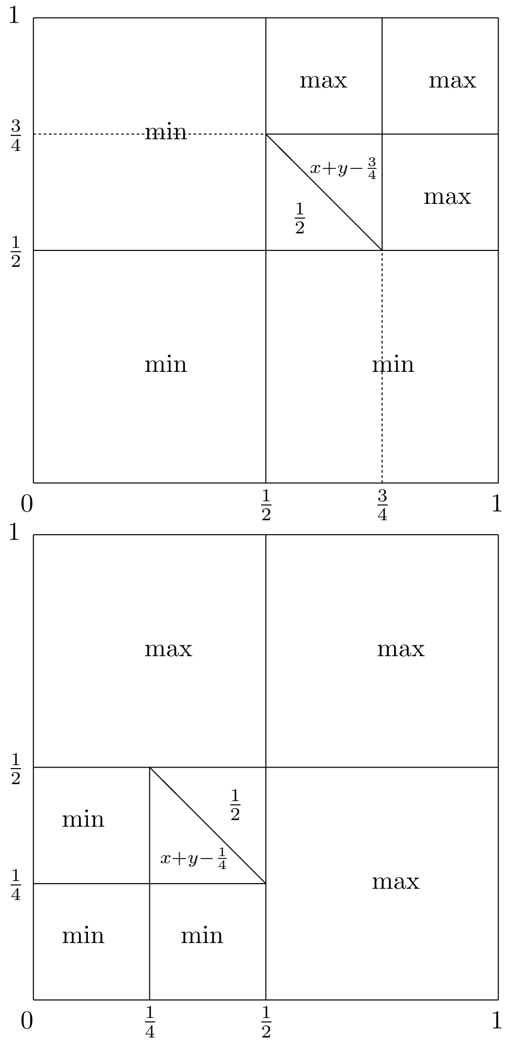

Fig. 1. The 2-uninorms U_1^2 (left) and U_2^2 (right) from Example III.1.

to define the i-th characterizing set-valued function in such a way that below its graph the n-uninorm will attain values smaller than e_i and above it will attain the values greater than e_i. However, in the following we will show that if $e_i = z_j$ for some $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, n-1\}$ then the n-uninorm can be reduced to an m-uninorm with $m < n$. First we recall the following result for idempotent points of an n-uninorm from [16].

Lemma III.2
Let $U^n \colon [0,1]^2 \longrightarrow [0,1]$ be an n-uninorm and let $U^n \in \mathcal{U}_n$. If $a \in [0,1]$ is an idempotent point of U^n then $U^n(a,x) \in \{x,a\} \cup \{z_i \mid z_i \in]\min(a,x), \max(a,x)[\}$ for all $x \in [0,1]$.

The previous result implies that for all $i, j \in \{0, \ldots, n\}$, $i \leq j$, there is $U^n(z_i, z_j) = z_m$ for some $m \in \{i, \ldots, j\}$. The monotonicity further implies that then z_m is the annihilator of U^n on $[z_i, z_j]^2$. Further, if for $i, j \in \{1, \ldots, n\}$, $i < j$, there is $U^n(z_i, z_{j-1}) = z_m$ for some $m \in \{i, \ldots, j-1\}$ then we have also $U^n(e_i, e_j) = z_m$ since $z_m = U^n(z_i, z_{j-1}) = U^n(z_i, U^n(z_{j-1}, e_j)) = U^n(U^n(z_i, z_{j-1}), e_j) = U^n(z_m, e_j)$ and similarly $z_m = U^n(z_m, e_i)$ and $e_i \leq z_i \leq z_m \leq z_{j-1} \leq e_j$ implies

$$z_m = U^n(e_i, z_m) \leq U^n(e_i, e_j) \leq U^n(z_m, e_j) = z_m.$$

Proposition III.3
Let $U^n \colon [0,1]^2 \longrightarrow [0,1]$ be an n-uninorm and let $U^n \in \mathcal{U}_n$. If $U^n(x, y_1) = U^n(x, y_2) = e_i$ for some $x, y_1, y_2 \in [0,1]$, $y_1 < y_2$, and $i \in \{1, \ldots, n\}$ then $e_i \in \{z_{i-1}, z_i\}$.

PROOF: First assume that $x \notin [z_{i-1}, z_i]$. Then there exists a $j \in \{1, \ldots, n\}$, $i \neq j$, such that $x \in [z_{j-1}, z_j]$. Then $U^n(x, y_1) = e_i$ implies

$$e_i = U^n(x, y_1) = U^n(U^n(e_j, x), y_1) = U^n(e_j, U^n(x, y_1)) = U^n(e_j, e_i) = z_m$$

for some $m \in \{1, \ldots, n-1\}$. Since $z_{i-2} < z_{i-1} \leq e_i \leq z_i < z_{i+1}$ we see that then $e_i \in \{z_{i-1}, z_i\}$. Similarly, if $y_1 \notin [z_{i-1}, z_i]$ or $y_2 \notin [z_{i-1}, z_i]$ we obtain $e_i \in \{z_{i-1}, z_i\}$. Now suppose that $x, y_1, y_2 \in [z_{i-1}, z_i]$. Then the associativity implies

$$y_1 = U^n(y_1, e_i) = U^n(y_1, U^n(x, y_2)) = U^n(U^n(y_1, x), y_2) = U^n(e_i, y_2) = y_2,$$

which is a contradiction. Thus in all possible cases $e_i \in \{z_{i-1}, z_i\}$. □

Theorem III.4
Let $U^n \colon [0,1]^2 \longrightarrow [0,1]$ be an n-uninorm, $U^n \in \mathcal{U}_n$. If for some $i \in \{1, \ldots, n\}$ there is $e_i = z_j$ for $j \in \{1, \ldots, n-1\}$ then U^n is an $(n-1)$-uninorm from $\mathcal{U}_{(n-1)}$ with the $(n-1)$-neutral element $\{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n\}_{z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{n-1}}$.

PROOF: The definition of the n-neutral element implies $e_i \in \{z_{i-1}, z_i\}$. We will suppose that $e_i = z_i$ as the other case is analogous. Then $U^n(z_i, e_{i+1}) = z_i$ and for all $x \in [z_{i-1}, z_i]$ there is

$$x = U^n(x, e_i) = U^n(x, z_i) = U^n(x, U^n(z_i, e_{i+1})) = U^n(U^n(x, z_i), e_{i+1}) = U^n(x, e_{i+1})$$

and therefore e_{i+1} is the neutral element of the restriction of U^n to $[z_{i-1}, z_{i+1}]^2$ and thus U^n restricted to $[z_{i-1}, z_{i+1}]^2$ is a uninorm. Moreover, for all $x \in [z_{i-1}, z_i]$, $y \in [z_i, e_{i+1}]$ there is $z_i = U^n(z_i, z_i) \leq U^n(z_i, y) \leq U^n(z_i, e_{i+1}) = z_i$, i.e., $U^n(z_i, y) = z_i$ and

$$U^n(x, y) = U^n(U^n(x, z_i), y) = U^n(x, U^n(z_i, y)) = U^n(x, z_i) = x.$$

Since U^n restricted to $[z_{i-1}, z_i]^2$ ($[z_i, e_{i+1}]^2$) is isomorphic to a continuous t-norm, we see that on $[z_{i-1}, e_{i+1}]^2$ is U^n isomorphic to an ordinal sum of continuous t-norms, i.e., to a continuous t-norm. Since $U^n \in \mathcal{U}_n$ we know that U^n restricted to $[e_{i+1}, z_{i+1}]^2$ is isomorphic to a continuous t-conorm. Thus on $[z_{i-1}, z_{i+1}]^2$ the n-uninorm U^n is isomorphic to a uninorm with continuous underlying functions. Summarizing, U^n is an

$(n-1)$-uninorm from $\mathcal{U}_{(n-1)}$ with the $(n-1)$-neutral element $\{e_1,\dots,e_{i-1},e_{i+1},\dots,e_n\}_{z_1,\dots,z_{i-1},z_{i+1},\dots,z_{n-1}}$. □

The previous result shows that for each e_i that is equal to z_j for $j \in \{1,\dots,n-1\}$ we can reduce the order of the n-uninorm by one. Using this procedure repeatedly we see that each n-uninorm U^n from $\mathcal{U}_n$ can be seen as an m-uninorm U^m from $\mathcal{U}_m$ such that if e_i is the i-th local neutral element of U^m then $e_i \in \{z_{i-1}, z_i\}$ implies $e_i \in \{0,1\}$. Then the m-uninorm U^m will be called the reduced form of the n-uninorm U^n (reduced m-uninorm for short). Therefore in the following section it is enough to focus just on reduced n-uninorms.

We will now continue with several useful definitions and results.

Definition III.5
For an n-uninorm $U^n\colon [0,1]^2 \longrightarrow [0,1]$ and each $x \in [0,1]$ we define a function $u_x^n\colon [0,1] \longrightarrow [0,1]$ by $u_x^n(z) = U^n(x,z)$ for $z \in [0,1]$. Further, for each $x \in [0,1]$ we denote $x_{U^n}^{(1)} = x$ and $x_{U^n}^{(m)} = U^n(x, x_{U^n}^{(m-1)})$ for all $m \in \mathbb{N}\setminus\{1\}$.

Lemma III.6
Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be an n-uninorm and let $U^n \in \mathcal{U}_n$. If $x \in]z_{i-1}, z_i[$ and $y \in]z_{j-1}, z_j[$ for some $i,j \in \{1,\dots,n\}$, $i \neq j$, $x<y$, then $U^n(x,y) \in [x, e_i[\cup \{U^n(e_i,e_j)\} \cup]e_j, y]$.

PROOF: Let $U^n(e_i,e_j) = z_k$ for some $k \in \{i,\dots,j\}$. If $U^n(x,z_k) = z_p$ for some $z_p \in]x, z_k[$ then $z_p \geq e_i$ and therefore $U^n(z_k,z_p) = z_k$. Then, however,

$$z_p = U^n(x,z_k) = U^n(x, U^n(z_k,z_k)) = U^n(U^n(x,z_k),z_k) = U^n(z_p,z_k) = z_k,$$

i.e., $p=k$ and thus $U^n(x,z_k) \in \{x, z_k\}$. Similarly we can show that $U^n(y,z_k) \in \{y,z_k\}$.

Then we have the following possibilities:

(i) If $U^n(x,z_k) = z_k$ and $U^n(y,z_k) = z_k$. Then
$$z_k = U^n(x,z_k) \leq U^n(x,y) \leq U^n(y,z_k) = z_k.$$
Thus $U^n(x,y) = z_k$.

(ii) If $U^n(x,z_k) = z_k$ and $U^n(y,z_k) = y$. Then
$$U^n(x,y) = U^n(x, U^n(z_k,y)) = U^n(U^n(x,z_k),y) = U^n(z_k,y) = y.$$

(iii) If $U^n(x,z_k) = x$ and $U^n(y,z_k) = z_k$. Then
$$U^n(y,x) = U^n(y, U^n(z_k,x)) = U^n(U^n(y,z_k),x) = U^n(z_k,x) = x.$$

(iv) If $U^n(x,z_k) = x$ and $U^n(y,z_k) = y$. First assume that $U^n(x,y) \in [e_i,e_j]$. Then
$$U^n(x,y) = U^n(x, U^n(y,z_k)) = U^n(U^n(x,y),z_k) = z_k.$$
If $U^n(x,y) < e_i$ then $x = U^n(x,e_i) \leq U^n(x,y) < e_i$ and if $U^n(x,y) > e_j$ then $y = U^n(y,e_j) \geq U^n(y,x) > e_j$.

Summarizing, $U^n(x,y) \in [x, e_i[\cup \{z_k\} \cup]e_j, y]$. □

In the following two results we will examine the continuity of the cuts u_x^n.

Lemma III.7
Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be an n-uninorm, $U^n \in \mathcal{U}_n$ and assume an $x \in [0,1]$.

(i) If $e_i \in \mathrm{Ran}(u_x^n)$ for some $i \in \{1,\dots,n\}$ then $[z_{i-1}, z_i] \subset \mathrm{Ran}(u_x^n)$.

(ii) If $e_i \notin \mathrm{Ran}(u_x^n)$ for some $i \in \{1,\dots,n\}$ and $p \in \mathrm{Ran}(u_x^n)$ for some $p \in [z_{i-1}, e_i[$ then $[z_{i-1}, p] \subset \mathrm{Ran}(u_x^n)$.

(iii) If $e_i \notin \mathrm{Ran}(u_x^n)$ for some $i \in \{1,\dots,n\}$ and $q \in \mathrm{Ran}(u_x^n)$ for some $q \in]e_i, z_i]$ then $[q, z_i] \subset \mathrm{Ran}(u_x^n)$.

PROOF:

(i) Assume $U^n(x,y) = e_i$ for some $y \in [0,1]$. Then for any $t \in [z_{i-1}, z_i]$ we have $t = U^n(e_i,t) = U^n(U^n(x,y),t) = U^n(x, U^n(y,t)) = u_x^n(U^n(y,t))$, i.e., $t \in \mathrm{Ran}(u_x^n)$.

(ii) Since U^n restricted to $[z_{i-1}, e_i]$ is isomorphic to a continuous t-norm for all $t \in [z_{i-1}, p]$ there exists a $p_t \in [z_{i-1}, p]$ such that $U^n(p, p_t) = t$. Then for $y \in [0,1]$ such that $U^n(x,y) = p$ we get
$$t = U^n(p,p_t) = U^n(U^n(x,y),p_t) = U^n(x, U^n(y,p_t)) = u_x^n(U^n(y,p_t)),$$
i.e., $t \in \mathrm{Ran}(u_x^n)$.

(iii) This result can be shown analogously as (ii).

□

Lemma III.8
Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be an n-uninorm and let $U^n \in \mathcal{U}_n$. Then for each $x \in [0,1]$ there are at most n points of discontinuity of u_x^n.

PROOF: First let us note that a monotone function is continuous whenever its range is a connected set. Similarly, a monotone function is non-continuous in exactly n points whenever its range is a union of $n+1$ connected sets.

Lemma III.7 implies that $\mathrm{Ran}(u_x^n) \cap [e_{i-1}, e_i]$ is a connected set for all $i = 1,\dots,n,n+1$, where $e_0 = 0$ and $e_{n+1} = 1$. Therefore the range of u_x^n can be expressed as a union of at most $n+1$ connected sets, i.e., u_x^n has at most n points of discontinuity.

□

Now we are going to define the characterizing functions of an n-uninorm $U^n \in \mathcal{U}_n$ which is for idempotent uninorms described in Theorem II.2. On each cut u_x^n for $x \in [0,1]$ and $i \in \{1,\dots,n\}$ we would like to select such a point $y \in [0,1]$ that $u_x^n(t) < e_i$ for all $t<y$ and $u_x^n(t) > e_i$ for all $t>y$. From Proposition III.3 we know that this is possible whenever $e_i \notin \{z_{i-1}, z_i\}$. In the other case we have a problem for $x \in [e_i, e_{i+1}[$ ($x \in]e_{i-1}, e_i]$) in the case when $e_i = z_i$ ($e_i = z_{i-1}$). However, from the previous discussion we know that it is enough to focus just on reduced n-uninorms. For reduced

n-uninorms there are only two anomalous cases: when $e_1 = 0$ and when $e_n = 1$. However, in the first case $e_1 = 0 < z_1$ and the monotonicity of U^n implies that $U^n(x,0) = U^n(0,x) > 0$ for all $x > 0$, i.e., $U^n(x,y) = e_1$ implies $x = y = 0$. In the second case $e_n = 1 > z_{n-1}$ and the monotonicity of U^n implies that $U^n(x,1) = U^n(1,x) < 1$ for all $x < 1$, i.e., $U^n(x,y) = e_n$ implies $x = y = 1$. Therefore for reduced n-uninorms and any $i \in \{1,\ldots,n\}$ there for each $x \in [0,1]$ exists at most one $y \in [0,1]$ such that $U^n(x,y) = e_i$.

Definition III.9
Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be a reduced n-uninorm, let $U^n \in \mathcal{U}_n$ and assume an $i \in \{1,\ldots,n\}$. Define a function $g_i\colon [0,1] \longrightarrow [0,1]$ by

$$g_i(x) = \sup\{t \in [0,1] \mid U^n(x,t) < e_i\},$$

where $\sup\emptyset = 0$. The function g_i will be called the i-th characterizing function of the n-uninorm U^n.

Note that evidently $g_i(e_i) = e_i$ for all $i \in \{1,\ldots,n\}$. Further, if $e_1 = 0$ then $g_1(x) = 0$ for all $x \in [0,1]$. Similarly, if $e_n = 1$ then $g_n(x) = 1$ for all $x \in [0,1]$.

Now we are going to show that the characterizing function g_i of an n-uninorm U^n from $\mathcal{U}_n$ is non-increasing for all $i \in \{1,\ldots,n\}$.

Proposition III.10
Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be a reduced n-uninorm and let $U^n \in \mathcal{U}_n$. Then the characterizing function g_i is non-increasing for all $i = 1,\ldots,n$.

PROOF: Assume $i \in \{1,\ldots,n\}$. Let $x_1, x_2 \in [0,1]$, $x_1 < x_2$. Then for all $t \in [0,1]$ such that $U^n(x_2,t) < e_i$ there is $U^n(x_1,t) < e_i$. Therefore $g_i(x_1) \geq g_i(x_2)$. $\square$

Remark III.11
(i) If $e_i \in \mathrm{Ran}(u^n_x)$ then Lemma III.7 implies that u^n_x is continuous in $g_i(x)$.
(ii) If $e_i \notin \mathrm{Ran}(u^n_x)$ then evidently u^n_x is non-continuous in $g_i(x)$.

Moreover, if for $x \in [0,1]$ we have $e_i \in \mathrm{Ran}(u^n_x)$ then $U^n(x, g_i(x)) = e_i$. Summarizing, either $U^n(x, g_i(x)) = e_i$, or u^n_x is non-continuous in $g_i(x)$.

From the definition of the characterizing function we see that $U^n(x,t) < e_i$ for all $t < g_i(x)$ and $U^n(x,t) > e_i$ for all $t > g_i(x)$.

Further we will define the characterizing set-valued functions of an n-uninorm $U^n \in \mathcal{U}_n$.

Definition III.12
Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be a reduced n-uninorm, $U^n \in \mathcal{U}_n$, and assume an $i \in \{1,\ldots,n\}$. We define a set-valued function $r_i\colon [0,1] \longrightarrow \mathcal{P}([0,1])$ by

$$r_i(x) = \begin{cases} \left[\lim\limits_{t\longrightarrow 0^+} g_i(t), 1\right] & \text{if } x = 0, \\ \left[0, \lim\limits_{t\longrightarrow 1^-} g_i(t)\right] & \text{if } x = 1, \\ \left[\lim\limits_{t\longrightarrow x^+} g_i(t), \lim\limits_{t\longrightarrow x^-} g_i(t)\right] & \text{otherwise.} \end{cases}$$

Observe that if g_i is continuous in x for some $i \in \{1,\ldots,n\}$ and $x \in \,]0,1[$ then $r_i(x) = \{g_i(x)\}$.

Since g_i is non-increasing, it is easy to show the following result for a characterizing set-valued function of an n-uninorm with continuous underlying functions.

Lemma III.13
The characterizing set-valued function r_i of a reduced n-uninorm $U^n \in \mathcal{U}_n$ is non-increasing for all $i = 1,\ldots,n$. Further, above (below) the graph of the characterizing set-valued function r_i the n-uninorm U^n attains values greater (smaller) than e_i.

Next we will show that a characterizing set-valued function is symmetric.

Lemma III.14
Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be a reduced n-uninorm, $U^n \in \mathcal{U}_n$, and assume an $i \in \{1,\ldots,n\}$. Then the characterizing set-valued function r_i of U^n is symmetric.

PROOF: Recall that r_i is symmetric if for all $x,y \in [0,1]$ there is $y \in r_i(x)$ if and only if $x \in r_i(y)$. Assume $x,y \in [0,1]$ and let $y \in r_i(x)$.

• If $x = 0$ then $y \in \left[\lim\limits_{t\longrightarrow 0^+} g_i(t), 1\right]$. For all $w > \lim\limits_{t\longrightarrow 0^+} g_i(t)$ there is $U^n(w,s) = U^n(s,w) > e_i$ for all $s > 0$, i.e., $g_i(w) = 0$. Thus if $y > \lim\limits_{t\longrightarrow 0^+} g_i(t)$ there is $0 \in r_i(y)$. If $y = \lim\limits_{t\longrightarrow 0^+} g_i(t)$ then either $0 = \lim\limits_{w\longrightarrow y^+} g_i(w)$ and $0 \in r_i(y)$, or $y = 1$. In the second case $0 \in r_i(1)$ follows from Definition III.12.

• If $x = 1$ then $y \in \left[0, \lim\limits_{t\longrightarrow 1^-} g_i(t)\right]$. For all $w < \lim\limits_{t\longrightarrow 1^-} g_i(t)$ there is $U^n(w,s) = U^n(s,w) < e_i$ for all $s < 1$, i.e., $g_i(w) = 1$. Thus if $y < \lim\limits_{t\longrightarrow 1^-} g_i(t)$ there is $1 \in r_i(y)$. If $y = \lim\limits_{t\longrightarrow 1^-} g_i(t)$ then either $1 = \lim\limits_{w\longrightarrow y^-} g_i(w)$ and $1 \in r_i(y)$, or $y = 0$. In the second case $1 \in r_i(0)$ follows from Definition III.12.

• Further we will assume that $x \in \,]0,1[$. If $r_i(x) = [a,b]$ and $a < b$ then since $g_i(s) \geq b$ for all $s < x$ we get $U^n(s,t) < e_i$ for all $t \in \,]a,b[$, $s < x$. Moreover, since $g_i(s) \leq a$ for all $s > x$ we get $U^n(s,t) > e_i$ for all $t \in \,]a,b[$, $s > x$. Therefore $g_i(t) = x$ for all $t \in \,]a,b[$. We have three possibilities, either $y = a$, or $y = b$, or $y \in \,]a,b[$. If $y \in \,]a,b[$ then $g_i(y) = x$ and therefore $x \in r_i(y)$. Assume $y \in \{a,b\}$. Then $x = \lim\limits_{t\longrightarrow y^+} g_i(t)$ if $y = a$ ($x = \lim\limits_{t\longrightarrow y^-} g_i(t)$ if $y = b$) and thus $x \in r_i(y)$.

Finally we will assume that $r_i(x)$ contains only one point. Then $U^n(x,t) > e_i$ for all $t > y$ and $U^n(x,s) < e_i$ for all

$s < y$. Therefore for all $t > y$ we have $g_i(t) \leq x$ and for all $s < y$ we have $g_i(s) \geq x$. If $y \in]0,1[$ there is

$$\lim_{t \longrightarrow y^+} g_i(t) \leq x \leq \lim_{t \longrightarrow y^-} g_i(t)$$

which means that $x \in r_i(y)$. If $y = 1$ there is $x \leq \lim_{t \longrightarrow 1^-} g_i(t)$, i.e., $x \in r_i(1)$. If $y = 0$ there is $x \geq \lim_{t \longrightarrow 0^+} g_i(t)$, i.e., $x \in r_i(0)$.

□

Since $g_i(x) \in r_i(x)$ for all $x \in [0,1]$, $i \in \{1,\ldots,n\}$, due to Lemma II.4 and Lemma III.14 we get the following result.

Lemma III.15
Let $U^n \colon [0,1]^2 \longrightarrow [0,1]$ be a reduced n-uninorm, $U^n \in \mathcal{U}_n$, and assume an $i \in \{1,\ldots,n\}$. Then the characterizing set-valued function r_i of U^n is u-surjective.

IV. Points of discontinuity of n-uninorms from $\mathcal{U}_n$.

Now we are going to show that all points of discontinuity of an n-uninorm with continuous underlying functions are covered by the graphs of $r_i(x)$ for $i = 1,\ldots,n$ (see Theorem IV.8). However, first we show several useful results.

Lemma IV.1
Let $U^n \colon [0,1]^2 \longrightarrow [0,1]$ be a reduced n-uninorm and let $U^n \in \mathcal{U}_n$. If u_x^n is non-continuous in $y \in [0,1]$, then there exists an $i \in \{1,\ldots,n\}$ such that $U(x,t) < e_i$ for all $t < y$ and $U(x,t) > e_i$ for all $t > y$.

PROOF: Assume that u_x^n is non-continuous in $y \in [0,1]$. Then

$$a = \lim_{t \longrightarrow y^-} u_x^n(t) < \lim_{t \longrightarrow y^+} u_x^n(t) = b.$$

If there exists $i,j \in \{1,\ldots,n\}$ such that $a \in]e_{i-1}, e_i[$, $b \in]e_{j-1}, e_j[$, then $i \neq j$ since $\mathrm{Ran}(u_x^n) \cap [e_{k-1}, e_k]$ is a connected set for all $k \in \{1,\ldots,n+1\}$. The monotonicity implies $i < j$ and therefore $a < e_i < b$ which implies $U(x,t) < e_i$ for all $t < y$ and $U(x,t) > e_i$ for all $t > y$.

If $a = 0$ then $U(x,t) > e_1$ for all $t > y$ since otherwise $e_1 > 0$ and u_x^n would be continuous in y by Lemma III.7. If $b = 1$ then similarly $U(x,t) < e_n$ for all $t < y$.

Finally assume that $a = e_i > 0$ (the case when $b = e_i < 1$ can be shown analogously) for some $i \in \{1,\ldots,n\}$. Then $b > e_i$, i.e., $U(x,t) > e_i$ for all $t > y$. If $u_x^n(t) = e_i$ for some $t < y$ then $u_x^n(s) = e_i$ for all $s \in [t,y[$. In such a case Proposition III.3 implies that $e_i \in \{z_{i-1}, z_i\}$ and since U^n is in the reduced form this implies $a = 0$, which is a contradiction. Therefore $U(x,t) < e_i$ for all $t < y$. □

Lemma IV.2
Let $U^n \colon [0,1]^2 \longrightarrow [0,1]$ be a reduced n-uninorm, $U \in \mathcal{U}_n$, and assume $i \in \{1,\ldots,n\}$. If $U^n(x,y) = e_i \in]0,1[$ for some $x,y \in [0,1]$ then $x,y \in]z_{i-1}, z_i[$.

PROOF: Without loss of generality assume that $x \notin]z_{i-1}, z_i[$. Then either $x = 0, i = 1$, or $x = 1, i = n$, or there exists a $j \in \{1,\ldots,n\}$, $i \neq j$, such that $U^n(e_j, x) = x$. In the last case we have

$$e_i = U^n(x,y) = U^n(U^n(e_j,x),y) = U^n(e_j, U^n(x,y)) = U^n(e_i, e_j) = z_k$$

for some $k \in \{1,\ldots,n-1\}$, which is a contradiction since U^n is in the reduced form and thus $e_i \in]0,1[$ implies $e_i \in]z_{i-1}, z_i[$. If $x = 0, i = 1$ (the case when $x = 1, i = n$ is analogous) then

$$e_1 = U^n(0,y) = U^n(U^n(0,0),y) = U^n(0, U^n(0,y)) = U^n(0, e_1) = 0,$$

which is a contradiction. □

From the previous result we see that if $U^n(0,y) = e_i$ for some $y \in [0,1]$ and $i \in \{1,\ldots,n\}$ then $e_i \in \{0,1\}$. Then either $y = 0 = e_i$, or $y = 1 = e_i$. However, in the second case $U^n(0,1) = 1 = U^n(1,1)$ which means that 1 is the annihilator of U^n and thus it cannot be a neutral element on $[z_{n-1},1]^2$, which is a contradiction. Similarly, $U^n(1,y) = e_i$ implies $y = 1 = e_i$.

Now we recall a result [23, Proposition 1] which shows a connection between continuity on cuts and joint continuity of a monotone function.

Proposition IV.3
Let $f(x,y)$ be a real valued function defined on an open set G in the plane. Suppose that $f(x,y)$ is continuous in x and y separately and is monotone in x for each y. Then $f(x,y)$ is (jointly) continuous on the set G.

Lemma IV.4
Let $U^n \colon [0,1]^2 \longrightarrow [0,1]$ be a reduced n-uninorm, $U \in \mathcal{U}_n$. Then U^n is continuous in (e_i, e_i) for all $i \in \{1,\ldots,n\}$.

For all e_i with $i \in \{2,\ldots,n-1\}$ and for $e_1 \neq 0$ and $e_n \neq 1$ the result follows from Lemma IV.2 and [26, Lemma 4]. Assume that $e_1 = 0$. Then U^n on $[0,z_1]^2$ is a continuous t-conorm and thus U^n is continuous in $(0,0)$. Similarly, for $e_n = 1$ we know that U^n on $[z_{n-1},1]^2$ is a continuous t-norm and thus U^n is continuous in $(1,1)$.

Proposition IV.5
Let $U^n \colon [0,1]^2 \longrightarrow [0,1]$ be a reduced n-uninorm, $U^n \in \mathcal{U}_n$. Then U^n is non-continuous in $(x_0,y_0) \in [0,1]^2$, $(x_0,y_0) \neq (e_i,e_i)$, if and only if one of the following is satisfied:

(i) $u_{x_0}^n$ is non-continuous in y_0,
(ii) $u_{y_0}^n$ is non-continuous in x_0,
(iii) there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that u_t^n is non-continuous in x_0 and u_s^n is non-continuous in y_0 either for all $t \in]y_0, y_0+\varepsilon_1]$, $s \in]x_0, x_0+\varepsilon_2]$, or for all $t \in [y_0-\varepsilon_1, y_0[$, $s \in [x_0-\varepsilon_2, x_0[$.

PROOF: Suppose that U^n is non-continuous in $(x_0,y_0) \in [0,1]^2$. If $U^n(x_0,y_0) = e_1 = 0$ then $x_0 = y_0 = 0$ and since U^n is continuous in $(0,0)$ we get a contradiction. Similarly, if $U^n(x_0,y_0) = e_n = 1$ then U^n is continuous in (x_0,y_0) and we get a contradiction. Therefore Lemma IV.2 implies that if

$U^n(x_0, y_0) = e_i$ for some $i \in \{1, \dots, n\}$ then $x, y \in]z_{i-1}, z_i[$ and since U^n is on $[z_{i-1}, z_i]^2$ isomorphic to a uninorm with continuous underlying functions [7, Proposition 8] implies that U^n is continuous in (x_0, y_0), which is a contradiction. Thus $U(x_0, y_0) \neq e_i$.

From Proposition IV.3 it follows that if U^n is non-continuous in $(x_0, y_0) \in [0, 1]^2$ then for all $\delta_1 > 0$ and all $\delta_2 > 0$ there exist an $x \in]x_0 - \delta_1, x_0 + \delta_1[$ and a $y \in]y_0 - \delta_2, y_0 + \delta_2[$ such that either u^n_x is non-continuous in y or u^n_y is non-continuous in x. Due to Lemma IV.1, since n is finite, there exists an $i \in \{1, \dots, n\}$ such that for all $\delta_1 > 0$ and all $\delta_2 > 0$ the n-uninorm U^n on $[x_0 - \delta_1, x_0 + \delta_1] \times [y_0 - \delta_2, y_0 + \delta_2]$ attain values smaller than e_i and bigger than e_i as well. Let W_i be a subset of $[0, 1]^2$ such that $(x, y) \in W_i$ if $U(x_1, y_1) < e_i$ for all $x_1 < x, y_1 < y$ and $U(x_2, y_2) > e_i$ for all $x_2 > x, y_2 > y$. Then the set $[x_0 - \delta_1, x_0 + \delta_1] \times [y_0 - \delta_2, y_0 + \delta_2] \cap W_i$ is non-empty for all $\delta_1 > 0$ and all $\delta_2 > 0$. Therefore due to the monotonicity of U^n we have $(x_0, y_0) \in W_i$.

The rest of the proof is analogous to the proof of [7, Proposition 11]. □

Lemma IV.6
Let $U^n\colon [0, 1]^2 \longrightarrow [0, 1]$ be a reduced n-uninorm and let $U^n \in \mathcal{U}_n$. If $U^n(x, y) = e_i$ for some $x, y \in [0, 1]$ and $i \in \{1, \dots, n\}$ then $(x, y) \in G(r_i)$.

PROOF: If $e_i \in \{0, 1\}$ then the result easily holds. Otherwise $x, y \in]0, 1[$ and the monotonicity of U^n implies $U^n(x, t) < e_i$ for all $t < y$ and $U^n(x, t) > e_i$ for all $t > y$. Thus $g_i(x) = y$. Therefore $y \in r_i(x)$ and $(x, y) \in G(r_i)$. □

Remark IV.7
From the previous results we can observe for a reduced n-uninorm U^n the following. If $(x_0, y_0) \in G(r_i) \cap]0, 1[^2$ then either $U^n(x_0, y_0) = e_i$, or $U^n(x_0, y_0)$ is a point of discontinuity of U^n. Further, on the lower boundary of the unit square (and similarly on the upper boundary of the unit square) we know that U^n is continuous in the point $(0, 0)$ and U^n is non-continuous in each point $(0, t), (t, 0)$ such that $t \in \left]\lim\limits_{t \longrightarrow 0^+} g_i(t), g_i(0)\right]$ for some $i \in \{1, \dots, n\}$, where $1 > e_i > 0$.

Now we show the main result.

Theorem IV.8
Let $U^n\colon [0, 1]^2 \longrightarrow [0, 1]$ be a reduced n-uninorm and let $U^n \in \mathcal{U}_n$. If $(x_0, y_0) \in [0, 1]^2$ is a point of discontinuity of U^n then $(x_0, y_0) \in \bigcup\limits_{i=1}^{n} G(r_i)$.

PROOF: In the case that $u^n_{x_0}$ is non-continuous in y_0, or $u^n_{y_0}$ is non-continuous in x_0 the claim follows from Lemma IV.1. In the opposite case Proposition IV.5 implies that there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that u^n_t is non-continuous in x_0 and u^n_s is non-continuous in y_0 either for all $t \in]y_0, y_0 + \varepsilon_1]$, $s \in]x_0, x_0 + \varepsilon_2]$, or for all $t \in [y_0 - \varepsilon_1, y_0[$, $s \in [x_0 - \varepsilon_2, x_0[$. Suppose the first case as the second is analogous. From the proof of Proposition IV.5 it further follows that there exists an $i \in \{1, \dots, n\}$ such that $(t, x_0) \in G(r_i)$ for all $t \in]y_0, y_0 + \varepsilon_1]$, and $(s, y_0) \in G(r_i)$ for all $s \in]x_0, x_0 + \varepsilon_2]$. Therefore $g_i(x) = y_0$ for all $x \in]x_0, x_0 + \varepsilon_2[$ and thus $\lim\limits_{t \longrightarrow x_0^+} g_i(t) = y_0$. Then $y_0 \in r_i(x_0)$ and $(x_0, y_0) \in \bigcup\limits_{i=1}^{n} G(r_i)$. □

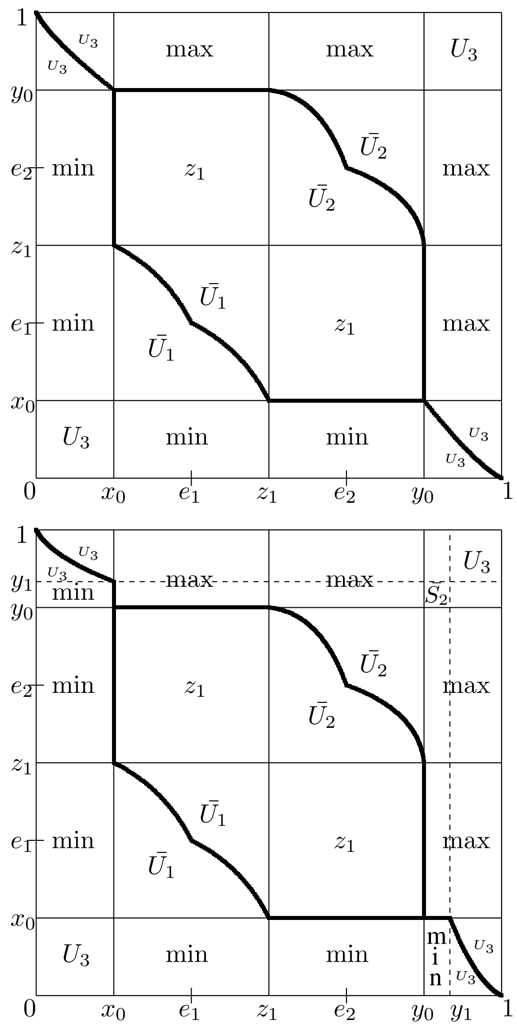

Fig. 2. The two 2-uninorms from [16, Figure 1]. The bold lines denote the characterizing set-valued functions r_1 and r_2.

Further we would like to show an opposite claim, i.e., if there exist n symmetric, non-increasing, u-surjective set-valued functions $r_i\colon [0, 1] \longrightarrow \mathcal{P}([0, 1])$ such that all points of discontinuity of U^n are covered by $\bigcup\limits_{i=1}^{n} G(r_i)$ and $U^n(x, y) = e_i$ implies $(x, y) \in G(r_i)$ for all $x, y \in [0, 1]$ then $U^n \in \mathcal{U}_n$. In the case of uninorms this was not enough to ensure that U has continuous underlying functions and it was necessary to suppose that U is in each point from the unit square either left-continuous, or right-continuous (or continuous). However, in the class of n-uninorms this is no longer true. Indeed, it can happen that there exists a point in which U^n is neither left-continuous, nor right-continuous.

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Example IV.9
Assume $0 < e_1 < z_1 < e_2 < 1$ an let a binary function $U^2\colon [0,1]^2 \longrightarrow [0,1]$ be given by:

$$U^n(x,y) = \begin{cases} \min(x,y) & \text{if } \min(x,y) < e_1, \\ \max(x,y) & \text{if } \min(x,y) > e_1, \max(x,y) > e_2, \\ \min(x,y) & \text{if } x,y \in [z_1, e_2]\,, \\ \max(x,y) & \text{if } x,y \in [e_1, z_1]\,, \\ z_1 & \text{otherwise.} \end{cases}$$

Then U^2 is a 2-uninorm with continuous underlying functions and $U^n(x,y) = e_i$ implies $x = y = e_i$, i.e., U^2 is in the reduced form. However, $U^2(e_1, e_2) = z_1$ and $U^n(s,t) = s < e_1$ for all $s < e_1$, $e_1 < t < e_2$ and $U^n(s,t) = t > e_2$ for all $e_2 > s > e_1$, $t > e_2$. Therefore in the point (e_1, e_2) the 2-uninorm U^2 is neither left-continuous, nor right-continuous.

Observe that in the previous example the point (e_1, e_2) belongs to graphs of both characterizing functions r_1 and r_2. This yields us to the following result.

Proposition IV.10
Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be a reduced n-uninorm, $U^n \in \mathcal{U}_n$, and assume a point $(x_0,y_0) \in [0,1]^2$. If there exists exactly one $i \in \{1,\dots,n\}$ such that $(x_0,y_0) \in G(r_i)$ then U^n is left/right continuous at the point (x_0,y_0).

PROOF: If there is exactly one $i \in \{1,\dots,n\}$ such that $(x_0,y_0) \in G(r_i)$ then there exists a $\delta > 0$ such that $G(r_j) \cap [x_0-\delta, x_0+\delta] \times [y_0-\delta, y_0+\delta] = \emptyset$ for all $j \in \{1,\dots,n\}$, $i \neq j$. If U^n is in (x_0,y_0) continuous we are done. Suppose that U^n is non-continuous in (x_0,y_0). Then, similarly as in Proposition IV.5, we can show that $U^n(x_0,y_0) \neq e_i$.

First let us assume that $U^n(x_0,y_0) < e_i$. Suppose that there exists a point $(s,t) \in [x_0-\delta, x_0] \times [y_0-\delta, y_0]$ such that $U^n(s,t) \leq e_{i-1}$. Then $g_{i-1}(s) \geq t$ and if $U^n(s,y_0) > e_{i-1}$ then there exists a $q \in [t, y_0]$ such that $g_{i-1}(s) = q$ and thus $(s,q) \in G(r_{i-1})$, which is a contradiction. Thus $U^n(s,y_0) \leq e_{i-1}$. If $U^n(x_0,y_0) > e_{i-1}$ then there exists a $q \in [s,x_0]$ such that $(q,y_0) \in G(r_{i-1})$, which is a contradiction. Thus $U^n(x_0,y_0) \leq e_{i-1}$. However, the n-uninorm U^n attains values greater than e_i above the graph of r_i and since $(x_0,y_0) \in G(r_i)$ it is easy to show that then also $(x_0,y_0) \in G(r_{i-1})$, which is a contradiction. Therefore for all $(s,t) \in [x_0-\delta, x_0] \times [y_0-\delta, y_0]$ we have $U^n(s,t) > e_{i-1}$. Lemma III.7 implies that $\mathrm{Ran}(u_x^n) \cap [e_{i-1}, e_i]$ is a connected set for all $x \in [0,1]$. Thus on $[x_0-\delta, x_0] \times [y_0-\delta, y_0]$ all cuts are continuous and since U^n is non-decreasing, similarly as in Proposition IV.3, we see that U^n is left-continuous in (x_0,y_0).

In the case when $U^n(x_0,y_0) > e_i$ we can analogously show that U^n is right-continuous in (x_0,y_0). □

Lemma IV.11
Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be a reduced n-uninorm, $U^n \in \mathcal{U}_n$, and assume $i \in \{1,\dots,n\}$. Then $[z_{i-1}, z_i]^2 \cap \bigcup_{j=1}^{n} G(r_j) = G(r_i)$.

PROOF: We will show that $G(r_j) \cap [z_{i-1}, z_i]^2 = \emptyset$ for all $j \in \{1,\dots,n\}$, $i \neq j$. We will suppose $i < j$ as the case $i > j$ is analogous. Since $U^n(e_j, e_j) = e_j$ we know that $g_j(x) \geq e_j$ for all $x \in [z_{i-1}, e_j[$ and then $r_j(x) \geq e_j$ for all $x \in [z_{i-1}, z_i]$. Thus $G(r_j) \cap [z_{i-1}, z_i]^2 = \emptyset$. Note that $e_j \neq z_i$ since U^n is in the reduced form. □

Now we can show the main result.

Theorem IV.12
Let $U^n\colon [0,1]^2 \longrightarrow [0,1]$ be a reduced n-uninorm. Suppose that U^n is continuous on $[0,1]^2 \setminus \bigcup_{i=1}^{n} G(r_i)$, where r_i is a symmetric, u-surjective, non-increasing set-valued function on $[0,1]$, such that $U^n(x,y) = e_i$ implies $(x,y) \in G(r_i)$ for $i = 1,\dots,n$. Further assume that U^n is either left-continuous, or right-continuous (or continuous) in each point $(x_0,y_0) \in [0,1]^2$ such that there is exactly one $i \in \{1,\dots,n\}$ for which $(x_0,y_0) \in G(r_i)$. Then $U^n \in \mathcal{U}_n$.

PROOF: Assume $i \in \{1,\dots,n\}$. Then similarly as in Lemma IV.11 we can show that all points of discontinuity of U^n from $[z_{i-1}, z_i]^2$ are covered exclusively by $G(r_i)$, i.e., U^n is in all these points either right-continuous, or left-continuous (or continuous). Thus U^n is either right-continuous, or left-continuous (or continuous) in each point $(x,y) \in [z_{i-1}, z_i]^2$. If $e_i = 0$ ($e_i = 1$) then U^n is continuous on $[z_{i-1}, z_i]^2$ since each t-norm (t-conorm) is continuous on the lower (the upper) boundary of the unit square. Further we will suppose that $e_i \in \,]0,1[$.

Let us define a set-valued function $r\colon [0,1] \longrightarrow \mathcal{P}([0,1])$ by

$$r(x) = \begin{cases} r_i(x) \cap [z_{i-1}, z_i] & \text{if } r_i(x) \cap [z_{i-1}, z_i] \neq \emptyset, x \in \,]z_{i-1}, z_i[\,, \\ \{z_i\} & \text{if } r_i(x) \cap [z_{i-1}, z_i] = \emptyset \text{ and } x < e_i, \\ \{z_{i-1}\} & \text{if } r_i(x) \cap [z_{i-1}, z_i] = \emptyset \text{ and } x > e_i, \\ [\min(r_i(x)), z_i] & \text{if } x = z_{i-1} \text{ and } r_i(x) \cap [z_{i-1}, z_i] \neq \emptyset, \\ [z_{i-1}, \max(r_i(x))] & \text{if } x = z_i \text{ and } r_i(x) \cap [z_{i-1}, z_i] \neq \emptyset. \end{cases}$$

Since r_i is non-increasing also r is non-increasing. Further we show that r is symmetric. Assume $x \in [z_{i-1}, z_i]$ and let $y \in r(x)$. Then we have the following cases:

(i) If $x \in \,]z_{i-1}, z_i[$ and $r(x) = r_i(x) \cap [z_{i-1}, z_i]$. Then $y \in r_i(x)$ and thus $x \in r_i(y)$. Since $r_i(y) \cap [z_{i-1}, z_i] \subseteq r(y)$ we get $x \in r(y)$.

(ii) If $r_i(x) \cap [z_{i-1}, z_i] = \emptyset$ and $x < e_i$. Then $t > z_i$ for all $t \in r_i(x)$ and $y = z_i$. Since r_i is u-surjective there exists an $s \in [0,1]$ such that $z_i \in r_i(s)$. Then either $\{z_i\} = r_i(v)$ for all $v \in \,]x,s[$, or there exists an $\varepsilon > 0$ such that $r_i(v) \cap [z_{i-1}, z_i] = \emptyset$ for all $v \in [x, x+\varepsilon[$. In the first case $v \in r_i(z_i)$ for all $v \in \,]x,s[$ and since $r_i(z_i)$ is a closed interval we get $x \in r_i(z_i)$ and then $x \in r(z_i)$. In the second case let $b = \max(r_i(z_i))$. Then $(z_i, b), (b, z_i) \in G(r_i)$, i.e., $b > x$. Since $r(z_i) = [z_{i-1}, b]$ we get $x \in r(z_i)$.

(iii) If $r_i(x) \cap [z_{i-1}, z_i] = \emptyset$ and $x > e_i$ the proof is analogous to the previous case.

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(iv) If $x = z_{i-1}$ and $r_i(x) \cap [z_{i-1}, z_i] \neq \emptyset$. Then there is $r(x) = [\min(r_i(z_{i-1})), z_i]$. If $y \in r_i(z_{i-1})$ then the result follows from the symmetry of r_i. Denote $a = \max(r_i(z_{i-1})) \geq e_i$ and suppose that $y > a \geq e_i$. Then $y \notin r_i(z_{i-1})$ and thus $t < z_{i-1}$ for all $t \in r_i(y)$. Therefore $r_i(y) \cap [z_{i-1}, z_i] = \emptyset$ and since $y > e_i$ there is $r(y) = \{z_{i-1}\}$, i.e., $x \in r(y)$.

(v) If $x = z_i$ and $r_i(x) \cap [z_{i-1}, z_i] \neq \emptyset$ the proof is analogous to the previous case.

Thus r is symmetric and Lemma II.4 implies that r is u-surjective. Further, $G(r)$ covers all points of discontinuity of U^n from $[z_{i-1}, z_i]^2$ and for all $(x, y) \in [z_{i-1}, z_i]^2$ there $U^n(x, y) = e_i$ implies $(x, y) \in G(r)$. Thus r is the linear transformation of the characterizing set-valued function of the uninorm U_i to the interval $[z_{i-1}, z_i]$. Moreover, U^n is in each point from $[z_{i-1}, z_i]^2$ either left-continuous, or right-continuous (or continuous). Then [7, Theorem 3] implies that U^n is isomorphic to a uninorm with continuous underlying functions on $[z_{i-1}, z_i]^2$. Since i was chosen arbitrarily we get $U^n \in \mathcal{U}_n$. □

V. Conclusions

We have studied characterizing (set-valued) functions of n-uninorms with continuous underlying functions. We have shown that similarly as in the case of uninorms, the set of points of discontinuity of such an n-uninorm can be characterized using the characterizing set-valued functions related to the local neutral element e_i for $i = 1, \dots, n$. Unlike in the case of uninorms, in the case of n-uninorms we had to cover also all anomalous cases when $e_i \in \{z_{i-1}, z_i\}$ for some $i \in \{1, \dots, n\}$. We have shown that in such a case the n-uninorm U^n can be reduced to an m-uninorm for which $e_i \in \{z_{i-1}, z_i\}$ implies $e_i \in \{0, 1\}$.

In the case of uninorms with continuous underlying functions it was shown that such a uninorm is in each point $(x, y) \in [0, 1]^2$ either left-continuous, or right continuous (or continuous). This no longer holds in the case of n-uninorms. However, each reduced n-uninorm satisfy this property in each point which is covered by exactly one characterizing set-valued function.

Our results offer a complete characterization of n-uninorms with continuous underlying functions by their set of discontinuity points.

In the future work we would like to use these results and show that each n-uninorm with continuous underlying functions can be decomposed into a z-ordinal sum of representable and idempotent semigroups.

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Characterization of n-uninorms with continuous underlying functions via z-ordinal sum construction

Andrea Mesiarová-Zemánková

Mathematical Institute, Slovak Academy of Sciences, Bratislava, Slovakia

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ABSTRACT

The n-uninorms with continuous underlying t-norms and t-conorms are characterized via the z-ordinal sum construction. We show that each n-uninorm with continuous underlying t-norms and t-conorms can be expressed as a z-ordinal sum of a countable number of Archimedean and idempotent semigroups with respect to the branching set $A \sim \{z_1, \ldots, z_{n-1}\}$, where the corresponding partial order has a tree structure.

1. Introduction

The uninorm aggregation operators (see [7,8,21]) generalize both t-norms and t-conorms ([3,9]), allowing the neutral element to be anywhere inside the unit interval, instead of in the point 1 (which is the case of t-norms) or in point 0 (which is the case of t-conorms). Therefore uninorms are able to model bipolar behavior and uninorms can be taken as bipolar t-conorms (see [13,22]). While continuous t-norms (t-conorms) were characterized as ordinal sums of continuous Archimedean t-norms (t-conorms), which are representable by a continuous additive generator, a similar result was shown in [14] also for uninorms with continuous underlying functions, where each such a uninorm was shown to be an ordinal sum of semigroups related to representable uninorms, continuous Archimedean t-norms, continuous Archimedean t-conorms and internal uninorms.

Another possible generalization of t-norms and t-conorms are nullnorms (see [5]), allowing the annihilator to be anywhere inside the unit interval. Nullnorms can be taken as bipolar t-norms [13]. Note that nullnorms were independently introduced under the name t-operators in [12] and in [11] it was shown that these two notions coincide on the unit interval.

The second step in the above mentioned generalizations yields n-uninorms, introduced by Akella in [1], which bring together uninorms and nullnorms. The basic properties of n-uninorms were described in [1,2,23]. Our work focuses on n-uninorms with continuous underlying functions. In [16] we have characterized idempotent n-uninorms and we have also shown that similarly as each idempotent uninorm can be expressed as an ordinal sum of trivial semigroups, each idempotent n-uninorm can be expressed as a z-ordinal sum of trivial semigroups $G_x = (\{x\}, \mathrm{Id})$ for $x \in [0,1]$, where $A = \{z_1, \ldots, z_{n-1}\}$. In [17] we have shown that each n-uninorm with continuous underlying functions can be expressed as an ordinal sum of a uninorm and an n-uninorm from Class 1 (and possibly also of a countable number of semigroups related to continuous

E-mail address: zemankova@mat.savba.sk.

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t-norms and t-conorms). In [18] we have discussed characterizing functions of n-uninorms with continuous underlying functions.

In this paper we would like to finish the characterization of n-uninorms with continuous underlying functions and show their complete decomposition into irreducible semigroups, i.e., we will show that each n-uninorm with continuous underlying t-norms and t-conorms can be expressed as a z-ordinal sum of a countable number of semigroups related to representable uninorms, continuous Archimedean t-norms, continuous Archimedean t-conorms and internal uninorms (including the min and the max operator and trivial semigroups) with respect to the set $A \sim \{z_1, \ldots, z_{n-1}\}$.

The paper is organized as follows: in the following section we recall all necessary basic notions and results. We discuss the relation of nullnorms with continuous underlying functions to the z-ordinal sum construction in Section 3. In Section 4 we introduce basic observations on n-uninorms with continuous underlying functions and in Section 5 we show the decomposition of all 2-uninorms with continuous underlying functions into Archimedean and idempotent semigroups. These results are then extended for n-uninorms with continuous underlying functions for $n \in \mathbb{N}$, $n > 2$, in Section 6. Section 7 is dedicated to our Conclusions.

2. Basic notions

A triangular norm ([9]) is a binary function $T\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and 1 is its neutral element. Due to the associativity, n-ary form of any t-norm is uniquely given and thus it can be extended to an aggregation function working on $\bigcup_{n\in\mathbb{N}}[0,1]^n$. Dual functions to t-norms are t-conorms. A triangular conorm ([9]) is a binary function $S\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and 0 is its neutral element.

The focus of this paper is on the ordinal sum and the z-ordinal sum construction. At first we recall the fundamental result of Clifford [6] as formulated in [9].

Theorem 1. *Let $A \neq \emptyset$ be a totally ordered set and $(G_\alpha)_{\alpha\in A}$ with $G_\alpha = (X_\alpha, *_\alpha)$ be a family of semigroups. Assume that for all $\alpha, \beta \in A$ with $\alpha < \beta$ the sets X_α and X_β are either disjoint or that $X_\alpha \cap X_\beta = \{x_{\alpha,\beta}\}$, where $x_{\alpha,\beta}$ is both the neutral element of G_α and the annihilator of G_β and where for each $\gamma \in A$ with $\alpha < \gamma < \beta$ we have $X_\gamma = \{x_{\alpha,\beta}\}$. Put $X = \bigcup\limits_{\alpha\in A} X_\alpha$ and define the binary operation $*$ on X by*

$$x * y = \begin{cases} x *_\alpha y & \text{if } (x, y) \in X_\alpha \times X_\alpha, \\ x & \text{if } (x, y) \in X_\alpha \times X_\beta \text{ and } \alpha < \beta, \\ y & \text{if } (x, y) \in X_\alpha \times X_\beta \text{ and } \alpha > \beta. \end{cases}$$

Then $G = (X, *)$ is a semigroup. The semigroup G is commutative if and only if for each $\alpha \in A$ the semigroup G_α is commutative.

Each continuous t-norm (t-conorm) is equal to an ordinal sum of continuous Archimedean t-norms (t-conorms). Note that a continuous t-norm (t-conorm) is Archimedean if and only if it has only trivial idempotent points 0 and 1. A continuous Archimedean t-norm T (t-conorm S) is either strict, i.e., strictly increasing on $]0,1]^2$ (on $[0,1[^2$), or nilpotent, i.e., there exists $(x, y) \in]0,1[^2$ such that $T(x, y) = 0$ ($S(x, y) = 1$). Moreover, each continuous Archimedean t-norm (t-conorm) has a continuous additive generator, which is uniquely determined up to a positive multiplicative constant. More details on t-norms and t-conorms can be found in [3,9].

A uninorm (introduced in [21]) is a binary function $U\colon [0,1]^2 \longrightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and have a neutral element $e \in [0,1]$ (see also [8]). Evidently, if $e = 1$ ($e = 0$) then we retrieve a t-norm (t-conorm).

For each uninorm the value $U(1,0) \in \{0,1\}$ is the annihilator of U. A uninorm is called conjunctive (disjunctive) if $U(1,0) = 0$ ($U(1,0) = 1$). For each uninorm U with the neutral element $e \in]0,1[$, the restriction of U to $[0,e]^2$ is a t-norm on $[0,e]^2$, i.e., a linear transformation of some t-norm T_U on $[0,1]^2$ and the restriction of U to $[e,1]^2$ is a t-conorm on $[e,1]^2$, i.e., a linear transformation of some t-conorm S_U. Moreover, $\min(x, y) \le U(x, y) \le \max(x, y)$ for all $(x, y) \in [0,e] \times [e,1] \cup [e,1] \times [0,e]$.

Similarly as in the case of t-norms and t-conorms we can construct uninorms using additive generators (see [8]). A uninorm which possesses a continuous additive generator is called representable. Note that in [19] (see also [13]) it was shown that a uninorm is representable if and only if it is continuous on $[0,1]^2 \setminus \{(0,1), (1,0)\}$.

A uninorm $U\colon [0,1]^2 \longrightarrow [0,1]$ is called internal if $U(x, y) \in \{x, y\}$ for all $(x, y) \in [0,1]^2$; and it is called idempotent if $U(x, x) = x$ for all $x \in [0,1]$.

Observe that if a uninorm U is internal then it is also idempotent and vice-versa. Let us recall the basic result from [20] that characterizes idempotent uninorms.

Theorem 2. *Let $U\colon [0,1]^2 \longrightarrow [0,1]$ be a binary function. Then U is an idempotent uninorm with the neutral element $e \in]0,1[$ if and only if there exists a non-increasing function $g\colon [0,1] \longrightarrow [0,1]$, which is Id-symmetric, with $g(e) = e$, such that*

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } y < g(x) \text{ or } (y = g(x) \text{ and } x < g(g(x))), \\ \max(x,y) & \text{if } y > g(x) \text{ or } (y = g(x) \text{ and } x > g(g(x))), \\ x \text{ or } y & \text{if } y = g(x) \text{ and } x = g(g(x)), \end{cases}$$

being commutative in the points (x, y) *such that* $y = g(x)$ *with* $x = g(g(x))$.

Note that the graph of the function g from Theorem 2 is a subset of the graph of the characterizing set-valued function of an idempotent uninorm (for more details see [15]). Therefore the completed graph of the function g divides the idempotent uninorm U into two parts: below the completed graph of g we have $U(x, y) = \min(x, y)$, i.e., $U(x, y) < e$, and above the completed graph of g there is $U(x, y) = \max(x, y)$, i.e., $U(x, y) > e$.

Definition 1. A uninorm $U : [0,1]^2 \longrightarrow [0,1]$ is called d-internal if it is internal and there exists a continuous and strictly decreasing function $g_U : [0,1] \longrightarrow [0,1]$ such that $U(x, y) = \min(x, y)$ if $y < g_U(x)$ and $U(x, y) = \max(x, y)$ if $y > g_U(x)$.

Uninorms with continuous underlying functions were completely characterized in [14,15]. In [14] it was shown that each uninorm with continuous underlying functions can be decomposed into an ordinal sum of a countable number of semigroups related to representable uninorms, continuous Archimedean t-norms, continuous Archimedean t-conorms and internal uninorms (including the min and the max operator). In [15] it was shown that the set of all points of discontinuity of a uninorm with continuous underlying functions is a subset of the graph of the characterizing set-valued function of such a uninorm.

The following is the definition of a nullnorm [5] (see also [11,12]).

Definition 2. A binary function $V : [0,1]^2 \longrightarrow [0,1]$ is called a nullnorm if it is commutative, associative, non-decreasing in each variable and has an annihilator $z \in [0, 1]$ such that $V(0, x) = x$ for all $x \le z$ and $V(1, x) = x$ for all $x \ge z$.

If $z = 0$ $(z = 1)$ then V is a t-norm (t-conorm). Note that for a commutative, associative and non-decreasing function $F : [0,1]^2 \longrightarrow [0,1]$, with $F(0,0) = 0$, $F(1,1) = 1$, the value $F(0,1)$ is always an annihilator of F. Thus for a nullnorm $z = V(0,1)$. In [5] the following result was shown.

Theorem 3. *Let* $z \in]0, 1[$. *Then* $V : [0,1]^2 \longrightarrow [0,1]$ *is a nullnorm with the annihilator* z *if and only if there exists a t-norm* T_V *and a t-conorm* S_V *such that*

$$V(x,y) = \begin{cases} z \cdot S_V(\frac{x}{z}, \frac{y}{z}) & \text{if } x, y \in [0,z]^2, \\ z + (1-z) \cdot T_V(\frac{x-z}{1-z}, \frac{y-z}{1-z}) & \text{if } x, y \in [z,1]^2, \\ z & \text{otherwise.} \end{cases}$$

Now let us recall the definition of an n-uninorm (see [1]).

Definition 3. Assume an $n \in \mathbb{N} \setminus \{1\}$. Let $V : [0,1]^2 \longrightarrow [0,1]$ be a commutative binary function. Then $\{e_1, \ldots, e_n\}_{z_1,\ldots,z_{n-1}}$ is called an n-neutral element of V if for $0 = z_0 < z_1 < \cdots < z_n = 1$ and $e_i \in [z_{i-1}, z_i]$, $i = 1, \ldots, n$ we have $V(e_i, x) = x$ for all $x \in [z_{i-1}, z_i]$.

Definition 4. A binary function $U^n : [0,1]^2 \longrightarrow [0,1]$ is an n-uninorm if it is commutative, associative, non-decreasing in each variable and has an n-neutral element $\{e_1, \ldots, e_n\}_{z_1,\ldots,z_{n-1}}$.

The basic structure of n-uninorms was described by Akella in [1] and the characterizations of the main five classes of 2-uninorms was given in [23]. Now we will recall these five exhaustive and mutually exclusive classes:

- Class 1: 2-uninorms with $U^2(0,1) = z_1$.
- Class 2a: 2-uninorms with $U^2(0,1) = 0$, $U^2(1, z_1) = z_1$.
- Class 2b: 2-uninorms with $U^2(0,1) = 1$, $U^2(0, z_1) = z_1$.
- Class 3a: 2-uninorms with $U^2(0,1) = 0$, $U^2(1, z_1) = 1$.
- Class 3b: 2-uninorms with $U^2(0,1) = 1$, $U^2(0, z_1) = 0$.

Each n-uninorm has the following building blocks around the main diagonal.

Proposition 1. *Let* $U^n : [0,1]^2 \longrightarrow [0,1]$ *be an n-uninorm with the n-neutral element* $\{e_1, \ldots, e_n\}_{z_1,\ldots,z_{n-1}}$. *Then*

(i) U^n *restricted to* $[z_{i-1}, e_i]^2$, *for* $i = 1, \ldots, n$, *is isomorphic to a t-norm. We will denote this t-norm by* T_i.

(ii) *U^n restricted to $[e_i, z_i]^2$ for $i = 1, \dots, n$, is isomorphic to a t-conorm. We will denote this t-conorm by S_i.*
(iii) *U^n restricted to $[z_{i-1}, z_i]^2$ for $i = 1, \dots, n$, is isomorphic to a uninorm. We will denote this uninorm by U_i.*
(iv) *U^n restricted to $\left[z_i, z_j\right]^2$ for $i, j \in \{0, 1, \dots, n\}$, $i < j$, is isomorphic to a $(j-i)$-uninorm.*

For $n \in \mathbb{N}$ we will denote the set of all n-uninorms U^n such that $T_1, \dots, T_n$ and $S_1, \dots, S_n$ are continuous by $\mathcal{U}_n$. Similarly, the set of all uninorms with continuous underlying functions will be denoted by $\mathcal{U}$.

Before we proceed with the main results of the paper we will recall several notions that we will use. Since we will use ordinal sums of trivial semigroups, let us recall that there exists only one operation on a trivial semigroup, namely the function $\mathrm{Id}\colon \{x\}^2 \longrightarrow \{x\}$, which is simply defined by $\mathrm{Id}(x, x) = x$.

If we will talk about linear transformation from interval $[a, b]$ to interval $[c, d]$ we mean a linear function $\varphi\colon [a, b] \longrightarrow [c, d]$ given by

$$\varphi(x) = \frac{(x-a)\cdot(d-c)}{b-a} + c,$$

which transforms a unary function $f\colon [a, b] \longrightarrow [a, b]$ to a function $g\colon [c, d] \longrightarrow [c, d]$ given by $g(x) = \varphi(f(\varphi^{-1}(x)))$, and transforms a binary function $V\colon [a, b]^2 \longrightarrow [a, b]$ to a function $U\colon [c, d]^2 \longrightarrow [c, d]$ given by $U(x, y) = \varphi(V(\varphi^{-1}(x), \varphi^{-1}(y)))$.

For a binary function $F\colon [0, 1]^2 \longrightarrow [0, 1]$ and an interval $[a, b]$ we will denote the linear transformation of F to $[a, b]$ by $F^{[a,b]}$. Similarly, for an open, or a half-open interval we will assume the linear transformation of the function restricted to corresponding open, or half-open unit square.

Further, for any $0 \le a < b \le c < d \le 1$, $v \in [b, c]$ and a uninorm $U\colon [0, 1]^2 \longrightarrow [0, 1]$ with the neutral element $e \in]0, 1[$ we will use the transformation $f\colon [0, 1] \longrightarrow [a, b[\cup \{v\} \cup]c, d]$, given by

$$f(x) = \begin{cases} (b-a)\cdot\frac{x}{e} + a & \text{if } x \in [0, e[, \\ v & \text{if } x = e, \\ d - \frac{(1-x)(d-c)}{(1-e)} & \text{otherwise.} \end{cases} \tag{1}$$

Then f is linear on $[0, e[$ and on $]e, 1]$ and thus it is an increasing, piece-wise linear isomorphism of $[0, 1]$ to $([a, b[\cup \{v\} \cup]c, d])$ which preserves the commutativity, the associativity, the monotonicity and the neutral element; and the binary function $U_v^{a,b,c,d}\colon ([a, b[\cup \{v\} \cup]c, d])^2 \longrightarrow ([a, b[\cup \{v\} \cup]c, d])$ given by

$$U_v^{a,b,c,d}(x, y) = f(U(f^{-1}(x), f^{-1}(y))) \tag{2}$$

is a uninorm on $([a, b[\cup \{v\} \cup]c, d])^2$. The backward transformation f^{-1} then transforms a uninorm defined on $([a, b[\cup \{v\} \cup]c, d])^2$ to a uninorm defined on $[0, 1]^2$.

For the rest of the paper if we say that two semigroups (X_1, F_1) and (X_2, F_2) are isomorphic we assume that there exists an increasing isomorphism $\varphi\colon X_1 \longrightarrow X_2$ such that $F_1(x, y) = \varphi^{-1}(F_2(\varphi(x), \varphi(y)))$ for all $x, y \in X_1$. Note that such an isomorphism preserves the commutativity, the associativity, the monotonicity, the (local) neutral element and the annihilator, as well. Moreover, we will use just isomorphisms that are either linear, or the functions f, f^{-1} given in (1). Observe that a linear function on an interval is also a homeomorphism and it is easy to show that if a restriction of an n-uninorm from $\mathcal{U}_n$ is a uninorm then it can be transformed to a uninorm with continuous underlying functions using f (f^{-1}).

As we mentioned above, each idempotent uninorm is an ordinal sum of trivial semigroups. Similarly, each idempotent n-uninorm is a z-ordinal sum of trivial semigroups [16]. In the following we recall the z-ordinal sum construction. Note that a meet semi-lattice (or lower semi-lattice) [4] is a partially ordered set which has a meet (or greatest lower bound) for any non-empty finite subset. Since the existence of the meet is required only for non-empty finite subsets this is equivalent to the existence of the meet between all pairs of arguments.

Theorem 4. *Let A and B be two index sets such that $A \cap B = \emptyset$ and $C = A \cup B \neq \emptyset$. Let $(G_\alpha)_{\alpha \in C}$ with $G_\alpha = (X_\alpha, *_\alpha)$ be a family of semigroups and let the set C be partially ordered by the binary relation $\preceq$ such that $(C, \preceq)$ is a meet semi-lattice. Further suppose that each semigroup G_α for $\alpha \in A$ possesses an annihilator z_α, and for all $\alpha, \beta \in C$ such that α and β are incomparable there is $\alpha \wedge \beta \in A$. Assume that for all $\alpha, \beta \in C$, $\alpha \neq \beta$, the sets X_α and X_β are either disjoint or that $X_\alpha \cap X_\beta = \{x_{\alpha,\beta}\}$. In the second case suppose that for all $\gamma \in C$ which is incomparable with $\alpha \wedge \beta$ there is $\alpha \wedge \gamma = \beta \wedge \gamma$ and for each $\gamma \in C$ with $\alpha \wedge \beta \prec \gamma \prec \alpha$ or $\alpha \wedge \beta \prec \gamma \prec \beta$ we have $X_\gamma = \{x_{\alpha,\beta}\}$. Further,*

(i) *in the case that $\alpha \wedge \beta \in A$ then $x_{\alpha,\beta} = z_{\alpha \wedge \beta}$ is the annihilator of both G_β and G_α;*
(ii) *in the case that $\alpha \wedge \beta = \alpha \in B$ then $x_{\alpha,\beta}$ is both the annihilator of G_β and the neutral element of G_α.*

Put $X = \bigcup_{\alpha\in C} X_\alpha$ *and define the binary operation* $*$ *on* X *by*

$$x * y = \begin{cases} x *_\alpha y & \text{if } (x, y) \in X_\alpha \times X_\alpha, \\ x & \text{if } (x, y) \in X_\alpha \times X_\beta, \alpha \neq \beta, \text{ and } \alpha \wedge \beta = \alpha \in B, \\ y & \text{if } (x, y) \in X_\alpha \times X_\beta, \alpha \neq \beta, \text{ and } \alpha \wedge \beta = \beta \in B, \\ z_\gamma & \text{if } (x, y) \in X_\alpha \times X_\beta, \alpha \neq \beta, \text{ and } \alpha \wedge \beta = \gamma \in A. \end{cases}$$

Then $G = (X, *)$ *is a semigroup. The semigroup* G *is commutative if and only if for each* $\alpha \in C$ *the semigroup* G_α *is commutative.*

Observe that sets X_α in z-ordinal sum construction need not to be disjoint, however, in [16] it was shown that $*$ is well defined and thus in order to obtain the value $x * y$ for $x \in X_\alpha \cap X_\beta$ we can select any of the two semigroups. The set A from Theorem 4 will be called the branching set of the respective z-ordinal sum.

Further, since we use several index sets in our proofs let us observe that these index sets can be chosen in such a way that they are mutually disjoint and we will suppose that they are, without mentioning it on every place. Similarly, when we add a new index we assume that it does not belong to any index set that is already in use.

3. Nullnorms as z-ordinal sums of semigroups related to t-norms and t-conorms

In this section we would like to briefly discuss the decomposition of nullnorms with continuous underlying functions into irreducible semigroups. Note that the set $X \subseteq [0, 1]$, or a semigroup $(X, V|_{X^2})$, is called irreducible with respect to a nullnorm V if $(X, V|_{X^2})$ cannot be expressed as a non-trivial z-ordinal sum of proper subsemigroups of $(X, V|_{X^2})$. Note that for the sake of simplicity we will write (X, V) instead of $(X, V|_{X^2})$.

First let us observe that due to Theorem 3 it is easy to show the following result.

Lemma 1. *Let* $V : [0, 1]^2 \longrightarrow [0, 1]$ *be a nullnorm with the annihilator* $z \in]0, 1[$. *Then* $([0, 1], V)$ *is a* z*-ordinal sum of* $G_1 = ([0, z], V)$, $G_2 = ([z, 1], V)$ *and* $G_3 = (\{z\}, \mathrm{Id})$, *where the corresponding partial order on* $\{1, 2, 3\}$ *is given by* $1 \wedge 2 = 3$. *Here* G_1 *is isomorphic to a t-conorm and* G_2 *is isomorphic to a t-norm.*

Proof. Theorem 3 implies that G_1 is isomorphic to a t-conorm and G_2 is isomorphic to a t-norm. Further, z is the annihilator of all three semigroups and it is easy to check that all conditions of Theorem 4 are fulfilled. Assume that $([0, 1], W)$ is a z-ordinal sum of G_1, G_2 and G_3 with respect to the partial order given by $1 \wedge 2 = 3$. Then $V(x, y) = W(x, y)$ for all $x, y \in [0, z]$, $V(x, y) = W(x, y)$ for all $x, y \in [z, 1]$ and for $x \in [0, z]$, $y \in [z, 1]$ we can assume that x is from G_1 and y from G_2 and thus $W(x, y) = z$. By Theorem 3 and the commutativity we get $V(x, y) = W(x, y)$ for all $x, y \in [0, 1]$. □

Since the ordinal sum decomposition of a t-norm (t-conorm) is generally known only for continuous t-norms (t-conorms) in the following we will focus on nullnorms with continuous underlying functions.

Archimedean nullnorms

First we will assume that a nullnorm $V : [0, 1]^2 \longrightarrow [0, 1]$ is Archimedean, i.e., that the underlying t-conorm S_V and the underlying t-norm T_V are continuous and Archimedean. In such a case V has exactly 3 idempotent points: 0, z and 1.

Case 1: If S_V and T_V are nilpotent.

Then $\lim_{n\longrightarrow\infty} x_V^{(n)} = z$ for all $x \in]0, 1[$, where for $x \in [0, 1]$, $n \in \mathbb{N}$, there is $x_V^{(1)} = x$ and $x_V^{(n)} = V(x, x_V^{(n-1)})$. Therefore the sets irreducible with respect to V are sets $\{0\}$, $]0, z]$, $[z, 1[$ and $\{1\}$. Thus we define semigroups $G_1 = (\{0\}, \mathrm{Id})$, $G_2(]0, z], V)$, $G_3([z, 1[, V)$ and $G_4 = (\{1\}, \mathrm{Id})$. However, since $V(0, x) = x$ for $x \in [0, z]$ there should be $2 \prec 1$ and $2 \in B$. Similarly, $3 \prec 4$ and $3 \in B$. If $3 \prec 2$ ($2 \prec 3$) then $V(x, y) = z$ for all $x \in [0, z]$, $y \in [z, 1]$ implies $3 \in A$ ($2 \in A$) which is a contradiction. Therefore 2 and 3 are incomparable and the semigroup $G_{2\wedge 3}$ should contain z. Hence, we have to add an additional semigroup $(\{z\}, \mathrm{Id})$. Then we have the following result.

Proposition 2. *Let* $V : [0, 1]^2 \longrightarrow [0, 1]$ *be a nullnorm with the annihilator* $z \in]0, 1[$ *and let* T_V *and* S_V *be nilpotent. Then* V *is a* z*-ordinal sum of semigroups* $G_1 = (\{0\}, \mathrm{Id})$, $G_2(]0, z], V)$, $G_3 = (\{z\}, \mathrm{Id})$, $G_4([z, 1[, V)$ *and* $G_5 = (\{1\}, \mathrm{Id})$. *Further,* G_2 *is isomorphic to a nilpotent t-conorm restricted to* $]0, 1]^2$ *and* G_4 *is isomorphic to a nilpotent t-norm restricted to* $[0, 1[^2$. *The partial order in the respective* z*-ordinal sum is given by* $3 \prec 2 \prec 1$, $3 \prec 4 \prec 5$ *and other pairs are incomparable.*

Proof. First let us observe that Theorem 3 implies that G_2 is isomorphic to a nilpotent t-conorm restricted to $]0, 1]^2$ and G_4 is isomorphic to a nilpotent t-norm restricted to $[0, 1[^2$. We define $A = \{3\}$, $C = \{1, 2, 3, 4, 5\}$, $B = C \setminus A$, and a partial order $\preceq$ on the set C given by $3 \prec 2 \prec 1$, $3 \prec 4 \prec 5$ and other pairs are incomparable. Then

- $(C, \preceq)$ is a meet semi-lattice and G_3 trivially possesses an annihilator z.
- It is easy to check that if $a, b \in C$ are incomparable then $a \wedge b = 3 \in A$.

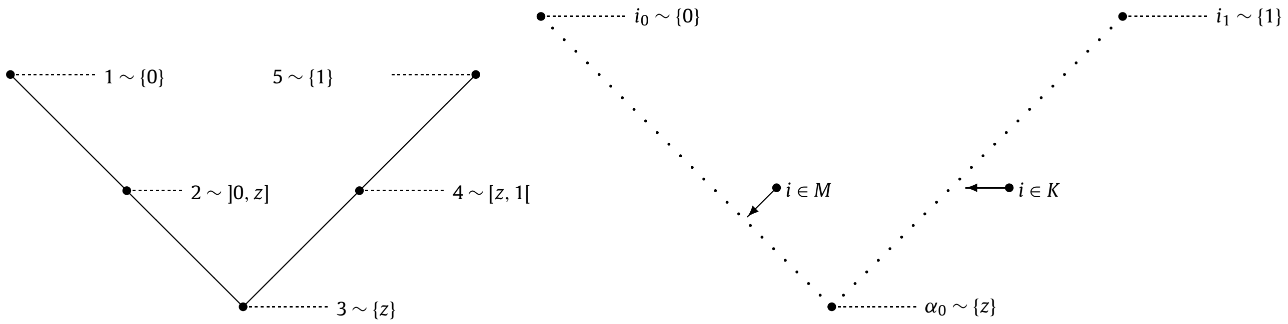

Fig. 1. A partial order corresponding to an Archimedean nullnorm such that T_V and S_V are nilpotent (left) and a partial order corresponding to a non-Archimedean nullnorm such that T_V and S_V are continuous (right). The left branch corresponds to the order used in the respective ordinal sum yielding S_V and the right branch corresponds to the order used in the respective ordinal sum yielding T_V.

- X_1 and X_5 have an empty intersection with all other semigroup carriers. $X_2 \cap X_3 = \{z\} = X_4 \cap X_3 = X_2 \cap X_4$ and since $2 \wedge 4 = 3$ we only have to show that z is the annihilator in all three G_2, G_3, G_4 which is evident since G_2 (G_4) is isomorphic to a nilpotent t-conorm (t-norm).

Thus the sets A, B, C, semigroups G_i for $i \in C$ and the partial order $\preceq$ fulfill all requirements of Theorem 4.

Let $([0,1], V^*)$ be the z-ordinal sum of our 5 semigroups with the partial order $\preceq$ with respect to the set A. Then since all semigroups G_i for $i \in C$ are commutative also V^* is commutative. $V = V^*$ on $]0,z]^2 \cup [z,1[^2$, $V^*(0,0) = 0$ and $V^*(1,1) = 1$.

Further, if $x \in]0,z]$ and $y \in [z,1[$ then since 2 and 4 are incomparable, $2 \wedge 4 = 3$, we get $V^*(x,y) = z$ and the commutativity together with previous findings implies $V(x,y) = V^*(x,y)$ for all $x, y \in]0,1[$. Since $2 \prec 1$ and $2 \in B$ for an $x \in]0,z]$ we get $V^*(0,x) = x$.

Since 1 and 4 are incomparable and $1 \wedge 4 = 3 \in A$ for a $y \in [z,1[$ we get $V^*(0,y) = z$ and finally, $V^*(0,1) = z$ since $1 \wedge 5 = 3 \in A$. Thus $V(0,x) = V^*(0,x)$ for all $x \in [0,1]$. Similarly we can show that $V^*(1,x) = V(1,x)$ for all $x \in [0,1]$. Summarizing, $V(x,y) = V^*(x,y)$ for all $x, y \in [0,1]$ (Fig. 1). □

Case 2: If S_V and T_V are strict.

For a nullnorm V such that T_V and S_V are strict we get a similar result. Since the proof is analogous we omit it here.

Proposition 3. *Let $V : [0,1]^2 \longrightarrow [0,1]$ be a nullnorm with the annihilator $z \in]0,1[$ and let T_V and S_V be strict. Then V is a z-ordinal sum of semigroups $G_1 = (\{0\}, \mathrm{Id})$, $G_2(]0,z[\,, V)$, $G_3 = (\{z\}, \mathrm{Id})$, $G_4(]z,1[\,, V)$ and $G_5 = (\{1\}, \mathrm{Id})$. Further, G_2 is isomorphic to a strict t-conorm restricted to $]0,1[^2$ and G_4 is isomorphic to a strict t-norm restricted to $]0,1[^2$. The partial order in the respective z-ordinal sum is given by $3 \prec 2 \prec 1$, $3 \prec 4 \prec 5$ and other pairs are incomparable.*

Case 3: If S_V is strict and T_V is nilpotent.

Proposition 4. *Let $V : [0,1]^2 \longrightarrow [0,1]$ be a nullnorm with the annihilator $z \in]0,1[$ and let T_V be nilpotent and S_V be strict. Then V is a z-ordinal sum of semigroups $G_1 = (\{0\}, \mathrm{Id})$, $G_2(]0,z[\,, V)$, $G_3 = (\{z\}, \mathrm{Id})$, $G_4([z,1[\,, V)$ and $G_5 = (\{1\}, \mathrm{Id})$. Further, G_2 is isomorphic to a strict t-conorm restricted to $]0,1[^2$ and G_4 is isomorphic to a nilpotent t-norm restricted to $[0,1[^2$. The partial order in the respective z-ordinal sum is given by $3 \prec 2 \prec 1$, $3 \prec 4 \prec 5$ and other pairs are incomparable.*

Case 4: If S_V is nilpotent and T_V is strict.

Proposition 5. *Let $V : [0,1]^2 \longrightarrow [0,1]$ be a nullnorm with the annihilator $z \in]0,1[$ and let T_V be strict and S_V be nilpotent. Then V is a z-ordinal sum of semigroups $G_1 = (\{0\}, \mathrm{Id})$, $G_2(]0,z]\,, V)$, $G_3 = (\{z\}, \mathrm{Id})$, $G_4(]z,1[\,, V)$ and $G_5 = (\{1\}, \mathrm{Id})$. Further, G_2 is isomorphic to a nilpotent t-conorm restricted to $]0,1]^2$ and G_4 is isomorphic to a strict t-norm restricted to $]0,1[^2$. The partial order in the respective z-ordinal sum is given by $3 \prec 2 \prec 1$, $3 \prec 4 \prec 5$ and other pairs are incomparable.*

Non-Archimedean nullnorms

Now assume that T_V and S_V are a continuous t-norm and a continuous t-conorm, respectively. Then T_V can be expressed as an ordinal sum of a countable number of semigroups G_i, $i \in K$, where K is an index set and each semigroup G_i for $i \in K$ corresponds either to an Archimedean t-norm (i.e., it is isomorphic to an Archimedean t-norm possibly restricted to open, or half open unit interval) or to the minimum t-norm (i.e., it is isomorphic to the minimum t-norm possibly restricted to open, or half open unit interval), or it is a trivial semigroup (see [9]). The corresponding linear order in the ordinal sum is given by $i \leq_T j$ if $x \leq y$ for all $x \in X_i$ and all $y \in X_j$, $i, j \in K$.

Similarly S_V can be expressed as an ordinal sum of a countable number of semigroups G_i, $i \in M$, where M is an index set, and each semigroup G_i for $i \in M$ corresponds either to an Archimedean t-conorm (i.e., it is isomorphic to an Archimedean t-conorm possibly restricted to open, or half open unit interval) or to the maximum t-conorm (i.e., it is isomorphic to the maximum t-conorm possibly restricted to open, or half open unit interval), or it is a trivial semigroup. The corresponding linear order in the ordinal sum is given by $i \leq_S j$ if $x \geq y$ for all $x \in X_i$ and all $y \in X_j$, $i, j \in M$.

We define the set C by $C = M \cup K \cup \{\alpha_0\}$ and a semigroup $G_{\alpha_0} = (\{z\}, \mathrm{Id})$. Further we define $A = \{\alpha_0\}$ and $B = C \setminus A$. Observe that if $z \in X_i$ for some $i \in M$ ($i \in K$) then z is the annihilator of G_i and if $z \in X_i \cap X_j$ for some $i, j \in M$, $i \leq_S j$ ($i, j \in K$, $i \leq_T j$) then the corresponding ordinal sum construction implies $X_i = \{z\}$.

Further we define a partial order $\preceq$ on C by:

(i) $\alpha_0 \preceq i$ for all $i \in C$.
(ii) $i \preceq j$ if $i, j \in K$ and $i \leq_T j$.
(iii) $i \preceq j$ if $i, j \in M$ and $i \leq_S j$.
(iv) If $i \in M$ and $j \in K$ then i and j are incomparable.

Then $(C, \preceq)$ is a meet semi-lattice with the bottom element α_0 and similarly as in Proposition 2 we can easily check that all requirements of Theorem 4 are fulfilled. Let V^* be a z-ordinal sum of semigroups G_i, $i \in C$ with respect to the set A and the partial order $\preceq$. Since z-ordinal sum reduces to the ordinal sum on a chain of semigroups from B, and the order on M (K) is a subset of the partial order $\preceq$ we can see that $V^* = V$ on $[0, z[^2$ (or on $[0, z]^2$ in the case that S_V has divisors of 1) and $V^* = V$ on $]z, 1]^2$ (or on $[z, 1]^2$ in the case that T_V has zero divisors). Further, since $\alpha_0 \in A$ is the bottom element of the meet semi-lattice $(C, \preceq)$, we get $V^*(z, x) = z$ for all $x \in [0, 1]$. Finally assume that $x \in [0, z[$, $y \in]z, 1]$. Then there exist $i \in K$ and $j \in M$ such that $x \in X_j$, $y \in X_i$ and thus i and j are incomparable, i.e., $i \wedge j = \alpha_0 \in A$. Therefore $V^*(y, x) = V^*(x, y) = z$. Then $V^*(x, y) = V(x, y)$ for all $x, y \in [0, 1]$.

Summarizing, we have shown the following result.

Theorem 5. *Let $V : [0, 1]^2 \longrightarrow [0, 1]$ be a nullnorm with the annihilator $z \in]0, 1[$ and let T_V and S_V be continuous. Then V is a z-ordinal sum of a countable number of semigroups related to continuous Archimedean t-norms, continuous Archimedean t-conorms and internal t-norms and t-conorms (including trivial semigroups).*

Observe that nullnorms are special 2-uninorms. In the following sections we will show similar results for n-uninorms with continuous underlying functions.

4. General remarks on *n*-uninorms with continuous underlying functions

First let us settle for this paper that if we say that a function is an n-uninorm we will suppose that it possesses the n-neutral element $\{e_1, \ldots, e_n\}_{z_1, \ldots, z_{n-1}}$.

For better understanding we begin with the description of the possible values of an n-uninorm $U^n \in \mathcal{U}_n$ when two points from different subintervals are taken. First recall the following result from [18].

Lemma 2. *Let $U^n : [0, 1]^2 \longrightarrow [0, 1]$ be an n-uninorm and let $U^n \in \mathcal{U}_n$. If $x \in]z_{i-1}, z_i[$ and $y \in \left]z_{j-1}, z_j\right[$ for some $i, j \in \{1, \ldots, n\}$, $i \neq j$, $x < y$, then $U^n(x, y) \in [x, e_i[\cup \{U^n(e_i, e_j)\} \cup \left]e_j, y\right]$.*

From [17] we know that $U^n(z_1, z_{n-1}) = z_k$ for some $k \in \{1, \ldots, n-1\}$. Then $U^n(x, z_k) \in \{x, z_k\}$ for all $x \in [0, 1]$ (see Lemma 5.1, 5.2 and 5.3 in [17]). Due to the monotonicity z_k is the annihilator of U^n on $[z_1, z_{n-1}]$. Further,

$$U^n(e_1, z_k) = U^n(e_1, U^n(z_1, z_k)) = U^n(U^n(e_1, z_1), z_k) = U^n(z_1, z_k) = z_k,$$

similarly $U^n(e_n, z_k) = z_k$ and the monotonicity implies $U^n(e_1, e_n) = z_k$. Since for any $i, j \in \{1, \ldots, n\}$, $i < j$, U^n restricted to $\left[z_{i-1}, z_j\right]^2$ is a $(j - i + 1)$-uninorm, similar observations can be shown for z_m, where $U^n(z_i, z_{j-1}) = z_m$. Here $U^n(e_i, e_j) = z_m$ and $U^n(x, z_m) \in \{x, z_m\}$ for all $x \in \left[z_{i-1}, z_j\right]^2$.

Proposition 6. *Let $U^n : [0, 1]^2 \longrightarrow [0, 1]$ be an n-uninorm and let $U^n \in \mathcal{U}_n$. Let $x \in]z_{i-1}, z_i[$ and $y \in \left]z_{j-1}, z_j\right[$ for some $i, j \in \{1, \ldots, n\}$, $i \neq j$, $x < y$, with $U^n(e_i, e_j) = z_k$. There is:*

(i) If $x \geq e_i$ and $y \leq e_j$ then $U^n(x, y) = z_k$.
(ii) If $x \leq e_i$ and $y \leq e_j$ then $U^n(x, y) = U^n(x, z_k) \in \{x, z_k\}$.
(iii) If $x \geq e_i$ and $y \geq e_j$ then $U^n(x, y) = U^n(y, z_k) \in \{y, z_k\}$.
(iv) If $x \leq e_i$ and $y \geq e_j$ then
- *$U^n(x, y) = z_k$, if $U^n(x, z_k) = z_k$ and $U^n(y, z_k) = z_k$,*
- *$U^n(x, y) = x$, if $U^n(x, z_k) = x$ and $U^n(y, z_k) = z_k$,*

- $U^n(x, y) = y$, *if* $U^n(x, z_k) = z_k$ *and* $U^n(y, z_k) = y$,
- $U^n(x, y) \in [x, e_i[\cup \{z_k\} \cup]e_j, y]$ *if* $U^n(x, z_k) = x$ *and* $U^n(y, z_k) = y$.

Proof. (i) If $x \geq e_i$ and $y \leq e_j$ then $[x, e_i[= \emptyset$, $]e_j, y] = \emptyset$ and then Lemma 2 implies $U^n(x, y) = U^n(e_i, e_j) = z_k$.

(ii) If $x \leq e_i$, $y \leq e_j$ there is

$$z_k = U^n(z_k, z_k) \leq U^n(z_k, y) \leq U^n(z_k, e_j) = z_k,$$

i.e., $U^n(y, z_k) = z_k$. If $U^n(x, z_k) = z_k$ then $z_k = U^n(x, z_k) \leq U^n(x, y) \leq U^n(z_k, y) = z_k$, i.e., $U^n(x, y) = z_k = U^n(x, z_k)$. If $U^n(x, z_k) = x$ then

$$U^n(x, y) = U^n(U^n(x, z_k), y) = U^n(x, U^n(z_k, y)) = U^n(x, z_k) = x,$$

i.e., $U^n(x, y) = x = U^n(x, z_k)$.

(iii) If $x \geq e_i$ and $y \geq e_j$ then we can show similarly as in previous case that $U^n(x, y) = z_k$, if $U^n(y, z_k) = z_k$ and $U^n(x, y) = y$ if $U^n(y, z_k) = y$.

(iv) If $x \leq e_i$ and $y \geq e_j$. If $U^n(x, z_k) = z_k$ and $U^n(y, z_k) = z_k$ then (similarly as above) the monotonicity implies $U^n(x, y) = z_k$. If $U^n(x, z_k) = x$ and $U^n(y, z_k) = z_k$ then the associativity implies $U^n(x, y) = x$. If $U^n(x, z_k) = z_k$ and $U^n(y, z_k) = y$ then the associativity implies $U^n(x, y) = y$. Finally, if $U^n(x, z_k) = x$ and $U^n(y, z_k) = y$ Lemma 2 implies $U^n(x, y) \in [x, e_i[\cup \{z_k\} \cup]e_j, y]$. □

From previous result we get the following example.

Example 1. Assume an n-uninorm $U^n \in \mathcal{U}_n$ and $i, j \in \{1, \ldots, n\}$, $i < j$.
Then we have the following.

(i) U^n restricted to $([e_i, z_i[\cup \{z_k\} \cup]z_{j-1}, e_j])^2$ is a z-ordinal sum of three semigroups $G_1 = ([e_i, z_i[\,, S_i^{[e_i,z_i[})$, $G_2 = (]z_{j-1}, e_j]\,, T_j^{]z_{j-1},e_j]})$ and $G_3 = (\{z_k\}, \mathrm{Id})$, where $1 \wedge 2 = 3$.

(ii) U^n restricted to $([e_i, z_i[\cup \{z_k\} \cup]e_j, z_j])^2$ is an z-ordinal sum of four semigroups. Assume that $y_0 = \sup\{y \in [e_j, z_j] \mid U^n(z_k, y) = z_k\}$. Then y_0 is an idempotent point of U^n and $S_j^{[e_j,z_j]}$ can be expressed as an ordinal sum of two t-conorms S_a on $[e_j, y_0]$ and S_b on $[y_0, z_j]$. We get $G_1 = ([e_i, z_i[\,, S_i^{[e_i,z_i[})$, $G_2 = (\{z_k\}, \mathrm{Id})$, $G_3 = (]e_j, y_0]\,, S_a)$ and $G_4 = (]y_0, z_j]\,, S_b)$ if $U^n(y_0, z_k) = z_k$ ($G_3 = (]e_j, y_0[\,, S_a)$ and $G_4 = ([y_0, z_j]\,, S_b)$ if $U^n(y_0, z_k) = y_0$) and the respective order in the z-ordinal sum is given by $1 \wedge 3 = 2$ and $4 \prec 2$.

(iii) U^n restricted to $([z_{i-1}, e_i[) \cup \{z_k\} \cup]z_{j-1}, e_j]))^2$ is an z-ordinal sum of four semigroups. Assume that $x_0 = \inf\{x \in [z_{i-1}, e_i] \mid U^n(z_k, x) = z_k\}$. Then x_0 is an idempotent point of U^n and $T_i^{[z_{i-1},e_i]}$ can be expressed as an ordinal sum of two t-norms T_a on $[z_{i-1}, x_0]$ and T_b on $[x_0, e_i]$. We get $G_1 = (]z_{j-1}, e_j]\,, T_j^{]z_{j-1},e_j]})$, $G_2 = (\{z_k\}, \mathrm{Id})$, $G_3 = ([z_{i-1}, x_0]\,, T_a)$ and $G_4 = (]x_0, e_i[\,, T_b)$ if $U^n(x_0, z_k) = x_0$ ($G_3 = ([z_{i-1}, x_0[\,, T_a)$ and $G_4 = ([x_0, e_i[\,, T_b)$ if $U^n(x_0, z_k) = z_k$) and the respective order in the z-ordinal sum is given by $1 \wedge 4 = 2$ and $3 \prec 2$.

(iv) If $U^n(x_0, z_k) = z_k = U^n(y_0, z_k)$ then U^n restricted to $([z_{i-1}, e_i[\cup \{z_k\} \cup]e_j, z_j])^2$ is an z-ordinal sum of four semigroups. As we will see later in other cases the situation can be quite complicated and therefore we focus just on the case when $U^n(x_0, y_0) = z_k$. Let x_0 and y_0, T_a, T_b and S_a and S_b be defined as above. Then U^n restricted to $([z_{i-1}, x_0[\cup \{z_k\} \cup]y_0, z_j])^2$ is a uninorm U with the neutral element z_k (see [17, Proposition 3.10]). We get $G_1 = ([x_0, e_i[\,, T_b)$, $G_2 = (\{z_k\}, \mathrm{Id})$, $G_3 = (]e_j, y_0]\,, S_a)$ and $G_4 = ([z_{i-1}, x_0[\cup \{z_k\} \cup]y_0, z_j]\,, U)$ and the order in the z-ordinal sum construction is $1 \wedge 3 = 2$ and $4 \prec 2$.

A sketch of the n-uninorm U^n on these regions can be seen on Fig. 2.

Our aim is to show the decomposition of each n-uninorm with continuous underlying functions using the z-ordinal sum construction. In the case of n-uninorms we will always use a z-ordinal sum with respect to the branching set A that contains only trivial semigroups corresponding to the points z_i for $i \in \{1, \ldots, n-1\}$. In order to make proofs easier we first introduce three useful results.

In the following sections we will show that for an n-uninorm with continuous underlying functions the structure of the respective partial order on C always resembles a tree. For such partial orders we have the following result.

Definition 5. Let $(C, \preceq)$ be a partially ordered set. We say that $(C, \preceq)$ has a tree structure if for each $p_1, p_2 \in C$ such that p_1 and p_2 are incomparable there is no upper bound for p_1 and p_2.

Lemma 3. *Let* $(C, \preceq)$ *be a partially ordered set which has a tree structure. For* $\alpha, \beta, \gamma \in C$, *if* γ *is incomparable with* $\alpha \wedge \beta$ *then* $\alpha \wedge \gamma = \beta \wedge \gamma$.

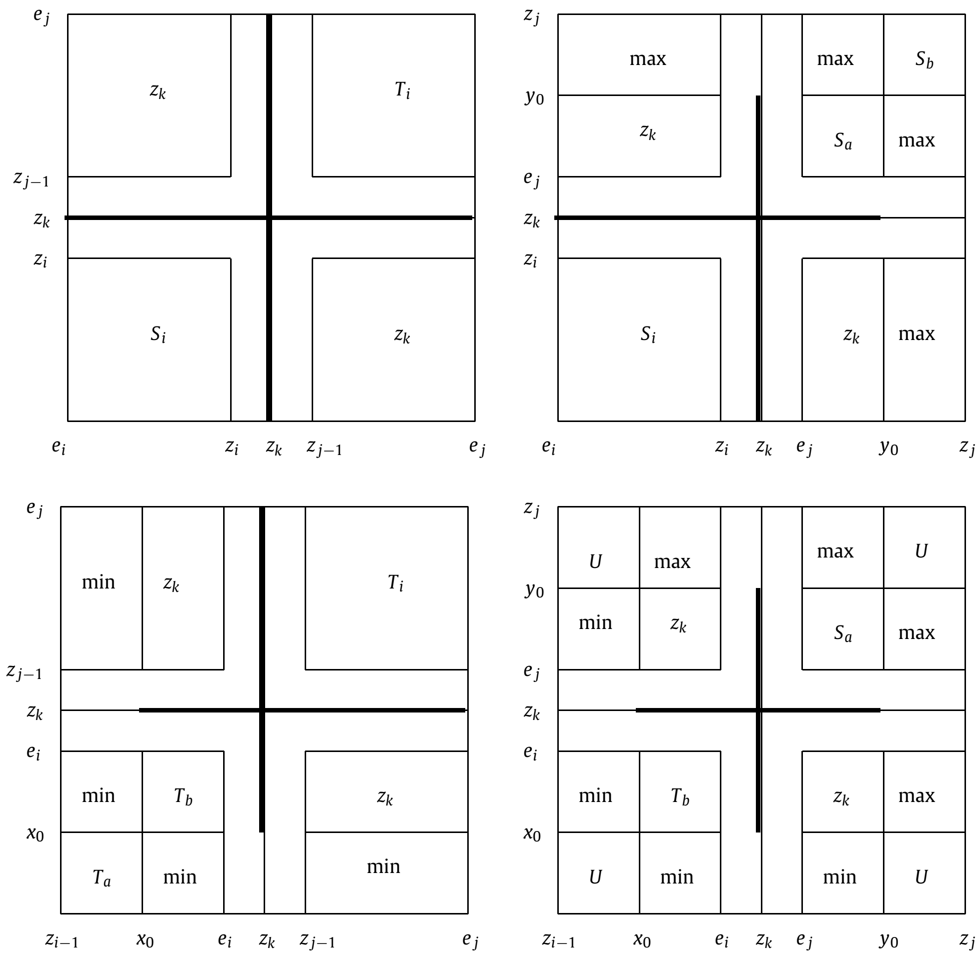

Fig. 2. Sketch of the n-uninorm U^n from Example 1 on respective regions. Bold lines denote the set where the functions $U^n(z_k, \cdot)$ and $U^n(\cdot, z_k)$ attain the value z_k.

Proof. Assume that γ is incomparable with $\alpha \wedge \beta$. If $\alpha \wedge \gamma$ is incomparable with $\alpha \wedge \beta$ then $\alpha \wedge \gamma \preceq \alpha$ and $\alpha \wedge \beta \preceq \alpha$ implies that α is an upper bound of $\alpha \wedge \gamma$ and $\alpha \wedge \beta$, which is a contradiction since $(C, \preceq)$ has a tree structure. Thus $\alpha \wedge \gamma$ is comparable with $\alpha \wedge \beta$ and similarly $\beta \wedge \gamma$ is comparable with $\alpha \wedge \beta$. If $\alpha \wedge \beta \preceq \alpha \wedge \gamma \preceq \gamma$ then γ is comparable with $\alpha \wedge \beta$, which is a contradiction. Thus $\alpha \wedge \gamma \preceq \alpha \wedge \beta \preceq \beta$ and similarly $\beta \wedge \gamma \preceq \alpha \wedge \beta \preceq \alpha$. Then, however, $\alpha \wedge \gamma \preceq \beta$ and $\beta \wedge \gamma \preceq \alpha$ and thus $\alpha \wedge \gamma = \alpha \wedge \gamma \wedge \gamma \preceq \beta \wedge \gamma$ and similarly $\beta \wedge \gamma \preceq \alpha \wedge \gamma$. Summarizing, $\beta \wedge \gamma = \alpha \wedge \gamma$. $\square$

Lemma 4. *Let $(X, *)$ be a z-ordinal sum of semigroups $(G_\alpha)_{\alpha \in C}$ with respect to sets A and B and a partial order $\preceq$. Assume that for each $\alpha \in B$ the semigroup G_α is an ordinal sum of semigroups $(H_\beta)_{\beta \in B_\alpha}$ for some linearly ordered index set $(B_\alpha, \leq_\alpha)$ and $H_\beta = G_\beta$ for all $\beta \in A$. Then $(X, *)$ is a z-ordinal sum of semigroups $(H_\beta)_{\beta \in A' \cup B'}$ with respect to sets $A' = A$, $B' = \bigcup_{\alpha \in B} B_\alpha$ and a partial order $\preceq'$ given by:*

(i) If $p_1, p_2 \in B_\alpha$ for some $\alpha \in B$ then $p_1 \preceq' p_2$ if $p_1 \leq_\alpha p_2$.
(ii) If $p_1 \in B_\alpha$ and $p_1 \in B_\beta$ for some $\alpha, \beta \in B$ then $p_1 \preceq' p_2$ if $\alpha \preceq \beta$ and $p_2 \preceq' p_1$ if $\beta \preceq \alpha$.
(iii) If $p_1 \in B_\alpha$ for some $\alpha \in B$ and $p_2 \in A$. Then $p_1 \preceq' p_2$ if $\alpha \preceq p_2$ and $p_2 \preceq' p_1$ if $p_2 \preceq \alpha$.
(iv) If $p_1, p_2 \in A$ then $p_1 \preceq' p_2$ if $p_1 \preceq p_2$.

Moreover, if $(C, \preceq)$ for $C = A \cup B$ has a tree structure then also $(C', \preceq')$ for $C' = A' \cup B'$ has a tree structure.

Proof. It is easy (but tedious) to check that $\preceq'$ is a partial order and $(C', \preceq')$ is a meet semi-lattice and therefore we leave it as an exercise for the reader.

Now we will check whether all conditions of Theorem 4 are fulfilled by A', B' and $\preceq'$. Since $A' = A$ we see that H_α possesses an annihilator for all $\alpha \in A'$. If $p_1, p_2 \in A' \cup B'$ are incomparable then either $p_1 \in B_\alpha$, $p_2 \in B_\beta$ and α and β are incomparable (with respect to $\preceq$), or at least one from p_1, p_2 belongs to the set A. Therefore $p_1 \wedge' p_2 \in A'$. Further assume that $x \in X_{p_1} \cap X_{p_2}$ for some $x \in [0, 1]$, $p_1, p_2 \in C'$.

1. If p_3 is incomparable with $p_1 \wedge p_2$ then $p_1 \wedge p_2$ and p_3 cannot belong to the same set B_α for some $\alpha \in B$. If $p_1, p_2 \in B_\alpha$ for some $\alpha \in B$ then p_3 is incomparable with α with respect to $\preceq$ if $p_3 \in A$ (β is incomparable with α with respect to $\preceq$ if $p_3 \in B_\beta$) and thus $p_1 \wedge' p_3 = p_2 \wedge' p_3 = \alpha \wedge p_3 \in A$ ($p_1 \wedge' p_3 = p_2 \wedge' p_3 = \alpha \wedge \beta \in A$). If p_1 and p_2 does not belong to the same set B_α for some $\alpha \in B$ then since A, B and $\preceq$ fulfill conditions of Theorem 4 we get $p_1 \wedge' p_3 = p_2 \wedge' p_3$.
2. If $p_1 \wedge' p_2 = p_1 \in B'$ ($p_1 \wedge' p_2 = p_2 \in B'$) then conditions of Theorem 1 (for all $(B_\alpha, \leq_\alpha), \alpha \in B$) and conditions of Theorem 4 (for A, B and $\preceq$) imply that x is both the annihilator of G_{p_2} (G_{p_1}) and the neutral element of G_{p_1} (G_{p_2}). If $p_1 \wedge' p_2 \in A'$ then conditions of Theorem 4 for A, B and $\preceq$ imply that x is the annihilator of both G_{p_1} and G_{p_2}.
3. We should check that for all $p_3 \in A' \cup B'$ with $p_1 \wedge' p_2 \prec' p_3 \prec' p_1$ or $p_1 \wedge' p_2 \prec' p_3 \prec' p_2$ we have $X_{p_3} = \{x\}$. If $p_1, p_2 \in B_\alpha$ for some $\alpha \in B$ then the claim follows from Theorem 1.
   Assume $p_1, p_3 \in B_\alpha$ for some $\alpha \in B$ (the case when $p_2, p_3 \in B_\alpha$ is analogous).
   - If $p_1 \wedge' p_2 \in A'$ then x is the annihilator of G_α. Since p_1 and p_3 are comparable and $p_1 \wedge' p_2 = p_3 \wedge' p_2 \in A'$ we have $p_1 \wedge' p_2 \prec' p_3 \prec' p_1$, i.e., $p_3 \leq_\alpha p_1$. Then for all $y \in X_{p_3}$ there is $y = y * x = x$, since x is the annihilator of G_α. Thus $X_{p_3} = \{x\}$.
   - If $p_1 \wedge' p_2 \in B'$ and $p_1 \prec' p_2$ then x is the neutral element of G_α and $p_1 \prec' p_3 \prec' p_2$, i.e., $p_1 \leq_\alpha p_3$. Then for all $y \in X_{p_3}$ there is $x = y * x = y$, since x is the neutral element of G_α. Thus $X_{p_3} = \{x\}$.
   - If $p_1 \wedge' p_2 \in B'$ and $p_2 \prec' p_1$. This case is analogous to previous one.

   If p_1, p_2 and p_3 belong to three distinct index sets then the claim follows from conditions of Theorem 4 for A, B and $\preceq$.

Summarizing, the sets A', B' and the partial order $\preceq'$ fulfill all requirements of Theorem 4.

Assume that $(X, \diamond)$ is a z-ordinal sum of semigroups $(H_\beta)_{\beta \in A' \cup B'}$ with respect to sets A', B' and the partial order $\preceq'$. We will show that $x * y = x \diamond y$ for all $x, y \in X$. If $x \in X_{p_1}$, $y \in X_{p_2}$ for some $p_1, p_2 \in B_\alpha$ then $p_1 \wedge p_2 \in B'$ and $x * y = x \diamond y$ since for $p_1 \wedge p_2 \in B'$ the ordinal sum and the z-ordinal sum coincide. In all other cases z-ordinal sum with respect to A, B and $\preceq$ and z-ordinal sum with respect to A', B' and $\preceq'$ coincide. Therefore $(X, *)$ is a z-ordinal sum of semigroups $(H_\beta)_{\beta \in A' \cup B'}$ with respect to sets $A' = A$ and $B' = \bigcup\limits_{\alpha \in B} B_\alpha$ and the partial order $\preceq'$. From the definition of $\preceq'$ it is easy to see that if $(C, \preceq)$ has a tree structure then also $(C', \preceq')$ has a tree structure. □

Lemma 5. *Let $(X, *)$ be a z-ordinal sum of semigroups G_1, G_2, G_3 and G_4, where $A = \{3\}$ and $\preceq$ is given by $1 \wedge 2 = 3$ and $4 \prec 3$. Assume that G_1 is a z-ordinal sum of semigroups H_α with respect to A_1, B_1 and $\preceq_1$, where $(C_1, \preceq_1)$ for $C_1 = A_1 \cup B_1$ has a tree structure. Similarly, assume that G_2 is a z-ordinal sum of semigroups H_α with respect to A_2, B_2 and $\preceq_2$, where $(C_2, \preceq_2)$ for $C_2 = A_2 \cup B_2$ has a tree structure; and $H_3 = G_3$, $H_4 = G_4$. Then $(X, *)$ is a z-ordinal sum of semigroups $(H_\alpha)_{\alpha \in C_1 \cup C_2 \cup \{3,4\}}$ with respect to $A' = A_1 \cup A_2 \cup \{3\}$, $B' = B_1 \cup B_2 \cup \{4\}$ and $\preceq'$ given by*

(i) $\alpha \preceq' \beta$ if $\alpha, \beta \in C_1$ and $\alpha \preceq_1 \beta$.
(ii) $\alpha \preceq' \beta$ if $\alpha, \beta \in C_2$ and $\alpha \preceq_2 \beta$.
(iii) $4 \prec' 3 \prec' \alpha$ for all $\alpha \in C_1 \cup C_2$.
(iv) If $\alpha \in C_1$ and $\beta \in C_2$ then α and β are incomparable.

Moreover, $(C', \preceq')$ has a tree structure.

Proof. Since $(C, \preceq), (C_1, \preceq_1)$ and $(C_2, \preceq_2)$ have a tree structure also $(C', \preceq')$ has a tree structure. Indeed, what we do is that in the tree with two branches we replace one branch with one sub-tree and the second branch with another sub-tree. We can easily verify that $(C', \preceq')$ fulfills all conditions of Theorem 4, since they follow from the corresponding properties of $(C, \preceq), (C_1, \preceq_1)$ and $(C_2, \preceq_2)$. Let $(X, \diamond)$ be a z-ordinal sum of $(H_\alpha)_{\alpha \in C_1 \cup C_2 \cup \{3,4\}}$ with respect to A', B' and $\preceq'$. Then similarly as before we can check that $x \diamond y = x * y$ for all $x \in X_\alpha$, $y \in X_\beta$, $\alpha, \beta \in C'$. Thus $(X, *)$ can be expressed as a z-ordinal sum of $(H_\alpha)_{\alpha \in C_1 \cup C_2 \cup \{3,4\}}$ with respect to $A' = A_1 \cup A_2 \cup \{3\}$, $B' = B_1 \cup B_2 \cup \{4\}$ and $\preceq'$. □

5. 2-uninorms with continuous underlying functions

In this section we will focus on 2-uninorms and we will show that each 2-uninorm with continuous underlying functions can be expressed as a z-ordinal sum of Archimedean and idempotent semigroups. In the following section, using induction, we will show similar results also for all n-uninorms with continuous underlying functions, where $n > 2$.

As we mentioned in Section 2, in [14] it was shown that each uninorm with continuous underlying functions is an ordinal sum of a countable number of semigroups related to representable uninorms, continuous Archimedean t-norms, continuous Archimedean t-conorms and internal uninorms (including the min and the max operator). In the following we recall the definition of these semigroups.

Definition 6. Let $a, b, c, d \in [0, 1]$ with $a < b < c < d$, $v \in [b, c]$. Then

(i) a semigroup $(]a,b[\,\cup\{v\}\cup\,]c,d[\,,*)$ will be called a *representable* semigroup if $*$ is isomorphic via (2) to a restriction of a representable uninorm on $[0,1]^2$ to $]0,1[^2$,
(ii) a semigroup $(]a,b[\,,*)$ will be called a *t-strict* semigroup if $*$ is linearly isomorphic to a restriction of a strict t-norm on $[0,1]^2$ to $]0,1[^2$,
(iii) a semigroup $(]c,d[\,,*)$ will be called an *s-strict* semigroup if $*$ is linearly isomorphic to a restriction of a strict t-conorm on $[0,1]^2$ to $]0,1[^2$,
(iv) a semigroup $([a,b[\,,*)$ will be called a *t-nilpotent* semigroup if $*$ is linearly isomorphic to a restriction of a nilpotent t-norm on $[0,1]^2$ to $[0,1[^2$,
(v) a semigroup $(]c,d]\,,*)$ will be called an *s-nilpotent* semigroup if $*$ is linearly isomorphic to a restriction of a nilpotent t-conorm on $[0,1]^2$ to $]0,1]^2$,
(vi) a semigroup $(]a,b[\,\cup\,]c,d[\,,*)$ will be called a *d-internal* semigroup if $*$ is isomorphic via (2) to a restriction of an d-internal uninorm on $[0,1]^2$ to $(]0,1[\setminus\{e\})^2$,
(vii) a semigroup $(]a,b[\,,*)$ will be called a *t-internal* semigroup if $*=\min$,
(viii) a semigroup $(]c,d[\,,*)$ will be called an *s-internal* semigroup if $*=\max$.

The set of semigroups from previous definition and trivial semigroups will be denoted by $\mathcal{H}$. Observe that a representable semigroup is not Archimedean, however, it is composed of an Archimedean t-norm and an Archimedean t-conorm and therefore in this work we include representable semigroups to the set of Archimedean semigroups.

Remark 1. Assume a 2-uninorm $U^2\colon[0,1]^2\longrightarrow[0,1]$, $U^2\in\mathcal{U}_2$. Then the restriction of U^2 to $[0,z_1]^2$ is a uninorm on $[0,z_1]$ with continuous underlying functions and therefore there exists a countable, linearly ordered index set $(A_1,\leq_{A_1})$ and a set of semigroups $(H_\alpha)_{\alpha\in A_1}$, where $H_\alpha\in\mathcal{H}$ for all $\alpha\in A_1$, such that $([0,z_1]\,,U^2)$ is an ordinal sum of $(H_\alpha)_{\alpha\in A_1}$ with respect to the linear order $\leq_{A_1}$ (see [14, Proposition 11]). Similarly, there exists a countable, linearly ordered index set $(A_2,\leq_{A_2})$, and a set of semigroups $(H_\alpha)_{\alpha\in A_2}$, where $H_\alpha\in\mathcal{H}$ for all $\alpha\in A_2$, such that $([z_1,1]\,,U^2)$ is an ordinal sum of $(H_\alpha)_{\alpha\in A_2}$ with respect to the linear order $\leq_{A_2}$.

In [17] it was shown that for each $U^2\in\mathcal{U}_2$ there exist idempotent points $x_0\in[0,e_1]$ and a $y_0\in[e_2,1]$ such that $U^2(x,z_1)=x$ for all $x<x_0$ and $U^2(x,z_1)=z_1$ for all $x_0<x\leq z_1$, $U^2(y,z_1)=y$ for all $y>y_0$ and $U^2(y,z_1)=z_1$ for all $z_1\leq y<y_0$. Then for all $x_0<x\leq z_1$ and $z_1\leq y<y_0$ there is $U^2(x,y)=z_1$. Since the structure of a 2-uninorm heavily depends on the value $U^2(x_0,y_0)$, following [17] we will distinguish the following five cases:

- if $U^2(x_0,y_0)=z_1$,
- if $U^2(x_0,y_0)=x_0$, $U^2(x_0,y)=y$ for all $y>y_0$,
- if $U^2(x_0,y_0)=x_0$, $U^2(x_0,y)\neq y$ for some $y>y_0$,
- if $U^2(x_0,y_0)=y_0$, $U^2(y_0,x)=x$ for all $x<x_0$,
- if $U^2(x_0,y_0)=y_0$, $U^2(y_0,x)\neq x$ for some $x<x_0$.

Since the second and the fourth (the third and the fifth) cases are analogous we will focus just on the first three cases. We will start with the simplest case when $U^2(x_0,y_0)=z_1$. First we introduce one definition and one useful lemma.

Definition 7. Let $(H_\alpha)_{\alpha\in I}$ be a family of trivial semigroups for an index set I. For a set S we denote $I\sim S$ if for all $x\in S$ there exists an $\alpha\in I$ such that H_α is defined on $\{x\}$ and for all $\alpha\in I$ there exists an $x\in S$ such that H_α is defined on $\{x\}$.

Lemma 6. *Let $U^2\colon[0,1]\longrightarrow[0,1]$ be a 2-uninorm from Class 1, $U^2\in\mathcal{U}_2$. Then U^2 can be expressed as a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$, where $A\sim\{z_1\}$ and $(C,\preceq)$ has a tree structure.*

Proof. First we will show that $([0,1],U^2)$ is a z-ordinal sum of three semigroups $G_1=([0,z_1]\,,U_1^{[0,z_1]})$, $G_2=([z_1,1]\,,U_2^{[z_1,1]})$ and $G_3=(\{z_1\},\mathrm{Id})$, with $A=\{3\}$. Observe that since U^2 is from Class 1 then z_1 is the annihilator of all three semigroups. Assume a partial order $\preceq$, where $1\wedge 2=3$. Then $(C,\preceq)$ has a tree structure and thus Lemma 3 implies that if γ is incomparable with $\alpha\wedge\beta$ for some α,β,γ then $\alpha\wedge\gamma=\beta\wedge\gamma$. Further, if $x\in X_\alpha\cap X_\beta$ for $\alpha\neq\beta$ then $x=z_1$ and since in our case $\alpha\wedge\beta\in A$ for all $\alpha,\beta\in C$, $\alpha\neq\beta$, it is enough to check that z_1 is the annihilator of G_α and G_β which clearly holds. Further, there is no α,β,γ such that $\alpha\wedge\beta\prec\gamma\prec\alpha$. Thus all conditions of Theorem 4 are fulfilled.

Assume that $([0,1],V)$ is a z-ordinal sum of $(G_\alpha)_{\alpha\in\{1,2,3\}}$, with respect to $A=\{3\}$, $B=\{1,2\}$ and $\preceq$. Then U^2 and V coincide on $[0,z_1]^2$ and $[z_1,1]^2$. For $x\in[0,z_1[$ and $y\in\,]z_1,1]$ we have

$$V(y,x)=V(x,y)=z_1=U^2(x,y)=U^2(y,x).$$

Thus V and U^2 coincide on the whole unit interval and therefore $([0,1],U^2)$ is a z-ordinal sum of $(G_\alpha)_{\alpha\in\{1,2,3\}}$, with respect to $A=\{3\}$, $B=\{1,2\}$ and $\preceq$. Assume sets A_1 and A_2 defined in Remark 1 and let $H_3=G_3$. Then by Lemma 4 we see that $([0,1],U^2)$ is a z-ordinal sum of $(H_\alpha)_{\alpha\in A'\cup B'}$, with respect to the partial order $\preceq'$, where $B'=A_1\cup A_2$, $A'=A$ and $\preceq'$ is given by:

(i) If $p_1, p_2 \in A_1$ then $p_1 \preceq' p_2$ if $p_1 \leq_{A_1} p_2$.
(ii) If $p_1, p_2 \in A_2$ then $p_1 \preceq' p_2$ if $p_1 \leq_{A_2} p_2$.
(iii) For all $p \in C \setminus \{3\}$ there is $3 \prec p$ (and trivially $3 \preceq 3$).

Observe that if $p_1 \in A_1$, $p_2 \in A_2$ then p_1 and p_2 are incomparable. Finally let us note that by Remark 1 we get $H_\alpha \in \mathcal{H}$ for all $\alpha \in A' \cup B'$ and by Lemma 4 the set $C' = A' \cup B'$ partially ordered by $\preceq'$ has a tree structure. Moreover, since A, A_1 and A_2 are countable we see that also C' is countable. $\square$

Observe that in previous result it is crucial that the underlying uninorm U_1 is disjunctive and the underlying uninorm U_2 is conjunctive. Otherwise z_1 would not be the annihilator of the first or the second semigroup and the respective condition of Theorem 4 would be violated.

Now we recall the basic result from [17] which will help us to describe the structure of U^2 in the case when $U^2(x_0, y_0) = z_1$.

Proposition 7 *([17]). Let $U^2 \colon [0,1] \longrightarrow [0,1]$ be a 2-uninorm, $U^2 \in \mathcal{U}_2$ and let $U^2(x_0, y_0) = z_1$. Then U^2 is an ordinal sum of semigroups $G_1 = ([0, x_0[\cup \{z_1\} \cup]y_0, 1], U^2)$ and $G_2 = ([x_0, y_0], U^2)$, where the order in the ordinal sum construction is $1 < 2$, and G_1 is isomorphic to $([0,1], U)$, where U is a uninorm from $\mathcal{U}$ and G_2 is isomorphic to $([0,1], V^2)$, where $V^2 \in \mathcal{U}_2$ is an 2-uninorm from Class 1.*

Using previous proposition we can show the following result.

Theorem 6. *Let $U^2 \colon [0,1] \longrightarrow [0,1]$ be a 2-uninorm, $U^2 \in \mathcal{U}_2$ and let $U^2(x_0, y_0) = z_1$. Then U^2 can be expressed as a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$, where $A \sim \{z_1\}$ and $(C, \preceq)$ has a tree structure.*

Proof. First we show that U^2 can be expressed as a z-ordinal sum, with a tree structure, of four semigroups $G_1 = ([0, x_0[\cup \{z_1\} \cup]y_0, 1], U^2)$, $G_2 = ([x_0, z_1], U^2)$, $G_3 = ([z_1, y_0], U^2)$ and $G_4 = (\{z_1\}, \mathrm{Id})$, with $A = \{4\}$, where the partial order $\preceq$ is given by $2 \wedge 3 = 4$ and $1 \prec 4$. Observe that z_1 is the annihilator of the last three semigroups and it is the neutral element of the first semigroup. Evidently, $(C, \preceq)$ has a tree structure and it is easy to check that if $x \in X_\alpha \cap X_\beta$ then all conditions of Theorem 4 are satisfied.

Assume that $([0,1], V)$ is a z-ordinal sum of $(G_\alpha)_{\alpha \in \{1,2,3,4\}}$, with respect to $A = \{4\}$, $B = \{1,2,3\}$ and $\preceq$. Then similarly as in Lemma 6 we can verify that V coincides with U^2 on $[x_0, y_0]^2$. Evidently, V coincides with U^2 also on $([0, x_0[\cup \{z_1\} \cup]y_0, 1])^2$. Further, if $x \in [x_0, y_0]$ and $y \in [0, x_0[\cup \{z_1\} \cup]y_0, 1]$ then $V(x,y) = y$. On the other hand, Proposition 7 implies that $U^2(x,y) = y$, i.e., summarizing, V coincides with U^2 on the whole unit square. Thus $([0,1], U^2)$ is a z-ordinal sum of our 4 semigroups.

Since x_0, y_0 are idempotent points, we know that U^2 on $[x_0, z_1]^2$ ($[z_1, y_0]^2$) is isomorphic to a uninorm with continuous underlying functions. Further, Proposition 7 implies that U^2 on $([0, x_0[\cup \{z_1\} \cup]y_0, 1])^2$ is isomorphic to a uninorm with continuous underlying functions, as well. Therefore there exist countable, linearly ordered index sets $(A_1, \leq_1)$, $(A_2, \leq_2)$ and $(A_3, \leq_3)$ such that G_i is an ordinal sum of semigroups $(H_\alpha)_{\alpha \in A_i}$, for $i = 1,2,3$, where $H_\alpha \in \mathcal{H}$ for all $\alpha \in A_i$, $i \in \{1,2,3\}$. Denote $H_4 = G_4$. Lemma 4 then implies that $([0,1], U^2)$ is a z-ordinal sum of semigroups $(H_\alpha)_{\alpha \in A' \cup B'}$, where $A' = A$ and $B' = A_1 \cup A_2 \cup A_3$, the set $C' = A' \cup B'$ is countable, it has a tree structure, and $H_\alpha \in \mathcal{H}$ for all $\alpha \in C'$. $\square$

Remark 2. If $x_0 = 0$ and $y_0 = 1$ then G_1 reduces to a trivial semigroup defined on $\{z_1\}$. However, it still can be taken as a uninorm with continuous underlying functions (defined on a single point). If $x_0 = 0$ and $y_0 < 1$ then G_1 is defined on $\{z_1\} \cup]y_0, 1]$, which is isomorphic to a continuous t-conorm, i.e., it is still a uninorm with continuous underlying functions. The case when $x_0 > 0$ and $y_0 = 1$ is analogous. Similar observations can be done also in the subsequent results.

In all other possible cases for a 2-uninorm from $\mathcal{U}_2$ the decomposition via the z-ordinal sum construction is similar and therefore we omit the detailed proof and just describe the respective semigroups and the corresponding partial order. Let us denote

$$y_1 = \sup\{y \in [y_0, 1] \mid U^2(x_0, y) = x_0\}.$$

Then following [17] there are these possible cases:

1. $U^2(x_0, y_0) = x_0$, $U^2(z_1, y_0) = z_1$ and $U^2(x_0, y) = y$ for all $y > y_0$.
2. $U^2(x_0, y_0) = x_0$, $U^2(z_1, y_0) = y_0$ and $U^2(x_0, y) = y$ for all $y > y_0$.
3. $U^2(x_0, y_0) = x_0$, $y_1 > y_0$ and $U^2(x_0, y_1) = x_0$, $U^2(z_1, y_0) = z_1$.
4. $U^2(x_0, y_0) = x_0$, $y_1 > y_0$ and $U^2(x_0, y_1) = x_0$, $U^2(z_1, y_0) = y_0$.
5. $U^2(x_0, y_0) = x_0$, $y_1 > y_0$ and $U^2(x_0, y_1) = y_1$, $U^2(z_1, y_0) = z_1$.
6. $U^2(x_0, y_0) = x_0$, $y_1 > y_0$ and $U^2(x_0, y_1) = y_1$, $U^2(z_1, y_0) = y_0$.

Remark 3. Since x_0 and y_0 are idempotent points, $x_0 \leq e_1$ and $e_2 \leq y_0$ we know that U^2 is closed on $[x_0, y_0]^2$. Moreover, if $U^2(x_0, y_0) = x_0$ then U^2 is closed on $]x_0, y_0]^2$. Indeed, in the opposite case there exist $s, t \in]x_0, y_0]$ such that $U^2(s,t) = x_0$. Then $x_0 = U^n(x_0, x_0) \leq U^2(s_1, t_1) \leq U^2(s,t) = x_0$ for all $s_1 \in]x_0, s]$, $t_1 \in]x_0, t]$. Since $s \leq y_0$, $t \leq y_0$, for any $s_2 \in]x_0, s[$, $t_2 \in]x_0, t[$ we get

$$U^2(x_0, z_1) = U^2(U^2(s_2, t_2), z_1) = U^2(s_2, U^2(t_2, z_1)) = U^2(s_2, z_1) = z_1 > x_0 = U^2(x_0, y_0),$$

which is a contradiction with the monotonicity of U^2.

Similarly we can show that if $U^2(x_0, y_0) = x_0$ and $U^2(z_1, y_0) = y_0$ then U^2 is closed on $]x_0, y_0[^2$.

Proposition 8. *Let $U^2\colon [0,1] \longrightarrow [0,1]$ be a 2-uninorm, $U^2 \in \mathcal{U}_2$ and let $U^2(x_0, y_0) = x_0$, $U^2(z_1, y_0) = z_1$ and $U^2(x_0, y) = y$ for all $y > y_0$. Then U^2 can be expressed as a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$, where $A \sim \{z_1\}$ and $(C, \preceq)$ has a tree structure.*

Proof. From [17, Theorem 4.4] we know that in this case U^2 is an ordinal sum of two semigroups $G_1 = ([0, x_0] \cup]y_0, 1], U^2)$ and $G_2 = (]x_0, y_0], U^2)$, where G_2 is isomorphic to a 2-uninorm from Class 1 restricted to $]0,1]^2$, G_1 is isomorphic to a uninorm with continuous underlying functions and $1 < 2$. Thus U^2 can be expressed as a z-ordinal sum of four semigroups G_1, $G_3 = (]x_0, z_1], U^2)$, $G_4 = ([z_1, y_0], U^2)$, $G_5 = (\{z_1\}, \mathrm{Id})$, where $A = \{5\}$ and the corresponding partial order is given by $3 \wedge 4 = 5$ and $1 \prec 5$. Since semigroups G_3, G_4 and G_1 are isomorphic to (restrictions of) uninorms with continuous underlying functions, Lemma 4 implies that U^2 can be expressed as a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$. □

Proposition 9. *Let $U^2\colon [0,1] \longrightarrow [0,1]$ be a 2-uninorm, $U^2 \in \mathcal{U}_2$ and let $U^2(x_0, y_0) = x_0$, $U^2(z_1, y_0) = y_0$ and $U^2(x_0, y) = y$ for all $y > y_0$. Then U^2 can be expressed as a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$, where $A \sim \{z_1\}$ and $(C, \preceq)$ has a tree structure.*

Proof. From [17, Theorem 4.5] we know that in this case U^2 is an ordinal sum of three semigroups $G_1 = ([0, x_0] \cup]y_0, 1], U^2)$, $G_2 = (]x_0, y_0[, U^2)$ and $G_3 = (\{y_0\}, \mathrm{Id})$, where G_1 is isomorphic to a uninorm with continuous underlying functions, G_2 is isomorphic to a 2-uninorm from Class 1 restricted to $]0,1[^2$ and $1 < 3 < 2$. Thus U^2 can be expressed as a z-ordinal sum of five semigroups G_1, G_3, $G_4 = (]x_0, z_1], U^2)$, $G_5 = ([z_1, y_0[, U^2)$, $G_6 = (\{z_1\}, \mathrm{Id})$, where $A = \{6\}$ and the corresponding partial order is given by $4 \wedge 5 = 6$ and $1 \prec 3 \prec 6$. Since semigroups G_4, G_5 and G_1 are isomorphic to (restrictions of) uninorms with continuous underlying functions, Lemma 4 implies that U^2 can be expressed as a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$ (see Fig. 3). □

Proposition 10. *Let $U^2\colon [0,1] \longrightarrow [0,1]$ be a 2-uninorm, $U^2 \in \mathcal{U}_2$ and let $U^2(x_0, y_0) = x_0$, $y_1 > y_0$ and $U^2(x_0, y_1) = x_0$, $U^2(z_1, y_0) = z_1$. Then U^2 can be expressed as a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$, where $A \sim \{z_1\}$ and $(C, \preceq)$ has a tree structure.*

Proof. From [17, Theorem 4.7] we know that in this case U^2 is an ordinal sum of three semigroups $G_1 = ([0, x_0] \cup]y_1, 1], U^2)$, $G_2 = (]x_0, y_0], U^2)$ and $G_3 = (]y_0, y_1], U^2)$, where G_1 is isomorphic to a uninorm with continuous underlying functions, G_2 is isomorphic to a 2-uninorm from Class 1 restricted to $]0,1]^2$ and G_3 is isomorphic to a continuous t-conorm restricted to $]0,1]^2$. For the respective linear order we have $1 < 3 < 2$. Thus U^2 can be expressed as a z-ordinal sum of five semigroups G_1, G_3, $G_4 = (]x_0, z_1], U^2)$, $G_5 = ([z_1, y_0], U^2)$, $G_6 = (\{z_1\}, \mathrm{Id})$, where $A = \{6\}$ and the corresponding partial order is given by $4 \wedge 5 = 6$ and $1 \prec 3 \prec 6$. Since semigroups G_4, G_5 and G_1 are isomorphic to (restrictions of) uninorms with continuous underlying functions, and semigroup G_3 is isomorphic to (a restriction of) a continuous t-conorm Lemma 4 implies that U^2 can be expressed as a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$. □

Proposition 11. *Let $U^2\colon [0,1] \longrightarrow [0,1]$ be a 2-uninorm, $U^2 \in \mathcal{U}_2$ and let $U^2(x_0, y_0) = x_0$, $y_1 > y_0$ and $U^2(x_0, y_1) = x_0$, $U^2(z_1, y_0) = y_0$. Then U^2 can be expressed as a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$, where $A \sim \{z_1\}$ and $(C, \preceq)$ has a tree structure.*

Proof. From [17, Theorem 4.8] we know that in this case U^2 is an ordinal sum of three semigroups $G_1 = ([0, x_0] \cup]y_1, 1], U^2)$, $G_2 = (]x_0, y_0[, U^2)$ and $G_3 = ([y_0, y_1], U^2)$, where G_1 is isomorphic to a uninorm with continuous underlying functions, G_2 is isomorphic to a 2-uninorm from Class 1 restricted to $]0,1[^2$ and G_3 is isomorphic to a continuous t-conorm. For the respective linear order we have $1 < 3 < 2$. Thus U^2 can be expressed as a z-ordinal sum of five semigroups G_1, G_3, $G_4 = (]x_0, z_1], U^2)$, $G_5 = ([z_1, y_0[, U^2)$, $G_6 = (\{z_1\}, \mathrm{Id})$, where $A = \{6\}$ and the corresponding partial order is given by $4 \wedge 5 = 6$ and $1 \prec 3 \prec 6$. Since semigroups G_4, G_5 and G_1 are isomorphic to (restrictions of) uninorms with continuous underlying functions, and semigroup G_3 is isomorphic to a continuous t-conorm Lemma 4 implies that U^2 can be expressed as a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$. □

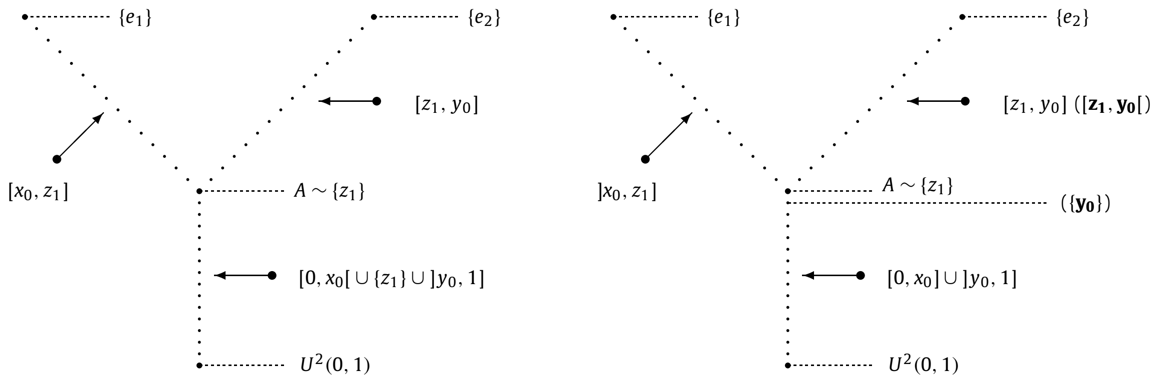

Fig. 3. A partial order from Theorem 6 (left) and Propositions 8 and 9 (right). The bold sets in brackets apply for Proposition 9. Labeled areas consist of semigroups which contain points from the given set. Note that e_1 and e_2 can belong to semigroups which are not trivial.

Further we will assume that $U^2(x_0, y_0) = x_0$, $y_1 > y_0$ and $U^2(x_0, y_1) = y_1$. Then [17, Lemma 4.9] implies that U^2 is closed on $([0, x_0[\cup]y_0, 1])^2$ and [17, Proposition 3.10] implies that U^2 restricted to $([0, x_0[\cup \{z_1\} \cup]y_0, 1])^2$ is isomorphic to a uninorm with continuous underlying functions.

Lemma 7. *Let $U^2 \colon [0, 1] \longrightarrow [0, 1]$ be a 2-uninorm, $U^2 \in \mathcal{U}_2$ and let $U^2(x_0, y_0) = x_0$, $y_1 > y_0$ and $U^2(x_0, y_1) = y_1$, $U^2(z_1, y_0) = z_1$. Then $([0, 1] \setminus \{x_0\}, U^2)$ can be expressed as a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$, where $A \sim \{z_1\}$ and $(C, \preceq)$ has a tree structure.*

Proof. From previous discussion we know that in this case U^2 is closed on $]x_0, y_0]^2$, as well as on $([0, x_0[\cup \{z_1\} \cup]y_0, 1])^2$. Since z_1 is the annihilator of U^2 on $]x_0, y_0]^2$ and the neutral element of U^2 on $([0, x_0[\cup \{z_1\} \cup]y_0, 1])^2$ we see that the semigroup $([0, 1] \setminus \{x_0\}, U^2)$ can be expressed as an ordinal sum of the semigroup $G_1 = ([0, x_0[\cup \{z_1\} \cup]y_0, 1], U^2)$, which is isomorphic to a uninorm with continuous underlying functions, and the semigroup $G_2 = (]x_0, y_0], U^2)$, which is isomorphic to a 2-uninorm from Class 1 restricted to $]0, 1]^2$.

Thus $([0, 1] \setminus \{x_0\}, U^2)$ is a z-ordinal sum of four semigroups G_1, $G_3 = (]x_0, z_1], U^2)$, $G_4 = ([z_1, y_0], U^2)$, $G_5 = (\{z_1\}, \mathrm{Id})$, where $A = \{5\}$ and the corresponding partial order is given by $3 \wedge 4 = 5$ and $1 \prec 5$. Since semigroups G_3, G_4 and G_1 are isomorphic to (restrictions of) uninorms with continuous underlying functions Lemma 4 implies that $([0, 1] \setminus \{x_0\}, U^2)$ can be expressed as a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$. □

Proposition 12. *Let $U^2 \colon [0, 1] \longrightarrow [0, 1]$ be a 2-uninorm, $U^2 \in \mathcal{U}_2$ and let $U^2(x_0, y_0) = x_0$, $y_1 > y_0$ and $U^2(x_0, y_1) = y_1$, $U^2(z_1, y_0) = z_1$. Then U^2 can be expressed as a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$, where $A \sim \{z_1\}$ and $(C, \preceq)$ has a tree structure.*

Proof. Lemma 7 shows that $([0, 1] \setminus \{x_0\}, U^2)$ can be expressed as a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$ with respect to the set $A' = \{5\}$. Assuming the semigroups from Lemma 7, to prove the result we have to add the trivial semigroup $G_0 = (\{x_0\}, \mathrm{Id})$ to the correct place. Evidently there is $0 \preceq 5$. Thus we only have to focus on the semigroups from the ordinal sum decomposition of G_1, i.e., semigroups $H_\alpha, \alpha \in A_1$, for the corresponding countable index set A_1 linearly ordered by $\leq_{A_1}$. Observe that $U^2(x_0, y) = x_0$ for all $y \in]y_0, y_1[$ and $U^2(x_0, z_1) = x_0$. On the other hand, $U^2(x_0, y) = y$ for all $y \in [0, x_0[\cup [y_1, 1]$. Together we get

$$U^2(x, y) = U^2(U^2(x, x_0), y) = U^2(x, U^2(x_0, y)) = U^2(x, x_0) = x$$

for all $x \in [0, x_0[\cup [y_1, 1]$ and $y \in]y_0, y_1[\cup \{z_1\}$. Thus if for $\alpha, \beta \in A_1$ there is $x \in X_\alpha$ for some $x \in [0, x_0[\cup [y_1, 1]$ and $y \in X_\beta$ for some $y \in]y_0, y_1[\cup \{z_1\}$ then $\alpha \leq_{A_1} \beta$.

Assume a non-trivial semigroup H_α from the decomposition of G_1. We will show that if $y \in X_\alpha$ for some $y \in]y_0, y_1[$ then $[0, x_0[\cap X_\alpha = \emptyset$, $z_1 \notin X_\alpha$. Indeed, [14, Proposition 11] (see Definition 6) implies that H_α is not defined on an interval only if it is either representable, or d-internal semigroup. However, for all $x \in [0, x_0[$ there is

$$U^2(x, y) = U^2(U^2(x, x_0), y) = U^2(x, U^2(x_0, y)) = U^2(x, x_0) = x,$$

which means that H_α is not a representable semigroup. Assume that H_α is d-internal. Then there exists points $x_1, x_2 \in [0, x_0[$ such that $U^2(x_1, y) = x_1$ and $U^2(x_2, y) = y$. However, as we showed above $U^2(x, y) = x$ for all $x \in [0, x_0[$ and thus H_α is not d-internal. Therefore H_α is s-strict, s-nilpotent, or s-internal semigroup.

Now there are two possibilities. If for all $y \in]y_0, y_1[$ there $y \in X_\alpha$ implies $y_1 \notin X_\alpha$ then U^2 is a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$, where $A'' = A'$, $B'' = B' \cup \{0\}$ and $\preceq''$ coincides with $\preceq'$ on $A' \cup B'$, $\preceq''$ is a linear order on $A_1 \cup \{0, 5\}$, $0 \prec'' 5$ and for $\alpha \in A_1$ there is $0 \prec'' \alpha$ if $y \in X_\alpha$ for some $y \in]y_0, y_1[$, $\alpha \prec'' 0$ if $y \in X_\alpha$ for some $y \in [0, x_0[\cup [y_1, 1]$.

If there exists $\alpha_1 \in A_1$ such that $y_1 \in X_{\alpha_1}$ and $y \in X_{\alpha_1}$ for some $y \in]y_0, y_1[$ then since y_1 is an idempotent element H_{α_1} cannot be an s-strict semigroup. Since H_{α_1} is not d-internal and it is not representable, the remaining two possibilities are that H_{α_1} is s-nilpotent, or s-internal. If H_{α_1} is an s-nilpotent semigroup then there exist $s, t \in]y_0, y_1[$ such that $U^2(s, t) = y_1$. However, then

$$y_1 = U^2(y_1, x_0) = U^2(U^2(s, t), x_0) = U^2(s, U^2(t, x_0)) = U^2(s, x_0) = x_0,$$

which is a contradiction. Thus H_{α_1} is an s-internal semigroup, which means that X_{α_1} is an (open) interval and $X_{\alpha_1} \subseteq]y_0, 1]$. Denote $X_{\alpha_1} =]a, b[$ and define two new semigroups $G_{\alpha_2} = (]a, y_1[\,, \max)$ and $G_{\alpha_3} = ([y_1, b[\,, \max)$. Then U^2 is a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$, where $A'' = A'$, $B'' = (B' \setminus \alpha_1) \cup \{0, \alpha_2, \alpha_3\}$ and $\preceq''$ coincides with $\preceq'$ on $A' \cup (B' \setminus \alpha_1)$ and $\alpha_3 \prec'' 0 \prec'' \alpha_2$, while for an $\alpha \in A_1$ there is $\alpha \prec'' \alpha_3$ if $\alpha \prec' \alpha_1$ and $\alpha_2 \prec'' \alpha$ if $\alpha_1 \prec' \alpha$. In other words we have inserted the semigroup G_0 between the semigroup G_{α_3} and the semigroup G_{α_2} while α_2 and α_3 inherited the position of α_1. □

Lemma 8. *Let $U^2 \colon [0, 1] \longrightarrow [0, 1]$ be a 2-uninorm, $U^2 \in \mathcal{U}_2$ and let $U^2(x_0, y_0) = x_0$, $y_1 > y_0$ and $U^2(x_0, y_1) = y_1$, $U^2(z_1, y_0) = y_0$. Then $([0, 1] \setminus \{x_0\}, U^2)$ can be expressed as a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$, where $A \sim \{z_1\}$ and $(C, \preceq)$ has a tree structure.*

Proof. The proof is similar to the proof of Lemma 7, we just have to replace G_2 by two semigroups $(]x_0, y_0[\,, U^2)$ and $(\{y_0\}, \mathrm{Id})$ (compare Propositions 8 and 9). □

Proposition 13. *Let $U^2 \colon [0, 1] \longrightarrow [0, 1]$ be a 2-uninorm, $U^2 \in \mathcal{U}_2$ and let $U^2(x_0, y_0) = x_0$, $y_1 > y_0$ and $U^2(x_0, y_1) = y_1$, $U^2(z_1, y_0) = y_0$. Then U^2 can be expressed as a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$, where $A \sim \{z_1\}$ and $(C, \preceq)$ has a tree structure.*

Proof. The proof is similar to the proof of Proposition 12, here we have just one additional index corresponding to the trivial semigroup $(\{y_0\}, \mathrm{Id})$, which is an immediate predecessor of $\alpha_0 \in A$ which corresponds to the trivial semigroup $(\{z_1\}, \mathrm{Id})$ (see Fig. 4). □

If we summarize Propositions 8–13 we obtain the following theorem.

Theorem 7. *Let $U^2 \colon [0, 1] \longrightarrow [0, 1]$ be a 2-uninorm, $U^2 \in \mathcal{U}_2$ and let $U^2(x_0, y_0) = x_0$. Then U^2 can be expressed as a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$, where $A \sim \{z_1\}$ and $(C, \preceq)$ has a tree structure.*

The case when $U^2(x_0, y_0) = y_0$ is analogous to the case when $U^2(x_0, y_0) = x_0$. We briefly describe the structure of U^2 in this case in the following remark.

Remark 4. Let $U^2 \colon [0, 1] \longrightarrow [0, 1]$ be a 2-uninorm, $U^2 \in \mathcal{U}_2$ and let $U^2(x_0, y_0) = y_0$. Denote

$$x_1 = \inf\{x \in [0, x_0] \mid U^2(y_0, x) = y_0\}.$$

Then U^2 can be expressed as a z-ordinal sum of the following semigroups.

(i) If $U^2(z_1, x_0) = z_1$ and $U^2(y_0, x) = x$ for all $x < x_0$. $G_1 = ([0, x_0[\cup [y_0, 1]\,, U^2)$, $G_2 = ([z_1, y_0[\,, U^2)$, $G_3 = ([x_0, z_1]\,, U^2)$, $G_4 = (\{z_1\}, \mathrm{Id})$, where $A = \{4\}$ and the corresponding partial order is given by $2 \wedge 3 = 4$ and $1 \prec 4$. Here semigroups G_1, G_2 and G_3 are isomorphic to (restrictions of) uninorms with continuous underlying functions.
(ii) If $U^2(z_1, x_0) = x_0$ and $U^2(y_0, x) = x$ for all $x < x_0$. $G_1 = ([0, x_0[\cup [y_0, 1]\,, U^2)$, $G_2 = (\{x_0\}, \mathrm{Id})$, $G_3 = ([z_1, y_0[\,, U^2)$, $G_4 = (]x_0, z_1]\,, U^2)$, $G_5 = (\{z_1\}, \mathrm{Id})$, where $A = \{5\}$ and the corresponding partial order is given by $3 \wedge 4 = 5$ and $1 \prec 2 \prec 5$. Here semigroups G_1, G_3, G_4 are isomorphic to (restrictions of) uninorms with continuous underlying functions.
(iii) If $x_1 < x_0$ and $U^2(y_0, x_1) = y_0$, $U^2(z_1, x_0) = z_1$. $G_1 = ([0, x_1[\cup [y_0, 1]\,, U^2)$, $G_2 = ([x_1, x_0[\,, U^2)$, $G_3 = ([z_1, y_0[\,, U^2)$, $G_4 = ([x_0, z_1]\,, U^2)$, $G_5 = (\{z_1\}, \mathrm{Id})$, where $A = \{5\}$ and the corresponding partial order is given by $3 \wedge 4 = 5$ and $1 \prec 2 \prec 5$. Here semigroups G_1, G_3 and G_4 are isomorphic to (restrictions of) uninorms with continuous underlying functions, and semigroup G_2 is isomorphic to (a restriction of) a continuous t-norm.
(iv) If $x_1 < x_0$ and $U^2(y_0, x_1) = y_0$, $U^2(z_1, x_0) = x_0$. $G_1 = ([0, x_1[\cup [y_0, 1]\,, U^2)$, $G_2 = ([x_1, x_0]\,, U^2)$, $G_3 = ([z_1, y_0[\,, U^2)$, $G_4 = (]x_0, z_1]\,, U^2)$, $G_5 = (\{z_1\}, \mathrm{Id})$, where $A = \{5\}$ and the corresponding partial order is given by $3 \wedge 4 = 5$ and $1 \prec 2 \prec 5$. Here semigroups G_1, G_3 and G_4 are isomorphic to (restrictions of) uninorms with continuous underlying functions, and semigroup G_2 is isomorphic to a continuous t-norm.

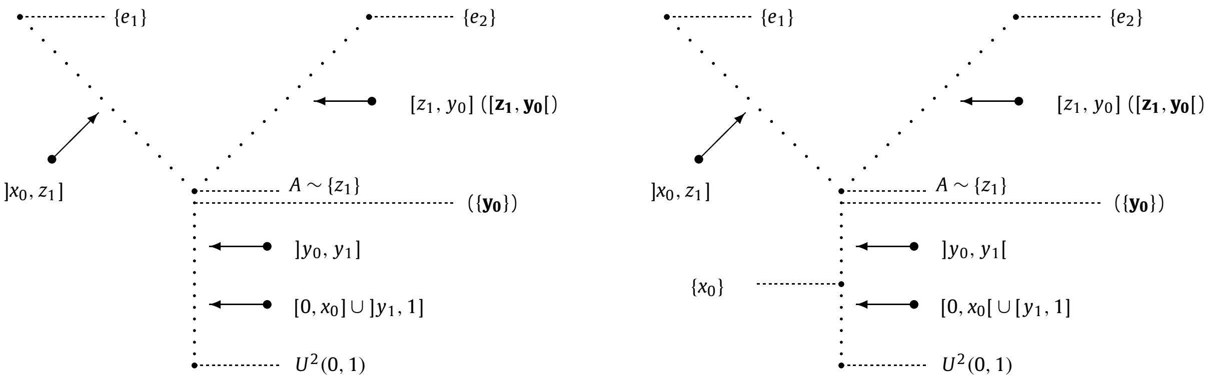

Fig. 4. A partial order from Propositions 10 and 11 (left) and Propositions 12 and 13 (right). The bold sets in brackets apply for Propositions 11 and 13. Labeled areas consist of semigroups which contain points from the given set.

(v) If $x_1 < x_0$ and $U^2(y_0, x_1) = x_1$, $U^2(z_1, x_0) = z_1$. $G_1 = ([0, x_0[\cup \{z_1\} \cup [y_0, 1], U^2)$, $G_2 = ([z_1, y_0[, U^2)$, $G_3 = ([x_0, z_1], U^2)$, $G_4 = (\{z_1\}, \mathrm{Id})$, where $A = \{4\}$ and the corresponding partial order is given by $2 \wedge 3 = 4$ and $1 \prec 4$. Here semigroups G_2 and G_3 are isomorphic to (restrictions of) uninorms with continuous underlying functions. The semigroup G_1 can be expressed as an ordinal sum of semigroups from $\mathcal{H}$ similarly as in Proposition 12.

(vi) If $x_1 < x_0$ and $U^2(y_0, x_1) = x_1$, $U^2(z_1, x_0) = x_0$. $G_1 = ([0, x_0[\cup \{z_1\} \cup [y_0, 1], U^2)$, $G_2 = ([z_1, y_0[, U^2)$, $G_3 = (]x_0, z_1], U^2)$, $G_4 = (\{z_1\}, \mathrm{Id})$, $G_5 = (\{x_0\}, \mathrm{Id})$, where $A = \{4\}$ and the corresponding partial order is given by $2 \wedge 3 = 4$ and $1 \prec 5 \prec 4$. Here semigroups G_2 and G_3 are isomorphic to (restrictions of) uninorms with continuous underlying functions. The semigroup G_1 can be expressed as an ordinal sum of semigroups from $\mathcal{H}$ similarly as in previous case.

Note that each non-trivial semigroup above can be further decomposed via the ordinal sum construction into semigroups from $\mathcal{H}$.

Remark 4 and Lemma 4 imply similar results to Propositions 8–13 in the case when $U^2(x_0, y_0) = y_0$. Then we get the following.

Theorem 8. *Let $U^2 \colon [0,1] \longrightarrow [0,1]$ be a 2-uninorm, $U^2 \in \mathcal{U}_2$ and let $U^2(x_0, y_0) = y_0$. Then U^2 can be expressed as a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$, where $A \sim \{z_1\}$ and $(C, \preceq)$ has a tree structure.*

Summarizing, we see that in all cases a 2-uninorm $U^2 \in \mathcal{U}_2$ can be expressed as a z-ordinal sum of semigroups from $\mathcal{H}$ with respect to set A which contains only one index. The corresponding partially ordered set C has a tree structure with two branches and the single node (which belongs to A) corresponding to the trivial semigroup $(\{z_1\}, \mathrm{Id})$.

Remark 5. The structure of a 2-uninorm from Class 1 was described in [1] and the structure of 2-uninorms from Classes 2a and 2b were completely characterized in [23] (see Theorems 5 and 6). From our results we can observe the following.

- If $U^2 \in \mathcal{U}_2$ is a 2-uninorm from Class 1 then $x_0 = 0$ and $y_0 = 1$ and $U^2(x_0, y_0) = U^2(0, 1) = z_1$. The z-ordinal sum structure of U^2 is described in Lemma 6. Here z_1 is the bottom element of the corresponding partial order and above we have two separate branches, each linearly ordered (corresponding to underlying uninorms).
- If $U^2 \in \mathcal{U}_2$ is a 2-uninorm from Class 2a then $y_0 = 1$ and $U^2(z_1, y_0) = U^2(z_1, 1) = z_1$. If $U^2(x_0, z_1) = x_0$ then U^2 is a z-ordinal sum (see Proposition 8) of a semigroup G_1 corresponding to a t-norm on $[0, x_0]^2$, a semigroup G_2 corresponding to a restriction of a uninorm on $]x_0, z_1]^2$, a semigroup G_3 corresponding to a uninorm on $[z_1, 1]^2$ and a semigroup $G_4 = (\{z_1\}, \mathrm{Id})$. The respective partial order is given by $1 \prec 4$ and $2 \wedge 3 = 4$. Each non-trivial semigroup can be further decomposed via the ordinal sum construction into semigroups from $\mathcal{H}$.
  If $U^2(x_0, z_1) = z_1$ the situation is similar (see Theorem 6), just the t-norm is restricted to $[0, x_0[^2$ and we have a uninorm on $[x_0, z_1]^2$. This covers also the special case when $x_0 = e_1$, which corresponds to [23, Corollary 2].
- If $U^2 \in \mathcal{U}_2$ is a 2-uninorm from Class 2b then $x_0 = 0$ and $U^2(z_1, x_0) = U^2(z_1, 0) = z_1$. If $U^2(y_0, z_1) = y_0$ then (see Remark 4(i)) U^2 is a z-ordinal sum of a semigroup G_1 corresponding to a t-conorm on $[y_0, 1]^2$, a semigroup G_2 corresponding to a restriction of a uninorm on $[z_1, y_0[^2$, a semigroup G_3 corresponding to a uninorm on $[0, z_1]^2$ and a semigroup $G_4 = (\{z_1\}, \mathrm{Id})$. The respective partial order is given by $1 \prec 4$ and $2 \wedge 3 = 4$. Each non-trivial semigroup can be further decomposed via the ordinal sum construction into semigroups from $\mathcal{H}$.
  If $U^2(y_0, z_1) = z_1$ the situation is similar (see Theorem 6), just the t-conorm is restricted to $]y_0, 1]^2$ and we have a uninorm on $[z_1, y_0]^2$. This covers also the special case when $y_0 = e_2$, which corresponds to [23, Corollary 4].

6. n-uninorms with continuous underlying functions.

In this section we will generalize the results from previous section for n-uninorms from $\mathcal{U}_n$ for all $n \in \mathbb{N}$, $n > 2$. We will prove these results by induction. Thus we will suppose that for some $m \in \mathbb{N}$, $m > 2$, each n-uninorm U^n for $n \in \mathbb{N}$, $n < m$, $U^n \in \mathcal{U}_n$, can be expressed as a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$ with tree structure, where $A \sim \{z_1, \dots, z_{n-1}\}$. Then we will show that the same holds also for all m-uninorms from $\mathcal{U}_m$.

Assume an m-uninorm $U^m \in \mathcal{U}_m$. Then $U^m(e_1, e_m) = z_k$ for some $k \in \{1, \dots, m-1\}$. Similarly as in the case of 2-uninorms there exist idempotent points $x_0 \in [0, e_1]$ and $y_0 \in [e_m, 1]$ such that $U^m(x, z_k) = x$ for all $x < x_0$ and $U^m(x, z_k) = z_k$ for all $x_0 < x \leq z_k$, and $U^m(y, z_k) = y$ for all $y > y_0$ and $U^m(y, z_k) = z_k$ for all $z_k \leq y < y_0$ (see [17]). We denote

$$y_1 = \sup\{y \in [y_0, 1] \mid U^m(x_0, y) = x_0\}.$$

Then we can distinguish the following cases.

1. $U^m(x_0, y_0) = z_k$.
2. $U^m(x_0, y_0) = x_0$, $U^m(z_1, y_0) = z_1$ and $U^m(x_0, y) = y$ for all $y > y_0$.
3. $U^m(x_0, y_0) = x_0$, $U^m(z_1, y_0) = y_0$ and $U^m(x_0, y) = y$ for all $y > y_0$.
4. $U^m(x_0, y_0) = x_0$, $y_1 > y_0$ and $U^m(x_0, y_1) = x_0$, $U^m(z_1, y_0) = z_1$.
5. $U^m(x_0, y_0) = x_0$, $y_1 > y_0$ and $U^m(x_0, y_1) = x_0$, $U^m(z_1, y_0) = y_0$.
6. $U^m(x_0, y_0) = x_0$, $y_1 > y_0$ and $U^m(x_0, y_1) = y_1$, $U^m(z_1, y_0) = z_1$.
7. $U^m(x_0, y_0) = x_0$, $y_1 > y_0$ and $U^m(x_0, y_1) = y_1$, $U^m(z_1, y_0) = y_0$.

Observe that for the dual case when $U^m(x_0, y_0) = y_0$ we can obtain a division dual to points 2.–7.

We will again show the detailed proof for the first case and then just show the respective decomposition for the other cases. Note that we say that an n-uninorm belongs to the Class 1 if $U^n(0, 1) = z_k$ for some $k \in \{1, \dots, n-1\}$. Note that in such a case $U^n(0, z_k) = z_k = U^n(z_k, 1)$ and thus $U^n(x, y) = z_k$ for all $x \in [0, z_k]$ and $y \in [z_k, 1]$.

Theorem 9. *Let $U^m\colon [0, 1] \longrightarrow [0, 1]$ be an m-uninorm, $U^m \in \mathcal{U}_m$ and let $U^m(x_0, y_0) = z_k$ for some $k \in \{1, \dots, m-1\}$. Then U^m can be expressed as a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$, where $A \sim \{z_1, \dots, z_{m-1}\}$ and $(C, \preceq)$ has a tree structure.*

Proof. From [17, Theorem 5.10] we know that in this case U^m is an ordinal sum of two semigroups $G_1 = ([0, x_0[\cup \{z_k\} \cup]y_0, 1], U^m)$ and $G_2 = ([x_0, y_0], U^m)$, where G_2 is isomorphic to an m-uninorm from Class 1 and G_1 is isomorphic to a uninorm with continuous underlying functions and $1 < 2$. Therefore similarly as in previous section (see Theorem 6) we can show that U^m can be expressed as a z-ordinal sum, with a tree structure, of four semigroups G_1, $G_3 = ([x_0, z_k], U^m)$, $G_4 = ([z_k, y_0], U^m)$ and $G_5 = (\{z_k\}, \mathrm{Id})$ with $A = \{5\}$, where the partial order $\preceq$ is given by $3 \wedge 4 = 5$ and $1 \prec 5$. Observe that z_k is the annihilator of the last three semigroups and it is the neutral element of the first semigroup. Further, G_3 is isomorphic to a k-uninorm from $\mathcal{U}_k$ and G_4 is isomorphic to an $(m-k)$-uninorm from $\mathcal{U}_{m-k}$.

Therefore, due to the induction assumption G_3 can be expressed as a z-ordinal sum of a countable number of semigroups $H_\alpha \in \mathcal{H}$ with respect to some index sets A_3, B_3 and a partial order $\preceq_3$, where $(C_3, \preceq_3)$ for $C_3 = A_3 \cup B_3$ has a tree structure. Similarly, G_4 can be expressed as a z-ordinal sum of a countable number of semigroups $H_\alpha \in \mathcal{H}$ with respect to A_4, B_4 and $\preceq_4$, where $(C_4, \preceq_4)$ has a tree structure. Since G_1 is isomorphic to a uninorm with continuous underlying functions it can be expressed as an ordinal sum of semigroups from $\mathcal{H}$ with respect to some linearly ordered set A_1. Thus Lemma 4 and Lemma 5 imply that $([0, 1], U^m)$ can be expressed as a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$ with respect to sets $A' = A_3 \cup A_4 \cup \{5\}$ and $B' = B_3 \cup B_4 \cup A_1$. Note that the induction assumption implies that $A_3 \sim \{z_1, \dots, z_{k-1}\}$ and $A_4 \sim \{z_{k+1}, \dots, z_{m-1}\}$. Therefore $A' \sim \{z_1, \dots, z_{m-1}\}$. The respective partial order $\preceq'$ on C' is given by

(i) $\alpha \preceq' \beta$ if $\alpha, \beta \in C_3$ and $\alpha \preceq_3 \beta$.
(ii) $\alpha \preceq' \beta$ if $\alpha, \beta \in C_4$ and $\alpha \preceq_4 \beta$.
(iii) $\alpha \preceq' \beta$ if $\alpha, \beta \in A_1$ and $\alpha \leq_1 \beta$.
(iv) $\alpha \prec' 5 \prec' \beta$ for all $\alpha \in A_1$ and $\beta \in C_3 \cup C_4$. □

In the following theorem, in the case when $U^m(x_0, y_0) = y_0$ we denote

$$x_1 = \inf\{x \in [0, x_0] \mid U^m(y_0, x) = y_0\}.$$

Theorem 10. *Let $U^m\colon [0, 1] \longrightarrow [0, 1]$ be an m-uninorm, $U^m \in \mathcal{U}_m$. Then U^m can be expressed as a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$, where $A \sim \{z_1, \dots, z_{m-1}\}$ and $(C, \preceq)$ has a tree structure.*

Proof. In the case when $U^m(x_0, y_0) = z_k$ the claim follows from Theorem 9.

• If $U^m(x_0, y_0) = x_0$, $U^m(z_k, y_0) = z_k$ and $U^m(x_0, y) = y$ for all $y > y_0$. Then U^m can be expressed as an ordinal sum of two semigroups $G_1 = ([0, x_0] \cup]y_0, 1], U^m)$ and $G_2 = (]x_0, y_0], U^m)$, where G_2 is isomorphic to m-uninorm from Class 1 restricted to $]0, 1]^2$, G_1 is isomorphic to a uninorm with continuous underlying functions and $1 < 2$ (see [17, Theorem 5.11]). Thus U^m is a z-ordinal sum of four semigroups G_1, $G_3 = (]x_0, z_k], U^m)$, $G_4 = ([z_k, y_0], U^m)$, $G_5 = (\{z_k\}, \mathrm{Id})$ with the partial order $3 \wedge 4 = 5$ and $1 \prec 5$, where G_3 is isomorphic to (a restriction of) a k-uninorm from $\mathcal{U}_k$ and G_4 is isomorphic to an $(m-k)$-uninorm from $\mathcal{U}_{m-k}$ and thus Lemma 4 and Lemma 5 imply the result.

• If $U^m(x_0, y_0) = x_0$, $U^m(z_k, y_0) = y_0$ and $U^m(x_0, y) = y$ for all $y > y_0$. Then U^m can be expressed as an ordinal sum of three semigroups $G_1 = ([0, x_0] \cup]y_0, 1], U^m)$, $G_2 = (]x_0, y_0[, U^m)$ and $G_3 = (\{y_0\}, \mathrm{Id})$, where G_1 is isomorphic to a uninorm with continuous underlying functions, G_2 is isomorphic to an m-uninorm from Class 1 restricted to $]0, 1[^2$ and $1 < 3 < 2$ (see [17, Theorem 5.12]). Thus U^m is a z-ordinal sum of five semigroups G_1, G_3, $G_4 = (]x_0, z_k], U^m)$, $G_5 = ([z_k, y_0[, U^m)$, $G_6 = (\{z_k\}, \mathrm{Id})$, where $A = \{6\}$ and the corresponding partial order is given by $4 \wedge 5 = 6$ and $1 \prec 3 \prec 6$. Since semigroups G_4, G_5 are isomorphic to (a restriction of) a k-uninorm from $\mathcal{U}_k$ and (a restriction of) an $(m-k)$-uninorm from $\mathcal{U}_{m-k}$, respectively, and G_1 is isomorphic to a uninorm with continuous underlying functions, Lemma 4 and Lemma 5 imply the result.

• If $U^m(x_0, y_0) = x_0$, $y_1 > y_0$ and $U^m(x_0, y_1) = x_0$, $U^m(z_k, y_0) = z_k$. Then U^m can be expressed as an ordinal sum of three semigroups $G_1 = ([0, x_0] \cup]y_1, 1], U^m)$, $G_2 = (]x_0, y_0], U^m)$ and $G_3 = (]y_0, y_1], U^m)$, where G_1 is isomorphic to a uninorm with continuous underlying functions, G_2 is isomorphic to an m-uninorm from Class 1 restricted to $]0, 1]^2$ and G_3 is isomorphic to a continuous t-conorm restricted to $]0, 1]^2$. For the corresponding linear order in the ordinal sum construction we have $1 < 3 < 2$ (see [17, Theorem 5.13]). Thus U^m can be expressed as a z-ordinal sum of five semigroups G_1, G_3, $G_4 = (]x_0, z_k], U^m)$, $G_5 = ([z_k, y_0], U^m)$, $G_6 = (\{z_k\}, \mathrm{Id})$, where $A = \{6\}$ and the corresponding partial order is given by $4 \wedge 5 = 6$ and $1 \prec 3 \prec 6$. Since semigroups G_4, G_5 are isomorphic to (restrictions of) a k-uninorm from $\mathcal{U}_k$ and an $(m-k)$-uninorm from $\mathcal{U}_{m-k}$, respectively, G_1 is isomorphic to a uninorm with continuous underlying functions, and the semigroup G_3 is isomorphic to (a restriction of) a continuous t-conorm, Lemma 4 and Lemma 5 imply the result.

• If $U^m(x_0, y_0) = x_0$, $y_1 > y_0$ and $U^m(x_0, y_1) = x_0$, $U^m(z_k, y_0) = y_0$. Then U^m can be expressed as an ordinal sum of three semigroups $G_1 = ([0, x_0] \cup]y_1, 1], U^m)$, $G_2 = (]x_0, y_0[, U^n)$ and $G_3 = ([y_0, y_1], U^m)$, where G_1 is isomorphic to a uninorm with continuous underlying functions, G_2 is isomorphic to an m-uninorm from Class 1 restricted to $]0, 1[^2$ and G_3 is isomorphic to a continuous t-conorm. For the corresponding linear order in the ordinal sum construction we have $1 < 3 < 2$ (see [17, Theorem 5.14]). Thus U^m can be expressed as a z-ordinal sum of five semigroups G_1, G_3, $G_4 = (]x_0, z_k], U^m)$, $G_5 = ([z_k, y_0[, U^m)$, $G_6 = (\{z_k\}, \mathrm{Id})$, where $A = \{6\}$ and the corresponding partial order is given by $4 \wedge 5 = 6$ and $1 \prec 3 \prec 6$. Since semigroups G_4, G_5 are isomorphic to (a restriction of) a k-uninorm from $\mathcal{U}_k$ and (a restriction of) an $(m-k)$-uninorm from $\mathcal{U}_{m-k}$, respectively, G_1 is isomorphic to a uninorm with continuous underlying functions, and the semigroup G_3 is isomorphic to a continuous t-conorm, Lemma 4 and Lemma 5 imply the result.

• If $U^m(x_0, y_0) = x_0$, $y_1 > y_0$ and $U^m(x_0, y_1) = y_1$, $U^m(z_k, y_0) = z_k$. Then similarly as in Lemma 7 we can show that $([0, 1] \setminus \{x_0\}, U^m)$ can be expressed as a z-ordinal sum of four semigroups $G_1 = ([0, x_0[\cup \{z_k\} \cup]y_0, 1], U^m)$, $G_2 = (]x_0, z_k], U^m)$, $G_3 = ([z_k, y_0], U^m)$, $G_4 = (\{z_k\}, \mathrm{Id})$, where $A = \{4\}$ and the corresponding partial order is given by $2 \wedge 3 = 4$ and $1 \prec 4$ and semigroups G_2, G_3 are isomorphic to (a restriction of) a k-uninorm from $\mathcal{U}_k$ and an $(m-k)$-uninorm from $\mathcal{U}_{m-k}$, respectively, and G_1 is isomorphic to a uninorm with continuous underlying functions. Lemma 4 and Lemma 5 then imply that $([0, 1] \setminus \{x_0\}, U^m)$ can be expressed as a z-ordinal sum of semigroups from $\mathcal{H}$, with respect to sets A, B and the partial order $\preceq$. Now we have to insert the trivial semigroup $G_0 = (\{x_0\}, \mathrm{Id})$ on the correct place. We will proceed as in Proposition 12. If for all $y \in]y_0, y_1[$ there $y \in X_\alpha$ implies $y_1 \notin X_\alpha$ then $([0, 1], U^m)$ is a z-ordinal sum with respect to $A' = A$ and $B' = B \cup \{0\}$, where 0 is below all semigroups containing points from $]y_0, y_1[$ and 0 is above all semigroups containing points from $[0, x_0[\cup [y_1, 1]$.

If there exists $\alpha_1 \in A_1$ such that $y_1 \in X_{\alpha_1}$ and $y \in X_{\alpha_1}$ for some $y \in]y_0, y_1[$ then H_{α_1} is an s-internal semigroup defined on an open interval $]a, b[$ and we assume two semigroups $G_{\alpha_2} = (]a, y_1[, \max)$ and $G_{\alpha_3} = ([y_1, b[, \max)$. Then U^m is a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$, where $A' = A$, $B' = (B \setminus \alpha_1) \cup \{0, \alpha_2, \alpha_3\}$ and $\alpha_3 \prec' 0 \prec' \alpha_2$, while α_2 and α_3 inherited the position with respect to all $\alpha \in C$ from the semigroup α_1.

• If $U^m(x_0, y_0) = x_0$, $y_1 > y_0$ and $U^m(x_0, y_1) = y_1$, $U^m(z_k, y_0) = y_0$. Then we can show that $([0, 1] \setminus \{x_0\}, U^m)$ can be expressed as a z-ordinal sum of five semigroups $G_1 = ([0, x_0[\cup \{z_k\} \cup]y_0, 1], U^m)$, $G_2 = (]x_0, z_k], U^m)$, $G_3 = ([z_k, y_0[, U^m)$, $G_4 = (\{z_k\}, \mathrm{Id})$ and $G_5 = (\{y_0\}, \mathrm{Id})$, where $A = \{4\}$ and the corresponding partial order is given by $2 \wedge 3 = 4$ and $1 \prec 5 \prec 4$ and semigroups G_2, G_3 are isomorphic to (a restriction of) a k-uninorm from $\mathcal{U}_k$ and (a restriction of) an $(m-k)$-uninorm from $\mathcal{U}_{m-k}$, respectively, and G_1 is isomorphic to a uninorm with continuous underlying functions. Lemma 4 and Lemma 5 then imply that $([0, 1] \setminus \{x_0\}, U^m)$ can be expressed as a z-ordinal sum of semigroups from $\mathcal{H}$, with respect to sets A, B and the partial order $\preceq$. The result can be then shown as in previous case, just assuming that 5 is an immediate predecessor of 4.

• If $U^m(x_0, y_0) = y_0$, $U^m(z_k, x_0) = z_k$ and $U^m(y_0, x) = x$ for all $x < x_0$. Then U^m is a z-ordinal sum of four semigroups $G_1 = ([0, x_0[\cup [y_0, 1], U^m)$, $G_2 = ([z_k, y_0[, U^m)$, $G_3 = ([x_0, z_k], U^m)$, $G_4 = (\{z_k\}, \mathrm{Id})$, where $A = \{4\}$ and the corresponding partial order is given by $2 \wedge 3 = 4$ and $1 \prec 4$. Here semigroup G_1 is isomorphic to a uninorm with continuous underlying functions, G_3 is isomorphic to a k-uninorm from $\mathcal{U}_k$ and G_2 is isomorphic to (a restriction of) an $(m-k)$-uninorm from $\mathcal{U}_{m-k}$. Then Lemma 4 and Lemma 5 imply the result.

• If $U^m(x_0, y_0) = y_0$, $U^m(z_k, x_0) = x_0$ and $U^m(y_0, x) = x$ for all $x < x_0$. Then U^m is a z-ordinal sum of five semigroups $G_1 = ([0, x_0[\cup [y_0, 1], U^m)$, $G_2 = (\{x_0\}, \mathrm{Id})$, $G_3 = ([z_k, y_0[, U^m)$, $G_4 = (]x_0, z_k], U^m)$, $G_5 = (\{z_k\}, \mathrm{Id})$, where $A = \{5\}$ and the corresponding partial order is given by $3 \wedge 4 = 5$ and $1 \prec 2 \prec 5$. Here semigroup G_1 is isomorphic to a uninorm with continuous underlying functions, G_4 is isomorphic to (a restriction of) a k-uninorm from $\mathcal{U}_k$ and G_3 is isomorphic to (a restriction of) an $(m-k)$-uninorm from $\mathcal{U}_{m-k}$. Then Lemma 4 and Lemma 5 imply the result.

- If $U^m(x_0, y_0) = y_0$, $x_1 < x_0$ and $U^m(y_0, x_1) = y_0$, $U^m(z_k, x_0) = z_k$. Then U^m is a z-ordinal sum of five semigroups $G_1 = ([0, x_1[\,\cup\,[y_0, 1], U^m)$, $G_2 = ([x_1, x_0[\,, U^m)$, $G_3 = ([z_k, y_0[\,, U^m)$, $G_4 = ([x_0, z_k], U^m)$, $G_5 = (\{z_k\}, \mathrm{Id})$, where $A = \{5\}$ and the corresponding partial order is given by $3 \wedge 4 = 5$ and $1 \prec 2 \prec 5$. Here semigroup G_1 is isomorphic to a uninorm with continuous underlying functions, G_4 is isomorphic to a k-uninorm from $\mathcal{U}_k$ and G_3 is isomorphic to (a restriction of) an $(m-k)$-uninorm from $\mathcal{U}_{m-k}$. Moreover, semigroup G_2 is isomorphic to (a restriction of) a continuous t-norm. Then Lemma 4 and Lemma 5 imply the result.
- If $U^m(x_0, y_0) = y_0$, $x_1 < x_0$ and $U^m(y_0, x_1) = y_0$, $U^m(z_k, x_0) = x_0$. Then U^m is a z-ordinal sum of five semigroups $G_1 = ([0, x_1[\,\cup\,[y_0, 1], U^m)$, $G_2 = ([x_1, x_0], U^m)$, $G_3 = ([z_k, y_0[\,, U^m)$, $G_4 = (]x_0, z_k], U^m)$, $G_5 = (\{z_k\}, \mathrm{Id})$, where $A = \{5\}$ and the corresponding partial order is given by $3 \wedge 4 = 5$ and $1 \prec 2 \prec 5$. Here semigroup G_1 is isomorphic to a uninorm with continuous underlying functions, G_4 is isomorphic to (a restriction of) a k-uninorm from $\mathcal{U}_k$ and G_3 is isomorphic to (a restriction of) an $(m-k)$-uninorm from $\mathcal{U}_{m-k}$. Moreover, semigroup G_2 is isomorphic to a continuous t-norm. Then Lemma 4 and Lemma 5 imply the result.
- If $U^m(x_0, y_0) = y_0$, $x_1 < x_0$ and $U^m(y_0, x_1) = x_1$, $U^m(z_k, x_0) = z_k$. Then U^m is a z-ordinal sum of four semigroups $G_1 = ([0, x_0[\,\cup\{z_k\}\cup[y_0, 1], U^m)$, $G_2 = ([z_k, y_0[\,, U^m)$, $G_3 = ([x_0, z_k], U^m)$, $G_4 = (\{z_k\}, \mathrm{Id})$, where $A = \{4\}$ and the corresponding partial order is given by $2 \wedge 3 = 4$ and $1 \prec 4$. Here semigroup G_3 is isomorphic to a k-uninorm from $\mathcal{U}_k$ and G_2 is isomorphic to (a restriction of) an $(m-k)$-uninorm from $\mathcal{U}_{m-k}$. The semigroup G_1 can be expressed as an ordinal sum of semigroups from $\mathcal{H}$ similarly as in Proposition 12. Then Lemma 4 and Lemma 5 imply the result.
- If $U^m(x_0, y_0) = y_0$, $x_1 < x_0$ and $U^m(y_0, x_1) = x_1$, $U^m(z_k, x_0) = x_0$. Then U^m is a z-ordinal sum of five semigroups $G_1 = ([0, x_0[\,\cup\{z_k\}\cup[y_0, 1], U^m)$, $G_2 = ([z_k, y_0[\,, U^m)$, $G_3 = (]x_0, z_k], U^m)$, $G_4 = (\{z_k\}, \mathrm{Id})$, $G_5 = (\{x_0\}, \mathrm{Id})$, where $A = \{4\}$ and the corresponding partial order is given by $2 \wedge 3 = 4$ and $1 \prec 5 \prec 4$. Here semigroup G_3 is isomorphic to (a restriction of) a k-uninorm from $\mathcal{U}_k$ and G_2 is isomorphic to (a restriction of) an $(m-k)$-uninorm from $\mathcal{U}_{m-k}$. The semigroup G_1 can be expressed as an ordinal sum of semigroups from $\mathcal{H}$ similarly as in previous case. Then Lemma 4 and Lemma 5 imply the result. $\square$

Summarizing, we see that in all cases an m-uninorm $U^m \in \mathcal{U}_m$ can be expressed as a z-ordinal sum of a countable number of semigroups from $\mathcal{H}$ with respect to the branching set $A \sim \{z_1, \ldots, z_{m-1}\}$. The corresponding partially ordered set C has a tree structure, where each node corresponds to a trivial semigroup $(\{z_i\}, \mathrm{Id})$ for $i \in \{1, \ldots, m-1\}$, while above each node there are two branches and at the end of each branch there is a semigroup containing the point e_i for $i \in \{1, \ldots, m\}$.

7. Conclusions

In this paper we have shown, in a constructive way, that each n-uninorm with continuous underlying functions can be expressed as a z-ordinal sum of Archimedean and idempotent semigroups. This result completely characterizes n-uninorms with continuous underlying functions. Note that the decomposition of an n-uninorm from $\mathcal{U}_n$ can be done also using the characterizing set-valued functions from [18], analogously as it was done for uninorms in [14]. Here the respective characterizing set-valued function pairs the t-norm and the t-conorm part of representable and d-internal semigroups. Further, the set of idempotent points of U^n together with horizontal, vertical and strictly decreasing segments of the corresponding characterizing set-valued function define the partition to the individual supports of the semigroups from $\mathcal{H}$.

In the future work we would like to study all binary functions on the unit square that can be obtained via a z-ordinal sum of semigroups from $\mathcal{H}$. Note that such a function need not be an n-uninorm. Further, we would like to study whether all commutative, asssociative binary functions on the unit square whose Archimedean components (see [10]) are semigroups from $\mathcal{H}$ (which means that their diagonal is a continuous function) can be expressed as a z-ordinal sum of these semigroups.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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